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Nodal Solutions for a Weighted (p,q)-Equation

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We consider a Dirichlet problem driven by a weighted (p,q)-Laplacian with a reaction that involves a critical term and a locally defined perturbation. Using variational tools and cut-off techniques, we show that the problem has a sequence of arbitrarily small nodal solutions.

Keywords: Weighted (p,q)-Laplacian, critical term, locally defined perturbation, nonlinear regularity, extremal constant sign solutions, nodal solutions, cut-off function.

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear Dirichlet problem:

$$\begin{cases} -\Delta_p^{a_1} u(z) - \Delta_q^{a_2} u(z) = f(z, u(z)) + |u(z)|^{p^*-2} u(z) \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad 1 < q < p. \end{cases} \tag{1}$$

Given $a \in C^{0,1}(\overline{\Omega})$ (that is, $a\colon \overline{\Omega} \to \mathbb{R}$ is Lipschitz continuous) and $1 < r < \infty$, by Δ_r^a we denote the weighted r-Laplace differential operator defined by

$$\Delta_r^a u = \operatorname{div}(a(z)|Du|^{r-2} Du) \quad \text{for all } u \in W_0^{1,r}(\Omega).$$

In problem (1) we have the sum of two such operators with different weights (a weighted (p,q)-Laplace differential operator). Such an operator is not homogeneous and this is a source of difficulties in the analysis of problem (1). Moreover, the fact that the weights a_1, a_2 are in general different, precludes the use of the nonlinear strong maximum principle of Pucci-Serrin [15, p.110]. In the reaction we have the effects of two terms. One is a critical term (recall p^* is the critical Sobolev exponent corresponding to p defined by $p^* = \frac{Np}{N-p}$ if $p < N$ and $+\infty$ if $p \geq N$) and the

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other is a Carathéodory perturbation (that is, $z \to f(z,x)$ is measurable, while $x \to f(z,x)$ is continuous). The distinguishing feature of this perturbation is that $f(z,\cdot)$ is defined only locally near 0, that is, all conditions concerning $f(z,\cdot)$, are imposed on an interval $[-\theta_0, \theta_0], \theta_0 > 0$. The presence of the critical term leads to an energy functional which dose not satisfy the necessary compactness condition to use the minimax results of critical point theory. Using cut-off techniques, we are able to overcome this difficultly, "neutralize" the critical term and produce nodal solutions.

In fact assuming a local symmetry condition on $f(z,\cdot)$, namely that $f(z,\cdot)|_{[-\theta_0,\theta_0]}$ is odd and using a generalized version of the symmetric mountain pass theorem due to Kajikiya [6], we produce a whole sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq C_0^1(\overline{\Omega})$ of distinct nodal solutions such that $u_n \to 0$ in $C_0^1(\overline{\Omega})$. By a "nodal solution", we mean a solution which changes sign, that is, it takes both positive and negative values in Ω. We point out that problem (1) is nonparametric. The presence of a parameter in the equation facilitates the analysis of the equation, since by varying the parameter, we can satisfy the geometric requirements of the theorems of critical point theory in order to use them. In (1) there is no parameter in the reaction.

The first author to consider equations with a locally defined reaction, is Wang [16]. In [16] the equation is driven by the Dirichlet p-Laplacian and has a reaction of the form $u \to f(z,u) + |u|^{q-2}u$ with $1 < q < p$ and $f \in C^1(\Omega \times (-\theta,\theta))$ $(\theta > 0)$, odd in the $x \in (-\theta,\theta)$ variable. Using cut-off techniques, Wang produced a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ of solutions of the problem such that $\|u_n\|_\infty \to 0$. However, these solutions are not nodal. Later Li-Wang [8], extended the work of [16] to semilinear Schrödinger equations and produced nodal solutions.

More recently there have been such results for more general parametric nonlinear elliptic equations. We refer to the works of Gasiński-Papageorgiou [3, 4] and Papageorgiou-Rădulescu-Repovš [11, 13]. In all these works, there is no critical term in the reaction.

The main result of this paper is the following theorem. For the hypotheses H_0, H_1 on the data of the problem we refer to Section 2.

Theorem. If the hypotheses H_0 and H_1 hold, then there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq C_0^1(\overline{\Omega})$ of nodal solutions of (1) such that $u_n \to 0$ in $C_0^1(\overline{\Omega})$.

As we already mentioned earlier, to prove this theorem we use a generalized version of the symmetric mountain pass theorem due to Kajikiya [6]. To do this we show that the problem admits extremal constant sign solutions, that is, there exists a smallest positive solution u_* and a biggest negative solution v_*. Then any nontrivial solution in the order interval distinct from u_* and v_* necessarily is nodal.

2. Mathematical background-hypotheses

The main spaces used in the study of the problem, are the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$. By $\|\cdot\|$ we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$. On account of the Poincaré inequality, we have

$$\|u\| = \|Du\|_p \text{ for all } u \in W_0^{1,p}(\Omega).$$

The space $C_0^1(\overline{\Omega})$ is an ordered Banach space with order (positive) cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \ for\ all\ z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$int\ C_+ = \{u \in C_+ : u(z) > 0 \ \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}|_{\partial\Omega} < 0\},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

By $\widehat{\lambda}_1(q)$ we denote the first eigenvalue of $(-\Delta_q^{a_2}, W_0^{1,q}(\Omega))$. We know that $\widehat{\lambda}_1(q) > 0$, is simple, isolated and all eigenfunctions corresponding to $\widehat{\lambda}_1(q)$ have fixed sign. By $\widehat{u}_1(q)$ we denote the positive, L^q-normalized (that is, $\|\widehat{u}_1(q)\|_q = 1$) eigenfunction corresponding to $\widehat{\lambda}_1(q)$. The nonlinear regularity theory and the nonlinear maximum principle imply that $\widehat{u}_1(q) \in int\ C_+$ (see[2]).

For $a \in C^{0,1}(\overline{\Omega})$, $a(z) > 0$ for all $z \in \overline{\Omega}, a \neq 0$ and $r \in (1, \infty)$, by

$$A_r^a : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega) = W_0^{1,r}(\Omega)^* \ (\tfrac{1}{r} + \tfrac{1}{r'} = 1)$$

we denote the nonlinear operator defined by

$$\langle A_r^a(u), h\rangle = \int_\Omega a(z)|Du|^{r-2}(Du, Dh)_{\mathbb{R}^N} dz \ \text{ for all } u, h \in W_0^{1,r}(\Omega).$$

This operator is continuous, strictly monotone (thus maximal monotone too) and of type $(S)_+$, that is

$$\text{If } u_n \xrightarrow{w} u \text{ in } W_0^{1,r}(\Omega) \text{ and } \limsup\langle A_r^\alpha(u_n), u_n - u\rangle \leq 0,$$
$$\text{then } u_n \to u \text{ in } W_0^{1,r}(\Omega).$$

If $u, v : \Omega \to \mathbb{R}$ are two measurable functions such that $u(z) \leq v(z)$ for a.a $z \in \Omega$, then we introduce the following order interval

$$[u, v] = \{h \in W_0^{1,p}(\Omega) : u(z) \leq h(z) \leq v(z) \ \text{ for a.a } z \in \Omega\}.$$

If $u \in W_0^{1,p}(\Omega)$, then $u^\pm = \max\{\pm u, 0\}$. We know that

$$u^\pm \in W_0^{1,p}(\Omega),\ u = u^+ - u^-,\ |u| = u^+ + u^-.$$

A set $S \subseteq W_0^{1,p}(\Omega)$ is said to be *downward directed* (*resp.upward directed*), if given $u_1, u_2 \in S$, we can find $u \in S$ such that $u \leq u_1, u \leq u_2$ (resp. if $v_1, v_2 \in S$, we can find $v \in S$ such that $v_1 \leq v, v_2 \leq v$).

The hypotheses on the data of (1) are the following:

H_0: $a_1, a_2 \in C^{0,1}(\overline{\Omega})$ and $a_1(z), a_2(z) \geq c_0 > 0$ for all $z \in \overline{\Omega}$.

H_1: $f : \Omega \times [-\theta_0, \theta_0]$ $(\theta_0 > 0)$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) there exists $a_0 \in L^\infty(\Omega)$ such that

$$|f(z, x)| \leq a_0(z) \text{ for a.a } z \in \Omega,\ \text{all } |x| \leq \theta_0$$

and for a.a. $z \in \Omega$, $f(z, \cdot)$ is odd;

(ii) $\lim_{x\to 0} \frac{f(z,x)}{|x|^{q-2}x} = +\infty$ uniformly for a. a. $z \in \Omega$ and there exists $\tau \in (1,q)$ such that $\lim_{x\to 0} \frac{f(z,x)}{|x|^{\tau-2}x} = 0$ uniformly for a.a. $z \in \Omega$.

On account of hypothesis H_1(ii), given $\eta > 0$, we can find $\delta = \delta(\eta) \in (0, \min\{\frac{\theta_0}{2}, 1\})$ such that

$$f(z,x)x \geq \eta |x|^q \quad \text{for a.a } z \in \Omega, \text{ all } |x| \leq \delta, \tag{2}$$

$$f(z,x)x \leq |x|^{\tau-1} \quad \text{for a.a } z \in \Omega, \text{ all } |x| \leq \delta. \tag{3}$$

Since $\tau < p^*$ and $\delta < 1$, we have also

$$|x|^{p^*} \leq |x|^\tau \quad \text{for all } |x| \leq \delta. \tag{4}$$

We consider an even cut-off function $\beta \in C^1_c(\mathbb{R})$ such that

$$supp\ \beta \subseteq [-\theta_0, \theta_0],\ \beta|_{[-\frac{\theta_0}{2}, \frac{\theta_0}{2}]} \equiv 1 \quad \text{and} \quad 0 < \beta \leq 1 \quad \text{on } (-\theta_0, \theta_0). \tag{5}$$

Using $\beta(\cdot)$ we introduce the function $g(z,x)$ defined by

$$g(z,x) = \beta(x)[f(z,x) + |x|^{p^*-2}x] + (1-\beta(x))|x|^{\tau-2}x \text{ for all } (z,x) \in \Omega \times \mathbb{R}. \tag{6}$$

This is a Carathéodory function, for a.a $z \in \Omega$, $g(x, \cdot)$ is odd and

$$|g(z,x)| \leq c_1[1 + |x|^{\tau-1}] \text{ for a.a } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_1 > 0 \tag{7}$$

(see (3),(4),(5),(6)). Then we consider the following nonlinear Dirichlet problem

$$-\Delta_p^{a_1} u(z) - \Delta_q^{a_2} u(z) = g(z, u(z)) \text{ in } \Omega, u|_{\partial\Omega} = 0. \tag{8}$$

The idea is to work on problem (8) and, using the result of Kajikiya [6], generate a sequence of solutions $\{u_n\}_{n\in\mathbb{N}}$ such that $u_n \to 0$ in $C^1_0(\overline{\Omega})$. To show that these solutions are nodal, we show first that (8) has extremal constant solutions, that is, $u_* \in int\ C_+$ a smallest positive solution and $v_* \in -int\ C_+$ a biggest negative solution. Then any nontrivial solution in the order interval $[v_*, u_*]$ distinct from u_* and v_* will be nodal. Moreover, on account of (5) and (6), we see that $\{u_n\}_{n\geq n_0}$ will be also nodal solutions of (1).

So, in the next section we examine problem (8) and produce extremal constant sign for this problem.

3. Constant sign solutions

In what follows by S_+ (resp S_-) we denote the set positive (resp. negative) solutions of (8).

First we prove a Lemma, which results from Proposition 2.3 of Papageorgiou-Vetro-Vetro [14]. We simply complete their proof, since now our hypothesis on $u(\cdot)$ is stronger.

Lemma 3.1. *If $\widehat{\xi} \in L^\infty(\Omega)$, $\widehat{\xi}(z) \geq 0$ for a.a. $z \in \Omega$ and $u \in C_+ \setminus \{0\}$ satisfies $-\Delta_p^{a_1} u - \Delta_q^{a_2} u + \widehat{\xi}(z) u^{p-1} \geq 0$ in Ω, then $u \in int\ C_+$.*

Proof. From [14] (Proposition 2.4), we know that $u(z) > 0$ for all $z \in \Omega$. Let $z_1 \in \partial\Omega$ and let $z_2 = z_1 - 2\rho n$ with $n = n(z_1)$ being the outward unit normal at $z_1 \in \partial\Omega$. As in the proof of Proposition 2.4 of [14], for $\rho > 0$ small, we consider the ring $R = \{z \in \Omega : \rho < |z - z_2| < 2\rho\}$ and let $y = \min\{u(z) : z \in \partial B_\rho(z_2)\} > 0$ small. From the proof in [14], we know that there exists a lower solution $y \in C^1(\overline{R}) \bigcap C^2(R)$ of the equation $-\Delta_p^{a_1} v - \Delta_q^{a_2} v + \widehat{\xi}(z)|v|^{p-2}v = 0$ such that

$$y \leq u, \quad y(z_1) = 0, \quad \tfrac{\partial u}{\partial n}(z_1) \leq \tfrac{\partial y}{\partial n}(z_1) < 0,$$

hence $u \in int\ C_+$. □

Proposition 3.2. *If hypotheses H_0, H_1 hold, then*

$$\phi \neq S_+ \subseteq int\ C_+ \text{ and } \phi \neq S_- \subseteq -int\ C_+.$$

Proof. Let $G(z, x) = \int_0^x g(z, s)ds$ and consider the C^1-functional $\varphi_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_+(u) = \frac{1}{p}\int_\Omega a_1(z)|Du|^p dz + \frac{1}{q}\int_\Omega a_2(z)|Du|^q dz - \int_\Omega G(z, u^+)dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

On account of hypotheses H_0 and (7) (recall that $\tau < q < p$), we see that the functional $\varphi_+(\cdot)$ is coercive. Also using the Sobolev embedding theorem, we see that $\varphi_+(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\varphi_+(u_0) = \min[\varphi_+(u) : u \in W_0^{1,p}(\Omega)]. \tag{9}$$

Since $\widehat{u}_1(q) \in int\ C_+$, we can choose $t \in (0, 1)$ small so that $0 \leq t\widehat{u}_1(q)(z) \leq \delta$ for all $z \in \overline{\Omega}$. Then from (2) and (5) (recall $\delta < \theta_0/2$), we have

$$\psi_+(t\widehat{u}_1(q)) \leq \frac{t^p}{p}\|a_1\|_\infty \|D\widehat{u}_1(q)\|_p^p + \frac{t^q}{q}[\widehat{\lambda}_1(q) - \eta] \quad \text{(recall that } \|\widehat{u}_1(q)\|_q = 1).$$

Choosing $\eta > \widehat{\lambda}_1(q)$, we obtain

$$\varphi_+(t\widehat{u}_1(q)) \leq c_2 t^p - c_3 t^q \quad \text{for some } c_2, c_3 > 0.$$

Since $q < p$, choosing $t \in (0, 1)$ even smaller if necessary, we have

$$\varphi_+(t\widehat{u}_1(q)) < 0 \ \Rightarrow\ \varphi_+(u_0) < 0 = \varphi_+(0) \text{ (see (9))} \ \Rightarrow\ u_0 \neq 0.$$

From (9) we have
$$\varphi_+'(u_0) = 0,$$

$$\Rightarrow\ \langle A_p^{a_1}(u_0), h\rangle + \langle A_q^{a_2}(u_0), h\rangle = \int_\Omega g(z, u_0^+)h dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{10}$$

In (10) we use the test function $h = -u_0^- \in W_0^{1,p}(\Omega)$ and obtain $c_0\|Du_0^-\|_p^p \leq 0$, (see hypotheses H_0 and note that $g(z, 0) = 0$ for all a.a $z \in \Omega$), hence $u_0 \geq 0$, $u_0 \neq 0$, and therefore $S_+ \neq \emptyset$.

From (10) we have

$$-\Delta_p^{a_1} u_0(z) - \Delta_q^{a_2} u(z) = g(z, u_0(z)) \text{ in } \Omega,\ u_0|_{\partial\Omega} = 0.$$

In consequence of Theorem 7.1, p. 286, of Ladyzhenskaya-Uraltseva [7] we have $u_0 \in L^\infty(\Omega)$. Then the nonlinear regularity theory of Lieberman [9] implies that $u_0 \in C_+ \setminus \{0\}$. From (7) we see that we can find $\widehat{\xi} > 0$ such that $g(z,x)x + \widehat{\xi}|x|^p \geq 0$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$. So, applying Lemma 3.1, we infer that $u_0 \in int\ C_+$. Therefore we conclude that

$$\phi \neq S_+ \subseteq int\ C_+.$$

In a similar way we show that $\phi \neq S_- \subseteq -int\ C_+$. This time we use the C^1-functional $\varphi_- : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined for all $u \in W_0^{1,p}(\Omega)$ by

$$\varphi_-(u) = \frac{1}{p}\int_\Omega a_1(z)|Du|^p dz + \frac{1}{q}\int_\Omega a_2(z)|Du|^q dz - \int_\Omega G(z, -u^-)dz. \qquad \square$$

We will show the existence of extremal constant sign solutions, that is, we will show that there exist $u_* \in S_+ \subseteq int\ C_+$ and $v_* \in S_- \subseteq -int\ C_+$ such that

$$u_* \leq u \text{ for all } u \in S_+,\ v \leq v_* \text{ for all } v \in S_-.$$

To produce these extremal solutions, we need to have a nontrivial lower bound for the set S_+ and a nontrivial upper bound for the set S_-.

From (5),(7) and hypothesis H_1(ii), we see that given $\eta > \widehat{\lambda}_1(q) > 0$ and $r \in (p, p^*)$, we can find $c_5 > 0$ such that

$$g(z,x)x \geq \eta|x|^\tau - c_5|x|^r \text{ for a.a } z \in \Omega,\ \text{all } x \in \mathbb{R}. \tag{11}$$

This unilateral growth condition on $g(z,\cdot)$, leads to the following auxiliary Dirichlet problem

$$\left\{ \begin{array}{l} -\Delta_p^{a_1} u(z) - \Delta_q^{a_2} u(z) = \eta|u(z)|^{q-2}u(z) - c_5|u(z)|^{r-2}u(z) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0,\ 1 < q < p. \end{array} \right\} \tag{12}$$

Proposition 3.3. *If hypotheses H_0 hold, then problem* (12) *has a unique positive solution $\overline{u} \in int\ C_+$ and since the equation is odd $\overline{v} = -\overline{u} \in -int\ C_+$ is the unique negative solution of* (12).

Proof. First we prove the existence of a positive solution for problem (12). To this end let $\psi_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined for all $u \in W_0^{1,p}(\Omega)$ by

$$\psi_+(u) = \frac{1}{p}\int_\Omega a_1(z)|Du|^p dz + \frac{1}{q}\int_\Omega a_2(z)|Du|^q dz + \frac{c_5}{r}\|u^+\|_r^r - \frac{\eta}{q}\|u^+\|_q^q.$$

Hypotheses H_0 and the fact that $q < p < r$, imply that $\psi_+(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $\overline{u} \in W_0^{1,p}(\Omega)$ such that

$$\psi_+(\overline{u}) = \min[\psi_+(u) : u \in W_0^{1,p}(\Omega)]. \tag{13}$$

As before (see the proof of Proposition 3.2), since $\eta > \widehat{\lambda}_1(q) > 0$ and $q < p < r$, choosing $t \in (0,1)$ small we have

$$\psi_+(t\widehat{u}_1(q)) < 0, \ \Rightarrow \ \psi_+(\overline{u}) < 0 = \psi_+(0) \quad \text{(see (13))} \ \Rightarrow \ \overline{u} \neq 0.$$

From (13) we have $$\psi'_+(\overline{u}) = 0,$$

$$\Rightarrow \langle A_p^{a_1}(\overline{u}), h\rangle + \langle A_q^{a_2}(\overline{u}), h\rangle = \int_\Omega \eta(\overline{u}^+)^{q-1} h dz - \int_\Omega c_5 (\overline{u}^+)^{r-1} h dz \ \text{ for all } h \in W_0^{1,p}(\Omega).$$

Choosing $h = -\overline{u}^- \in W_0^{1,p}(\Omega)$, we obtain

$$c_0\|D\overline{u}^-\|_p^p \le 0, \ \text{(see hypotheses } H_0), \quad \Rightarrow \ \overline{u} \ge 0, \ \overline{u} \neq 0,$$
$$\Rightarrow \ \overline{u} \in W_0^{1,p}(\Omega) \ \text{ is a positive solution of (12).}$$

As before (see the proof of Proposition 3.2), using the nonlinear regularity theory and Lemma 1, we have that $\overline{u} \in int\ C_+$.

Next we show the uniqueness of this positive solution of problem (12). To this end, let $j : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be the integral functional defined by

$$j(u) = \begin{cases} \frac{1}{p}\int_\Omega a_1(z)|Du^{1/q}|^p dz + \frac{1}{q}\int_\Omega a_2(z)|Du^{1/q}|^q dz & \text{if } u \ge 0, \ u^{1/q} \in W_0^{1,p}(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Consider the function $\widehat{G}_0(z,t) = \frac{a_1(z)}{p}t^p + \frac{a_2(z)}{q}t^q$ for all $t \ge 0$. Evidently for all $z \in \Omega$, $\widehat{G}_0(z,\cdot)$ is strictly increasing and strictly convex. We set $\widehat{G}(z,y) = \widehat{G}_0(z,|y|)$ for all $(z,y) \in \Omega \times \mathbb{R}^N$. Also, let $dom\ j = \{u \in L^1(\Omega) : j(u) < +\infty\}$ (effective domain $j(\cdot)$). From Diaz-Saa [1] (Lemma 1), we know that $j(\cdot)$ is convex.

Suppose $\overline{y} \in W_0^{1,p}(\Omega)$ is another positive solution of problem (12). Again we have $\overline{y} \in int\ C_+$. On account of Proposition 4.1.22, p. 234, of [12], we have

$$\frac{\overline{u}}{\overline{y}} \in L^\infty(\Omega), \quad \frac{\overline{y}}{\overline{u}} \in L^\infty(\Omega).$$

So, if $h = \overline{u}^q - \overline{y}^q \in W_0^{1,p}(\Omega)$, then for $|t| < 1$ small we have $\overline{u}^q + th \in dom\ j$ and $\overline{v}^q + th \in dom\ j$.

Then the convexity of $j(\cdot)$ implies the Gateaux differentiability of $j(\cdot)$ at $\overline{u}^q$ and $\overline{v}^q$ in the direction h. From the chain rule and the nonlinear Green's identity (see [12], p.35), we have

$$j'(\overline{u}^q)(h) = \frac{1}{q}\int_\Omega \frac{-\Delta_p^{a_1}\overline{u} - \Delta_q^{a_2}\overline{u}}{\overline{u}^{q-1}}(\overline{u}^q - \overline{y}^q)dz = \frac{1}{q}\int_\Omega [\eta - c_5\overline{u}^{r-q}](\overline{u}^q - \overline{v}^q)dz,$$

$$j'(\overline{y}^q)(h) = \frac{1}{q}\int_\Omega \frac{-\Delta_p^{a_1}\overline{y} - \Delta_q^{a_2}\overline{y}}{\overline{y}^{q-1}}(\overline{u}^q - \overline{y}^q)dz = \frac{1}{q}\int_\Omega [\eta - c_5\overline{y}^{r-q}](\overline{u}^q - \overline{y}^q)dz,$$

The convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$. Hence

$$0 \le \int_\Omega c_5[\overline{y}^{r-q} - \overline{u}^{r-q}](\overline{u}^q - \overline{y}^q)dz \le 0 \quad \Rightarrow \quad \overline{u} = \overline{y}.$$

We conclude that $\overline{u} \in int\ C_+$ is the unique positive solution of problem (12).

Finally since the equation is odd, $\overline{v} = -\overline{u} \in -int\ C_+$ is the unique negative solution of (12). $\square$

These solutions will be the bounds for S_+ and S_- respectively.

Proposition 3.4. *If the hypotheses H_0, H_1 hold, then $\overline{u} \leq u$ for all $u \in S_+$ and $v \leq \overline{v}$ for all $v \in S_-$.*

Proof. Let $u \in S_+$ and consider the Carathéodory function $k_+(z,x)$ defined by

$$k_+(z,x) = \begin{cases} \eta(x^+)^{q-1} - c_5(x^+)^{r-1} & \text{if } x \leq u(z) \\ \eta u(z)^{q-1} - c_5 u(z)^{r-1} & \text{if } u(z) < x. \end{cases} \tag{14}$$

Let $K_+(z,x) = \int_0^x k_+(z,s)ds$ and consider the C^1-functional $\sigma_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined for all $u \in W_0^{1,p}(\Omega)$ by

$$\sigma_+(u) = \frac{1}{p}\int_\Omega a_1(z)|Du|^p dz + \frac{1}{q}\int_\Omega a_2(z)|Du|^q dz - \int_\Omega K_+(z,u)dz.$$

From hypothesis H_0 and (14) it follows that $\sigma_+(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u} \in W_0^{1,p}(\Omega)$ such that

$$\sigma_+(\widetilde{u}) = \min[\sigma_+(u) : u \in W_0^{1,p}(\Omega)].$$

Let $t \in (0,1)$ be small so that $t\widehat{u}_1(q) \leq u$ (this is possible since $u \in int\ C_+$). Then since $\eta > \widehat{\lambda}_1(q)$ and $q < p < r$, taking $t \in (0,1)$ even smaller if necessary, we have $\sigma_+(t\widehat{u}_1(q)) < 0$ and so

$$\sigma_+(\widetilde{u}) < 0 = \sigma_+(0), \quad \Rightarrow \quad \widetilde{u} \neq 0.$$

We have
$$\sigma'_+(\widetilde{u}) = 0,$$

$$\Rightarrow \langle A_p^{a_1}(\widetilde{u}), h\rangle + \langle A_q^{a_2}(\widetilde{u}), h\rangle = \int_\Omega k_+(z,\widetilde{u})h dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{15}$$

In (15) we choose $h = -\widetilde{u}^- \in W_0^{1,p}(\Omega)$. Then

$$c_0\|D\widetilde{u}^-\|_p^p \leq 0 \quad \Rightarrow \quad \widetilde{u} \geq 0, \ \ \widetilde{u} \neq 0.$$

Next in (15) we choose $h = [\widetilde{u} - u]^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} &\langle A_p^{a_1}(\widetilde{u}), (\widetilde{u}-u)^+\rangle + \langle A_q^{a_2}(\widetilde{u}), (\widetilde{u}-u)^+\rangle \\ &= \int_\Omega [\eta u^{p-1} - c_5 u^{r-1}](\widetilde{u}-u)^+ dz \quad \text{(see (14))} \\ &\leq \int_\Omega g(z,u)(\widetilde{u}-u)^+ dz \quad \text{(see (11))} \\ &= \langle A_p^{a_1}(u), (\widetilde{u}-u)^+\rangle + \langle A_q^{a_2}(u), (\widetilde{u}-u)^+\rangle \quad (\text{since } u \in S_+), \\ &\Rightarrow \ \widetilde{u} \leq u. \end{aligned}$$

So, we have proved that

$$\widetilde{u} \in [0, u], \quad \widetilde{u} \neq 0. \tag{16}$$

From (16),(14),(15) it follows that $\widetilde{u}$ is a positive solution cf(12). So, by Proposition 3.3 we have

$$\widetilde{u} = \overline{u} \in int\ C_+, \quad \Rightarrow \quad \overline{u} \leq u \quad \text{for all } u \in S_+.$$

Similarly we show that $v \leq \overline{v}$ for all $v \in S_-$. □

These bounds allow us to produce extremal constant sign solutions of problem (8).

Proposition 3.5. *If the hypotheses H_0, H_1 hold, then there exist $u_* \in S_+$ and $v_* \in S_-$ such that $u_* \leq u$ for all $u \in S_+$ and $v \leq v_*$ for all $v \in S_-$.*

Proof. From Papageorgiou-Rădulescu-Repovš [10] (see the proof of Proposition 7) we have that S_+ is downward directed. Then Lemma 3.10, p. 178, of Hu-Papageorgiou [5], implies that we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq S_+$ decreasing such that

$$\inf S_+ = \inf_{n\in\mathbb{N}} u_n.$$

We have

$$\langle A_p^{a_1}(u_n), h\rangle + \langle A_q^{a_2}(u_n), h\rangle = \int_\Omega g(z, u_n)hdz \text{ for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N}, \tag{17}$$

$$\overline{u} \leq u_n \leq u_1 \quad \text{ for all } n \in \mathbb{N}. \tag{18}$$

In (17) we choose $h = u_n \in W_0^{1,p}(\Omega)$ and use (7), (18) and hypothesis H_0. We infer that $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u_* \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u_* \text{ in } L^p(\Omega) \text{ as } n \to \infty. \tag{19}$$

In (17) we choose $h = u_n - u_* \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ as use (19). We obtain

$$\begin{aligned}
&\lim_{n\to\infty} [\langle A_p^{a_1}(u_n), u_n - u_*\rangle + \langle A_q^{a_2}(u_n), u_n - u_*\rangle] = 0, \\
\Rightarrow\ &\limsup_{n\to\infty} [\langle A_p^{a_1}(u_n), u_n - u_*\rangle + \langle A_q^{a_2}(u_*), u_n - u_*\rangle] \leq 0 \ \ (\text{since } A_q^{a_2}(\cdot) \text{ is monotone}), \\
\Rightarrow\ &\limsup_{n\to\infty} \langle A_p^{a_1}(u_n), u_n - u_*\rangle \leq 0, \\
\Rightarrow\ &u_n \to u_* \text{ in } W_0^{1,p}(\Omega).
\end{aligned} \tag{20}$$

So, if in (17) we pass to the limit as $n \to \infty$ and use (20), then

$$\langle A_p^{a_1}(u_*), h\rangle + \langle A_q^{a_2}(u_*), h\rangle = \int_\Omega g(z, u_*)hdz \text{ for all } h \in W_0^{1,p}(\Omega), \tag{21}$$

$$\overline{u} \leq u_* \quad (\text{ see } (18)). \tag{22}$$

From (21) and (22) it follows that $u_* \in S_+$, $u_* = \inf S_+$.

Similarly we show that there exists $v_* \in S_-$, $v_* = \sup S_-$. Note that S_- is upward directed. □

4. Nodal solutions

We consider the energy functional $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ of problem (8) defined by

$$\varphi(u) = \frac{1}{p}\int_\Omega a_1(z)|Du|^p dz + \frac{1}{q}\int_\Omega a_2(z)|Du|^q dz - \int_\Omega G(z,u)dz \text{ for all } u \in W_0^{1,p}(\Omega)$$

where $G(z,x) = \int_0^x g(z,s)ds$ we know that $\varphi \in C^1(W_0^{1,p}(\Omega))$.

Proposition 4.1. *If hypotheses H_0, H_1 hold, and $V \subseteq W_0^{1,p}(\Omega)$ is a finite dimensional subspace, then there exists $\rho_V > 0$ such that*

$$\sup[\varphi(u) : u \in V, \|u\| = \rho_V] < 0.$$

Proof. Let $\delta > 0$ be as in (2),(3),(4). Since V is finite dimensional, all norms are equivalent. So, we can find $\rho_V \in (0,1)$ such that

$$w \in V,\ \|w\| \le \rho_V \Rightarrow |w(z)| \le \delta \text{ for a.a.} z \in \Omega. \tag{23}$$

Recall that $\delta < \frac{\eta_0}{2}$. So, from (5) it follows that $\beta(w(z)) = 1$ for a.a. $z \in \Omega$. Hence from (6) and (2), for all $w \in V$, with $\|w\| \le \rho_V$, we have

$$\begin{aligned}\varphi(w) &\le \frac{1}{p}\int_\Omega a_1(z)|Dw|^p dz + \frac{1}{q}[\int_\Omega a_2(z)|Dw|^q dz - \eta\|w\|_q^q] \\ &\le c_6\|w\|^p - c_7\|w\|^q\end{aligned}$$

(since all norms on V are equivalent and choosing $\eta > 0$ big).

But $q < p$. So, choosing $\rho_V \in (0,1)$ even smaller if necessary, we conclude that

$$\sup[\varphi(w) : w \in V,\ \|w\| = \rho_V] < 0. \qquad \square$$

Now we are ready to produce a whole sequence of arbitrarily small nodal solutions.

Theorem 4.2. *If the hypotheses H_0, H_1 hold, then problem* (1) *has a whole sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq C_0^1(\overline{\Omega})$ of nodal solutions such that $u_n \to 0$ in $C_0^1(\overline{\Omega})$.*

Proof. For hypotheses H_1(i), (6) and (8), we see that $\varphi(\cdot)$ is even and coercive. Therefore $\varphi(\cdot)$ is bounded below and satisfies the Palais-Smale condition (see [12], Proposition 5.1.15, p. 369). These facts and Proposition 4.1, permit the use of Theorem 1 of Kajikiya [6]. So, we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ such that

$$u_n \in K_\varphi \text{ for all } n \in \mathbb{N},\ \|u_n\| \to 0. \tag{24}$$

Then the nonlinear regularity theory implies that there exist $\alpha \in (0,1)$ and $c_8 > 0$ such that

$$u_n \in C_0^{1,\alpha}(\overline{\Omega}) = C^{1,\alpha}(\overline{\Omega}) \cap C_0^1(\overline{\Omega}),\ \|u_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \le c_8 \text{ for all } n \in \mathbb{N}.$$

The compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ and (24) imply that

$$u_n \to 0 \text{ in } C_0^1(\overline{\Omega}), \quad \Rightarrow \quad u_n \in [v_*, u_*] \text{ for all } n \ge n_0.$$

The extremality of u_* and v_* implies that $\{u_n\}_{n\geq n_0}$ are nodal.

Taking $n_0 \in \mathbb{N}$ even bigger if necessary, we can also have

$$|u_n(z)| \leq \theta_0/2 \ \text{ for all } z \in \overline{\Omega}, \text{ all } n \geq n_0. \tag{25}$$

Then (5),(6) and (25) imply that

$$\{u_n\}_{n\geq n_0} \text{ are nodal solutions of (1).} \qquad \square$$

Remark 4.3. If for some $\delta_0 \in (0, \theta_0)$ we have $pF(z,x) \leq f(z,x)x$ for a.a $z \in \Omega$, all $|x| \leq \delta_0$, then we can also have that the nodal solutions $\{u_n\}_{n\in\mathbb{N}}$ have negative energy, that is $\varphi(u_n) < 0$ (see [6]).

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