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Epsilon Subdifferential Method and Integrability

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We develop a novel variant of the epsilon subdifferential method and use it to give a new proof of Moreau-Rockafellar theorem that a proper lower semicontinuous convex function on Banach space is determined up to a constant by its subdifferential.

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1. Introduction

The Epsilon Subdifferential Method is well known and widely used for minimizing convex functions, see e.g. [2, 3]. In this note we develop a novel Epsilon Subdifferential Method (ESM). Let us outline it here.

The ESM applies to a given proper convex lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$, defined on a Banach space X, such that $0 = f(0) = \min_{x \in X} f(x)$ with fixed in advance parameters $\varepsilon > 0$ and $\delta \in (0, \varepsilon)$.

Starting at arbitrary $x_0 \in \operatorname{dom} f$, for $i = 0, 1, \dots$

- if $0 \in \partial_\varepsilon f(x_i)$, then STOP;
- if $0 \notin \partial_\varepsilon f(x_i)$, for $\varphi_{x_i}(K) := \inf_{x \in X} F_{x_i}(K, x)$, where

$$F_{x_i}(K, x) := f(x) - f(x_i) + \varepsilon + K\|x - x_i\|,$$

  find $K_i > 0$ such that $\varphi_{x_i}(K_i) = 0$. Take any x_{i+1} satisfying

$$0 \le f(x_{i+1}) - f(x_i) + \varepsilon + K_i\|x_{i+1} - x_i\| \le \delta.$$

In the finite dimensional case $\delta = 0$ works, and ESM is much more simple.

The immediate estimate for the number of iterations n is $n \le \operatorname{const} \varepsilon^{-1}$. But when f satisfies $f(x) \ge c\|x\|$ for all $x \in X$ and some $c > 0$, the parameter δ is appropriately chosen, and the starting point $x_0 \in \operatorname{dom} \partial f$, then the number of iterations n of ESM has the more precise estimate $n \le \operatorname{const} \varepsilon^{-\frac{1}{2}}$, see Lemma 3.3. The proof relies on Lemma 2.3. Note that in this case, $n\varepsilon \le \operatorname{const} \varepsilon^{\frac{1}{2}}$ which yields that $n\varepsilon$ tends to 0 as ε tends to 0. This is the key argument in the presented here new proof of the famous Moreau-Rockafellar Theorem, see e.g. [10, 11]:

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Theorem 1.1. *Let X be a Banach space. Let g and h be proper lower semicontinuous convex functions from X to $\mathbb{R} \cup \{+\infty\}$. If*

$$\partial g \subset \partial h, \tag{1}$$

then $h = g + \text{const}$.

This result has numerous important implications, see e.g. Section 3 of Phelps [9].

Let us make a short historical overview. The integrability of the subdifferential of proper lower semicontinuous convex function on Hilbert space is stated and proved first by Moreau in [7] by using Moreau-Yosida regularisation. The proof also works in reflexive Banach space as mentioned at p. 87 of [8]. The first complete proof in Banach space – that of Rockafellar in [11] – uses strong duality arguments. Another approach is to approximate the directional derivative and to reduce to the one-dimensional case. The latter was taken by Rockafellar in his original proof in [10]. Though there are some gaps in this proof, Taylor [12] fills them and provides a different proof, cf. [4]. The idea of directional derivative approximation/one dimensional reduction is most clearly outlined in the proof of Thibault [13]. A different proof using the mean-value theorem of Zagrodny is due to Thibault and Zagrodny [14], see also [15]. In [16] the result is proved by using regularization (and approximation) techniques which was the initial idea of Moreau.

In [6] Ivanov and Zlateva give a proof similar to the proof of the classical calculus theorem that a monotone function is Riemann integrable which uses neither duality nor explicit one-dimensional arguments. The main step in their proof is to show directly that a proper lower semicontinuous convex function on Banach space differs by a constant from the *Rockafellar function* (see [1]) of its subdifferential, see [6, Theorem 1.2]. The proof relies on a technical lemma [6, Lemma 3.3] proved by Ekeland variational principle.

Here we use the novel Epsilon Subdifferential Method (ESM) to prove in a different way to the one used in [6] the following

Theorem 1.2. (Rockafellar [10, 11]) *Let $g\colon X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Let $\bar{x} \in \operatorname{dom} \partial g$ and $\bar{p} \in \partial g(\bar{x})$. Then for all $x \in X$ we have*

$$g(x) = g(\bar{x}) + R_{\partial g,(\bar{x},\bar{p})}(x),$$

$$\textit{where} \qquad R_{\partial g,(\bar{x},\bar{p})}(x) := \sup \left\{ \sum_{i=0}^{n-1} \langle q_{i+1}, x_i - x_{i+1} \rangle : \begin{array}{l} x_0 = x,\ x_n = \bar{x},\ q_n = \bar{p}, \\ q_i \in \partial g(x_i),\ n \in \mathbb{N} \end{array} \right\}. \tag{2}$$

A distinctive feature of the new proof here is that it reveals the relationship between a natural optimization method and Moreau-Rockafellar Theorem. By use of the ESM the sequences realizing supremum in (2) are kind of constructed. Thereafter, the proof of Theorem 1.1 continues exactly as in [6]. That is why, we only sketch it here: it readily follows that (1) implies

$$g(x) - g(\overline{x}) \leq h(x) - h(\overline{x})$$

for any $\overline{x} \in \operatorname{dom} \partial g$ and all $x \in X$.

In particular, $g - h \equiv$ const on $\operatorname{dom} \partial g$. To conclude, we use lower semicontinuity of h and graphical density of points of subdifferentiability to g, i.e. that for any $\bar{x} \in \operatorname{dom} g$ and any $\varepsilon > 0$ there exists $x \in \operatorname{dom} \partial g$ such that $\|x - \bar{x}\| + |g(x) - g(\bar{x})| < \varepsilon$, see [5] and [4].

Let us also note that tools used in the proof had been known by 1970.

The rest of the paper is organized as follows. After a short Section 2 on notations and preliminaries, in Section 3 we dwell on some of the basic properties of the novel Epsilon Subdifferential Method (ESM). In the last Section 4 we give the proof of Theorem 1.2.

2. Preliminaries and notations

The notation used throughout the paper is standard. Usually $(X, \|\cdot\|)$ denotes a real Banach space, that is, complete normed space over $\mathbb{R}$. The dual space X^* of X is the Banach space of all continuous linear functionals p from X to $\mathbb{R}$. The natural norm of X^* is again denoted by $\|\cdot\|$. The value of $p \in X^*$ at $x \in X$ is denoted by $\langle p, x\rangle$.

The *effective domain* $\operatorname{dom} f$ of an extended real-valued function $f : X \to \mathbb{R} \cup \{+\infty\}$ is the set of points x where $f(x) \in \mathbb{R}$. The function f is *proper* if $\operatorname{dom} f \neq \varnothing$. It is *lower semicontinuous* if $f(\bar{x}) \le \liminf_{x \to \bar{x}} f(x)$ for all $\bar{x} \in X$.

Let us recall that for $\varepsilon \ge 0$, the ε-subdifferential of a proper, convex and lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ at $x \in \operatorname{dom} f$ is the set

$$\partial_\varepsilon f(x) = \{p \in X^* \,:\, -\varepsilon + \langle p, y - x\rangle \le f(y) - f(x), \quad \forall y \in X\},$$

and $\partial_\varepsilon f = \varnothing$ on $X \setminus \operatorname{dom} f$. Of course, for $\varepsilon = 0$, $\partial_0 f(x)$ coincides with the subdifferential of f at x in the sense of convex analysis $\partial f(x)$. The *domain* $\operatorname{dom} \partial_\varepsilon f$ consists of all points $x \in X$ such that $\partial_\varepsilon f(x)$ is non-empty. But while $\partial f(x)$ could be empty, for $\varepsilon > 0$, the sets $\partial_\varepsilon f(x)$ are non-empty for any $x \in \operatorname{dom} f$. For any real numbers ε_1 and ε_2 such that $0 < \varepsilon_1 \le \varepsilon_2$ one has $\partial_{\varepsilon_1} f(x) \subset \partial_{\varepsilon_2} f(x)$ and

$$\partial f(x) = \bigcap_{\varepsilon > 0} \partial_\varepsilon f(x).$$

Moreover, if $f, g : X \to \mathbb{R} \cup \{+\infty\}$ are two proper lower semicontinuous convex functions with $x \in \operatorname{dom} f \cap \operatorname{dom} g$ and one of them is continuous at x, then the following sum rule holds, see e.g. [15, Theorem 2.8.7],

$$\partial_\varepsilon (f + g)(x) = \bigcup \{\partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} g(x) : \varepsilon_1 \ge 0,\ \varepsilon_2 \ge 0\ \varepsilon = \varepsilon_1 + \varepsilon_2\}.$$

We will use it further in its weaker form

$$\partial_\varepsilon (f + g)(x) \subset \partial_\varepsilon f(x) + \partial_\varepsilon g(x).$$

The Brøndsted-Rockafellar Theorem, saying that the graph of $\partial_\varepsilon f$ is close to the graph of ∂f, is well known:

Theorem 2.1. (Brøndsted-Rockafellar [5]) *Let $f : X \to \mathbb{R} \cup \{+\infty\}$, be a proper, convex and lower semicontinuous function, let $\varepsilon > 0$ and $p \in \partial_\varepsilon f(x)$. Then there exists $q \in \partial f(z)$ such that*

$$\|z - x\| \le \sqrt{\varepsilon}, \quad \textit{and} \quad \|q - p\| \le \sqrt{\varepsilon}.$$

Another result of Brøndsted and Rockafellar [5] will also be used:

Proposition 2.2. *Let f be a proper lower semicontinuous convex function from a Banach space X into $\mathbb{R} \cup \{+\infty\}$. Then for all $x \in X$*

$$f(x) = \sup\{f(\bar{x}) + \bar{p}(x - \bar{x});\ (\bar{x}, \bar{p}) \in \operatorname{gph} \partial f\}. \tag{3}$$

To estimate the number of iteration of the novel ESM we prove the following lemma of its own interest.

Lemma 2.3. *Let $A > 0$, $B > 0$ and $\varepsilon > 0$ be real numbers. If there exist real numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$ satisfying the conditions*

$$a_i > 0,\ b_i > 0, \qquad i = 1, \dots, n, \tag{4}$$

$$a_i b_i \geq \varepsilon, \qquad i = 1, \dots, n, \tag{5}$$

$$\sum_{i=1}^{n} a_i \leq A, \tag{6}$$

$$\sum_{i=1}^{n} b_i \leq B, \tag{7}$$

then the inequality $n \leq \sqrt{\dfrac{AB}{\varepsilon}}$ *holds.*

Proof. From (4) and (5) follow the inequalities

$$b_i \geq \frac{\varepsilon}{a_i}, \quad i = 1, \dots, n. \tag{8}$$

Summing the inequalities (8) for $i = 1, \dots, n$ we get that by (7)

$$\sum_{i=1}^{n} \frac{\varepsilon}{a_i} \leq \sum_{i=1}^{n} b_i \leq B.$$

Hence, $\displaystyle\sum_{i=1}^{n} \frac{1}{a_i} \leq \frac{B}{\varepsilon}$ and, equivalently, $\displaystyle\frac{\varepsilon}{B} \leq \Big(\sum_{i=1}^{n} \frac{1}{a_i}\Big)^{-1}$. Multiplying by n we get

$$\frac{n\varepsilon}{B} \leq n\Big(\sum_{i=1}^{n} \frac{1}{a_i}\Big)^{-1}. \tag{9}$$

From (6) it follows that

$$\frac{\sum_{i=1}^{n} a_i}{n} \leq \frac{A}{n}. \tag{10}$$

By (9), Cauchy inequality and (10) we get the following chain of inequalities

$$\frac{n\varepsilon}{B} \leq \frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}} \leq \frac{\sum_{i=1}^{n} a_i}{n} \leq \frac{A}{n}$$

which yields that $n^2 \leq \dfrac{AB}{\varepsilon}$. Therefore, $n \leq \sqrt{\dfrac{AB}{\varepsilon}}$. □

3. Properties of the novel ESM

We have outlined the method in the introduction. In this section we will consider some of its properties. Throughout this section we work with a proper lower semi-continuous convex function $f : X \to \mathbb{R} \cup \{+\infty\}$, such that $\min_{x\in X} f(x) = f(0) = 0$, and fixed $\varepsilon > 0$, and $\varepsilon > \delta > 0$. Next lemma describes what happens at one of ESM iterations.

Lemma 3.1. *Let $x_0 \in \operatorname{dom} f$. The function $\varphi_{x_0} : \mathbb{R} \to \mathbb{R}$ defined by*

$$\varphi_{x_0}(K) := \inf_{x\in X} F_{x_0}(K, x),$$

where
$$F_{x_0}(K, x) := f(x) - f(x_0) + \varepsilon + K\|x - x_0\|,$$

is strictly monotone increasing and locally Lipschitz on $(0, \infty)$.
Assume in addition that $0 \notin \partial_\varepsilon f(x_0)$. Then

(i) *there exists $K_0 > 0$ such that $\varphi_{x_0}(K_0) = 0$;*

(ii) *for any $x_1 \in X$ such that*
$$0 \le f(x_1) - f(x_0) + \varepsilon + K_0\|x_1 - x_0\| \le \delta, \tag{11}$$
there is $p_1 \in \partial_\delta f(x_1)$ such that
$$K_0 \ge \|p_1\| - \delta, \tag{12}$$
and,
$$\langle p_1, x_1 - x_0\rangle \le f(x_1) - f(x_0) + \varepsilon + \delta. \tag{13}$$

Moreover,
$$K_0 \le \min\{\|p\| : p \in \partial_\varepsilon f(x_0)\}, \tag{14}$$
and if $p_0 \in \partial_\delta f(x_0)$, then
$$\varepsilon \le (\|p_0\| - \|p_1\|)\|x_1 - x_0\| + \delta\left(2 + \frac{f(x_0)}{K_0}\right). \tag{15}$$

Proof. It is straightforward to prove the strict monotonicity and the local Lipschitz continuity of φ_{x_0} on $(0, \infty)$.

The existence of $K' > 0$ such that $\varphi_{x_0}(K') < 0$ follows by $0 \notin \partial_\varepsilon f(x_0)$, while the existence of $K'' > 0$ such that $\varphi_{x_0}(K'') > 0$ is a consequence of lower semicontinuity of f at x_0. Bolzano Theorem ensures the existence of $K_0 > 0$ with $\varphi_{x_0}(K_0) = 0$ which thanks to the strict monotonicity of φ_{x_0} on $(0, \infty)$ have to be unique.

Now, let $x_1 \in X$ be a point of δ-infimum of the function $F_{x_0}(K_0, \cdot)$, i.e. satisfying (11). Equivalently, $0 \in \partial_\delta \left(f(\cdot) - f(x_0) + \varepsilon + K_0\|\cdot - x_0\|\right)(x_1)$. By the weaker form of the sum rule for the δ-subdifferential, we have that there exist $p_1 \in \partial_\delta f(x_1)$, and $\xi_1 \in \partial_\delta K_0\|\cdot - x_0\|(x_1)$ such that $0 = p_1 + \xi_1$. Since $\xi_1 \in \partial_\delta K_0\|\cdot - x_0\|(x_1)$,

$$\langle \xi_1, x - x_1\rangle \le K_0\|x - x_0\| - K_0\|x_1 - x_0\| + \delta \le K_0\|x - x_1\| + \delta,\ \forall x \in X. \tag{16}$$

Hence, $|\langle \xi_1, x - x_1\rangle| \le K_0\|x - x_1\| + \delta$ for all $x \in X$, and (12) holds. From (16) it easily follows that

$$K_0\|x_1 - x_0\| \le \langle \xi_1, x_1 - x_0\rangle + \delta = \langle p_1, x_0 - x_1\rangle + \delta$$

which combined with the left inequality in (11) yields (13).

Take an arbitrary $p \in \partial_\varepsilon f(x_0)$. By definition of the ε-subdifferential,

$$\langle p, x - x_0 \rangle \le f(x) - f(x_0) + \varepsilon, \ \forall x \in X.$$

Hence,

$$\langle p, x - x_0 \rangle + K_0\|x - x_0\| \le f(x) - f(x_0) + \varepsilon + K_0\|x - x_0\|, \ \forall x \in X,$$

$$\langle p, \frac{x - x_0}{\|x - x_0\|}\rangle + K_0 \le \frac{f(x) - f(x_0) + \varepsilon + K_0\|x - x_0\|}{\|x - x_0\|}, \ \forall x \in X, \ x \ne x_0,$$

$$K_0 + \inf_{x\in X, x\ne x_0} \langle p, \frac{x - x_0}{\|x - x_0\|}\rangle \le \inf_{x\in X, x\ne x_0} \left(\frac{f(x) - f(x_0) + \varepsilon + K_0\|x - x_0\|}{\|x - x_0\|}\right) = 0.$$

Finally, $K_0 \le \|p\|$, and (14) holds.

Take any $p_0 \in \partial_\delta f(x_0)$. Using (12), and $\|x_1 - x_0\| \le \dfrac{f(x_0)}{K_0}$, which is an easy consequence of (11) and $\delta < \varepsilon$, we get

$$\begin{aligned}
\varepsilon &\le f(x_0) - f(x_1) - K_0\|x_1 - x_0\| + \delta \\
&\le \langle p_0, x_0 - x_1\rangle - \|p_1\|\|x_1 - x_0\| + \delta\|x_1 - x_0\| + 2\delta \\
&\le \|p_0\|\|x_1 - x_0\| - \|p_1\|\|x_1 - x_0\| + \delta\left(\frac{f(x_0)}{K_0} + 2\right) \\
&= (\|p_0\| - \|p_1\|)\|x_1 - x_0\| + \delta\left(\frac{f(x_0)}{K_0} + 2\right),
\end{aligned}$$

which is (15). The proof is complete. □

In the context of the ESM, Lemma 3.1 ensures the existence of $K_i > 0$. As x_{i+1} can be taken any point of δ-minimum, i.e. such that

$$0 \le f(x_{i+1}) - f(x_i) + \varepsilon + K_i\|x_{i+1} - x_i\| \le \delta. \tag{17}$$

From the lemma we also have the existence of $p_{i+1} \in \partial_\delta f(x_{i+1})$ such that

$$K_i \ge \|p_{i+1}\| - \delta, \ i \ge 0, \tag{18}$$

$$\langle p_{i+1}, x_{i+1} - x_i\rangle \le f(x_{i+1}) - f(x_i) + \varepsilon + \delta, \ i \ge 0, \tag{19}$$

as well as,

$$\varepsilon \le (\|p_i\| - \|p_{i+1}\|)\|x_{i+1} - x_i\| + \delta\left(2 + \frac{f(x_i)}{K_i}\right), \ i \ge 1. \tag{20}$$

The next lemma shows that ESM is finite.

Lemma 3.2. *ESM ends after a finite number of iterations n such that*

$$n \le \frac{f(x_0)}{\varepsilon - \delta} + 1$$

at the point x_{n-1} of the ε-minimum of f.

Proof. Let us assume the contrary, i.e. that the number of iterations satisfies

$$n > \frac{f(x_0)}{\varepsilon - \delta} + 1$$

and fix such $n \in \mathbb{N}$.

This means that ESM generates at least x_i, $i = 0, \ldots, n-1$ such that

$$0 \notin \partial_\varepsilon f(x_i), \quad i = 0, \ldots, n-2.$$

Then from (17) we will have that

$$f(x_i) - f(x_{i+1}) \geq \varepsilon + K_i \|x_{i+1} - x_i\| - \delta \geq \varepsilon - \delta > 0, \qquad i = 0, \ldots, n-2.$$

Summing up the inequalities we obtain

$$f(x_0) - f(x_{n-1}) = \sum_{i=0}^{n-2} (f(x_i) - f(x_{i+1})) \geq (n-1)(\varepsilon - \delta) > f(x_0),$$

hence $0 > f(x_{n-1})$ which contradicts to $f(x_{n-1}) \geq f(0) = 0$. $\square$

It is possible to obtain a better estimate of the number of iteration for a strictly convex function with more precise choice of the parameter δ.

Lemma 3.3. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex function satisfying $f(x) \geq 2c\|x\|$ for all $x \in X$ and some $c > 0$.*
Applied to f with $\varepsilon > 0$ and $\delta > 0$ such that

$$\delta \leq \frac{c}{2}, \quad \delta \leq 1, \quad \delta \left(1 + \frac{f(x_0)}{c}\right) \leq \frac{\varepsilon}{4}, \tag{21}$$

the ESM ends after n iterations, and

$$\sum_{i=0}^{n-2} \|x_{i+1} - x_i\| \leq \frac{2f(x_0)}{c}. \tag{22}$$

Moreover, for the number of iterations n we have the estimation

$$n \leq 2\sqrt{\frac{f(x_0)(\|p_0\| + 1)}{c\varepsilon}} + 2, \tag{23}$$

where $p_0 \in \partial_\varepsilon f(x_0)$ is arbitrary.

Proof. Since $f(x) \geq 2c\|x\|$, it is easy to see that if $p \in \partial_\delta f(x)$, then

$$0 = f(0) \geq f(x) - \langle p, x\rangle - \delta \geq 2c\|x\| - \langle p, x\rangle - \delta$$

yields

$$\|p\|\|x\| \geq \langle p, x\rangle \geq 2c\|x\| - \delta. \tag{24}$$

We have three cases: (a) $\|p\| \geq 2c$; (b) $\|p\| < 2c$ and $\|x\| > \delta/c$, and (c) $\|p\| < 2c$ and $\|x\| \leq \delta/c$.

In case (b), by (24) we have $\|p\| \geq 2c - \dfrac{\delta}{\|x\|} > 2c - \dfrac{\delta c}{\delta} = c$.

In case (c) we have $f(x) \leq \langle p, x\rangle + \delta \leq \|p\|\|x\| + \delta \leq 3\delta < \varepsilon$, and x should be a point of ε-minimum for f.

As x_i, $i = 0, \dots, n-2$, are not ε-minimum points for f, the latter implies, see (18),

$$K_i \geq \|p_{i+1}\| - \delta \geq c - \delta \geq c - \frac{c}{2} = \frac{c}{2}.$$

To establish (22) we sum up inequalities (17) from 0 to $n-2$ to get that

$$f(x_{n-1}) - f(x_0) + (n-1)\varepsilon + \sum_{i=0}^{n-2} K_i \|x_{i+1} - x_i\| \leq (n-1)\delta.$$

Hence, $$\sum_{i=0}^{n-2} K_i \|x_{i+1} - x_i\| + (n-1)(\varepsilon - \delta) \leq f(x_0) - f(x_{n-1}).$$

Since $K_i \geq \dfrac{c}{2}$ for all i in the above sum, and $\delta < \varepsilon$,

$$\frac{c}{2} \sum_{i=0}^{n-2} \|x_{i+1} - x_i\| \leq f(x_0),$$

and (22) holds. Since $f(x_{i+1}) \leq f(x_i)$ for all i, see (17), we have that $f(x_i) \leq f(x_0)$ for all i. Using this and $K_i \geq \dfrac{c}{2}$ in (20) we obtain that

$$\varepsilon \leq (\|p_i\| - \|p_{i+1}\|)\|x_{i+1} - x_i\| + 2\delta \left(1 + \frac{f(x_0)}{c}\right), \ i \geq 1,$$

hence, having in mind the choice of δ,

$$\frac{\varepsilon}{2} \leq (\|p_i\| - \|p_{i+1}\|)\|x_{i+1} - x_i\|, \ i \geq 1. \tag{25}$$

To apply Lemma 2.3, set

$$a_i := \|p_i\| - \|p_{i+1}\|, \quad b_i := \|x_{i+1} - x_i\|, \quad i = 1, \dots, n-2.$$

From (25) we have that $a_i b_i \geq \varepsilon/2$, hence, $a_i > 0$, $b_i > 0$, $i = 1, \dots, n-2$. From (22)

$$\sum_{i=1}^{n-2} b_i \leq \frac{2f(x_0)}{c}.$$

On the other hand,

$$\sum_{i=1}^{n-2} a_i = \|p_1\| - \|p_{n-1}\| \leq \|p_1\| \leq K_0 + \delta \leq \|p_0\| + \delta \leq \|p_0\| + 1,$$

where $p_0 \in \partial_\varepsilon f(x_0)$ is arbitrary (see (14)). Setting $A := \|p_0\| + 1$ and $B := 2f(x_0)/c$ we obtain that the conditions of Lemma 2.3 hold. Hence,

$$n - 2 \leq \sqrt{\frac{2AB}{\varepsilon}} = 2\sqrt{\frac{f(x_0)(\|p_0\| + 1)}{c\varepsilon}}$$

and (23) holds. The proof is complete. □

Let us note that p_0 in (23) as an arbitrary element in $\partial_\varepsilon f(x_0)$ depends on ε. But when $x_0 \in \operatorname{dom} \partial f$, then p_0 could be taken in $\partial f(x_0)$ and in this case, the estimation (23) is of the type $n\sqrt{\varepsilon} \leq \text{const}$.

4. Proof of Theorem 1.2

We will establish first that $g(x) = g(\bar{x}) + R_{\partial g,(\bar{x},\bar{p})}(x)$ for $x \in \operatorname{dom} \partial g$.

To prove that
$$g(x) - g(\bar{x}) \geq R_{\partial g,(\bar{x},\bar{p})}(x) \tag{26}$$
is easy. Indeed, for any sequence $\{(x_i, q_i)\}_{i=1}^n \subset \operatorname{gph} \partial g$ with $x_0 = x$, $x_n = \bar{x}$, and $q_n = \bar{p}$, by the definition of subdifferential we have that

$$\langle q_{i+1}, x_i - x_{i+1} \rangle \leq g(x_i) - g(x_{i+1}), \quad i = 0, \ldots, (n-1).$$

After summing these inequalities we immediately get

$$\sum_{i=0}^{n-1} \langle q_{i+1}, x_i - x_{i+1} \rangle \leq g(x) - g(\bar{x})$$

and (26) follows. To obtain that $g(x) - g(\bar{x}) \leq R_{\partial g,(\bar{x},\bar{p})}(x)$ it is enough for any fixed $\varepsilon' > 0$ to find a sequence $\{(x_i, q_i)\}_{i=1}^n \subset \operatorname{gph} \partial g$ such that $x_0 = x$, $x_n = \bar{x}$, $q_n = \bar{p}$, and
$$g(x) - g(\bar{x}) - \sum_{i=0}^{n-1} \langle q_{i+1}, x_i - x_{i+1} \rangle < \varepsilon'. \tag{27}$$

To this end we consider the function $f : X \to \mathbb{R} \cup \{+\infty\}$, defined as

$$f(x) := g(x + \bar{x}) - \langle \bar{p}, x \rangle - g(\bar{x}) + 2c\|x\|, \tag{28}$$

where $c \in (0,1)$ is a fixed constant. It is easy to see that f is proper lower semicontinuous and convex, $f(0) = 0$, $0 \in \partial f(0)$, $f(x) \geq 2c\|x\|$ for all $x \in X$ and $\operatorname{dom} \partial f \equiv \operatorname{dom} \partial g - \bar{x}$. Set $x_0 := x$, take arbitrary $p_0 \in \partial g(x_0)$ and set

$$M := 4 \left(\sqrt{\frac{f(x_0)(\|p_0\| + 1)}{c}} + 1 \right).$$

Take $\varepsilon \in (0, c)$ such that $M\sqrt{\varepsilon} < \varepsilon'$ and then apply ESM for f with this ε and $\delta > 0$ such that $\eta(\delta) < \varepsilon/3$, where

$$\eta(\delta) := 2\sqrt{\delta} \left(1 + 2c + \|p_0\| + \|\overline{p}\| + \frac{f(x_0)}{c} \right).$$

It is easy to check that if δ is such that $\eta(\delta) < \varepsilon/3$, then δ satisfies (21). When such a δ is chosen, set $\eta := \eta(\delta)$.

Denote $y_0 := x - \bar{x}$. Observe that $p_0 \in \partial f(y_0)$.

Starting at $y_0' := y_0$ ESM generates a finite sequence $p_{i+1} \in \partial_\delta f(y_{i+1}')$, $i = 0, \ldots, n-2$.

By the weaker version of the δ-subdifferential sum rule we have that

$$\partial_\delta f(\cdot) \subset \partial_\delta g(\cdot + \overline{x}) + \partial_\delta \langle -\bar{p}, \cdot \rangle + \partial_\delta 2c\| \cdot \|,$$

therefore,
$$p_{i+1} = q_{i+1}' - \overline{p}_{i+1} + \xi_{i+1}, \tag{29}$$

for some $q_{i+1}' \in \partial_\delta g(\cdot + \bar{x})(y_{i+1}')$, $\xi_{i+1} \in \partial_\delta 2c\|\cdot\|(y_{i+1}')$, and $\overline{p}_{i+1}$ such that $\|\overline{p}_{i+1} - \overline{p}\| \leq \delta$, $i = 0, \ldots, n-2$.

From (19) we have that

$$\langle p_{i+1}, y'_{i+1} - y'_i \rangle \le f(y'_{i+1}) - f(y'_i) + \varepsilon + \delta, \quad i = 0, \dots, n-2.$$

Summing these equalities and using that $\delta < \eta$, we get

$$\sum_{i=0}^{n-2} \langle p_{i+1}, y'_{i+1} - y'_i \rangle \le f(y'_{n-1}) - f(y_0) + (n-1)(\varepsilon + \eta),$$

and from (29) we obtain that

$$\begin{aligned} \sum_{i=0}^{n-2} \langle q'_{i+1}, y'_{i+1} - y'_i \rangle &\le \sum_{i=0}^{n-2} \langle \overline{p}_{i+1}, y'_{i+1} - y'_i \rangle + \sum_{i=0}^{n-2} \langle \xi_{i+1}, y'_i - y'_{i+1} \rangle \\ &+ f(y'_{n-1}) - f(y_0) + (n-1)(\varepsilon + \eta). \end{aligned} \tag{30}$$

To estimate the right hand side of (30) we use, first, that

$$\begin{aligned} &\sum_{i=0}^{n-2} \langle \overline{p}_{i+1}, y'_{i+1} - y'_i \rangle \le \langle \overline{p}, y'_{n-1} - y_0 \rangle + \delta \sum_{i=0}^{n-2} \|y'_{i+1} - y'_i\| \\ &\le \langle \overline{p}, y'_{n-1} - y_0 \rangle + 2\delta \frac{f(x_0)}{c} \le \langle \overline{p}, y'_{n-1} - y_0 \rangle + \eta, \end{aligned}$$

second, that $\xi_{i+1} \in \partial_\delta 2c\| \cdot \|(y'_{i+1})$, hence

$$\begin{aligned} &\sum_{i=0}^{n-2} \langle \xi_{i+1}, y'_i - y'_{i+1} \rangle \le \sum_{i=0}^{n-2} \left(2c\|y'_i\| - 2c\|y'_{i+1}\| + \delta \right) \\ &= 2c\|y_0\| - 2c\|y'_{n-1}\| + (n-1)\delta \le 2c\|y_0\| + (n-1)\eta, \end{aligned}$$

and, third, that y'_{n-1} is an ε-minimum of f, hence $f(y'_{n-1}) \le \varepsilon$.
Incorporating all these in (30) we obtain that

$$\begin{aligned} &\sum_{i=0}^{n-2} \langle q'_{i+1}, y'_{i+1} - y'_i \rangle \\ &\le \langle \bar{p}, y'_{n-1} - y_0 \rangle + 2c\|y_0\| - f(y_0) + (n-1)(\varepsilon + 2\eta) + \varepsilon + \eta. \end{aligned} \tag{31}$$

By the Brøndsted-Rockafellar Theorem there exist points $q_{i+1} \in \partial g(\overline{x} + y_{i+1})$ such that $\|q_{i+1} - q'_{i+1}\| \le \sqrt{\delta}$, and $\|y_{i+1} - y'_{i+1}\| \le \sqrt{\delta}$. Then

$$\begin{aligned} &\langle q_{i+1}, y_{i+1} - y_i \rangle - \langle q'_{i+1}, y'_{i+1} - y'_i \rangle \\ &= \langle q_{i+1} - q'_{i+1}, y_{i+1} - y_i \rangle + \langle q'_{i+1}, y_{i+1} - y_i - y'_{i+1} + y'_i \rangle \\ &\le \|q_{i+1} - q'_{i+1}\|\|y_{i+1} - y_i\| + \|q'_{i+1}\|(\|y_{i+1} - y'_{i+1}\| + \|y_i - y'_i\|). \end{aligned}$$

Since $\|p_{i+1}\| \le \|p_1\|$ for all i, which follows from (20), and since $\|p_1\| \le \|p_0\|$, see (15), we easily derive that $\|q'_{i+1}\| \le 2\delta + 2c + \|\overline{p}\| + \|p_0\|$ for all i. Using the latter and $\|y_{i+1} - y_i\| \le 2\sqrt{\delta} + \|y'_{i+1} - y'_i\|$ we obtain that

$$\begin{aligned} &\langle q_{i+1}, y_{i+1} - y_i \rangle - \langle q'_{i+1}, y'_{i+1} - y'_i \rangle \\ &\le \sqrt{\delta}(2\sqrt{\delta} + \|y'_{i+1} - y'_i\|) + 2\sqrt{\delta}(2\delta + 2c + \|\overline{p}\| + \|p_0\|) \le \eta + \sqrt{\delta}\|y'_{i+1} - y'_i\|. \end{aligned}$$

Hence,
$$\sum_{i=0}^{n-2}\langle q_{i+1}, y_{i+1}-y_i\rangle - \sum_{i=0}^{n-2}\langle q'_{i+1}, y'_{i+1}-y'_i\rangle$$
$$\le (n-1)\eta + \sqrt{\delta}\sum_{i=0}^{n-2}\|y'_{i+1}-y'_i\| \le (n-1)\eta + 2\sqrt{\delta}\frac{f(x_0)}{c} \le (n-1)\eta+\eta.$$

Using this in (31), as well as $\sqrt{\delta}\|\overline{p}\|\le\eta$, and $\eta\le\varepsilon/3$, we get

$$\begin{aligned}&\sum_{i=0}^{n-2}\langle q_{i+1}, y_{i+1}-y_i\rangle\\ &\le \langle \bar p, y_{n-1}-y_0\rangle + 2c\|y_0\| - f(y_0) + (n-1)(\varepsilon+3\eta)+\varepsilon+2\eta+\sqrt{\delta}\|\overline{p}\|\\ &\le \langle \bar p, y_{n-1}-y_0\rangle + 2c\|y_0\| - f(y_0) + 2n\varepsilon. \qquad (32)\end{aligned}$$

But $f(y_0) = f(x-\bar x) = g(x) - \langle \bar p, y_0\rangle - g(\bar x) + 2c\|y_0\|$, see (28), which combined with (32) yields

$$\sum_{i=0}^{n-2}\langle q_{i+1}, y_{i+1}-y_i\rangle \le \langle \bar p, y_{n-1}\rangle + g(\overline{x}) - g(x) + 2n\varepsilon. \qquad (33)$$

Now, let us denote $x_{i+1} := y_{i+1}+\bar x$, $i=0,\dots,n-2$. Then $q_{i+1}\in\partial g(x_{i+1})$, and $x_i - x_{i+1} = y_i - y_{i+1}$, $i=0,\dots,n-2$.

Setting $x_n=\bar x$, $y_n = 0$ and $q_n=\bar p$ from (33) we obtain that

$$\begin{aligned}&g(x)-g(\bar x)-\sum_{i=0}^{n-1}\langle q_{i+1}, x_i - x_{i+1}\rangle = g(x)-g(\bar x)-\sum_{i=0}^{n-1}\langle q_{i+1}, y_i-y_{i+1}\rangle\\ &\le \langle q_n, y_n - y_{n-1}\rangle + \langle \bar p, y_{n-1}\rangle + 2n\varepsilon = 2n\varepsilon \quad (\text{since } y_n=0 \text{ and } q_n=\bar p)\\ &\le 4\left(\sqrt{\frac{f(x_0)(\|p_0\|+1)}{c\varepsilon}}+1\right)\varepsilon \quad (\text{by (23)}) \le 4\left(\sqrt{\frac{f(x_0)(\|p_0\|+1)}{c}}+1\right)\sqrt{\varepsilon}\\ &= M\sqrt{\varepsilon} < \varepsilon',\end{aligned}$$

and (27) follows.

So far we have shown that $g(x) = g(\bar x) + R_{\partial g,(\bar x,\bar p)}(x)$ for $x\in\operatorname{dom}\partial g$.

Now, fix any $x\in X$ and a real number r such that $r<g(x)$. By Proposition 2.2 we can find $(y,p)\in\operatorname{gph}\partial g$ such that $r<g(y)+\langle p, x-y\rangle$. Since $y\in\operatorname{dom}\partial g$ for a fixed $\varepsilon>0$ we find a sequence $\{(x_i,q_i)\}_{i=2}^n\in\operatorname{gph}\partial g$ with $x_1=y$, $x_n=\bar x$ and $q_n=\bar p$ such that

$$g(y)-g(\bar x)-\sum_{i=1}^{n-1}\langle q_{i+1}, x_i-x_{i+1}\rangle < \varepsilon.$$

Then,

$$r < g(\overline{x}) + \langle p, x-y\rangle + \sum_{i=1}^{n-1}\langle q_{i+1}, x_i - x_{i+1}\rangle + \varepsilon = g(\overline{x}) + \sum_{i=0}^{n-1}\langle q_{i+1}, x_i-x_{i+1}\rangle + \varepsilon,$$

where $q_1 := p$. Since $r<g(x)$ and $\varepsilon>0$ were arbitrary, the proof is completed. $\square$

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