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Uniform λ-Property in Orlicz Spaces

Adam Bohonos
School of Mathematics, West Pomeranian University of Technology, Szczecin, Poland
abohonos@zut.edu.pl

Ryszard Płuciennik*
Institute of Mathematics, University of Technology, Poznań, Poland
ryszard.pluciennik@put.poznan.pl

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We discuss Orlicz spaces with uniform λ-property. These spaces are equipped with the Orlicz norm and they are generated by arbitrary Orlicz functions (even Orlicz functions which vanish outside zero and which attain infinite values to the right of some point $u > 0$ are not excluded).

Keywords: Orlicz spaces, extreme points, λ-points, λ-property, uniform λ-property.

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1. Introduction

Let $(X, \|\cdot\|_X)$ be a real Banach space and $B(X)$ $(S(X))$ be the closed unit ball (the unit sphere) of X, respectively. Denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of reals.

In [4] we studied λ-points and λ-property in generalized Orlicz spaces. In this paper, we study the criteria for uniform λ-property in generalized Orlicz function space equipped with the Orlicz norm. Before we start with our results, we need to recall some notions.

A point $x \in S(X)$ is said to be an *extreme point of* $B(X)$ if x cannot be written as the arithmetic mean $\frac{1}{2}(y+z)$ of two distinct points $y, z \in S(X)$. The set of all extreme points of $B(X)$ is denoted by $\mathrm{ext}B(X)$. When every point of the unit sphere is an extreme point than we say that Banach space is strictly convex. The notion of strictly convex spaces was introduced by J. A. Clarkson in [10]. The notion of an extreme point play an important role in some branches of mathematics. For example, the Krein-Milman theorem, Choquet integral representation theorem, Rainwater theorem on convergence in the weak topology, Bessaga Pełczyński theorem and Elton test for unconditional convergence are strongly connected with this notion. All these theorems we can find in [13], chapter IX.

Define a so called *λ-function* $\lambda : B(X) \to [0,1]$ by the formula

$$\lambda(x) = \sup\{\lambda \in [0,1] : x = \lambda e + (1-\lambda)y,\ e \in \mathrm{ext}B(X),\ y \in B(X)\}$$

if $\mathrm{ext}B(X) \neq \varnothing$, and $\lambda(x) = 0$ otherwise.

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A point $x \in B(X)$ is said to be a *λ-point of* $B(X)$ if $\lambda(x) > 0$. The space X is said to have the *λ-property* if every $x \in B(X)$ is a λ-point. Moreover, if *λ-characteristic of* X

$$\lambda_X = \inf \{\lambda(x) : x \in S(X)\} > 0,$$

then X is said to have the *uniform λ-property.* Orlicz spaces equipped with the Orlicz norm possessing the λ-property are described in [4].

The λ-property was introduced by Aron and Lohman [1, 17]. This notion is strictly connected with so called Krein-Milman property (KMP in short). A Banach space X is said to have KMP if every closed bounded convex set of it is a closed convex hull of its extreme points. Aron and Lohman proved that for Banach spaces with the λ-property we have that $B(X) = \overline{co}(\mathrm{ext}B(X))$, so X satisfies a restricted version of KMP. However, in [1] we may additionally find, that the converse is false. It was shown in the same paper that in finite-dimensional space $x \in \mathrm{ext}B(X)$ if and only if $\lambda(x) = 1$ and consequently, in this case X is rotund if and only if $\lambda(x) = 1$ for all $x \in S(x)$. In [17] we may find that in infinite-dimensional Banach space situation is different. In such spaces a point $x \in S(x)$ can be found such that $\lambda(x) = 1$ and $x \notin \mathrm{ext}B(X)$.

Additionally, spaces with the λ-property have geometric feature which is very useful in applications. In such spaces every real-valued convex function on $B(X)$ may attains its maximum value only at points $x \in \mathrm{ext}B(X)$.

Moreover, Aron, Lohman and Granero proved in [2] that a Banach space X has the λ-property if and only if it has the *convex series representation property,* i.e. for each $x \in B(X)$, there is a sequence (e_k) of extreme points of $B(X)$ and a sequence of non-negative real numbers (λ_k) such that $\sum_{k=1}^{\infty} \lambda_k = 1$ and $x = \sum_{k=1}^{\infty} \lambda_k e_k$. It has been also shown in [2] that the uniform λ-property of a Banach space X is equivalent to the *uniform convex series representation property* of X, i.e. convex series representation property in which the sequence (λ_k) does not depend on x.

Moreover it is known that $\lambda(0) = \frac{1}{2}$ and

$$\lambda(x) \geq \max\left\{\frac{1}{2}(1 - \|x\|_X),\ \lambda\left(\frac{x}{\|x\|_X}\right)\|x\|_X\right\} \tag{1}$$

for any $x \in B(X)$ whenever $\mathrm{ext}B(X) \neq \varnothing$ (see Proposition 2.12 in [6]). It implies that any interior point of $B(X)$ is a λ-point whenever $\mathrm{ext}B(X) \neq \varnothing$. Hence the definition of the λ-property in X can be restricted to the unit sphere $S(X)$. Namely, a Banach space X has the λ-property if and only if $\mathrm{ext}B(X) \neq \varnothing$ and $\lambda(x) > 0$ for any $x \in S(X)$.

Let (T, Σ, μ) be a measure space with a σ-finite non-atomic and complete measure μ and $L_0 = L_0(T, \Sigma, \mu)$ be the set of all μ-equivalence classes of real and Σ-measurable functions defined on T.

By a *Banach function space* (or a *Köthe space*) $(E, \|\cdot\|_E)$ we mean a subspace of L_0 which is an ideal in L_0 that is, if $x \in L_0(T, \Sigma, \mu)$, $y \in E$ and $|x(t)| \leq |y(t)|$ μ-a.e. T, then $x \in E$ and $\|x\|_E \leq \|y\|_E$ as well as there exists $x \in E$ such that $x(t) > 0$ for any $t \in T$.

The set $E_+ = \{x \in E : x \geq 0\}$ is called the *positive cone of* E. For any subset $A \subset E$ define $A_+ = A \cap E_+$.

For $x \in L_0$ we denote its *distribution function* by

$$d_x(\eta) = \mu \{s \in T : |x(s)| > \eta\}, \qquad \eta \geq 0,$$

and its *decreasing rearrangement* by

$$x^*(t) = \inf \{\eta > 0 : d_x(\eta) \leq t\}, \quad t \in [0, \mu(T)].$$

Two functions $x, y \in L_0$ are called *equimeasurable* ($x \sim y$ for short) if $d_x = d_y$. We say that a Banach function space $(E, \|\cdot\|_E)$ is *rearrangement invariant* or *symmetric* if whenever $x \in L_0$ and $y \in E$ with $x \sim y$, then $x \in E$ and $\|x\|_E = \|y\|_E$. For basic properties and some applications of symmetric spaces we refer to [3, 8, 9].

The following simple fact will be useful for our considerations.

Proposition 1.1. (see [4], Proposition 2.1) *Let E be a Köthe space. Then*

(a) $\lambda(x) = \lambda(|x|)$ *for any* $x \in S(E)$;

(b) $\lambda(x) = \lambda_+(x)$ *for any* $x \in S(E)_+$, *where* λ_+ *is defined on* $S(E)_+$ *by the formula*

$$\lambda_+(x) = \sup \left\{\lambda \in [0,1] : x = \lambda e + (1-\lambda)y,\ e \in \operatorname{ext} B(X)_+,\ y \in B(X)\right\}.$$

A map $\Phi : \mathbb{R} \to [0, \infty]$ is said to be an *Orlicz function* if it is even, convex, left continuous on the whole $\mathbb{R}_+$, $\Phi(0) = 0$ and Φ is not identically equal to zero. Since Φ is even, without loss of generality we often consider Φ with the domain restricted to the interval $[0, \infty)$. It follows that every Orlicz function Φ is non-decreasing on $\mathbb{R}_+$.

We say that a point w is a *point of strict convexity* of Φ (and we write $w \in SC(\Phi)$) if

$$\Phi(w) < \frac{1}{2}(\Phi(u) + \Phi(v))$$

for every $u, v \in \mathbb{R}$ such that $u \neq v$ and $w = \frac{1}{2}(u+v)$. The set $AF(\Phi) := \mathbb{R} \setminus SC(\Phi)$ is called the *set of all structural affine intervals of* Φ, or simply, SAI of Φ.

For any Orlicz function Φ we define

$$a_\Phi := \sup \{u \geq 0 : \Phi(u) = 0\}, \quad b_\Phi := \sup \{u > 0 : \Phi(u) < \infty\},$$

$$u_\Phi := \sup \{u \geq a_\Phi : u \in SC(\Phi)\}, \quad v_\Phi = \sup \{\alpha > 0 : \Phi(u) \text{ is affine on } [0, \alpha]\}.$$

Obviously, any Orlicz function Φ is strictly increasing on the interval $[a_\Phi, b_\Phi]$.

Denote by p the right hand side derivative of Φ with the domain restricted to the interval $[0, \infty)$. For any Orlicz function Φ we define function Ψ by the formula

$$\Psi(u) = \sup \{|u| v - \Phi(v) : v \geq 0\}.$$

The function Ψ is called *complementary to* Φ *in the sense of Young.* We write q for the right hand side derivative of Ψ.

For any Orlicz function Φ we define on L_0 the modular I_Φ by

$$I_\Phi(x) = \int_T \Phi(x(t))\, d\mu.$$

By the *Orlicz function space* L_Φ we mean the set of all $x \in L_0$ such that $I_\Phi(cx) < \infty$ for some $c > 0$. The space L_Φ is equipped with the so called *Orlicz norm* that can be expressed by the following Amemiya formula

$$\|x\|_\Phi = \inf_{k>0} \frac{1}{k}\left(1 + I_\Phi(kx)\right)$$

(see [16]). We refer to [11] for some basic geometric properties in Orlicz spaces with more general p-Amemiya norm. The set of all $k > 0$ at which the infimum in the Amemiya formula for $\|x\|_\Phi$ is attained (for fixed $x \in L_\Phi$) will be denoted by $K(x)$. Moreover, for any $x \in L_\Phi \setminus \{0\}$, we define

$$k^*(x) = \inf\left\{k > 0 : I_\Psi\left(p \circ k\,|x|\right) \geq 1\right\},$$
$$k^{**}(x) = \sup\left\{k > 0 : I_\Psi\left(p \circ k\,|x|\right) \leq 1\right\}.$$

Obviously, $k^*(x) \leq k^{**}(x)$ for any $x \in L_\Phi \setminus \{0\}$. It is well known that

$$K(x) = \begin{cases} \varnothing & \text{if } \; k^*(x) = \infty \\ \left[k^*(x), k^{**}(x)\right] & \text{if } \; k^{**}(x) < \infty \\ \left[k^*(x), \infty\right) & \text{if } \; k^*(x) < \infty \text{ and } k^{**}(x) = \infty \end{cases}$$

(see [6]). Orlicz spaces are important examples of Köthe spaces. For more details on Orlicz spaces and their applications we refer to [6, 18, 19].

The Orlicz spaces generated by arbitrary Orlicz function (that is Orlicz functions which vanish outside zero and which attain infinite values to the right of some point $u > 0$) share many properties with the usual Orlicz spaces generated by finitely valued and vanishing only at zero N-functions (i.e. Orlicz functions satisfying $\lim_{u\to 0} \Phi(u)/u = 0$ and $\lim_{u\to\infty} \Phi(u)/u = \infty$), but in many aspects they are very different. The class of Orlicz spaces generated by N-functions does not contain spaces such as L^1 and L^∞ or classic interpolation spaces like $L^1 \cap L^\infty$ and $L^1 + L^\infty$. If the Orlicz space is generated by finitely valued and vanishing only at zero N-functions in the case of both (the Luxemburg and the Orlicz) norms, then it has the λ-property [7, 15, 20]. A characterization of λ-points in such kind of Orlicz spaces does not have sense, because every point of $B(L_\Phi)$ is a λ-point. The situation becomes more interesting, when Orlicz spaces are generated by arbitrary Orlicz functions. Then the Orlicz space need not have the λ-property. In the case of the Luxemburg norm, the λ-property in Orlicz spaces generated by finitely valued Orlicz functions has been characterized by Granero [15]. The λ-property in Orlicz spaces equipped with the Orlicz norm was described in [4]. The uniform λ-property in such Orlicz spaces generated by arbitrary Orlicz functions were not considered till now except some particular cases e.g. $L^1 \cap L^\infty$ (see [5]). In this note we describe Orlicz spaces will close the gap.

We will use the following criterion for extreme points of the unit ball in Orlicz spaces equipped with the Orlicz norm.

Proposition 1.2. (see [12], Theorem 1) *Let (T, Σ, μ) be a non-atomic and complete measure space and Φ be an arbitrary Orlicz function. Then $x \in S(L_\Phi)$ is an extreme point of $B(L_\Phi)$ if and only if:*

(a) *the set $K(x)$ is a singleton in $(0, \infty)$,*

(b) *$kx(t) \in SC(\Phi)$ for μ-a.e. $t \in T$, where $\{k\} = K(x)$.*

Moreover, we will use the following characterization of Orlicz function spaces L_Φ without extreme points on its unit sphere.

Proposition 1.3. (see [14], Theorem 1) *The set* $\mathrm{ext}B(L_\Phi)$ *is empty if and only if*

$$I_\Psi\left(p \circ u_\Phi \chi_T\right) \leq 1.$$

The above characterization works under the convention $\infty \cdot 0 = 0$. Moreover, if $u_\Phi = \infty$, then $p(u_\Phi) := \lim_{u\to\infty} p(u)$.

Finally we use the following characterization of Orlicz spaces with λ-property.

Corollary 1.4. (see [4], Corollary 3.1) *The following conditions are equivalent:*

(a) *The space* L_Φ *has the* λ*-property;*

(b) *Either* $\lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty$ *or* $\lim_{u\to\infty} \frac{\Phi(u)}{u} < \infty$ *and* $\sup_u R(u) = \infty$, *where* $R(u) = A|u| - \Phi(u)$ *and* $A = \lim_{u\to\infty} \frac{\Phi(u)}{u}$;

(c) *The space* L_Φ *does not contain an order isometric copy of* l^1.

In this result we can observe that the conditions

$$\textit{either} \ \lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty \ \ \textit{or} \ \ \lim_{u\to\infty} \frac{\Phi(u)}{u} < \infty \text{ and } \sup_u R(u) = \infty$$

are necessary for the λ-property of L_Φ. Therefore we will always assume them without loss of generality. It follows by Proposition 2.2 in [4] that $k(x) \neq \varnothing$ for any $x \in L_\Phi$.

Throughout the paper we will use the following construction. For any Orlicz function Φ define

$$AF(\Phi)_+ = [0, a_\Phi) \cup \left(\bigcup_{i=1}^{r} (a_i, b_i)\right) \cup (u_\Phi, \infty), \tag{2}$$

where $r \in \mathbb{N}$ or $r = \infty$ and $a_i, b_i \in SC(\Phi)$. If $a_\Phi = 0$ ($u_\Phi = \infty$), then we assume that $[0, a_\Phi) = \varnothing$ ($(u_\Phi, \infty) = \varnothing$), respectively. For any $x \in S(L_\Phi)$ define

$$T_1^i = \left\{ t \in T : a_i < k_0 x(t) \leq \frac{a_i + b_i}{2} \right\}, \tag{3}$$

$$T_2^i = \left\{ t \in T : \frac{a_i + b_i}{2} < k_0 x(t) < b_i \right\}. \tag{4}$$

Divide the set T into five disjoint subsets T_1, T_2, T_3, T_4, T_5 such that

$$T_1 = \bigcup_{i=1}^{r} T_1^i,\ T_2 = \bigcup_{i=1}^{r} T_2^i,\ T_3 = \{t \in T : k_0 x(t) < a_\Phi\},\ T_4 = \{t \in T : k_0 x(t) > u_\Phi\} \tag{5}$$

and $T_5 = T \setminus \bigcup_{j=1}^{4} T_j$. If $a_\Phi = 0$ ($u_\Phi = \infty$), then $T_3 = \varnothing$ ($T_4 = \varnothing$), respectively. Based on the function $x \in S(L_\Phi)$ we construct $\widetilde{e} \in \mathrm{ext}B\left(L_\Phi\right)_+$. To this end define

$$e(t) = \begin{cases} \frac{1}{k_0} a_i & \text{if} \quad t \in T_1^i \text{ and } 1 \leq i \leq r, \\ \frac{1}{k_0} b_i & \text{if} \quad t \in T_2^i \text{ and } 1 \leq i \leq r \\ \frac{1}{k_0} a_\Phi & \text{if} \quad t \in T_3, \\ \frac{1}{k_0} u_\Phi & \text{if} \quad t \in T_4, \\ x(t) & \text{if} \quad t \in T_5. \end{cases} \qquad \text{and} \quad \widetilde{e} = \frac{e}{||e||_\Phi}. \tag{6}$$

Clearly, $\widetilde{e}$ defined in this way is an extreme point of the unit ball of L_Φ (see [4]).

Lemma 1.5. *For any $x \in S(L_\Phi)$ we have $\lambda(x) \geq \frac{||e||_\Phi}{2}$ and $\mu(\operatorname{supp} e) > 0$.*

Proof. The result follows immediately from the proof of Proposition 2.3 in [4]. □

2. Results

Lemma 2.1. *Let (T, Σ, μ) be a non-atomic and complete measure space and let Φ be an Orlicz function such that $0 < a_\Phi \leq b_\Phi < \infty$.*

(a) *If $x \in L^0_\Phi \setminus \{0\}$ and $k \in K(x)$, then*

$$\frac{a_\Phi}{\|x\|_\infty} \leq k \leq \frac{b_\Phi}{\|x\|_\infty}, \quad \textit{where } \|x\|_\infty = \operatorname*{supess}_{t\in T}|x(t)|.$$

(b) $\frac{1}{b_\Phi} \leq \|\chi_T\|_\Phi \leq \frac{1}{a_\Phi}$.

(c) *If either $\mu(T) = \infty$ or $p(a_\Phi) > 0$ and $\mu(T) > \frac{1}{a_\Phi p(a_\Phi)}$, then $\|\chi_T\|_\Phi = \frac{1}{a_\Phi}$.*

Proof. (a) Let $x \in L_\Phi \setminus \{0\}$. Then there is $c > 0$ such that

$$I_\Phi(cx) = \int_T \Phi\left(cx(t)\right) d\mu < \infty$$

and consequently $\|x\|_\infty = \operatorname*{supess}_{t\in T}|x(t)| \leq \frac{b_\Phi}{c} < \infty$. Take $k > \frac{b_\Phi}{\|x\|_\infty}$. Define

$$T_k = \{t \in T : k|x(t)| > b_\Phi\}.$$

Obviously, $\mu(T_k) > 0$ and $p_-(k|x(t)|) = \infty$ a.e. in T_k. Consequently,

$$I_\Psi(p_- \circ k|x|) = \int_T \Psi\left(p_-(k|x(t)|)\right) d\mu = \infty$$

for any $k > ||x||^{-1}_\infty b_\Phi$, whence

$$k^{**}(x) = \sup\{k > 0 : I_\Psi(p_- \circ k|x|) \leq 1\} \leq \frac{b_\Phi}{\|x\|_\infty}.$$

Moreover $p(k|x(t)|) = 0$ a.e. in T for any $k < ||x||^{-1}_\infty a_\Phi$. Hence

$$I_\Psi(p \circ k|x|) = \int_T \Psi\left(p(k|x(t)|)\right) d\mu = 0.$$

Consequently, $$k^*(x) = \inf\{k > 0 : I_\Psi(p \circ k|x|) \geq 1\} \geq \frac{a_\Phi}{\|x\|_\infty}.$$

Thus $K(x) = [k^*(x), k^{**}(x)] \subset [||x||^{-1}_\infty a_\Phi, ||x||^{-1}_\infty b_\Phi]$, whence the desired inequality follows immediately.

(b) By the proof of (a), the set $K(\chi_T) \neq \varnothing$ and $a_\Phi \leq k \leq b_\Phi$ for any $k \in K(\chi_T)$.

If $k \in K(\chi_T)$, then

$$\begin{aligned}\frac{1}{b_\Phi} \le \frac{1}{k} \le \|\chi_T\|_\Phi &= \frac{1}{k}\left(1+\int_T \Phi(a_\Phi)\,d\mu\right)\\ &= \inf_{l\ge a_\Phi}\frac{1}{l}\left(1+\int_T \Phi(l)\,d\mu\right) \le \frac{1}{a_\Phi}\left(1+\int_T \Phi(a_\Phi)\,d\mu\right) = \frac{1}{a_\Phi},\end{aligned}$$

which finishes the proof of (b).

(c) First suppose that $\mu(T)=\infty$. Then

$$\int_T \Phi(k)\,d\mu = \begin{cases} 0 & \text{for } \ 0<k\le a_\Phi \\ \infty & \text{for } \ \ k>a_\Phi \end{cases}.$$

Hence $$\|\chi_T\|_\Phi = \inf_{0<k\le a_\Phi}\frac{1}{k}\left(1+\int_T \Phi(k)\,d\mu\right) = \inf_{0<k\le a_\Phi}\frac{1}{k} = \frac{1}{a_\Phi}.$$

Now suppose that $p(a_\Phi)>0$ and $\mu(T)>\frac{1}{a_\Phi p(a_\Phi)}$. It is easy to prove that we have $\Psi(u)\ge a_\Phi|u|$ for any $u\in\mathbb{R}$. For any $k\ge a_\Phi$ we have

$$I_\Psi(p_-(k)) = \int_T \Psi(p(a_\Phi))\,d\mu \ge a_\Phi p(a_\Phi)\mu(T) > 1,$$

whence $$k^{**}(\chi_T) = \sup\{k>0 : I_\Psi(p_-(k))\le 1\} \le a_\Phi.$$

Hence $K(\chi_T)\ne\varnothing$ and $k\le a_\Phi$ for any $k\in K(\chi_T)$. On the other hand, by (a), $a_\Phi<k$ for any $k\in K(\chi_T)$. Hence $K(\chi_T)=\{a_\Phi\}$. Therefore

$$\|\chi_T\|_\Phi = \frac{1}{a_\Phi}\left(1+\int_T \Phi(a_\Phi)\,d\mu\right) = \frac{1}{a_\Phi},$$

which finishes the proof of the lemma. □

Lemma 2.2. *Let* $y,z\in S(L_\Phi)$ *be such that* $K(y)\ne\varnothing$ *and* $K(z)\ne\varnothing$. *If* $x\in S(L_\Phi)$ *and* $x=\lambda z+(1-\lambda)y$ *for some* $\lambda\in(0,1)$, *then*

(a) $K(x)\ne\varnothing$ *and* $k_x=\frac{k_yk_z}{(1-\lambda)k_z+\lambda k_y}\in K(x)$, *where* $k_y\in K(y)$, $k_z\in K(z)$,

(b) *either* $k_x|x(t)|=k_y|y(t)|=k_z|z(t)|$ *or* $k_x|x(t)|$, $k_y|y(t)|$, *and* $k_z|z(t)|$ *are on the same SAI of* Φ *for a.e.* $t\in T$,

(c) *if* $a_\Phi=0$, *then* $\operatorname{sign}x(t)=\operatorname{sign}y(t)=\operatorname{sign}z(t)$ *for a.e.* $t\in T$.

Proof. (a) Since $K(y)\ne\varnothing$ and $K(z)\ne\varnothing$, there are k_y and k_z such that

$$\|y\|_\Phi = \frac{1}{k_y}\left(1+I_\Phi(k_yy)\right) \text{ and } \|z\|_\Phi = \frac{1}{k_z}\left(1+I_\Phi(k_zz)\right).$$

By the definition of Orlicz norm and convexity of the modular, we have

$$\begin{aligned}
1 = \|x\|_\Phi &\le \frac{(1-\lambda)k_z+\lambda k_y}{k_yk_z}\left(1+I_\Phi\left(\frac{k_yk_z}{(1-\lambda)k_z+\lambda k_y}\left((1-\lambda)\,y+\lambda z\right)\right)\right)\\
&= \frac{(1-\lambda)k_z+\lambda k_y}{k_yk_z}\left(1+I_\Phi\left(\frac{(1-\lambda)\,k_z}{(1-\lambda)k_z+\lambda k_y}k_yy+\frac{\lambda k_y}{(1-\lambda)k_z+\lambda k_y}k_zz\right)\right)\\
&= \frac{(1-\lambda)k_z+\lambda k_y}{k_yk_z}\left(1+\frac{(1-\lambda)\,k_z}{(1-\lambda)k_z+\lambda k_y}I_\Phi\left(k_yy\right)+\frac{\lambda k_y}{(1-\lambda)k_z+\lambda k_y}I_\Phi\left(k_zz\right)\right)\\
&= \frac{(1-\lambda)k_z+\lambda k_y}{k_yk_z}+\frac{(1-\lambda)}{k_y}I_\Phi\left(k_yy\right)+\frac{\lambda}{k_z}I_\Phi\left(k_zz\right)\\
&= \frac{(1-\lambda)}{k_y}\left(1+I_\Phi\left(k_yy\right)\right)+\frac{\lambda}{k_z}\left(1+I_\Phi\left(k_zz\right)\right) = (1-\lambda)\,||y||_\Phi+\lambda||z||_\Phi \le 1.
\end{aligned}$$

Hence $k_x=\frac{k_yk_z}{(1-\lambda)k_z+\lambda k_y}\in K(x)$, which finishes the proof of (a).

(b) By the above inequalities, we conclude

$$\begin{aligned}
I_\Phi(k_xx) &= I_\Phi\left(\frac{(1-\lambda)\,k_z}{(1-\lambda)k_z+\lambda k_y}k_yy+\frac{\lambda k_y}{(1-\lambda)k_z+\lambda k_y}k_zz\right)\\
&= \frac{(1-\lambda)\,k_z}{(1-\lambda)k_z+\lambda k_y}I_\Phi\left(k_yy\right)+\frac{\lambda k_y}{(1-\lambda)k_z+\lambda k_y}I_\Phi\left(k_zz\right).
\end{aligned}$$

It follows from the convexity of Φ that

$$\begin{aligned}
\Phi\left(k_xx(t)\right) &= \Phi\left(\frac{(1-\lambda)\,k_z}{(1-\lambda)k_z+\lambda k_y}k_yy(t)+\frac{\lambda k_y}{(1-\lambda)k_z+\lambda k_y}k_zz(t)\right)\\
&= \frac{(1-\lambda)\,k_z}{(1-\lambda)k_z+\lambda k_y}\Phi\left(k_yy(t)\right)+\frac{\lambda k_y}{(1-\lambda)k_z+\lambda k_y}\Phi\left(k_zz(t)\right)
\end{aligned}$$

for a.e. $t\in T$. Consequently, either $k_x|x(t)|=k_y|y(t)|=k_z|z(t)|$ or $k_x|x(t)|,\ k_y|y(t)|$ and $k_z|z(t)|$ are on the same SAI of Φ for a.e. $t\in T$, completing the proof of (b).

(c) Suppose that $a_\Phi=0$. Then the Orlicz space L_Φ is strictly monotone. Since

$$1=||x||_\Phi\le||(1-\lambda)|y|+\lambda|z|||_\Phi\le(1-\lambda)||y||_\Phi+\lambda||z||_\Phi\le 1,$$

by the strict monotonicity of L_Φ, we conclude

$$|(1-\lambda)y(t)+\lambda z(t)|=(1-\lambda)|y(t)|+\lambda|z(t)|$$

for a.e. $t\in T$. Hence $\operatorname{sign}x(t)=\operatorname{sign}y(t)=\operatorname{sign}z(t)$ for a.e. $t\in T$. □

Lemma 2.3. *Let (T,Σ,μ) be a non-atomic and complete measure space and let Φ be an Orlicz function with $0<v_\Phi\le b_\Phi<\infty$ and $\Phi(u)=au$ for $u\in[0,v_\Phi]$. Then*

(a) $L_\Phi=L^1\cap L^\infty$ *and for any* $x\in L_\Phi$:

$$a\,\|x\|_1+\frac{1}{b_\Phi}\,\|x\|_\infty\le\|x\|_\Phi\le a\,\|x\|_1+\frac{1}{v_\Phi}\,\|x\|_\infty\,;$$

(b) $\frac{v_\Phi}{ab_\Phi\mu(T)+1}\le\|x\|_\infty<b_\Phi$ *for any* $x\in S(L_\Phi)$;

(c) $K(x)\subset\left(1,\frac{b_\Phi(ab_\Phi\mu(T)+1)}{v_\Phi}\right]$ *for any* $x\in S(L_\Phi)$.

Proof. (a) For $x = 0$ the inequality is trivial. Let $x \in L_\Phi \setminus \{0\}$. Then there is $c > 0$ such that

$$I_\Phi(cx) = \int_T \Phi\left(cx(t)\right) d\mu < \infty$$

and consequently $\|x\|_\infty = \operatorname*{supess}_{t\in T} |x(t)| \leq \frac{b_\Phi}{c} < \infty$. Since $\Phi(u) = au$ for $u \in [0, v_\Phi]$, we have

$$ac \int_T |x(t)|\, d\mu \leq \int_T \Phi\left(cx(t)\right) d\mu < \infty,$$

i.e. $x \in L^1$. Consequently, $L_\Phi = L^1 \cap L^\infty$ because $L^1 \cap L^\infty$ is the smallest symmetric function space.

For any $k > \|x\|_\infty^{-1} b_\Phi$ define $T_k = \{t \in T : k\,|x(t)| > b_\Phi\}$. Then $\mu(T_k) > 0$ and

$$\frac{1}{k}\left(1 + \int_T \Phi(kx(t))d\mu\right) \geq \frac{1}{k}\left(1 + \int_{T_k} \Phi(kx(t))d\mu\right) = \infty,$$

for any $k > \|x\|_\infty^{-1} b_\Phi$. Moreover, we have

$$\begin{aligned}\inf_{0<k\leq\|x\|_\infty^{-1}v_\Phi} \frac{1}{k}\left(1 + \int_T \Phi(kx(t))d\mu\right) &= \inf_{0<k\leq\|x\|_\infty^{-1}v_\Phi} \frac{1}{k}\left(1 + ak\int_T |x(t)|\, d\mu\right) \\ &= a\,\|x\|_1 + \frac{1}{v_\Phi}\,\|x\|_\infty .\end{aligned}$$

Therefore, by the definition of Orlicz norm, we conclude that

$$\|x\|_\Phi \leq a\,\|x\|_1 + \frac{1}{v_\Phi}\,\|x\|_\infty$$

for any $x \in L_\Phi$. On the other hand we have for any $x \in L_\Phi$:

$$\begin{aligned}\|x\|_\Phi &= \inf_{k>0} \frac{1}{k}\left(1 + \int_T \Phi(kx(t))d\mu\right) \\ &= \inf_{\|x\|_\infty^{-1}v_\Phi\leq k\leq\|x\|_\infty^{-1}b_\Phi} \frac{1}{k}\left(1 + \int_T \Phi(kx(t))d\mu\right) \\ &\geq \inf_{\|x\|_\infty^{-1}v_\Phi\leq k\leq\|x\|_\infty^{-1}b_\Phi} \frac{1}{k}\left(1 + ak\int_T |x(t)|\, d\mu\right) \geq a\,\|x\|_1 + \frac{1}{b_\Phi}\,\|x\|_\infty\end{aligned}$$

(b) Let $x \in S(L_\Phi)$. By the proof of (a),

$$a\,\|x\|_1 + \frac{1}{b_\Phi}\,\|x\|_\infty \leq \|x\|_\Phi = 1 \leq a\,\|x\|_1 + \frac{1}{v_\Phi}\,\|x\|_\infty .$$

Consequently, $0 < a\,\|x\|_1 < 1$ and

$$v_\Phi\left(1 - a\,\|x\|_1\right) \leq \|x\|_\infty . \tag{7}$$

Notice that the largest possible value of $a\,\|x\|_1$ is in the case, when the function x is of the form $z = c\chi_T$.

Moreover, there is a number $d_z \in [v_\Phi, b_\Phi]$ such that

$$1 = a\|z\|_1 + \frac{1}{d_z}\|z\|_\infty = ac\mu(T) + \frac{c}{d_z},$$

whence $c = \dfrac{d_z}{ad_z\mu(T)+1}$. Therefore

$$1 - a\|x\|_1 \geq 1 - a\|z\|_1 = \frac{1}{ad_z\mu(T)+1} \geq \frac{1}{ab_\Phi\mu(T)+1}$$

for any $x \in S(L_\Phi)$. Hence, by the inequality (7), we get

$$\frac{v_\Phi}{ab_\Phi\mu(T)+1} \leq \|x\|_\infty .$$

If $b_\Phi < \infty$, then $K(x) \neq \varnothing$ for any $x \in S(L_\Phi)$. Then there is $k_0 > 0$ such that

$$1 = \|x\|_\Phi = \frac{1}{k_0}(1 + I_\Phi(k_0 x)) > \frac{1}{k_0},$$

whence $k_0 > 1$ and $k_0 x(t) < b_\Phi$, which implies that $\|x\|_\infty < b_\Phi$.

(c) By the proof of (b) it follows that $1 < k \leq \frac{b_\Phi}{\|x\|_\infty}$ for any $k \in K(x)$. Taking into account the lower estimation of $\|x\|_\infty$ given in Lemma 2.3(a), we have

$$\frac{b_\Phi}{\|x\|_\infty} \leq \frac{b_\Phi(ab_\Phi\mu(T)+1)}{v_\Phi}$$

whence the desired inclusion follows immediately. □

Theorem 2.4. *Let Φ be an Orlicz function such that $0 < v_\Phi \leq b_\Phi < \infty$ and $\Phi(u) = au$ for $u \in [0, v_\Phi]$ and $a > 0$. If $\mu(T) < \infty$ then the space $(L_\Phi, \|\cdot\|_\Phi)$ has the uniform λ-property.*

Proof. By Corollary 3.1 in [4], the space $(L_\Phi, \|\cdot\|_\Phi)$ has the λ-property.

Suppose that $\mu(T) < \infty$. Let $x \in S(L_\Phi) \setminus \mathrm{Ext}S(L_\Phi)$. By Proposition 1.1, without loss of generality, we can assume that $x(t) \geq 0$ for μ-a.e $t \in T$. Since $b_\Phi < \infty$, we have that $K(x) \neq \varnothing$ for any $x \in L_\Phi$. Then there is $k_0 > 0$ such that

$$1 = \|x\|_\Phi = \frac{1}{k_0}(1 + I_\Phi(k_0 x)) > \frac{1}{k_0},$$

whence $k_0 > 1$ and $k_0 x(t) \leq b_\Phi$. We have

$$AF(\Phi)_+ = \left(\bigcup_{i=1}^{r}(a_i, b_i)\right),$$

where $r \in \mathbb{N}$ or $r = \infty$ and $a_i, b_i \in SC(\Phi)$ for any $i \in \mathbb{N}$. Obviously, $(a_1, b_1) = (0, v_\Phi)$. Since x is not an extreme point of $B(L_\Phi)$, it follows from Theorem 1 in [12] that

$$\mu\{t \in T : k_0 x(t) \in AF(\Phi)_+\} > 0.$$

Let $T_1 = \bigcup_{i=1}^r T_1^i$, $T_2 = \bigcup_{i=1}^r T_2^i$, where the sets T_1^i and T_2^i $(1 \le i \le r)$ are defined by (3) and (4), respectively. Define $T_3 = T \setminus (T_1 \cup T_2)$ and

$$e(t) = \begin{cases} \frac{1}{k_0} a_i & \text{if} \quad t \in T_1^i \text{ and } 1 \le i \le r, \\ \frac{1}{k_0} b_i & \text{if} \quad t \in T_2^i \text{ and } 1 \le i \le r \\ x(t) & \text{if} \quad t \in T_3. \end{cases}$$

First we claim that $\mu(\operatorname{supp} e) > 0$. Suppose for the contrary that $\mu(\operatorname{supp} e) = 0$. Then $0 \le k_0 x(t) \le \frac{1}{2} v_\Phi$ for μ-a.e. $t \in T$. Take $k_1 = \frac{3}{2} k_0$. Hence $0 \le k_1 x(t) \le \frac{3}{4} v_\Phi$ and by the linearity of Φ on $(0, v_\Phi)$, we have

$$\frac{1}{k_1}\left(1 + I_\Phi(k_1 x)\right) = \frac{1}{k_1} + \int_T x(t) d\mu < \frac{1}{k_0} + \int_T x(t) d\mu = \frac{1}{k_0}\left(1 + I_\Phi(k_0 x)\right),$$

whence $k_0 \notin K(x)$. A contradiction.

It will now be shown that $\lambda(x) \ge \frac{\|e\|_\Phi}{2}$. Obviously, $e \in L_\Phi$ and $K(e) \ne \varnothing$. Choose $\varepsilon \in (0,1)$. Clearly, $e(t) \ge x(t)$ for $t \in T \setminus T_1$, whence

$$I_\Psi\left[p\left((1+\varepsilon) k_0 e\right) \chi_{T \setminus T_1}\right] \ge I_\Psi\left[p\left(\left(1 + \frac{\varepsilon}{4}\right) k_0 x\right) \chi_{T \setminus T_1}\right]. \tag{8}$$

If $t \in T_1$, then either $(1+\varepsilon) a_i < b_i$ or $(1+\varepsilon) a_i \ge b_i$ for any $i \in \mathbb{N}$. If the first inequality is satisfied, then

$$\left(1+\frac{\varepsilon}{4}\right) k_0 x(t) \le \left(1+\frac{\varepsilon}{4}\right) \frac{a_i+b_i}{2} < \frac{1}{2}\left(1+\frac{\varepsilon}{4}\right)\left(\frac{b_i}{1+\varepsilon} + b_i\right) = \frac{(4+\varepsilon)(2+\varepsilon)}{8(1+\varepsilon)} b_i < b_i.$$

Suppose that $(1+\varepsilon) a_i \ge b_i$. We have

$$\left(1+\frac{\varepsilon}{4}\right) k_0 x(t) \le \left(1+\frac{\varepsilon}{4}\right) \frac{a_i+b_i}{2} \le \frac{(4+\varepsilon)(2+\varepsilon)}{8} a_i < (1+\varepsilon) a_i,$$

Since p is constant on each (a_i, b_i), we conclude that

$$p\left((1+\varepsilon) k_0 e\right) \chi_{T_1} \ge p\left(\left(1+\frac{\varepsilon}{4}\right) k_0 x\right) \chi_{T_1}. \tag{9}$$

in both considered cases. Combining (8) with (9), we get

$$I_\Psi\left[p\left((1+\varepsilon) k_0 e\right)\right] \ge I_\Psi\left[p\left(\left(1+\frac{\varepsilon}{4}\right) k_0 x\right)\right] \ge 1, \tag{10}$$

whence, $k_0 \ge k^{**}(e)$. Analogously, by $e(t) \le x(t)$ for $t \in T \setminus T_2$, we have

$$p\left((1-\varepsilon) k_0 e\right) \chi_{T \setminus T_2} \le p\left(\left(1-\frac{\varepsilon}{4}\right) k_0 x\right) \chi_{T \setminus T_2}. \tag{11}$$

Now, assume that $t \in T_2$. Then either

$$(1-\varepsilon) b_i > a_i \quad \text{or} \quad (1-\varepsilon) b_i \le a_i.$$

If $(1-\varepsilon) b_i > a_i$, then

$$\left(1-\frac{\varepsilon}{4}\right) k_0 x(t) \ge \left(1-\frac{\varepsilon}{4}\right) \frac{a_i+b_i}{2} > \frac{1}{2}\left(1-\frac{\varepsilon}{4}\right)\left(\frac{a_i}{1-\varepsilon} + a_i\right) = \frac{(4-\varepsilon)(2-\varepsilon)}{8(1-\varepsilon)} a_i > a_i.$$

When the second inequality holds, we get

$$\left(1-\frac{\varepsilon}{4}\right)k_0x(t)\geq\left(1-\frac{\varepsilon}{4}\right)\frac{a_i+b_i}{2}\geq\frac{(4-\varepsilon)(2-\varepsilon)}{8}b_i>(1-\varepsilon)b_i.$$

Therefore, taking into account both cases, we obtain

$$p\left((1-\varepsilon)k_0e\right)\chi_{T_2}\leq p\left(\left(1-\frac{\varepsilon}{4}\right)k_0x\right)\chi_{T_2}. \tag{12}$$

The inequalities (11) and (12) imply that

$$I_\Psi\left[p\left((1-\varepsilon)k_0e\right)\right]\leq I_\Psi\left[p\left(\left(1-\frac{\varepsilon}{4}\right)k_0x\right)\right]\leq 1, \tag{13}$$

whence $k_0\leq k^*(e)$. It follows from the definition of the set $K(e)$ that $K(e)=\{k_0\}$. Define $\widetilde{e}=\frac{e}{\|e\|_\Phi}$. Then

$$K(\widetilde{e})=K\left(\frac{e}{\|e\|_\Phi}\right)=\{\|e\|_\Phi k_0\}=\left\{\widetilde{k}_0\right\}.$$

Moreover, $\widetilde{k}_0\widetilde{e}(t)=k_0e(t)\in SC(\Phi)$ for μ-a.e. $t\in T$. Hence, by Theorem 1 in [12], $\widetilde{e}$ is an extreme point of $B\left(L_\Phi\right)$.
Let $y=2x-e$ and $\widetilde{y}=\frac{y}{\|y\|_\Phi}$. Then

$$e\chi_{T_3}=y\chi_{T_3}=x\chi_{T_3},\quad a_i\chi_{T_1}=k_0e\chi_{T_1^i}<k_0x\chi_{T_1^i}<k_0y\chi_{T_1^i}\leq b_i\chi_{T_1^i},$$

and
$$a_i\chi_{T_2}<k_0y\chi_{T_2}<k_0x\chi_{T_2}<k_0e\chi_{T_2}=b_i\chi_{T_2}$$

for $i=1,2,...,r$. Since k_0e differs from k_0y only on $AF(\Phi)$, it follows that

$$\Phi\left(\frac{k_0e(t)+k_0y(t)}{2}\right)=\frac{1}{2}\Phi\left(k_0e(t)\right)+\frac{1}{2}\Phi\left(k_0y(t)\right)$$

for all $t\in T$. Hence

$$\begin{aligned}\frac{1}{2}\|e\|_\Phi+\frac{1}{2}\|y\|_\Phi\geq\left\|\frac{e+y}{2}\right\|_\Phi&=\|x\|_\Phi=\frac{1}{k_0}\left(1+I_\Phi(k_0x)\right)\\&=\frac{1}{k_0}\left(1+I_\Phi\left(k_0\frac{e+y}{2}\right)\right)=\frac{1}{2k_0}\left(1+I_\Phi\left(k_0e\right)\right)+\frac{1}{2k_0}\left(1+I_\Phi\left(k_0y\right)\right)\\&=\frac{1}{2}\|e\|_\Phi+\frac{1}{2k_0}\left(1+I_\Phi\left(k_0y\right)\right)\end{aligned}$$

and consequently $\frac{1}{k_0}\left(1+I_\Phi\left(k_0y\right)\right)\leq\|y\|_\Phi=\inf\limits_{k>0}\frac{1}{k}\left(1+I_\Phi\left(ky\right)\right)$.

Therefore $\|y\|_\Phi=\frac{1}{k_0}\left(1+I_\Phi\left(k_0y\right)\right)$, and $k_0\in K(y)$. Moreover,

$$1=\|x\|_\Phi=\frac{1}{2}\|e\|_\Phi+\frac{1}{2}\|y\|_\Phi.$$

Taking $\lambda = \frac{1}{2}\left\|e\right\|_\Phi$, we have

$$\lambda\widetilde{e} + (1-\lambda)\,\widetilde{y} = \frac{1}{2}\left\|e\right\|_\Phi \frac{e}{\left\|e\right\|_\Phi} + \left(1 - \frac{1}{2}\left\|e\right\|_\Phi\right)\frac{y}{\left\|y\right\|_\Phi} = \frac{1}{2}e + \frac{1}{2}y = x,$$

whence $\lambda(x) \geq \frac{1}{2}\left\|e\right\|_\Phi$ as desired.

Since $\mu(\operatorname{supp} e) > 0$ and $k_0 e(t) \in SC(\Phi)$ for μ-a.e. $t \in T$, the inequality $k_0 e(t) \geq v_\Phi$ holds for μ-a.e. $t \in \operatorname{supp} e$. Hence, by Lemma 2.3(a) and (c), we have

$$\lambda(x) \geq \frac{1}{2}\left\|e\right\|_\Phi \geq \frac{1}{2b_\Phi}\left\|e\right\|_\infty \geq \frac{v_\Phi}{2k_0 b_\Phi} \geq \frac{v_\Phi^2}{2b_\Phi^2\left(ab_\Phi\mu(T)+1\right)}$$

for any $x \in S(L_\Phi)$. Therefore the space $(L_\Phi, \left\|\cdot\right\|_\Phi)$ has the uniform λ-property. $\square$

Theorem 2.5. *Let $\mu(T) = \infty$. If Φ is defined by the formula*

$$\Phi(u) = \begin{cases} a|u| & \text{for} \quad |u| \leq b \\ \infty & \text{for} \quad |u| > b \end{cases}$$

for $a > 0$ and $b > 0$, then the Orlicz space L_Φ has no uniform λ-property.

Proof. It is easy to see from Lemma 2.3 a) that for such Φ we have

$$||\cdot||_\Phi = \frac{1}{b}||\cdot||_\infty + a||\cdot||_1.$$

By the divergence of harmonic series, for any $n \in \mathbb{N}$ the smallest number $h(n) \in \mathbb{N}$ can be chosen such that

$$n - 1 \leq \sum_{i=1}^{h(n)} \frac{1}{i} \leq n.$$

Define
$$x_n = \frac{b}{n}\sum_{i=1}^{h(n)} \frac{1}{i}\chi_{T_i^{(n)}}, \tag{14}$$

where $\left(T_i^{(n)}\right)$ is a sequence such that for any $n \in \mathbb{N}$ the sets $T_i^{(n)}$ $(i = 1, 2, ..., h(n))$ are pairwise disjoint and $\mu\left(T_i^{(n)}\right) = \frac{1}{ab}$. Hence

$$\left\|x_n\right\|_\Phi = \frac{1}{b}\frac{b}{n} + a\frac{b}{n}\sum_{i=1}^{h(n)}\frac{1}{i}\mu\left(\chi_{T_i^{(n)}}\right) = \frac{1}{n} + \frac{1}{n}\sum_{i=1}^{h(n)}\frac{1}{i}$$

for any $n \in \mathbb{N}$. It follows from the definition of $h(n)$, that

$$1 \leq \left\|x_n\right\|_\Phi \leq 1 + \frac{1}{n}$$

for any $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ define

$$\widetilde{x}_n = \frac{x_n}{\left\|x_n\right\|_\Phi}.$$

Since the space L_Φ has the λ-property (see Corollary 3.1 in [4]), there are $\lambda_n \in (0,1)$, $y_n \in S(L_\Phi)$ and $e_n \in \mathrm{ext}B(L_\Phi)$ such that

$$\widetilde{x}_n = \lambda_n e_n + (1-\lambda_n) y_n \tag{15}$$

for any $n \in \mathbb{N}$. By the definition of the norm $\|\cdot\|_\Phi$ and Lemma 2.2 *b*),

$$k_{e_n} e_n(t) \in [0,b] \quad \text{and} \quad k_{y_n} y_n(t) \in [0,b]$$

for μ-a.e. $t \in T$, any $n \in \mathbb{N}$ and $k_{e_n} \in K(e_n)$ and $k_{y_n} \in K(y_n)$. Since for any $n \in \mathbb{N}$ we have $e_n \in \mathrm{ext}B\left(L_\Phi\right)$, it follows from Corollary 6 in [12] that e_n $(n = 1,2,...)$ is in the following form: $k_{e_n} e_n(t) = 0$ or $k_{e_n} e_n(t) = b$. Moreover,

$$\begin{aligned} 1 = \|e_n\|_\Phi = \frac{1}{b}||e_n||_\infty + a||e_n||_1 = \frac{1}{b}\frac{b}{k_{e_n}} + a \int\limits_{\operatorname{supp} e_n} |e_n| d\mu \\ = \frac{1}{k_{e_n}} + a\frac{b}{k_{e_n}}\mu\left(\operatorname{supp} e_n\right) = \frac{1}{k_{e_n}}\left(1 + ab\mu\left(\operatorname{supp} e_n\right)\right), \end{aligned}$$

for any $n \in \mathbb{N}$, whence $k_{e_n} = 1 + ab\mu\left(\operatorname{supp} e_n\right)$ and

$$e_n(t) = \frac{b_\Phi}{1 + ab\mu\left(\operatorname{supp} e_n\right)} \chi_{\operatorname{supp} e_n}$$

for any $n \in \mathbb{N}$. It follows from (14) and (15) that

$$\frac{b}{n||x_x||_\Phi} \sum_{i=1}^{h(n)} \frac{1}{i} \chi_{T_i^{(n)}}(t) = \frac{\lambda_n}{k_{e_n}} e_n(t) + \frac{(1-\lambda_n)}{k_{y_n}} k_{y_n} y_n(t) \geq \frac{\lambda_n}{k_{e_n}} b \chi_{\operatorname{supp} e_n}(t)$$

for μ-a.e. $t \in T$ and any $n \in \mathbb{N}$. Consequently,

$$\begin{aligned} \lambda_n \chi_{\operatorname{supp} e_n} &\leq \frac{1}{||x_n||_\Phi} \frac{k_{e_n}}{b} \left(\frac{b}{n} \sum_{i=1}^{h(n)} \frac{1}{i} \chi_{T_i^{(n)}} \right) \\ &= \frac{1 + ab\mu(\operatorname{supp} e_n)}{b\,\|x_n\|_\Phi} \left(\frac{b}{n} \sum_{i=1}^{h(n)} \frac{1}{i} \chi_{T_i^{(n)}} \right). \end{aligned} \tag{16}$$

By Lemma 2.2, $\operatorname{supp} e_n \subset \operatorname{supp} x_n$, whence $\mu(\operatorname{supp} e_n) \leq \frac{h(n)}{ab}$ and at least $\left\lceil \frac{\mu(\operatorname{supp} e_n)}{ab} \right\rceil$ sets of the collection $\left\{T_i^{(n)}\right\}_{i=1}^{h(n)}$ has a non-empty intersection with $\operatorname{supp} e_n$, where $\lceil \cdot \rceil$ denotes the ceiling function. Let j_0 be the smallest integer such that $j_0 \geq \frac{\mu(\operatorname{supp} e_n)}{ab}$ and $\mu\left(T_{j_0}^{(n)} \cap \operatorname{supp} e_n\right) \neq \varnothing$. Then, by the inequality (16), we have

$$\begin{aligned} \lambda_n \chi_{T_{j_0}^{(n)} \cap \operatorname{supp} e_n} &\leq \frac{1 + ab\mu(\operatorname{supp} e_n)}{b\,\|x_n\|_\Phi} \cdot \frac{b}{n} \cdot \frac{1}{j_0} \chi_{T_{j_0}^{(n)} \cap \operatorname{supp} e_n} \\ &\leq \frac{1 + ab\mu(\operatorname{supp} e_n)}{n j_0\,\|x_n\|_\Phi} \cdot \frac{1}{\frac{\mu(\operatorname{supp} e_n)}{ab}} \chi_{T_{j_0}^{(n)} \cap \operatorname{supp} e_n}. \end{aligned}$$

By the definition of the λ-characteristic of L_Φ, we have

$$\lambda_{L_\Phi} = \inf\{\lambda(x) : x \in S(L_\Phi)\} \le \lambda(x_n) \to 0 \text{ as } n \to \infty,$$

whence $\lambda_{L_\Phi} = 0$. Thus the space L_Φ fails to have the uniform λ-property whenever $\mu(T) = \infty$. $\square$

Theorem 2.6. *Let Φ be an Orlicz function for which $b_\Phi = \infty$ and let $\chi_T \in L_\Phi$. Then L_Φ has the uniform λ-property if and only if $\sup\limits_{a_i>0}\left\{\frac{b_i}{a_i}\right\} < \infty$ and one of the following conditions holds*

(a) $\lim\limits_{u\to\infty} \frac{\Phi(u)}{u} = \infty$;

(b) $\lim\limits_{u\to\infty} \frac{\Phi(u)}{u} < \infty$ *and* $\sup\limits_u R(u) = \infty$.

Proof. If $R(u)$ is upper bounded, then, by Corollary 1.4, L_Φ fails λ-property. Consequently, it also fails uniform λ-property. Assume now that $R(u)$ is not upper bounded. It follows from Proposition 2.2 in [4] that for any $x \in L_\Phi$ we have $K(x) \neq \varnothing$.

Sufficiency. Assume that $\sup\limits_{a_i>0}\left\{\frac{b_i}{a_i}\right\} < \infty$ and define $\alpha = \frac{1}{2||\chi_T||_\Phi}$. If $\mu(T) < \infty$ or $\mu(T) = \infty$ and $a_\Phi > 0$, then Lemma 2.1 assures that $0 < ||\chi_T||_\Phi < \infty$. Take $x \in S(L_\Phi)$. Without loss of the generality we may assume that $x(t) \ge 0$ for μ-a.e. $t \in T$.

Define $E = \{t \in T : x(t) > \alpha\}$. Then

$$\left|\left|x\chi_{T\setminus E}\right|\right|_\Phi \le \left|\left|\alpha\chi_{T\setminus E}\right|\right|_\Phi \le \frac{1}{2\,||\chi_T||_\Phi}\,||\chi_T||_\Phi = \frac{1}{2},$$

Hence, by the triangle inequality,

$$||x\chi_E||_\Phi = \left|\left|x - x\chi_{T\setminus E}\right|\right|_\Phi \ge ||x||_\Phi - \left|\left|x\chi_{T\setminus E}\right|\right|_\Phi \ge \frac{1}{2}. \tag{17}$$

Let $e(t)$ be defined by the formula (6). By Lemma 1.5,

$$\lambda(x) \ge \frac{||e||_\Phi}{2} \tag{18}$$

Define $\beta = 1 + \sup\limits_i\left\{\frac{b_i}{a_i}\right\}$. Observe that $e(t) \ge x(t)$ for μ-a.e. $t \in T \setminus (T_1 \cup T_4)$. The assumption $\sup\limits_i\left\{\frac{b_i}{a_i}\right\} < \infty$, assures that $u_\Phi = \infty$, whence $T_4 = \varnothing$. Moreover $T_3 = \varnothing$ because $b_\Phi = \infty$. By the formula (3), we have

$$a_i < k_0 x(t) \le \frac{a_i + b_i}{2}.,$$

whence $a_i \ge \dfrac{2k_0 x(t)}{1 + \frac{b_i}{a_i}} \ge \dfrac{2}{\beta}k_0 x(t)$ for μ-a.e. $t \in T_1^i$ $(1 \le i \le r)$. Hence

$$e(t) = \frac{1}{k_0}a_i \ge \frac{1}{k_0}\frac{2}{\beta}k_0 x(t) = \frac{2}{\beta}x(t) \tag{19}$$

for μ-a.e. $t \in T_1^i$ $(1 \le i \le r)$. Combining (17), (18) and (19) we have

$$\lambda(x) \ge \frac{||e||_\Phi}{2} \ge \frac{||e\chi_E||_\Phi}{2} \ge \frac{1}{2}\frac{2}{\beta}\,||x\chi_E||_\Phi \ge \frac{1}{2\beta} > 0$$

for any $x \in S(L_\Phi)$. Hence

$$\lambda_{L_\Phi} = \inf\{\lambda(x) : x \in S(L_\Phi)\} \ge \frac{1}{2\beta} > 0,$$

which proves that L_Φ has uniform λ-property.

Necessity. Suppose for the contrary that $\sup_i \left\{\frac{b_i}{a_i}\right\} = \infty$. Passing to a subsequence if necessary, we may assume that $b_n > n^3 a_n$ and $\Psi(p(a_n))\mu(T) \ge 1$. Take disjoint A_n, B_n, $C_n \subset T$ for $n \in \mathbb{N}$ such that

$$\Psi(p(a_n))\mu(A_n) = \Psi(p(b_1))\mu(B_n) = \frac{1}{2n}, \tag{20}$$

and
$$\Psi(p(a_n))\mu(C_n) = 1 - \frac{1}{n}. \tag{21}$$

Clearly
$$\mu(A_n) \to 0 \text{ and } \mu(B_n) \to 0 \tag{22}$$

as $n \to \infty$. Every set C_n $(n \in \mathbb{N})$ decompose into disjoints sets $(C_n^i)_{i=1}^n$ such that $\mu(C_n^i) = \frac{1}{n}\mu(C_n)$ for any $i \le n$. Define $x_1 = a_1\chi_{A_1} + b_1\chi_{B_1}$ and

$$x_n = a_1\chi_{A_n} + b_1\chi_{B_n} + \sum_{i=2}^{n}\left[\left(1 - \frac{1}{i\ln n}\right)a_n + \frac{1}{i\ln n}b_n\right]\chi_{C_n^i}.$$

for every $n > 1$. Then $x_n(t) \in \overline{AF(\Phi)_+}$ for any $n \in \mathbb{N}$ and μ-a.e. $t \in \operatorname{supp} x_n(t)$. For any $k < 1$ and $n \in \mathbb{N}$ we get the following implications

$$\begin{aligned}
&\text{for } \mu\text{-a.e. } t \in C_n \ \ (kx_n(t) < b_n) \Longrightarrow p(kx_n(t)) \le p(a_n),\\
&\text{for } \mu\text{-a.e. } t \in B_n \ \ (kx_n(t) < b_1) \Longrightarrow p(kx_n(t)) \le p(a_1),\\
&\text{for } \mu\text{-a.e. } t \in A_n \ \ (kx_n(t) < a_1) \Longrightarrow p(kx_n(t)) \le p(a_1).
\end{aligned}$$

It follows, by (20) and (21), that

$$\begin{aligned}
\int_T \Psi(p(kx_n(t))d\mu &= \int_{C_n} \Psi(p(kx_n(t))d\mu + \int_{B_n} \Psi(p(kx_n(t))d\mu + \int_{A_n} \Psi(p(kx_n(t))d\mu\\
&\le \Psi(p(a_n)\mu(C_n) + \Psi(p(a_1)\mu(B_n) + \Psi(p(a_1)\mu(A_n) \le 1
\end{aligned}$$

for any $n \in \mathbb{N}$. On the other hand for any $k > 1$ and $n \in \mathbb{N}$ we get

$$\begin{aligned}
&\text{for } \mu\text{-a.e. } t \in C_n \ \ (kx_n(t) > a_n) \Longrightarrow p(kx_n(t)) \ge p(a_n),\\
&\text{for } \mu\text{-a.e. } t \in B_n \ \ (kx_n(t) > b_1) \Longrightarrow p(kx_n(t)) \ge p(b_1),\\
&\text{for } \mu\text{-a.e. } t \in A_n \ \ (kx_n(t) > a_1) \Longrightarrow p(kx_n(t)) \ge p(a_1).
\end{aligned}$$

By (20) and (21), it follows that

$$\int_T \Psi(p(kx_n(t))d\mu = \int_{C_n} \Psi(p(kx_n(t))d\mu + \int_{B_n} \Psi(p(kx_n(t))d\mu + \int_{A_n} \Psi(p(kx_n(t))d\mu$$
$$\geq \Psi(p(a_n)\mu(C_n) + \Psi(p(b_1)\mu(B_n) + \Psi(p(a_1)\mu(A_n) \geq 1$$

for any $n \in \mathbb{N}$. It follows from the definition of k^* and k^{**} that $k^*(x_n) = k^{**}(x_n) = 1$. Hence $K(x_n) = \{1\}$ for any $n \in \mathbb{N}$.

Now, let $\lambda_n \in (0,1)$, $y_n \in B(L_\Phi)$, $e_n \in \operatorname{ext} B(L_\Phi)$ be such that

$$\frac{x_n}{\|x_n\|_\Phi} = \lambda_n e_n + (1-\lambda_n) y_n.$$

for any $n \in \mathbb{N}$. By the definition of the Orlicz norm we conclude that

$$\left\| \frac{x_n}{\|x_n\|_\Phi} \right\|_\Phi \leq \frac{1}{k}\left(1 + I_\Phi\left(k\left(\lambda_n e_n + (1-\lambda_n) y_n\right)\right)\right) \tag{23}$$

for any $k > 0$.

The next point in the order of business is to show that $\lambda_n \to 0$ as $n \to \infty$. By Proposition 2.2 in [4] we have $K(e_n) \neq \varnothing$ and $K(y_n) \neq \varnothing$. Take $k_{e_n} \in K(e_n)$ and $k_{y_n} \in K(y_n)$. Then we obtain

$$1 = ||e_n||_\Phi = \frac{1}{k_{e_n}}\left(1 + I_\Phi\left(k_{e_n} e_n\right)\right), \quad \text{and} \quad 1 \geq ||y_n||_\Phi = \frac{1}{k_{y_n}}\left(1 + I_\Phi\left(k_{y_n} y_n\right)\right) \tag{24}$$

for any $n \in \mathbb{N}$. Hence, by Lemma 2.2, we get

$$\frac{k_{e_n} k_{y_n}}{(1-\lambda_n) k_{e_n} + \lambda_n k_{y_n}} \in K\left(\frac{x_n}{\|x_n\|_\Phi}\right)$$

for any $n > 1$. It is easy to see that if $k \in K(x_n)$, then $k\|x_n\|_\Phi \in K\left(\frac{x_n}{\|x_n\|_\Phi}\right)$. Since $K(x_n) = \{1\}$, we have

$$\|x_n\|_\Phi = \frac{k_{e_n} k_{y_n}}{(1-\lambda_n) k_{e_n} + \lambda_n k_{y_n}}$$

for any $n \in \mathbb{N}$. Then $\dfrac{1}{\|x_n\|_\Phi} = \dfrac{1-\lambda_n}{k_{y_n}} + \dfrac{\lambda_n}{k_{e_n}} > \dfrac{\lambda_n}{k_{e_n}}$ for any $n \in \mathbb{N}$, whence

$$\lambda_n < \frac{k_{e_n}}{\|x_n\|_\Phi} \tag{25}$$

for any $n \in \mathbb{N}$. By Lemma 2.2, we get

$$\begin{aligned} &k_{e_n} e_n(t) \in [a_n, b_n] \text{ for } t \in C_n, \\ &k_{y_n} y_n(t) \in [a_n, b_n] \text{ for } t \in C_n, \\ &k_{e_n} e_n(t) = k_{y_n} y_n(t) = x(t) \text{ for } t \in A_n \cup B_n. \end{aligned} \tag{26}$$

for any $n \in \mathbb{N}$.

Since $e_n \in \operatorname{ext}B\left(L_\Phi\right)$, it follows from Theorem 1 in [12] that either

$$k_{e_n}e_n(t) = a_n \text{ or } k_{e_n}e_n(t) = b_n.$$

Combining the inequalities

$$\Phi(p(a_n)) \geq (b_n - a_n)\, p\left(a_n\right)$$

and

$$\Psi\left(p\left(a_n\right)\right) = a_n p\left(a_n\right) - \Phi\left(a_n\right) < a_n p\left(a_n\right)$$

with the equality (21), we get

$$\begin{aligned}
\Phi\left(b_n\right)\mu\left(C_n\right) &\geq \frac{\frac{b_n - a_n}{a_n} p\left(a_n\right) a_n \left(1 - \frac{1}{n}\right)}{\Psi\left(p\left(a_n\right)\right)} \\
&\geq \left(\frac{b_n}{a_n} - 1\right) \frac{p\left(a_n\right) a_n}{\Psi\left(p\left(a_n\right)\right)} \left(1 - \frac{1}{n}\right) \\
&> \left(\frac{b_n}{a_n} - 1\right)\left(1 - \frac{1}{n}\right) \geq \left(n^3 - 1\right)\left(1 - \frac{1}{n}\right). \qquad (27)
\end{aligned}$$

For any $n \in \mathbb{N}$ define the set $H_n = \left\{t \in C_n : k_{e_n} e_n\left(t\right) = b_n\right\}$ and the positive integer

$$i(n) = \max\left\{i \leq n : \mu\left(C_n^i \cap H_n\right) > 0\right\}.$$

Then

$$\begin{aligned}
&\left(1 - \frac{1}{i\left(n\right)\ln n}\right) a_n + \frac{1}{i\left(n\right)\ln n} b_n = x_n\left(t\right) \\
&= \frac{\lambda_n \left\|x_n\right\|_\Phi}{k_{e_n}} k_{e_n} e_n\left(t\right) + \frac{\left(1 - \lambda_n\right)\left\|x_n\right\|_\Phi}{k_{y_n}} k_{y_n} y_n\left(t\right) > \frac{\lambda_n \left\|x_n\right\|_\Phi}{k_{e_n}} b_n. \qquad (28)
\end{aligned}$$

for μ-a.e $t \in C_n^i \cap H_n$. But $\sum\limits_{i=1}^{n} \frac{1}{n} > \ln n$, so we may observe that

$$\begin{aligned}
\left\|x_n\right\|_\Phi &= 1 + I_\Phi\left(x_n\right) \\
&= 1 + \int_{A_n} \Phi\left(x_n\left(t\right)\right) d\mu + \int_{B_n} \Phi\left(x_n\left(t\right)\right) d\mu + \int_{C_n} \Phi\left(x_n\left(t\right)\right) d\mu \\
&> \int_{C_n} \Phi\left(x_n\left(t\right)\right) d\mu = \sum_{i=1}^{n} \int_{C_n^i} \Phi\left(x_n\left(t\right)\right) d\mu \\
&> \sum_{i=1}^{n} \int_{C_n^i} \Phi\left(\frac{1}{i\ln n} b_n\right) d\mu = \sum_{i=1}^{n} \Phi\left(\frac{1}{i\ln n} b_n\right) \mu\left(C_n^i\right) \qquad (29) \\
&> \sum_{i=1}^{n} \frac{1}{i\ln n} \Phi\left(b_n\right) \mu\left(C_n^i\right) = \frac{1}{\ln n} \Phi\left(b_n\right) \mu\left(C_n^i\right) \sum_{i=1}^{n} \frac{1}{i} \\
&> \frac{1}{\ln n} \Phi\left(b_n\right) \mu\left(C_n^i\right) \ln n = \frac{1}{n} \Phi\left(b_n\right) \mu\left(C_n^i\right).
\end{aligned}$$

Additionally, by (24) and (26), we have

$$\begin{aligned}
k_{e_n} &= 1 + I_\Phi\left(k_{e_n} e_n\right)\\
&= 1 + \int_{A_n} \Phi\left(k_{e_n} e_n(t)\right) d\mu + \int_{B_n} \Phi\left(k_{e_n} e_n(t)\right) d\mu + \int_{C_n} \Phi\left(k_{e_n} e_n(t)\right) d\mu\\
&= 1 + \int_{A_n} \Phi\left(a_1\right) d\mu + \int_{B_n} \Phi\left(b_1\right) d\mu + \int_{C_n} \Phi\left(k_{e_n} e_n(t)\right) d\mu\\
&< 1 + \int_{A_n} \Phi\left(a_1\right) d\mu + \int_{B_n} \Phi\left(b_1\right) d\mu + \int_{C_n} \Phi\left(a_n\right) d\mu +\\
&\quad + \sum_{i<i(n)} \int_{C_n^i \cap H_n} \Phi\left(b_n\right) d\mu\\
&< 1 + \Phi\left(a_1\right) \mu\left(A_n\right) + \Phi\left(b_1\right) \mu\left(B_n\right) + \Phi\left(a_n\right) \mu\left(C_n\right) +\\
&\quad + \sum_{i<i(n)} \Phi\left(b_n\right) \mu\left(C_n^i \cap H_n\right)\\
&< 1 + \Phi\left(a_1\right) \mu\left(A_n\right) + \Phi\left(b_1\right) \mu\left(B_n\right) + \Phi\left(a_n\right) \mu\left(C_n\right) +\\
&\quad + \frac{1}{n} i(n) \Phi\left(b_n\right) \mu\left(C_n\right).
\end{aligned} \tag{30}$$

Consider two cases:

(a) Let $i(n) = 0$. We know that $\Phi(b_n) \geq \Phi(n^3 a_n) > n^3 \Phi(a_n)$, so combining inequality (25) with inequalities (27), (29), (30) and (22), we have

$$\begin{aligned}
\lim_{n\to\infty} \lambda_n &\leq \lim_{n\to\infty} \frac{k_{e_n}}{\|x_n\|_\Phi}\\
&\leq \lim_{n\to\infty} \left(\frac{1 + \Phi\left(a_1\right) \mu\left(A_n\right) + \Phi\left(b_1\right) \mu\left(B_n\right)}{\frac{1}{n}\Phi\left(b_n\right) \mu\left(C_n\right)} + \right.\\
&\qquad \left. + \frac{\Phi\left(a_n\right) \mu\left(C_n\right) + \frac{1}{n} i(n) \Phi\left(b_n\right) \mu\left(C_n\right)}{\frac{1}{n}\Phi\left(b_n\right) \mu\left(C_n\right)} \right)\\
&= \lim_{n\to\infty} \frac{1 + \Phi\left(a_1\right) \mu\left(A_n\right) + \Phi\left(b_1\right) \mu\left(B_n\right) + \Phi\left(a_n\right) \mu\left(C_n\right)}{\frac{1}{n}\Phi\left(b_n\right) \mu\left(C_n\right)}\\
&\leq \lim_{n\to\infty} \frac{n}{\left(n^3 - 1\right)\left(1 - \frac{1}{n}\right)} \left(1 + \Phi\left(a_1\right) \mu\left(A_n\right) + \Phi\left(b_1\right) \mu\left(B_n\right)\right) + \frac{1}{n^2}\\
&= 0.
\end{aligned}$$

(b) Now let $i(n) > 0$. Combining conditions (28), (29), (30) with the fact that $i(n) \leq n$ and $\frac{a_n}{b_n} < \frac{1}{n^3}$, we conclude

$$\begin{aligned}
\lambda_n &< \left(\left(1 - \frac{1}{i(n)\ln n}\right) a_n + \frac{1}{i(n)\ln n} b_n \right) \frac{k_{e_n}}{b_n \|x_n\|_\Phi}\\
&< \left(\left(\frac{i(n)\ln n - 1}{i(n)\ln n}\right) \frac{a_n}{b_n} + \frac{1}{i(n)\ln n} \right) \frac{k_{e_n}}{\|x_n\|_\Phi}
\end{aligned}$$

$$
\begin{aligned}
&< \frac{n\ln n + n^3}{i(n)\, n^3 \ln n} \frac{k_{e_n}}{\|x_n\|_\Phi} \\
&< \frac{n\ln n + n^3}{i(n)\, n^3 \ln n} \left(\frac{n\left(1 + \Phi(a_1)\,\mu(A_n) + \Phi(b_1)\,\mu(B_n)\right)}{(n^3-1)\left(1-\frac{1}{n}\right)} + \frac{1}{n^2} + i(n) \right) \\
&= \frac{n\ln n + n^3}{i(n)\, n^3 \ln n} \left(\frac{n\left(1 + \Phi(a_1)\,\mu(A_n) + \Phi(b_1)\,\mu(B_n)\right)}{(n^3-1)\left(1-\frac{1}{n}\right)} + \frac{1}{n^2} \right) + \\
&\quad + \frac{1}{n^2} + \frac{1}{\ln n.}
\end{aligned}
$$

Thus $\lim\limits_{n\to\infty} \lambda_n = 0$, whence L_Φ fails the uniform λ-property. □

Theorem 2.7. *Let Φ be an Orlicz function such that $b_\Phi < \infty$. If $\mu(T) < \infty$ or $\mu(T) = \infty$ with $a_\Phi > 0$, then L_Φ^0 has the uniform λ-property.*

Proof. Let $x \in S(L_\Phi)$. Without loss of generality, we can assume that $x(t) \geq 0$ for μ-a.e $t \in T$. Define

$$
\alpha = \frac{1}{2\,||\chi_T||_\Phi} \quad \text{and} \quad E = \{t \in T \colon x(t) > \alpha\}\,.
$$

Repeating the same arguments as at the beginning of the proof of Theorem 2.6, we obtain

$$
||x\chi_T||_\Phi \geq \frac{1}{2}. \tag{31}
$$

Define the function $e(t)$ by the formula (6). Then, by Lemma 1.5, we have

$$
\lambda(x) \geq \frac{||e||_\Phi}{2}. \tag{32}
$$

Consider two cases.

Case 1. Assume that the function Φ is affine on $[0, b_1]$. Then $a_\Phi = a_1 = 0$. Similarly as in the proof of sufficiency of Theorem 2.5 in [4] we observe that there is a set H of positive measure such that $k_0 x(t) \geq \frac{b_1}{2}$ for μ-a.e. $t \in H$. Then $e(t) = \frac{b_1}{k_0}$ for μ-a.e. $t \in H$. Hence, by Lemma 2.1, it follows that

$$
\begin{aligned}
\lambda(x) &\geq \frac{||e||_\Phi}{2} \geq \frac{||e\chi_H||_\Phi}{2} = \frac{b_1}{2k_0}\,||\chi_H||_\Phi \geq \frac{b_1\,||x||_\infty}{2b_\Phi}\,||\chi_H||_\Phi \\
&\geq \frac{b_1 \left|\left|\frac{1}{2||\chi_T||_\Phi}\chi_E\right|\right|_\infty}{2b_\Phi}\,||\chi_h||_\Phi = \frac{b_1}{b_\Phi 4\,||\chi_T||_\Phi}\,||\chi_H||_\Phi \\
&\geq \frac{b_1}{4b_\Phi^2\,||\chi_T||_\Phi} > 0.
\end{aligned}
$$

Case 2. Suppose that $a_1 \neq 0$. Define

$$
\beta = 1 + \sup_i \left\{ \frac{b_i}{a_i} \right\}.
$$

Similarly as in the proof of sufficiency of Theorem 2.6, we have

$$\lambda(x) \geq \frac{||e||_\Phi}{2} \geq \frac{||e\chi_E||_\Phi}{2} \geq \frac{1}{2}\frac{2}{\beta}\,||x\chi_E||_\Phi \geq \frac{1}{2\beta} > 0.$$

Combining both cases, by the inequality (1), we conclude that

$$\lambda(L_\phi^0) \geq \min\left\{\frac{1}{4\beta}, \frac{b_1}{8b_\Phi^2\,||\chi_T||_\Phi}\right\} > 0.$$

Hence L_ϕ^0 has the uniform λ-property. □

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