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Dual Formulations of the Elasticity Problem for a Homogeneous Elastic Body with Fractures

Riuji Sato*, **Bogdan Vernescu**
Dept. of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, U.S.A.
risato@wpi.edu, vernescu@wpi.edu

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We extend previous results on the dual formulations for an elastic body without fractures to a model of a homogeneous elastic body with fractures. In particular, in the framework of Legendre-Fenchel duality, we were able to provide three equivalent formulations for the problem where the displacement, the stress, and the strain are the unknowns respectively. We also provide a characterization of the image of the convex cone of admissible displacements under the linearized strain tensor.

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1. Introduction

The duality of displacement, stress, and strain formulations in elasticity without fractures was studied by Ciarlet et al., in [4]. In particular, they considered an homogeneous elastic body Ω in $\mathbb{R}^3$ with a body force $\boldsymbol{f}$ acting on it and surface traction $\boldsymbol{F}$ on a part of the boundary Γ_1. The three-dimensional linearized elasticity problem is then written as the following minimization problem:

Problem 1.1. (Displacement formulation) *Find* $\boldsymbol{u} \in \boldsymbol{V}$ *such that*

$$J(\boldsymbol{u}) = \inf_{\boldsymbol{v} \in \boldsymbol{V}} J(\boldsymbol{v}),$$

where, $J(\boldsymbol{v}) := \dfrac{1}{2}\displaystyle\int_\Omega A\nabla_S(\boldsymbol{v}) : \nabla_S(\boldsymbol{v})\, dx - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\, dx - \int_{\Gamma_1} \boldsymbol{F}\cdot\boldsymbol{v}\, d\Gamma.$

Here, $J(\cdot)$ may be interpreted as the potential energy of the body and the above minimization problem can be thought of as a modern analog of the classical principle of minimum potential energy. $\boldsymbol{V}$ is the set of admissible displacements. Its definition is similar to the one presented in Section 2, but without any fractures.

The problem can also be formulated as a another minimization problem, for which the stress is the unknown:

Problem 1.2. (Stress formulation) *Find* $\boldsymbol{\sigma} \in \mathbb{S}$ *such that*

$$g(\boldsymbol{\sigma}) = \inf_{\boldsymbol{\mu} \in \mathbb{S}} g(\boldsymbol{\mu}), \quad \textit{where} \quad g(\boldsymbol{\mu}) = \frac{1}{2}\int_\Omega B\boldsymbol{\mu} : \boldsymbol{\mu}\, dx.$$

* Corresponding author.

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Here B is the compliance tensor, i.e., $AB = I$. The function $g(\cdot)$ is the complimentary energy, and the above problem can be thought of as a modern version of the classical principle of minimum complimentary energy. Here $\mathbb{S}$ is the set of admissible stresses. We present a similar definition in Section 2.

Lastly, the authors present a different approach to the problem where the strain tensor field is the unknown. This is known as the intrinsic approach in some sources (see [4] and [5] and the references within):

Problem 1.3. (Strain formulation) *Find* $\boldsymbol{\mu} \in \mathbb{M}^{\perp}$ *such that*

$$\tilde{J}(\boldsymbol{\mu}) = \inf_{\boldsymbol{\mu} \in \mathbb{M}^*} \tilde{J}(\boldsymbol{\mu}),$$

where $\tilde{J}(\boldsymbol{\mu}) := \dfrac{1}{2} \displaystyle\int_{\Omega} A\nabla_S(\boldsymbol{\mu}) : \nabla_S(\boldsymbol{\mu})\, dx - \int_{\Omega} \boldsymbol{f} \cdot \mathcal{L}(\boldsymbol{\mu})\, dx - \int_{\Gamma_1} \boldsymbol{F} \cdot \mathcal{L}(\boldsymbol{\mu}).$

Here, $\mathbb{M}$ is a set of tensors that has divergence $\mathbf{0}$ in $\boldsymbol{H}^{-1}(\Omega)$ such that the linear functionals acting on the trace space $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ defined by these tensors is zero. Here $\mathbb{M}^{\perp}$ refers to the orthogonal complement of $\mathbb{M}$ in $\mathbb{L}^2_S(\Omega_F)$. $\mathbb{L}^2_S(\Omega_F)$ is defined in Section 2. See [4] for more details.

Ciarlet et al. were able to show using Legendre-Fenchel duality that the displacement and stress formulations and the strain and stress formulations are dual formulations. The arguments for strong duality, i.e., the primal and dual problems attains the same objective value, relies on known results on elasticity. While it is already known that these are dual problems, the novelty lies in obtaining dual formulations through Legendre-Fenchel duality theory.

In this paper, we extend the results in [4] to the case of a fractured elastic body. It is a nonlinear extension of the results of that paper, since the space of admissible displacements for this case is not a linear space. A similar nonlinear extension can also be found in [8].

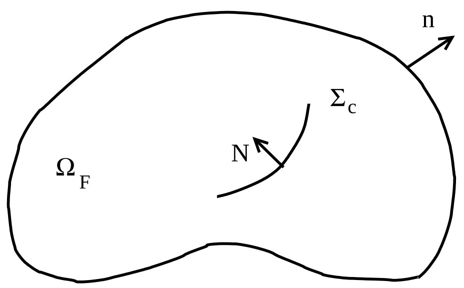

Figure 1.1: Elastic solid with fracture

A model of the elasticity problem with fractures can be found in [14]. Here, the author assumed that the elastic body, Ω having a fixed boundary $\partial\Omega$ is homogeneous and contains a fracture inside its interior. The fracture is thought to be a smooth orientable surface which may or may not be connected, and is denoted by Σ_c. We write as Ω_F the set $\Omega \backslash \Sigma_c$. The formulation of the problem is written in the following form:

Find $\boldsymbol{u}$ such that:

$$\operatorname{div} \boldsymbol{\sigma} + \boldsymbol{f} = 0 \text{ in } \Omega_F \tag{1}$$

$$\boldsymbol{\sigma} = \boldsymbol{A}\nabla_S(\boldsymbol{u}) \text{ in } \Omega_F \tag{2}$$

$$\boldsymbol{u} = \boldsymbol{0} \text{ on } \partial\Omega \tag{3}$$

$$[\boldsymbol{u} \cdot \boldsymbol{N}] \geq 0 \text{ on } \Sigma_c \tag{4}$$

$$\boldsymbol{\sigma n}|_1 = \sigma_{NN}\boldsymbol{N};\ \boldsymbol{\sigma n}|_2 = -\sigma_{NN}\boldsymbol{N};\ \sigma_{NN} \leq 0 \text{ on } \Sigma_c \tag{5}$$

$$\text{if } [\boldsymbol{u} \cdot \boldsymbol{N}] > 0 \text{ on } F, \text{then } \sigma_{NN} = 0. \tag{6}$$

Here, $\boldsymbol{N}$ refers to the unit normal on Σ_c, n is the outward unit normal on the boundary of Ω_F, $[\phi] = \phi_1 - \phi|_2$ refers to the jump of the field ϕ across the fracture Σ_c, where the subscripts 1 and 2 denote the faces of Σ_c, $\sigma_{NN} = \boldsymbol{\sigma N} \cdot \boldsymbol{N}$. $A = [a_{ijkl}]$ is the elasticity tensor, assumed to have symmetry and positivity properties, i.e.,

$$AB \cdot B > 0, \text{ for all } B \in \mathbb{R}^{3\times3}, \tag{7}$$

$$a_{ijkl} = a_{ijlk} = a_{jikl} = a_{jilk}, \tag{8}$$

$\boldsymbol{f}$ represents the body forces acting on the body. $\nabla_S(\cdot)$ is the linearized strain tensor.

Equation (2) gives the constitutive relation for the elastic body, (3) says that on the outer boundary, the displacement is fixed, (4) implies that the body cannot penetrate itself on the crack, (5) shows that there is no friction on the crack and there is compression on it. Finally, (6) says that if the crack if open, there are no stresses on Σ_c.

Introducing the following spaces:

$$\boldsymbol{V}_F = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega_F) \mid \boldsymbol{v} = 0 \text{ on } \partial\Omega\},$$

$$\boldsymbol{K}_F = \{\boldsymbol{v} \in \boldsymbol{V} \mid [v_n] \geq 0 \text{ on } \Sigma_c\}.$$

the problem is shown to be equivalent to the following variational formulation:

Find $\boldsymbol{u} \in K_F$ such that

$$\int_{\Omega_F} A\nabla_S(\boldsymbol{u}) : \nabla_S(\boldsymbol{v} - \boldsymbol{u})\,\mathrm{dx} \geq \int_{\Omega_F} \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{u})\,\mathrm{dx} \ \ \forall \boldsymbol{v} \in K_F. \tag{9}$$

Using classical results on variational inequalities, the author shows that the above problem has a unique solution. Kovtunenko in [12] studied the problem with the assumption that there is Coulomb friction on the fracture. The author introduced appropriate trace spaces for functions defined on the fracture as well as Green formulas. Using fixed point methods, the author was able to show existence of a solution to the variational formulation of the problem. We also mention [15] where the authors prove a homogenization result for an elastic solid with periodically distributed fractures using Γ-convergence and Mosco convergence.

We need a suitable characterization of symmetric tensors as strain of admissible displacements. The difficulty lies in the fact that the set of admissible displacements is a convex cone rather than a linear space as in [4]. Ciarlet et al. used the classical Banach closed range theorem to obtain this characterization. However, these are not directly applicable to our case.

Craven and Koliha in [6] obtained a generalization of Farkas' theorem. They provide necessary and sufficient conditions on the solvability of a linear problem posed in locally convex spaces. We use this characterization to write a suitable strain formulation to the elasticity problem with fractures.

In [4], Ciarlet et al. worked with a minimzation of the form

$$\inf_{x\in X}\left(f(v)+(g\circ h)(x)\right).$$

They were able to apply results from classical convex analysis. In particular, they considered the case where $f: X\to\mathbb{R}\cup\{+\infty\}$, $g: Y\to\mathbb{R}\cup\{+\infty\}$, $h: X\to Y$ is a continuous linear map, and X and Y are Banach spaces.

Under certain conditions, we have strong duality, i.e.,

$$\inf_{x\in X}\left(f(x)+(g\circ h)(x)\right)=\sup_{y^*\in Y^*}\left(-f^*\left(h^*(y^*)\right)-g^*(-y^*)\right).$$

Here f^* and g^* are the Fenchel conjugates of f and g, respectively, and Y^* is the dual space of Y.

However, in this paper, h is not linear. Moreover, the supremum problem is posed in a dual cone of a convex set. In order to obtain strong duality, we make use of [3], where the author considered dual formulations to minimzation problems of the same form but with h not necessarily linear, and X and Y are locally separated convex spaces. More details are provided in the following discussions.

In Section 2 we introduce some functional analytic preliminaries and the notation used in the paper. Section 3 gives the displacement, stress, and the strain formulations of the elasticity problem with fractures. These form analog formulations to those in [4]. Section 4 provides some preliminary results needed to obtain dual formulations of the stress problem in away similar to [4]. Section 5 talks about the characterization of the image of the strain operator under the set of admissible displacements. In Section 6, we obtain dual formulations to the stress formulation of the problem. We show that these dual problems are equivalent to the displacement and strain formulations. Furthermore, we prove strong duality. Lastly, we prove a relationship between the solutions of these problems.

2. Notation and preliminaries

Let Ω be a bounded domain in $\mathbb{R}^3$ with boundary denoted by Γ. Let Σ_c, the fracture, be an open oriented surface contained inside Ω, without self-intersection, and may not necessarily be connected, i.e., there may be several fractures. We denote $\Omega_F=\Omega\backslash\overline{\Sigma}_c$, $\overline{\Sigma}_c=\Sigma_c\cup\partial\Sigma_c$, where $\partial\Sigma_c$ is the boundary of the fracture. We assume that there is an extension Σ of Σ_c such that it divides the domain Ω into two subdomains Ω_1 and Ω_2 such that $\partial\Omega_1=\Sigma^-$ and $\partial\Omega_2=\Gamma\cup\Sigma^+$, where $\partial\Omega_1$ and $\partial\Omega_2$ are the boundaries of Ω_1 and Ω_2, respectively. We denote by N the unit normal vector on Σ and define by $\Sigma^\pm$ the opposite faces of Σ, with N pointing outwards of Σ^+. We let $\Sigma_c^\pm$ be the corresponding sections of $\Sigma^\pm$. We say that the boundary $\partial\Omega_F$ belongs to $C^{k,1}$ if $\partial\Omega_1$ and $\partial\Omega_2$ belong to $C^{k,1}$. Hereafter, we assume that $\partial\Omega_F$ belongs to $C^{1,1}$, so that the Green's formulas and trace theorems from [12] hold.

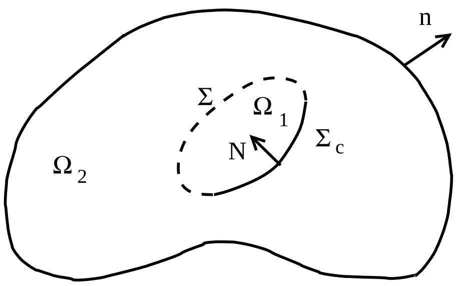

Figure 2.1: Extension of the fracture to Σ

We denote by $A = [a_{ijkh}]_{1\leq i,j,k,h\leq 3}$ the fourth-order elasticity tensor. We assume that

$$a_{ijkh} \in L^\infty(\Omega_F) \tag{10}$$

$$a_{ijkh} = a_{jikh} = a_{khij}, \quad \forall i,j,k,h = 1,2,3 \tag{11}$$

$$\exists\alpha > 0 \text{ such that } \alpha|\boldsymbol{\sigma}|^2 \leq A\boldsymbol{\sigma} : \boldsymbol{\sigma} \quad \forall\boldsymbol{\sigma} \in \mathbb{R}^{3\times 3} \tag{12}$$

$$\exists\beta > 0 \text{ such that } |A\boldsymbol{\sigma}| \leq \beta|\boldsymbol{\sigma}| \quad \forall\boldsymbol{\sigma} \in \mathbb{R}^{3\times 3} \tag{13}$$

These imply the existence of the tensor $B = [b_{ijkh}]_{1\leq i,j,k,h\leq 3}$ such that $AB = I$ and having similar boundedness, coercivity, and symmetry properties.

Vector fields are denoted by bold lowercase Roman letters. Matrix fields are written using bold Greek letters. Sets and subsets of vector fields in $\mathbb{R}^3$ are denoted using capital, boldface Roman letters. Sets and subsets of matrix fields are written using special Roman capitals. We append a subscript S to denote spaces of symmetric matrix fields. We use the Einstein convention on repeated indices.

We use the conventional notations for the Sobolev spaces such as $H^1(\Omega)$, $H^{\frac{1}{2}}(\Sigma)$, and $H^{-\frac{1}{2}}(\Sigma)$. The set of infinitely differential functions with compact support defined on a set Ω is denoted by $D(\Omega)$. We apply the notation convention on sets of vector and tensor fields described above to these spaces whenever appropriate. We write the H^1-norm on a set Ω as $\|\cdot\|_{1,\Omega}$, where we omit Ω in the subscript when the context is clear. For the L^2-norm on a set Ω, we write the norm as $\|\cdot\|_{0,\Omega}$, where similarly, we omit Ω in the subscript when the context is clear.

We will make use of the following space to describe functions defined on the fracture:

$$\boldsymbol{H}^{\frac{1}{2}}_{00}(\Sigma_c) := \{\boldsymbol{v} \in \boldsymbol{H}^{\frac{1}{2}}(\Sigma_c) \mid d^{-\frac{1}{2}}\boldsymbol{v} \in \boldsymbol{L}^2(\Sigma_c)\},$$

where $d \in C^{1,1}(\overline{\Sigma}_c)$, $d > 0$, $d = 0$ on $\partial\Sigma_c$, and

$$\lim_{x\to x_0} \frac{d(x)}{\operatorname{dist}(x,\partial\Sigma_c)} = \alpha \neq 0 \quad \text{for every } x_0 \in \partial\Sigma_c.$$

$\operatorname{dist}(x,\partial\Sigma_c)$ refers to the distance from $x \in \Sigma_c$ to $\partial\Sigma_c$. This space is a Hilbert space with the norm

$$\|\boldsymbol{v}\|^2_{00,\Sigma_c} = \|\boldsymbol{v}\|^2_{\frac{1}{2},\Sigma_c} + \left\|d^{-\frac{1}{2}}\boldsymbol{v}\right\|^2_{0,\Sigma_c}.$$

We write the duality pairing on $\boldsymbol{H}^{\frac{1}{2}}(\Sigma)$ and its dual by $\langle\cdot,\cdot\rangle_{\frac{1}{2},\Sigma}$. Similarly, the duality pairing between $\boldsymbol{H}^{\frac{1}{2}}_{00}(\Sigma_c)$ and its dual is denoted by $\langle\cdot,\cdot\rangle_{00,\Sigma_c}$. We denote the jump across Σ_c by $[\boldsymbol{v}] = \boldsymbol{v}^+ - \boldsymbol{v}^-$, where $\boldsymbol{v}^\pm$ refers to the trace of $\boldsymbol{v}$ on corresponding faces of $\Sigma_c^\pm$.

We recall some trace theorems on these spaces. For details, see [12].

Proposition 2.1. (Trace Theorem 1) *Let the boundary Γ belong to the class $C^{0,1}$, and let a function $\boldsymbol{u}$ belong to the space $\boldsymbol{H}^1(\Omega)$. Then there exists a linear continuous operator $\gamma : \boldsymbol{H}^1(\Omega) \to \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$, which uniquely defines the trace $\gamma\boldsymbol{u} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ of $\boldsymbol{u}$ at Γ. Conversely, there exists a linear continuous operator $\boldsymbol{H}^{\frac{1}{2}} \to \boldsymbol{H}^1(\Omega)$ such that for any given $\boldsymbol{\varphi} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$, a function $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ can be found such that $\gamma\boldsymbol{u} = \boldsymbol{\varphi}$ on Γ.*

Proposition 2.2. (Trace Theorem 2) *Let the boundary $\partial\Omega_F$ belong to the class $C^{0,1}$, and let a function $\boldsymbol{u}$ belong to $\boldsymbol{H}^1(\Omega_F)$. Then there exists a linear continuous operator which uniquely defines at $\partial\Omega_F$ the values*

$$\boldsymbol{u}|_\Gamma \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma),\ \ \boldsymbol{u}^\pm \in \boldsymbol{H}^{\frac{1}{2}}(\Sigma_c),\ \ [\boldsymbol{u}] \in \boldsymbol{H}^{\frac{1}{2}}_{00}(\Sigma_c).$$

Conversely, there exists a linear continuous operator such that for any given

$$\boldsymbol{\psi} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma),\ \ \boldsymbol{\varphi}^\pm \in \boldsymbol{H}^{\frac{1}{2}}(\Sigma_c),\ \ [\boldsymbol{\varphi}] \in \boldsymbol{H}^{\frac{1}{2}}_{00}(\Sigma_c),$$

a function $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega_F)$ can be found such that

$$\boldsymbol{u} = \boldsymbol{\psi} \text{ on } \Gamma,\ \ \boldsymbol{u}^\pm = \boldsymbol{\varphi}^\pm \text{ on } \Sigma_c.$$

We now define the following sets which will be important in the following discussions.

$$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega_F) \mid \boldsymbol{v} = 0 \,\text{on}\, \boldsymbol{H}^{\frac{1}{2}}(\Gamma)\},$$

$$\boldsymbol{K} = \{\boldsymbol{v} \in \boldsymbol{V} \mid [v_n] \geq 0 \,\text{on}\, \boldsymbol{H}^{\frac{1}{2}}_{00}(\Sigma_c)\}.$$

For a subset U of a Banach space V, we define the *polar cone* of A as

$$U^- := \{v^* \in V^* \mid \langle v^*, u\rangle_{V,V^*} \leq 0\ \ \forall u \in U\}.$$

Similarly, the *dual cone* of A is defined to be the following set:

$$U^+ := \{v^* \in V^* \mid \langle v^*, u\rangle_{V,V^*} \geq 0\ \ \forall u \in U\}.$$

We denote the duality pairing between V and its dual by $\langle \cdot, \cdot \rangle_{V,V^*}$.

Let X, Z be linear spaces, and C a nonempty convex subset of Z. Then C induces a partial order $\leq_C$ on Z. Indeed, we say $x \leq_C y$ if $y - x \in C$ for $x, y \in Z$.

We say that a function $g : X \to Z \cup \infty_C$ is *C-convex* if for all $x, y \in X$ and $\lambda \in [0,1]$,

$$g(\lambda x + (1-\lambda)y) \leq_C \lambda g(x) + (1-\lambda)g(y).$$

The linearized strain tensor $\nabla_S : \boldsymbol{D}(\Omega_F) \to \mathbb{D}_S(\Omega_F)$ is defined as

$$\nabla_S(\boldsymbol{v}) := \frac{1}{2}\left(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T\right).$$

We also define the *divergence operator* $\operatorname{div} : \mathbb{D}(\Omega_F) \to \boldsymbol{D}(\Omega_F)$:

$$(\operatorname{div} \boldsymbol{\sigma})_i := \frac{\partial \sigma_{ij}}{\partial x_j}.$$

We now define the following space:

$$\mathbb{H}_S(\mathrm{div}, \Omega_F) := \{\boldsymbol{\sigma} \in \mathbb{L}^2_S(\Omega_F) \mid \mathrm{div}\ \boldsymbol{\sigma} \in \boldsymbol{L}^2(\Omega_F)\}.$$

We equip this space with the norm:

$$\|\boldsymbol{\sigma}\|^2_{\mathbb{H}_S(\mathrm{div},\Omega_F)} := \|\boldsymbol{\sigma}\|^2_{\mathbb{L}^2_S(\Omega_F)} + \|\mathrm{div}\ \boldsymbol{\sigma}\|^2_{\boldsymbol{L}^2_S(\Omega_F)} \quad \forall \boldsymbol{\sigma} \in \mathbb{H}_S(\mathrm{div}, \Omega_F).$$

We recall some Green's formulas relevant to our discussions, for details see [12].

Proposition 2.3. (Green Formula 1) *Let the boundary Γ belong to the class $C^{1,1}$ and let a function $\boldsymbol{\sigma}$ belong to $\mathbb{H}(div, \Omega_F)$. There exists a linear continuous operator $\mathbb{H}(div, \Omega_F) \to \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$ which uniquely defines at the boundary Γ the values*

$$\sigma_n, \in H^{-\frac{1}{2}}(\Gamma), \ \ \boldsymbol{\sigma}_\tau \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma), \ \ \boldsymbol{\sigma}_\tau \cdot \boldsymbol{n} = 0,$$

and for all $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ the generalized Green formula holds:

$$\int_\Omega \boldsymbol{\sigma} : \nabla_S(\boldsymbol{v})\, dx = -\int_\Omega div\ \boldsymbol{\sigma} \cdot \boldsymbol{v}\, dx + \langle \sigma_n, v_n \rangle_{\frac{1}{2},\Gamma} + \langle \boldsymbol{\sigma}_\tau, \boldsymbol{v}_\tau \rangle_{\frac{1}{2},\Gamma} \tag{14}$$

Proposition 2.4. (Green Formula 2) *Let the boundary $\partial\Omega_F$ belong to the class $C^{1,1}$, let $\boldsymbol{\sigma}$ belong to $\mathbb{H}(div, \Omega_F)$ such that $[\boldsymbol{\sigma n}] = 0$ on Σ. Then there is a linear continuous operator $\mathbb{H}(div, \Omega_F) \to \left(\boldsymbol{H}^{\frac{1}{2}}_{00}(\Sigma_c)\right)^*$ which uniquely defines at the crack Σ_c the values*

$$\sigma_n \in \left(H^{\frac{1}{2}}_{00}(\Sigma_c)\right)^*, \ \ \boldsymbol{\sigma}_\tau \in \left(\boldsymbol{H}^{\frac{1}{2}}_{00}(\Sigma_c)\right)^*, \ \ \boldsymbol{\sigma}_\tau \cdot \boldsymbol{n} = 0,$$

and for all $\boldsymbol{v} \in V$, the generalized Green formula holds:

$$\int_{\Omega_F} \boldsymbol{\sigma} : \nabla_S(\boldsymbol{v})\, dx = -\int_{\Omega_F} div\ \boldsymbol{\sigma} \cdot \boldsymbol{v}\, dx - \langle \sigma_n, [v_n] \rangle_{00,\Sigma_c} - \langle \boldsymbol{\sigma}_\tau, [\boldsymbol{v}_\tau] \rangle_{00,\Sigma_c}. \tag{15}$$

We now define the set of admissible stresses as:

$$\mathbb{S} = \left\{ \boldsymbol{\sigma} \in \mathbb{H}_S(\mathrm{div}, \Omega_F) \;\middle|\; \begin{array}{l} \mathrm{div}\ \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0} \text{ in } \boldsymbol{D}'(\Omega_F),\ \sigma_n \le 0 \text{ on } H^{\frac{1}{2}}_{00}(\Sigma_c) \\ \boldsymbol{\sigma}_\tau = \boldsymbol{0} \text{ on } \boldsymbol{H}^{\frac{1}{2}}_{00}(\Sigma_c),\ [\sigma_n] = 0 \text{ on } H^{\frac{1}{2}}(\Sigma) \end{array} \right\}.$$

In the coming discussion, it will be important to obtain a characterization of symmetric tensor fields as the strain of vectors coming from the set of admissible displacements. The following set plays an important role in this.

$$\mathbb{M} := \{\boldsymbol{\sigma} \in \mathbb{L}^2_S(\Omega_F) \mid \langle \nabla^*_S(\boldsymbol{\sigma}), \boldsymbol{v} \rangle_{V^*,V} \ge 0\ \forall \boldsymbol{v} \in \boldsymbol{K}\}.$$

3. Formulations of the problem with fractures

We present three formulations of the elasticity problem with fractures. Analogs of these for the case without fractures present are in [4]. We begin with the displacement formulation which can be posed as a minimization problem.

Problem 3.1. (Displacement Formulation) *Find* $\boldsymbol{u} \in \boldsymbol{K}$ *such that*

$$J(\boldsymbol{u}) = \inf_{\boldsymbol{v} \in \boldsymbol{K}} J(\boldsymbol{v}),$$

$$\textit{where} \quad J(\boldsymbol{v}) := \frac{1}{2} \int_{\Omega_F} A \nabla_S(\boldsymbol{v}) : \nabla_S(\boldsymbol{v})\, dx - \int_{\Omega_F} \boldsymbol{f} \cdot \boldsymbol{v}\, dx \quad \textit{for all } \boldsymbol{v} \in \boldsymbol{V}.$$

There is an alternate formulation to this problem in which the stress is the unknown:

Problem 3.2. (Stress Formulation) *Find* $\boldsymbol{\sigma} \in \mathbb{S}$ *such that*

$$g(\boldsymbol{\sigma}) = \inf_{\boldsymbol{\mu} \in \mathbb{S}} g(\boldsymbol{\mu}), \quad \textit{where} \quad g(\boldsymbol{\mu}) = \frac{1}{2} \int_{\Omega_F} B\boldsymbol{\mu} : \boldsymbol{\mu}\, dx \quad \textit{for all } \boldsymbol{\mu} \in \mathbb{L}^2_S(\Omega_F).$$

Finally, we can recast the problem in which the strain is the unknown. This formulation is known in literature as the intrinsic formulation [5]. The use of the set $\mathbb{M}^+$ will be apparent in the coming sections.

Problem 3.3. (Strain Formulation) *Find* $\boldsymbol{\pi} \in \mathbb{M}^+$ *such that*

$$\tilde{J}(\boldsymbol{\pi}) = \inf_{\boldsymbol{\mu} \in \mathbb{M}^+} \tilde{J}(\boldsymbol{\mu}),$$

$$\textit{where} \quad \tilde{J}(\boldsymbol{\mu}) := \frac{1}{2} \int_{\Omega_F} A\boldsymbol{\mu} : \boldsymbol{\mu}\, dx - \int_{\Omega_F} \boldsymbol{f} \cdot \mathcal{L}(\boldsymbol{\mu})\, dx \quad \textit{for all } \boldsymbol{\mu} \in \mathbb{M}^+.$$

Here $\mathcal{L}(\boldsymbol{\mu})$ is the unique element in K such that $\nabla_S(\mathcal{L}(\boldsymbol{\mu})) = \boldsymbol{\mu}$. The existence of such an element in $\boldsymbol{K}$ will be discussed in Section 5.

It will be shown that indeed, up to a change of sign, the displacement and stress formulations are dual problems, and similarly the strain and stress formulations.

4. Auxiliary results

We first show that the stress formulation of the problem can be written as a minimization over the entire space $\mathbb{L}^2_S(\Omega_F)$. To do this, we introduce the following functions. Let $h : \boldsymbol{V}^* \to \overline{\mathbb{R}}$ such that

$$h(\boldsymbol{v}^*) = \mathbb{1}_{\boldsymbol{K}^-}(\boldsymbol{v}^*). \tag{16}$$

We also define $\Lambda : \mathbb{L}^2_S(\Omega_F) :\to \boldsymbol{V}^*$ by

$$\langle \Lambda\boldsymbol{\sigma}, \boldsymbol{v} \rangle_{V^*,V} = \int_{\Omega_F} \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{dx} - \int_{\Omega_F} \boldsymbol{\sigma} : \nabla_S(\boldsymbol{v})\, \mathrm{dx}\ . \tag{17}$$

Proposition 4.1.

$$\inf_{\boldsymbol{\sigma} \in \mathbb{S}} g(\boldsymbol{\sigma}) = \inf_{\boldsymbol{\sigma} \in \mathbb{L}^2_S(\Omega_F)} \left(g(\boldsymbol{\sigma}) + \mathbb{1}_{\mathbb{S}}(\boldsymbol{\sigma})\right) = \inf_{\boldsymbol{\sigma} \in \mathbb{L}^2_S(\Omega_F)} \left(g(\boldsymbol{\sigma}) + h(\Lambda\boldsymbol{\sigma})\right).$$

Proof. Clearly, $\inf_{\boldsymbol{\sigma} \in \mathbb{S}} g(\boldsymbol{\sigma}) = \inf_{\boldsymbol{\sigma} \in \mathbb{L}^2_S(\Omega_F)} \left(g(\boldsymbol{\sigma}) + \mathbb{1}_{\mathbb{S}}(\boldsymbol{\sigma})\right).$

Thus, it suffices to show that $h(\Lambda\boldsymbol{\sigma}) = \mathbb{1}_{\mathbb{S}}(\boldsymbol{\sigma}) \quad \forall \boldsymbol{\sigma} \in \mathbb{L}^2_S(\Omega_F).$

Indeed, let $\boldsymbol{\sigma} \in \mathbb{S}$. Then $\mathbb{1}_{\mathbb{S}}(\boldsymbol{\sigma}) = 0$. Now, for $\boldsymbol{v} \in \boldsymbol{K}$, using Proposition 4,

$$\begin{aligned}\langle \Lambda\boldsymbol{\sigma}, \boldsymbol{v}\rangle_{V^*,V} &= \int_{\Omega_F} \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{dx} - \int_{\Omega_F} \boldsymbol{\sigma} : \nabla_S(\boldsymbol{v})\, \mathrm{dx} \\ &= \int_{\Omega_F} \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{dx} + \int_{\Omega_F} \mathrm{div}\ \boldsymbol{\sigma} \cdot \boldsymbol{v}\, \mathrm{dx} + \langle \sigma_n, [v_n]\rangle_{00,\Sigma_c} \leq 0.\end{aligned}$$

So that, $\Lambda\boldsymbol{\sigma} \in \boldsymbol{K}^-$ and hence, $h(\Lambda\boldsymbol{\sigma}) = 0$.

Now, let $\boldsymbol{\sigma} \in \mathbb{L}^2_S(\Omega_F)$ such that $h(\Lambda\boldsymbol{\sigma}) = 0$, then $\Lambda\boldsymbol{\sigma} \in \boldsymbol{K}^-$ and thus

$$\int_{\Omega_F} \boldsymbol{\sigma} : \nabla_S(\boldsymbol{v})\, \mathrm{dx} \geq \int_{\Omega_F} \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{dx} \quad \forall \boldsymbol{v} \in \boldsymbol{K}. \tag{18}$$

Now, we let $\boldsymbol{\varphi} \in \boldsymbol{D}(\Omega_F)$. Then $\boldsymbol{v} := \pm\boldsymbol{\varphi} \in \boldsymbol{K}$. Substituting in (18), we obtain

$$\int_{\Omega_F} \boldsymbol{\sigma} : \nabla_S(\boldsymbol{\varphi})\, \mathrm{dx} = \int_{\Omega_F} \boldsymbol{f} \cdot \boldsymbol{\varphi}\, \mathrm{dx}.$$

Thus,

$$\mathrm{div}\ \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0} \quad \text{in } \boldsymbol{D}'(\Omega_F). \tag{19}$$

As $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, we have that $\boldsymbol{\sigma} \in \mathbb{H}_S(\mathrm{div}, \Omega_F)$.

We now show that

$$\sigma_n \leq 0 \quad \text{in } \left(H^{\frac{1}{2}}_{00}(\Sigma_c)\right)^*. \tag{20}$$

Let $\boldsymbol{v} \in \boldsymbol{K}$ such that $[\boldsymbol{v}_{\boldsymbol{\tau}}] = 0$ in $\boldsymbol{H}^{\frac{1}{2}}_{00}(\Sigma_c)$. Then by (18) and the second Green's formula (15), we obtain

$$\int_{\Omega_F} \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{dx} \leq \int_{\Omega_F} \boldsymbol{\sigma} : \nabla_S \boldsymbol{v}\, \mathrm{dx} = -\int_{\Omega_F} \mathrm{div}\ \boldsymbol{\sigma} \cdot \boldsymbol{v}\, \mathrm{dx} - \langle \sigma_n, [v_n]\rangle_{00,\Sigma_c}$$

so that by (19), we get $\langle \sigma_n, [v_n]\rangle_{00,\Sigma_c} \leq 0 \quad \forall \boldsymbol{v} \in \boldsymbol{K}$. By the second trace theorem,

$$\langle \sigma_n, \psi\rangle_{00,\Sigma_c} \leq 0 \quad \forall \psi \in H^{\frac{1}{2}}_{00}(\Sigma_c),\ \psi \geq 0,$$

which proves (20). Next, we prove that

$$[\sigma_n] = 0 \quad \text{in } H^{-\frac{1}{2}}(\Sigma). \tag{21}$$

Let $\boldsymbol{\varphi} \in \boldsymbol{D}(\Omega)$ such that $\boldsymbol{\varphi}_\tau = 0$ in $H^{\frac{1}{2}}(\Sigma)$. Since $[\boldsymbol{\varphi}] = 0$ on Σ_c, we have that $\boldsymbol{\varphi} \in \boldsymbol{K}$. Thus, by (18) and the first Green's formula (14),

$$\begin{aligned}\int_{\Omega_F} \boldsymbol{f} \cdot \boldsymbol{\varphi}\, \mathrm{dx} &\leq \int_{\Omega_F} \boldsymbol{\sigma} : \nabla_S(\boldsymbol{\varphi}) \\ &= -\int_{\Omega_F} \mathrm{div}\ \boldsymbol{\sigma} \cdot \boldsymbol{\varphi}\, \mathrm{dx} - \left[\langle \sigma_n, \varphi_n\rangle_{\frac{1}{2},\Sigma}\right] - \left[\langle \boldsymbol{\sigma}_\tau, \boldsymbol{\varphi}_\tau\rangle_{\frac{1}{2},\Sigma}\right] \\ &= -\int_{\Omega_F} \mathrm{div}\ \boldsymbol{\sigma} \cdot \boldsymbol{\varphi}\, \mathrm{dx} - \left[\langle \sigma_n, \varphi_n\rangle_{\frac{1}{2},\Sigma}\right].\end{aligned}$$

By (19) we obtain $\left[\langle \sigma_n, \varphi_n\rangle_{\frac{1}{2},\Sigma}\right] \leq 0$. Using $\pm\boldsymbol{\varphi}$ as test function, we have that

$$\left[\langle \sigma_n, \varphi_n\rangle_{\frac{1}{2},\Sigma}\right] = 0.$$

As $[\boldsymbol{\varphi}] = 0$ on Σ_c, we obtain

$$\langle [\sigma_n], \varphi_n \rangle_{\frac{1}{2},\Sigma} = 0 \quad \forall \boldsymbol{\varphi} \in \boldsymbol{D}(\Omega),\ \boldsymbol{\varphi}_\tau = 0 \text{ in } \boldsymbol{H}^{\frac{1}{2}}(\Sigma),$$

which proves the claim. Lastly, we show that

$$\boldsymbol{\sigma}_\tau = 0 \quad \text{in } \left(\boldsymbol{H}_{00}^{\frac{1}{2}}(\Sigma_c)\right)^*. \tag{22}$$

Indeed, let $\boldsymbol{v} \in \boldsymbol{V}$ such that $[v_n] = 0$ in $H_{00}^{\frac{1}{2}}(\Sigma_c)$. Then, $\boldsymbol{v} \in \boldsymbol{K}$. By (18) and Green's formula (15),

$$\int_{\Omega_F} \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{dx} \le \int_{\Omega_F} \boldsymbol{\sigma} : \nabla_S(\boldsymbol{v}) \, \mathrm{dx} = -\int_{\Omega_F} \operatorname{div} \boldsymbol{\sigma} \cdot \boldsymbol{v} \, \mathrm{dx} - \langle \boldsymbol{\sigma}_\tau, [\boldsymbol{v}_\tau] \rangle_{00,\Sigma_c}.$$

Using (19), we have that $\langle \boldsymbol{\sigma}_\tau, [\boldsymbol{v}_\tau] \rangle_{00,\Sigma_c} \le 0$. So, using $\pm \boldsymbol{v}$ as a test function, we obtain that

$$\langle \boldsymbol{\sigma}_\tau, [\boldsymbol{v}_\tau] \rangle_{00,\Sigma_c} = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V},\ [v_n] = 0.$$

Using the Trace Theorem as in (20), the claim follows. Hence, from (19), (20), (21), and (22), we have that $\boldsymbol{\sigma} \in \mathbb{S}$, i.e., $\mathbb{1}_{\mathbb{S}}(\boldsymbol{\sigma}) = 0$. □

Next, we calculate h^* and h^{**}.

Proposition 4.2. *$h^* = \mathbb{1}_{\boldsymbol{K}}$ and $h^{**} = h$*

Proof. Firstly, since $\boldsymbol{K}$ is a convex set, h is convex on the reflexive space V. Thus, by the Fenchel-Moreau theorem, $h \equiv h^{**}$. We now show that $(\boldsymbol{K}^-)^- = \boldsymbol{K}$. As $\boldsymbol{V}$ is reflexive, we have that

$$(\boldsymbol{K}^-)^- = \left\{\boldsymbol{v} \in \boldsymbol{V} \mid \langle \boldsymbol{v}^*, \boldsymbol{v} \rangle_{\boldsymbol{V}^*,\boldsymbol{V}} \le 0 \quad \forall \boldsymbol{v}^* \in \boldsymbol{K}^-\right\}.$$

If $\boldsymbol{u} \in \boldsymbol{K}$, clearly $\langle \boldsymbol{v}^*, \boldsymbol{v} \rangle_{\boldsymbol{V}^*,\boldsymbol{V}} \le 0$ for all $\boldsymbol{v}^* \in \boldsymbol{K}^-$. Thus, $\boldsymbol{K} \subset (\boldsymbol{K}^-)^-$.

Suppose that $\boldsymbol{u} \notin \boldsymbol{K}$. Since $\boldsymbol{K}$ is a closed and convex subset of $\boldsymbol{V}$, by the Hahn-Banach theorem, there exists a hyperplane which strictly separates $\{\boldsymbol{u}\}$ and $\boldsymbol{K}$, i.e., there is some $\boldsymbol{f} \in \boldsymbol{V}^*$ and $\alpha \in \mathbb{R}$ such that

$$\langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\boldsymbol{V}^*,\boldsymbol{V}} < \alpha < \langle \boldsymbol{f}, \boldsymbol{u} \rangle_{\boldsymbol{V}^*,\boldsymbol{V}} \quad \forall \boldsymbol{v} \in \boldsymbol{K}.$$

Fix $\boldsymbol{v} \in \boldsymbol{K}$ and let $\lambda > 0$. Since $\boldsymbol{K}$ is a convex cone, $\lambda \boldsymbol{v} \in \boldsymbol{K}$. Then,

$$\langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\boldsymbol{V}^*,\boldsymbol{V}} < \frac{1}{\lambda}\alpha \quad \forall \lambda > 0.$$

Letting $\lambda \to +\infty$ we get $\langle \boldsymbol{f}, \boldsymbol{v} \rangle_{V^*,V} \le 0 \quad \forall \boldsymbol{v} \in \boldsymbol{K}$. Thus, $\boldsymbol{f} \in \boldsymbol{K}^-$.

Similarly, $\lambda \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\boldsymbol{V}^*,\boldsymbol{V}} < \alpha$ for all $\lambda > 0$. As $\lambda \to 0^+$, we obtain the inequalities $0 \le \alpha < \langle \boldsymbol{f}, \boldsymbol{u} \rangle_{\boldsymbol{V}^*,\boldsymbol{V}}$, i.e., there is some $\boldsymbol{f} \in \boldsymbol{K}^-$ such that $\langle \boldsymbol{f}, \boldsymbol{u} \rangle_{\boldsymbol{V}^*,\boldsymbol{V}} > 0$. Hence, $\boldsymbol{u} \notin (\boldsymbol{K}^-)^-$ and the claim follows.

Next, we show that $(\mathbb{1}_{\boldsymbol{K}})^* = \mathbb{1}_{\boldsymbol{K}^-}$. As $\boldsymbol{V}$ is reflexive, we have that for $\boldsymbol{v}^* \in \boldsymbol{V}^*$,

$$(\mathbb{1}_{\boldsymbol{K}})^* = \sup_{\boldsymbol{v} \in \boldsymbol{V}} \left(\langle \boldsymbol{v}^*, \boldsymbol{v} \rangle_{\boldsymbol{V}^*,\boldsymbol{V}} - \mathbb{1}_{\boldsymbol{K}}(\boldsymbol{v})\right) = \sup_{\boldsymbol{v} \in \boldsymbol{K}} \left(\langle \boldsymbol{v}^*, \boldsymbol{v} \rangle_{\boldsymbol{V}^*,\boldsymbol{V}}\right).$$

Suppose that $\boldsymbol{v}^* \in \boldsymbol{K}^-$. Then $\langle \boldsymbol{v}^*, \boldsymbol{v}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} \leq 0$ for all $\boldsymbol{v} \in \boldsymbol{K}$. Thus, $(\mathbb{1}_{\boldsymbol{K}})^*(\boldsymbol{v}^*) \leq 0$. Since $\boldsymbol{0} \in \boldsymbol{K}$, we have that

$$(\mathbb{1}_{\boldsymbol{K}})^*(\boldsymbol{v}^*) \geq \langle \boldsymbol{v}^*, \boldsymbol{0}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} = 0.$$

Hence, $(\mathbb{1}_{\boldsymbol{K}})^*(\boldsymbol{v}^*) = \mathbb{1}_{\boldsymbol{K}^-}(\boldsymbol{v}^*) = 0$. Now, assume that $\boldsymbol{v}^* \notin \boldsymbol{K}^-$. Then, there exists some $\boldsymbol{v} \in \boldsymbol{K}$ such that $\langle \boldsymbol{v}^*, \boldsymbol{v}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} > 0$. Hence,

$$(\mathbb{1}_{\boldsymbol{K}})^*(\boldsymbol{v}^*) \geq \sup_{\lambda>0} \left(\langle \boldsymbol{v}^*, \lambda\boldsymbol{v}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}}\right) = +\infty.$$

Thus, $\mathbb{1}_{\boldsymbol{K}^-}(\boldsymbol{v}^*) = (\mathbb{1}_{\boldsymbol{K}})^*(\boldsymbol{v}^*) = +\infty$ and the claim is proved.

Since $\boldsymbol{0} \in \boldsymbol{K}^-$, $\boldsymbol{K}^-$ is a convex cone, and $\boldsymbol{V}^*$ is reflexive, arguing as previously, we obtain that $h^* = (\mathbb{1}_{\boldsymbol{K}^-})^* = \mathbb{1}_{(\boldsymbol{K}^-)^-} = \mathbb{1}_{\boldsymbol{K}}$. □

5. Characterization of the range of the strain operator

The adjoint of the strain operator is the linear map $\nabla_S^* : \mathbb{L}^2_S(\Omega_F) \to \boldsymbol{V}^*$ such that for $\boldsymbol{\sigma} \in \mathbb{L}^2_S(\Omega_F)$, $\nabla_S^*(\boldsymbol{\sigma})$ is defined by,

$$\langle \nabla_S^*(\boldsymbol{\sigma}), \boldsymbol{v}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} = \int_{\Omega_F} \boldsymbol{\sigma} : \nabla_S(\boldsymbol{v})\, \mathrm{dx} \quad \forall \boldsymbol{v} \in \boldsymbol{V}.$$

It is important to obtain a suitable characterization of the image of the convex cone $\boldsymbol{K}$ under the strain operator. This will allow us to obtain a relationship between the stress and the strain formulations of the elasticity problem with fractures. We mention that the first result on the characterization of the space of admissible displacements can be found in [9]. For a characterization of matrix fields as linearized strain tensor fields, we refer to [1]. In their paper, since their space of admissible displacements is a linear space, the classical Banach Closed Range Theorem allows them this suitable characterization. In our case, we use results from [6], in particular Theorem 5, which is a generalization of Farkas' theorem that we tailor to our case in the following proposition.

Proposition 5.1. *If $\nabla_S : \boldsymbol{V} \to \mathbb{L}^2_S(\Omega_F)$ is strongly continuous and $\nabla_S(\boldsymbol{K})$ is strongly closed, then the following are equivalent for $\boldsymbol{\mu} \in \mathbb{L}^2_S(\Omega_F)$:*

- *$\nabla_S(\boldsymbol{v}) = \boldsymbol{\mu}$ has a solution $\boldsymbol{v}$ in $\boldsymbol{K}$*
- *If $\boldsymbol{\sigma} \in \mathbb{L}^2_S(\Omega_F)$ such that $\nabla_S^*(\boldsymbol{\sigma}) \in \boldsymbol{K}^+$, then $\int_{\Omega_F} \boldsymbol{\mu} : \boldsymbol{\sigma}\, dx \geq 0$*

Remark 5.2. The previous lemma implies that $\nabla_S(\boldsymbol{K}) = \mathbb{M}^+$.

In order to utilize the previous remark, we have to show that the hypotheses of Proposition 5.1 hold true. Indeed we have the following result:

Proposition 5.3. *$\nabla_S : \boldsymbol{V} \to \mathbb{L}^2_S(\Omega_F)$ is strongly continuous and $\nabla_S(\boldsymbol{K})$ is strongly closed.*

Proof. There exists some $C > 0$ such that $\|\nabla_S(\boldsymbol{v})\|_0 \leq C \|\boldsymbol{v}\|_1$ $\forall \boldsymbol{v} \in \boldsymbol{V}$. Thus, as ∇_S is a linear map, it is strongly continuous.

To show that $\nabla_S(\boldsymbol{K})$ is strongly closed, let $\{\nabla_S(\boldsymbol{v}_n)\}_{n=1}^{\infty}$ be a sequence of tensors in $\mathbb{L}^2_S(\Omega_F)$ such that $\boldsymbol{v}_n \in \boldsymbol{K}$ for each n and $\nabla_S(\boldsymbol{v}_n) \to \boldsymbol{D}$ for some $\boldsymbol{D} \in \mathbb{L}^2_S(\Omega_F)$. By Korn's inequality, we have that

$$\|\boldsymbol{v}_n - \boldsymbol{v}_m\|_1 \leq C\|\nabla_S(\boldsymbol{v}_n - \boldsymbol{v}_n)\|_0 \to 0,$$

as $n, m \to \infty$. Thus the sequence $\{\boldsymbol{v}_n\}_{n=1}^{\infty}$ is Cauchy in $\boldsymbol{V}$. As $\boldsymbol{K}$ is closed in $\boldsymbol{V}$, $\boldsymbol{v}_n \to \boldsymbol{v}$ in $\boldsymbol{V}$ for some $\boldsymbol{v} \in \boldsymbol{K}$. By strong continuity of ∇_S, we have that $\boldsymbol{D} = \nabla_S(\boldsymbol{v})$. Hence, $\nabla_S(\boldsymbol{K})$ is strongly closed. □

6. Duality

We present a treatment of primal-dual problems taken from [3]. Let X, Z be separated locally convex spaces and $F : X \to \bar{\mathbb{R}}$ be a proper function. To the primal problem

$$\inf_{x \in X} F(x),$$

we can assign a dual problem through the use of perturbation functions. Indeed, let $\Phi : X \times Y \to \bar{\mathbb{R}}$ such that $\Phi(x, 0) = F(x)$ for all $x \in X$. Then the primal problem may be written as

$$\inf_{x \in X} \Phi(x, 0).$$

The dual problem is
$$\sup_{z^* \in Z^*} (-\Phi^*(0, z^*)).$$

By specific choices of the perturbation function $\Phi(\cdot, \cdot)$, we arrive at different dual formulations of a given primal problem.

6.1. Stress-displacement duality

Let X, Z be separable locally convex spaces. Let $C \subset Z$ be a nonempty convex cone that partially orders Z, i.e., for $x, y \in Z$ such that $x \leq_C y$, we have that $y - x \in C$. We attach to Z a greatest element with respect to $\leq_C$, ∞_C, which is not in Z. We have that $x \leq_C \infty_C$ for all $x \in Z \cup \{\infty_C\}$. Let $f : X \to \bar{\mathbb{R}}$ be a proper function, $g : Z \to \bar{\mathbb{R}}$ be a proper, C-increasing function, i.e., $dom\, g := \{x \in Z | g(x) \in \mathbb{R}\} \neq \emptyset$, $g(x) > -\infty$ for all $x \in Z$, and for $x \leq_C y$, $g(x) \leq g(y)$. Let $h : X \to Z \cup \{\infty_C\}$ be a proper function such that $h(dom\, f \cap dom\, h) \cap dom\, g \neq \emptyset$.

We define the primal problem as

$$\inf_{x \in X} \{f(x) + (g \circ h)(x)\}. \tag{23}$$

We define $\Phi : X \times Z \to \bar{\mathbb{R}}$ such that $\Phi(x, z) := f(x) + g(h(x) + z)$ as the perturbation function. It can be shown (see [3]) that $\Phi^* : X^* \times Z^* \to \bar{\mathbb{R}}$ has for $(x^*, z^*) \in X^* \times Z^*$,

$$\Phi(x^*, z^*)^* = g^*(z^*) + (f + (z^* h))^*(x^*) + \mathbb{1}_{C^*}(z^*).$$

Hence, the corresponding dual problem is given by

$$\sup_{z \in C^+} \{-g^*(z^*) - (f + (z^* h))^* (0)\}, \tag{24}$$

Sufficient conditions that guarantee *strong duality*, i.e., the optimal values of the primal and dual problems coincide, are given in the following proposition taken from Chapter 1, Theorem 4.1 of [3].

Proposition 6.1. *Let X, Z be Fréchet spaces, C a nonempty convex cone contained in Z, $g : X \to \overline{\mathbb{R}}$ be proper and convex, $h : Z \to \overline{\mathbb{R}}$ be a $C-$increasing function such that if $z^* \notin C^*$, then $h^*(z^*) = +\infty$, $\Lambda : X \to Z \cup \{\infty_C\}$ be proper, $C-$convex such that $\Lambda(dom\, g \cap dom\, \Lambda) \cap dom\, h \neq \emptyset$. Suppose the following regularity conditions are satisfied:*

- *g and h are lower semicontinuous*
- *Λ is star-C lower semicontinuous*
- *$0 \in core\,(dom\, h - \Lambda(dom\, g \cap\, dom\, \Lambda))$.*

Then,
$$\inf_{x\in X} \left(g(x) + (h \circ \Lambda)\right)(x) = \sup_{z^*\in C^+} \left(-h^*(z^*) - (g + z^*\Lambda)^*(0)\right). \tag{25}$$

We go back to the stress formulation of the elasticity problem. Here, we define $X := \mathbb{L}^2_S(\Omega_F)$, $Z := \boldsymbol{V}^*$, $C := \boldsymbol{K}^+$, and g, h, and Λ to be the functions defined in the preliminaries. From Proposition 4.1, the stress formulation gives rise to the following primal problem

$$\text{(P)} \qquad \inf_{\boldsymbol{\sigma}\in\mathbb{L}^2_S(\Omega_F)} \{g(\boldsymbol{\sigma}) + (\mathbb{1}_{\boldsymbol{K}^-} \circ \Lambda)(\boldsymbol{\sigma})\}. \tag{26}$$

From Proposition 4.2 and (24), we have that the dual problem is:

$$\text{(D1)} \qquad \sup_{\boldsymbol{v}\in\boldsymbol{K}} \{-\mathbb{1}_{\boldsymbol{K}}(\boldsymbol{v}) - (g + \boldsymbol{v}\Lambda)^*(\boldsymbol{0})\}. \tag{27}$$

We show that the dual problem is equivalent to the displacement formulation of the elasticity problem.

Proposition 6.2. *For $\boldsymbol{v} \in \boldsymbol{K}$, we have that*
$$\mathbb{1}_{\boldsymbol{K}}(\boldsymbol{v}) + (g + \boldsymbol{v}\Lambda)^*(\boldsymbol{0}) = J(\boldsymbol{v}). \tag{28}$$

Moreover,
$$\sup_{\boldsymbol{v}\in\boldsymbol{K}} \{-\mathbb{1}_{\boldsymbol{K}}(\boldsymbol{v}) - (g + \boldsymbol{v}\Lambda)^*(\boldsymbol{0})\} = -\inf_{\boldsymbol{v}\in\boldsymbol{K}} J(\boldsymbol{v}). \tag{29}$$

Proof. Let $\boldsymbol{v} \in \boldsymbol{K}$. Then

$$\begin{aligned}
\mathbb{1}_{\boldsymbol{K}}(\boldsymbol{v}) + (g + \boldsymbol{v}\Lambda)^*(\boldsymbol{0}) &= \sup_{\boldsymbol{\sigma}\in\mathbb{L}^2_S(\boldsymbol{\Omega_F})} \{-g(\boldsymbol{\sigma}) - (\boldsymbol{v}\Lambda)(\boldsymbol{\sigma})\} \\
&= \sup_{\boldsymbol{\sigma}\in\mathbb{L}^2_S(\boldsymbol{\Omega_F})} \{-g(\boldsymbol{\sigma}) - \langle\Lambda\boldsymbol{\sigma}, \boldsymbol{v}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}}\} \\
&= \sup_{\boldsymbol{\sigma}\in\mathbb{L}^2_S(\boldsymbol{\Omega_F})} \{-g(\boldsymbol{\sigma}) - \int_{\Omega_F} \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{dx} + \int_{\Omega_F} \boldsymbol{\sigma} : \nabla_S(\boldsymbol{v})\,\mathrm{dx}\} \\
&= g^*\left(\nabla_S(\boldsymbol{v})\right) - \int_{\Omega_F} \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{dx} = J(\boldsymbol{v}),
\end{aligned}$$

where the last equality due to $A^{-1} = B$. Lastly,

$$\sup_{\boldsymbol{v}\in\boldsymbol{K}} \{-\mathbb{1}_{\boldsymbol{K}}(\boldsymbol{v}) - (g + \boldsymbol{v}\Lambda)^*(\boldsymbol{0})\} = -\inf_{\boldsymbol{v}\in\boldsymbol{K}} \{\mathbb{1}_{\boldsymbol{K}}(\boldsymbol{v}) + (g + \boldsymbol{v}\Lambda)^*(\boldsymbol{0})\} = -\inf_{\boldsymbol{v}\in\boldsymbol{K}} J(\boldsymbol{v}). \qquad \square$$

Remark 6.3. The previous proposition shows that, up to a change in sign, the (D1) and the displacement formulation are equivalent. In addition, the solution to (D1) is also a solution to the displacement problem.

We now prove strong duality. We show that the hypotheses of Proposition 6.1 are satisfied.

Lemma 6.4. $\mathbb{1}_{\boldsymbol{K}^-}$ *is* $\boldsymbol{K}^+$*-increasing.*

Proof. Let $\boldsymbol{v}_1^*, \boldsymbol{v}_2^* \in \boldsymbol{V}^*$ such that $\boldsymbol{v}_1^* \leq_{\boldsymbol{K}^+} \boldsymbol{v}_2^*$. To prove that $\mathbb{1}_{\boldsymbol{K}^-}(\boldsymbol{v}_1^*) \leq \mathbb{1}_{\boldsymbol{K}^-}(\boldsymbol{v}_2^*)$, it suffices to show that if $\boldsymbol{v}_2^* \in \boldsymbol{K}^-$, then $\boldsymbol{v}_1^* \in \boldsymbol{K}^-$. Indeed, suppose that $\boldsymbol{v}_2^* \in \boldsymbol{K}^-$.

Then, $$0 \geq \langle \boldsymbol{v}_2^*, \boldsymbol{v} \rangle_{\boldsymbol{V},\boldsymbol{V}^*} \geq \langle \boldsymbol{v}_1^*, \boldsymbol{v} \rangle_{\boldsymbol{V},\boldsymbol{V}^*}, \quad \forall \boldsymbol{v} \in \boldsymbol{K}.$$

where the last inequality is because $\boldsymbol{v}_2^* - \boldsymbol{v}_1^* \in \boldsymbol{K}^+$. Thus, $\boldsymbol{v}_1^* \in \boldsymbol{K}^-$. □

Proposition 6.5. $\mathbf{0} \in$ *core*$(\textit{dom}\,\mathbb{1}_{\boldsymbol{K}^-} - \Lambda(\textit{dom}\, g \cap \Lambda))$.

Proof. Since $\textit{dom}\, g \cap \textit{dom}\, \Lambda = \mathbb{L}^2_S(\Omega_F)$, we have that

$$W := \{\boldsymbol{v}^* - \lambda \boldsymbol{\sigma} \mid \boldsymbol{v}^* \in \boldsymbol{K}^-, \, \boldsymbol{\sigma} \in \mathbb{L}^2_S(\Omega_F)\} = \textit{dom}\,\mathbb{1}_{\boldsymbol{K}^-} - \Lambda(\textit{dom}\, g \cap \textit{dom}\, \Lambda).$$

We show that $\mathbf{0} \in \mathrm{core}(W)$. Recall that for a given a linear space X and a nonempty subset A,

$$\mathrm{core}(A) := \{x_0 \in A \mid \forall x \in X, \, \exists t_x > 0 \text{ such that for all } t \in [0, t_x], \, x_0 + tx \in A\}.$$

Let $\boldsymbol{u}^* \in \boldsymbol{V}^*$. Set $t_{\boldsymbol{u}^*} = 1$. We claim that for each $t \in [0,1]$, we can find some $\boldsymbol{v}_t^* \in \boldsymbol{K}^-$ and $\boldsymbol{\sigma}_t \in \mathbb{L}^2_S(\Omega_F)$ such that

$$\langle t\boldsymbol{u}^* - (\boldsymbol{v}_t^* - \lambda \boldsymbol{\sigma}_t), \boldsymbol{v} \rangle_{\boldsymbol{V}^*,\boldsymbol{V}} = 0 \ \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{30}$$

We choose $\boldsymbol{v}_t^* = \mathbf{0} \in \boldsymbol{K}^-$. Consider the following problem:

Find $\boldsymbol{u}_t \in \boldsymbol{V}$ such that

$$\int_{\Omega_F} A \nabla_S(\boldsymbol{u}_t) : \nabla_S(\boldsymbol{v}) \, \mathrm{dx} = \langle t\boldsymbol{u}^*, \boldsymbol{v} \rangle_{\boldsymbol{V}^*,\boldsymbol{V}} + \int_{\Omega_F} \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{dx} \ \ \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{31}$$

To show that this has a unique solution, first observe that the bilinear form defined by $a(\boldsymbol{u}, \boldsymbol{v}) := \int_{\Omega_F} A \nabla_S(\boldsymbol{u}) : \nabla_S(\boldsymbol{v}) \, \mathrm{dx}$ for all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$ is coercive and bounded.

Now, $F(\boldsymbol{v}) := \langle t\boldsymbol{u}^*, \boldsymbol{v} \rangle_{\boldsymbol{V}^*,\boldsymbol{V}} + \int_{\Omega_F} \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{dx} \ \ \forall \boldsymbol{v} \in \boldsymbol{V}$ defines a linear function on $\boldsymbol{V}$. Moreover, it is bounded. Indeed,

$$|F(\boldsymbol{v})| \leq (t \, \|\boldsymbol{u}^*\|_V^* + \|\boldsymbol{f}|\|_0) \, \|\boldsymbol{v}\|_1 \leq (\|\boldsymbol{u}^*\|_V^* + \|\boldsymbol{f}|\|_0) \, \|\boldsymbol{v}\|_1 \ \ \forall \boldsymbol{v} \in \boldsymbol{V},$$

so that $\|F\|_{V^*} \leq \|\boldsymbol{u}^*\|_V^* + \|\boldsymbol{f}|\|_0 < +\infty$, i.e., $F \in \boldsymbol{V}^*$.

Thus, by the Lax-Milgram theorem, a unique solution exists to (31).

We now let $\boldsymbol{\sigma}_t := A\nabla_S(\boldsymbol{u}_t)$. Then we have that,

$$\langle t\boldsymbol{u}^*, \boldsymbol{v}\rangle_{V^*,V} = \int_{\Omega_F} \boldsymbol{\sigma}_t : \nabla_S(\boldsymbol{v})\,\mathrm{dx} - \int_{\Omega_F} \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{dx} = \langle \mathbf{0} - \Lambda\boldsymbol{\sigma}_t, \boldsymbol{v}\rangle_{V^*,V} \ \ \forall \boldsymbol{v} \in \boldsymbol{V},$$

i.e., $t\boldsymbol{u}^* \in W$. Thus, $\mathbf{0} \in \mathrm{core}(W)$. □

Proposition 6.6. Λ *is* $\boldsymbol{K}^+$ *-convex and star-*$\boldsymbol{K}^+$ *lower semicontinuous.*

Proof. Let $\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2 \in \mathbb{L}^2_S(\Omega_F)$ and $t\in[0,1]$. To show that Λ is $\boldsymbol{K}^+$-convex, we must have

$$\Lambda\left(t\boldsymbol{\sigma}_1 + (1-t)\boldsymbol{\sigma}_2\right) \leq_{\boldsymbol{K}^+} t\Lambda\boldsymbol{\sigma}_1 + (1-t)\Lambda\boldsymbol{\sigma}_2,$$

i.e., $t\Lambda\boldsymbol{\sigma}_1 + (1-t)\Lambda\boldsymbol{\sigma}_2 - \Lambda\left(t\boldsymbol{\sigma}_1 + (1-t)\boldsymbol{\sigma}_2\right) \in \boldsymbol{K}^+$. But this follows immediately, as it is easy to verify that

$$t\Lambda\boldsymbol{\sigma}_1 + (1-t)\Lambda\boldsymbol{\sigma}_2 - \Lambda\left(t\boldsymbol{\sigma}_1 + (1-t)\boldsymbol{\sigma}_2\right) = \mathbf{0} \in \boldsymbol{K}^+.$$

Now, since $(\boldsymbol{K}^+)^+ = \boldsymbol{K}$, to show that Λ is star-$\boldsymbol{K}^+$ lower semicontinuous, it suffices to show that for all $\boldsymbol{v} \in \boldsymbol{K}$, $\boldsymbol{v}\Lambda$ is lower semicontinuous.

Indeed, let $\{\boldsymbol{\sigma}_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{L}^2_S(\Omega_F)$ such that $\boldsymbol{\sigma}_n \to \boldsymbol{\sigma}$ in $\mathbb{L}^2_S(\Omega_F)$ for some $\boldsymbol{\sigma} \in \mathbb{L}^2_S(\Omega_F)$. Then,

$$\begin{aligned}|(\boldsymbol{v}\Lambda)(\boldsymbol{\sigma}_n) - (\boldsymbol{v}\Lambda)(\boldsymbol{\sigma})| &= |\langle \Lambda\boldsymbol{\sigma}_n, \boldsymbol{v}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} - \langle \Lambda\boldsymbol{\sigma}, \boldsymbol{v}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}}| \\ &= \left|\int_{\Omega_F} (\boldsymbol{\sigma} - \boldsymbol{\sigma}_n) : \nabla_S(\boldsymbol{v})\,\mathrm{dx}\right| \leq C\,\|\boldsymbol{v}\|_1\,\|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}\|_0 \to 0.\end{aligned}$$

Thus, $\boldsymbol{v}\Lambda$ is continuous for all $\boldsymbol{v}\in\boldsymbol{K}$. Hence, Λ is star-$\boldsymbol{K}^+$ lower semicontinuous. □

We now have the duality between the stress and the displacement formulations.

Theorem 6.7.
$$\inf_{\boldsymbol{\sigma}\in\mathbb{S}} g(\boldsymbol{\sigma}) = -\inf_{\boldsymbol{v}\in\boldsymbol{K}} J(\boldsymbol{v}). \tag{32}$$

Proof. Propositions 6.4, 6.5, and 6.6 guarantee that the conditions of Proposition 6.1 are satisifed. Hence, we have that

$$\inf_{\boldsymbol{\sigma}\in\mathbb{L}^2_S(\Omega_F)} \{g(\boldsymbol{\sigma}) + (\mathbb{1}_{\boldsymbol{K}^-} \circ \Lambda)(\boldsymbol{\sigma})\} = \sup_{\boldsymbol{v}\in\boldsymbol{K}} \{-\mathbb{1}_{\boldsymbol{K}}(\boldsymbol{v}) - (g + \boldsymbol{v}\Lambda)^*(\mathbf{0})\}. \tag{33}$$

The result now follows from Propositions 4.1 and 6.2. □

Remark 6.8. The above result shows that, up to a change in sign, the stress and displacement formulations are equivalent.

6.2. Stress-strain duality

We follow a treatment on dual problems with cone constraints as presented in [3]. We adopt the definitions for the sets X, Z, C from the previous section. The primal problem we look at is

$$\inf_{x\in X} (f(x) + \mathbb{1}_{\mathcal{A}}(x)), \tag{34}$$

where $\mathcal{A} = \{x \in S|\, G(x) \in -C\}$. Here $S \subseteq X$ is a given nonempty set, $f : X \to \overline{\mathbb{R}}$ and $G : X \to Z \cup \{\infty_C\}$ are proper functions such that $\mathit{dom}\, f \cap S \cap G^{-1}(-C) \neq \emptyset$.

We define the perturbation function $\Phi : X \times X \to \bar{\mathbb{R}}$ as

$$\Phi(x, y) := f(x + y) + \mathbb{1}_{\mathcal{A}}(x).$$

It can be shown that its Fenchel conjugate, $\Phi^* : X^* \times X^* \to \bar{\mathbb{R}}$, is

$$\Phi^*(x^*, y^*) = f^*(y^*) + \sup_{z\in\mathcal{A}} \langle x^* - y^*, z\rangle_{X,X^*}.$$

The *Fenchel dual problem* can then be written as

$$\sup_{y^*\in X^*} \{-f^*(y^*) - \sup_{x\in\mathcal{A}} \langle y^*, x\rangle\} \tag{35}$$

To obtain strong duality, we make use of the following result taken from Chapter 1, Theorem 3.5 of [3].

Proposition 6.9. *Let $S \subseteq X$ be a nonempty convex set. $f : X \to \overline{\mathbb{R}}$ be a proper and convex function and $g : X \to Z \cup \{\infty_C\}$ a proper and C-convex function such that $dom\, f \cap S \cap G^{-1}(-C) \neq \emptyset$. If there exists some $x' \in dom\, f \cap \mathcal{A}$ such that f is continuous at x', then* (34) *and* (35) *agree and the dual has an optimal solution.*

We look at the stress formulation. Here, we take $X = Z := \mathbb{L}^2_S(\Omega_F)$, and the map $g : \mathbb{L}^2_S(\Omega_F) \to \mathbb{R}$ as defined in the preliminaries. We define the sets

$$S := \{\boldsymbol{\sigma} \in \mathbb{H}_S(\mathrm{div}, \Omega_F)|\, \mathrm{div}\, \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0} \,\mathrm{in}\, \boldsymbol{D}'(\Omega_F)\},$$

$$C := \{\boldsymbol{\sigma} \in \mathbb{L}^2_S(\Omega_F)|\, \sigma_n \geq 0 \text{ on } H^{\frac{1}{2}}_{00}(\Sigma_c),\ \boldsymbol{\sigma}_\tau = \boldsymbol{0} \text{ on } \boldsymbol{H}^{\frac{1}{2}}_{00}(\Sigma_c),\ [\sigma_n] = 0 \text{ on } H^{\frac{1}{2}}(\Sigma)\},$$

$$\mathcal{A} := \{\boldsymbol{\sigma} \in S|\, I\boldsymbol{\sigma} \in -C\}, \quad \text{where } I : \mathbb{L}^2_S(\Omega_F) \to \mathbb{L}^2_S(\Omega_F) \text{ is the identity map.}$$

It is easy to see that C is a convex cone in Z and that $\mathcal{A} = \mathbb{S}$. The primal problem can be written as

$$\text{(P2)} \qquad \inf_{\boldsymbol{\sigma}\in\mathbb{L}^2_S(\Omega_F)} (g(\boldsymbol{\sigma}) + \mathbb{1}_{\mathcal{A}}(\boldsymbol{\sigma})). \tag{36}$$

Following (35), the dual problem is

$$\text{(D2)} \qquad \sup_{\boldsymbol{\mu}\in\mathbb{L}^2_S(\Omega_F)} \Big(- g^*(\boldsymbol{\mu}) - \sup_{\boldsymbol{\sigma}\in\mathbb{S}} \int_{\Omega_F} \boldsymbol{\sigma} : \boldsymbol{\mu}\, \mathrm{dx}\Big) \tag{37}$$

We show that (D2) is equivalent to the strain formulation of the elasticity problem.

Proposition 6.10. *For $\boldsymbol{\mu} \in \mathbb{M}^-$, we have that*

$$g^*(\boldsymbol{\mu}) + \sup_{\boldsymbol{\sigma}\in\mathbb{S}} \int_{\Omega_F} \boldsymbol{\sigma} : \boldsymbol{\mu}\, dx = \tilde{J}(-\boldsymbol{\mu}). \tag{38}$$

Moreover,

$$\sup_{\boldsymbol{\mu}\in\mathbb{L}^2_S(\Omega_F)} \left(-g^*(\boldsymbol{\mu}) - \sup_{\boldsymbol{\sigma}\in\mathbb{S}} \int_{\Omega_F} \boldsymbol{\sigma} : \boldsymbol{\mu}\, dx\right) = - \inf_{\boldsymbol{\mu}\in\mathbb{M}^+} \tilde{J}(\boldsymbol{\mu}). \tag{39}$$

Proof. Fix some $\boldsymbol{\pi} \in \mathbb{S}$ such that $\pi_n = 0$ in $H^{\frac{1}{2}}_{00}(\Sigma_c)$. We claim that for each $\boldsymbol{\sigma} \in \mathbb{S}$, there exists some $\boldsymbol{\tau} \in \mathbb{M}$ such that $\boldsymbol{\sigma} = \boldsymbol{\pi} + \boldsymbol{\tau}$. Indeed, it suffices to show that $\boldsymbol{\sigma} - \boldsymbol{\pi} \in \mathbb{M}$. Let $\boldsymbol{v} \in \boldsymbol{K}$. Then,

$$\begin{aligned}&\langle \nabla^*_S(\boldsymbol{\sigma} - \boldsymbol{\pi}), \boldsymbol{v}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} = \langle \boldsymbol{\sigma} - \boldsymbol{\pi}, \nabla_S(\boldsymbol{v})\rangle_{\boldsymbol{V}^*,\boldsymbol{V}}\\ &= - \int_{\Omega_F} \mathrm{div}\ (\boldsymbol{\sigma} - \boldsymbol{\pi}) \cdot \boldsymbol{v}\, \mathrm{dx} - \langle \sigma_n - \pi_n, [v_n]\rangle_{00,\Sigma_c} = -\langle \sigma_n, [v_n]\rangle_{00,\Sigma_c} \geq 0.\end{aligned}$$

Hence, $\boldsymbol{\sigma} - \boldsymbol{\pi} \in \mathbb{M}$. Now, let $\boldsymbol{\mu} \in \mathbb{M}^-$. Thus, $-\boldsymbol{\mu} \in \mathbb{M}^+$. As $\nabla_S(\boldsymbol{K}) = \mathbb{M}^+$, there is some $\boldsymbol{v} \in \boldsymbol{K}$ such that $-\boldsymbol{\mu} = \nabla_S(\boldsymbol{v})$. Observe that since $\mathbf{0} \in \mathbb{M}$, we have that

$$0 \leq \sup_{\boldsymbol{\tau}\in\mathbb{M}} \int_{\Omega_F} \boldsymbol{\mu} : \boldsymbol{\tau} \,\mathrm{dx} \leq 0.$$

Hence,

$$\begin{aligned} g^*(\boldsymbol{\mu}) + \sup_{\boldsymbol{\sigma}\in\mathbb{S}} \int_{\Omega_F} \boldsymbol{\sigma} : \boldsymbol{\mu} \,\mathrm{dx} &= g^*(\boldsymbol{\mu}) + \sup_{\boldsymbol{\tau}\in\mathbb{M}} \int_{\Omega_F} \boldsymbol{\tau} : \boldsymbol{\mu} \,\mathrm{dx} + \int_{\Omega_F} \boldsymbol{\pi} : \boldsymbol{\mu} \,\mathrm{dx} \\ = g^*(-\nabla_S(\boldsymbol{v})) - \int_{\Omega_F} \boldsymbol{\pi} : \nabla_S(\boldsymbol{v}) \,\mathrm{dx} &= g^*(\nabla_S(\boldsymbol{v})) + \int_{\Omega_F} \operatorname{div} \boldsymbol{\pi} \cdot \boldsymbol{v} \,\mathrm{dx} - \langle \pi_n, [v_n] \rangle_{00,\Sigma_c} \\ = \frac{1}{2} \int_{\Omega_F} A\nabla_S(\boldsymbol{v}) : \nabla_S(\boldsymbol{v}) \,\mathrm{dx} - \int_{\Omega_F} \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{dx} &= \tilde{J}(-\boldsymbol{\mu}). \end{aligned}$$

For the second assertion, we first claim that

$$\text{if } \boldsymbol{\mu} \notin \mathbb{M}^-, \text{ then } \sup_{\boldsymbol{\sigma}\in\mathbb{S}} \int_{\Omega_F} \boldsymbol{\sigma} : \boldsymbol{\mu} \,\mathrm{dx} = +\infty.$$

Indeed, suppose $\boldsymbol{\mu} \notin \mathbb{M}^-$. Then there is some $\boldsymbol{\omega} \in \mathbb{M}$ such that $\int_{\Omega_F} \boldsymbol{\mu} : \boldsymbol{\omega} \,\mathrm{dx} > 0$. As $\mathbb{M}$ is a convex cone, we have that

$$\begin{aligned} \sup_{\boldsymbol{\sigma}\in\mathbb{S}} \int_{\Omega_F} \boldsymbol{\sigma} : \boldsymbol{\mu} \,\mathrm{dx} &= \sup_{\boldsymbol{\tau}\in\mathbb{M}} \left(\int_{\Omega_F} \boldsymbol{\tau} : \boldsymbol{\mu} \,\mathrm{dx} \right) + \int_{\Omega_F} \boldsymbol{\pi} : \boldsymbol{\mu} \,\mathrm{dx} \\ &\geq \sup_{t>0} \left(t \int_{\Omega_F} \boldsymbol{\omega} : \boldsymbol{\mu} \,\mathrm{dx} \right) + \int_{\Omega_F} \boldsymbol{\pi} : \boldsymbol{\mu} \,\mathrm{dx} = +\infty. \end{aligned}$$

From (38) and the above claim, we have that

$$\begin{aligned} \sup_{\boldsymbol{\mu}\in\mathbb{L}^2_S(\Omega_F)} \left(-g^*(\boldsymbol{\mu}) - \sup_{\boldsymbol{\sigma}\in\mathbb{S}} \int_{\Omega_F} \boldsymbol{\sigma} : \boldsymbol{\mu} \,\mathrm{dx} \right) &= - \inf_{\boldsymbol{\mu}\in\mathbb{L}^2_S(\Omega_F)} \left(g^*(\boldsymbol{\mu}) + \sup_{\boldsymbol{\sigma}\in\mathbb{S}} \int_{\Omega_F} \boldsymbol{\sigma} : \boldsymbol{\mu} \,\mathrm{dx} \right) \\ = - \inf_{\boldsymbol{\mu}\in\mathbb{M}^-} \left(g^*(\boldsymbol{\mu}) + \sup_{\boldsymbol{\sigma}\in\mathbb{S}} \int_{\Omega_F} \boldsymbol{\sigma} : \boldsymbol{\mu} \,\mathrm{dx} \right) &= - \inf_{\boldsymbol{\mu}\in\mathbb{M}^-} \tilde{J}(-\boldsymbol{\mu}) = - \inf_{\boldsymbol{\mu}\in\mathbb{M}^+} \tilde{J}(\boldsymbol{\mu}). \end{aligned}$$ □

We now have the strong duality result.

Theorem 6.11.

$$\inf_{\boldsymbol{\sigma}\in\mathbb{S}} g(\boldsymbol{\sigma}) = - \inf_{\boldsymbol{\mu}\in\mathbb{M}^+} \tilde{J}(\boldsymbol{\mu}). \tag{40}$$

Proof. Clearly, the identity operator, $I : \mathbb{L}^2_S(\Omega_F) \to \mathbb{L}^2_S(\Omega_F)$ is C-convex. Also, we have that $\operatorname{dom} g \cap S \cap I^{-1}(C) = \mathbb{L}^2_S(\Omega_F) \cap S \cap (-C) = \mathbb{S} \neq \emptyset$.

We now show that $g : \mathbb{L}^2_S(\Omega) \to \mathbb{R}$ is continuous. Indeed, if $\boldsymbol{\sigma}_n \to \boldsymbol{\sigma}$ in $\mathbb{L}^2_S(\Omega_F)$, then

$$\begin{aligned} |g(\boldsymbol{\sigma}_n) - g(\boldsymbol{\sigma})| &= \frac{1}{2} \left| \int_{\Omega_F} B\boldsymbol{\sigma}_n : \boldsymbol{\sigma}_n \,\mathrm{dx} - \int_{\Omega_F} B\boldsymbol{\sigma} : \boldsymbol{\sigma} \,\mathrm{dx} \right| \\ &= \frac{1}{2} \left| \int_{\Omega_F} B(\boldsymbol{\sigma}_n - \boldsymbol{\sigma}) : (\boldsymbol{\sigma}_n + \boldsymbol{\sigma}) \,\mathrm{dx} \right| \leq \tilde{\beta} \|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}\|_{\mathbb{L}^2_S(\Omega_F)} \to 0. \end{aligned}$$

Thus, from Proposition 6.9, we have that

$$\inf_{\boldsymbol{\sigma}\in\mathbb{S}} g(\boldsymbol{\sigma}) = \inf_{\boldsymbol{\sigma}\in\mathcal{A}} g(\boldsymbol{\sigma}) = \sup_{\boldsymbol{\mu}\in\mathbb{L}^2_S(\Omega_F)} \left(-g^*(\boldsymbol{\mu}) - \sup_{\boldsymbol{\sigma}\in\mathbb{S}} \int_{\Omega_F} \boldsymbol{\sigma} : \boldsymbol{\mu} \,\mathrm{dx} \right).$$

Finally, from (38), the result follows. □

6.3. Solutions of the primal and dual problems

In this subsection, we look at how the minimizers of the displacement, stress, and strain formulations are related to each other.

Theorem 6.12. *Let $\bar{\boldsymbol{\sigma}} \in \mathbb{S}$, $\bar{\boldsymbol{v}} \in \boldsymbol{K}$, and $\bar{\boldsymbol{\mu}}$ such that*

$$g(\bar{\boldsymbol{\sigma}}) = \inf_{\boldsymbol{\sigma}\in\mathbb{S}} g(\boldsymbol{\sigma}), \quad J(\bar{\boldsymbol{v}}) = \inf_{\boldsymbol{v}\in\boldsymbol{K}} J(\boldsymbol{v}), \quad \tilde{J}(\bar{\boldsymbol{\mu}}) = \inf_{\boldsymbol{\mu}\in\mathbb{M}^+} \tilde{J}(\boldsymbol{\mu}).$$

Then

$$\bar{\boldsymbol{\sigma}} = A\nabla_S(\bar{\boldsymbol{v}}) = A\bar{\boldsymbol{\mu}}. \tag{41}$$

Proof. The perturbation function used to obtain (D1) is $\Phi : \mathbb{L}^2_S(\Omega_F) \times \boldsymbol{V}^* \to \bar{\mathbb{R}}$,

$$\Phi(\boldsymbol{\sigma}, \boldsymbol{z}^*) := g(\boldsymbol{\sigma}) + \mathbb{1}_{\boldsymbol{K}^-}(\Lambda\boldsymbol{\sigma} + \boldsymbol{z}^*).$$

From classical results in convex analysis (see for instance [7] and [16]), we have that $(\mathbf{0}, \bar{\boldsymbol{v}}) \in \partial\Phi(\bar{\boldsymbol{\sigma}}, \mathbf{0})$, i.e., for all $\boldsymbol{\mu} \in \mathbb{L}^2_S(\Omega_F)$ and $\boldsymbol{u}^* \in \boldsymbol{V}^*$,

$$\langle(\boldsymbol{\mu}, \boldsymbol{u}^*) - (\bar{\boldsymbol{\sigma}}, \mathbf{0}), (\mathbf{0}, \bar{\boldsymbol{v}})\rangle_{(\mathbb{L}^2_S(\Omega_F)\times\boldsymbol{V}^*)^*, \mathbb{L}^2_S(\Omega_F)\times\boldsymbol{V}^*} \le \Phi(\boldsymbol{\mu}, \boldsymbol{u}^*) - \Phi(\bar{\boldsymbol{\sigma}}, \mathbf{0}).$$

This means that

$$\int_{\Omega_F} (\boldsymbol{\mu} - \bar{\boldsymbol{\sigma}}) : \mathbf{0}\,\mathrm{dx} + \langle \boldsymbol{u}^*, \bar{\boldsymbol{v}}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} \le g(\boldsymbol{\mu}) + \mathbb{1}_{\boldsymbol{K}^-}(\Lambda\boldsymbol{\mu} + \boldsymbol{u}^*) - g(\bar{\boldsymbol{\sigma}}) - \mathbb{1}_{\boldsymbol{K}^-}(\Lambda\bar{\boldsymbol{\sigma}}).$$

As $\mathbb{1}_{\boldsymbol{K}^-}(\Lambda\bar{\boldsymbol{\sigma}}) = \mathbb{1}_{\mathbb{S}}(\bar{\boldsymbol{\sigma}}) = 0$, we have that

$$\langle \boldsymbol{u}^*, \bar{\boldsymbol{v}}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} \le g(\boldsymbol{\mu}) + \mathbb{1}_{\boldsymbol{K}^-}(\Lambda\boldsymbol{\mu} + \boldsymbol{u}^*) - g(\bar{\boldsymbol{\sigma}}).$$

If we set $\boldsymbol{\mu} := \bar{\boldsymbol{\sigma}}$ and $\boldsymbol{u}^* := -\Lambda\bar{\boldsymbol{\sigma}}$, and noting that $\mathbf{0} \in \boldsymbol{K}^-$, then $\langle \Lambda\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{v}}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} \ge 0$. As $\bar{\boldsymbol{v}} \in \boldsymbol{K}$ and $\Lambda\bar{\boldsymbol{\sigma}} \in \boldsymbol{K}^-$, we have that

$$0 = \langle \Lambda\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{v}}\rangle_{\boldsymbol{V}^*,\boldsymbol{V}} = \int_{\Omega_F} \boldsymbol{f}\cdot\bar{\boldsymbol{v}}\,\mathrm{dx} - \int_{\Omega_F} \bar{\boldsymbol{\sigma}} : \nabla_S(\bar{\boldsymbol{v}})\,\mathrm{dx}.$$

Hence,

$$\int_{\Omega_F} \bar{\boldsymbol{\sigma}} : \nabla_S(\bar{\boldsymbol{v}})\,\mathrm{dx} = \int_{\Omega_F} \boldsymbol{f}\cdot\bar{\boldsymbol{v}}\,\mathrm{dx}. \tag{42}$$

From Theorem 6.7, we get

$$\begin{aligned} 0 &= g(\bar{\boldsymbol{\sigma}}) + J(\bar{\boldsymbol{v}}) \\ &= \frac{1}{2}\int_{\Omega_F} B\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{\sigma}}\,\mathrm{dx} + \frac{1}{2}\int_{\Omega_F} A\nabla_S(\bar{\boldsymbol{v}}) : \nabla_S(\bar{\boldsymbol{v}})\,\mathrm{dx} - \int_{\Omega_F} \boldsymbol{f}\cdot\bar{\boldsymbol{v}}\,\mathrm{dx} \\ &= \frac{1}{2}\int_{\Omega_F} B\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{\sigma}}\,\mathrm{dx} + \frac{1}{2}\int_{\Omega_F} A\nabla_S(\bar{\boldsymbol{v}}) : \nabla_S(\bar{\boldsymbol{v}})\,\mathrm{dx} - \int_{\Omega_F} \bar{\boldsymbol{\sigma}} : \nabla_S(\bar{\boldsymbol{v}})\,\mathrm{dx}, \end{aligned}$$

where the last equality came from (42). Hence, we have that

$$g(\bar{\boldsymbol{\sigma}}) + g^*(\nabla_S(\bar{\boldsymbol{v}})) = \int_{\Omega_F} \bar{\boldsymbol{\sigma}} : \nabla_S(\bar{\boldsymbol{v}})\,\mathrm{dx}.$$

Looking at g as a convex, proper function on $\mathbb{L}^2_S(\Omega_F)$, this implies $\nabla_s(\bar{\boldsymbol{v}}) \in \partial g(\bar{\boldsymbol{\sigma}})$, i.e., for all $\boldsymbol{\mu} \in \mathbb{L}^2_S(\Omega_F)$,

$$\int_{\Omega_F} (\boldsymbol{\mu} - \bar{\boldsymbol{\sigma}}) : \nabla_S(\bar{\boldsymbol{v}})\,\mathrm{dx} \leq g(\boldsymbol{\mu}) - g(\bar{\boldsymbol{\sigma}}).$$

If we set $\boldsymbol{\mu} := A\nabla_S(\bar{\boldsymbol{v}})$, then

$$\begin{aligned} g(A\nabla_S(\bar{\boldsymbol{v}})) - g(\bar{\boldsymbol{\sigma}}) &\geq \int_{\Omega_F} (A\nabla_S(\bar{\boldsymbol{v}}) - \bar{\boldsymbol{\sigma}}) : \nabla_S(\bar{\boldsymbol{v}})\,\mathrm{dx} \\ = \int_{\Omega_F} (A\nabla_S(\bar{\boldsymbol{v}}) - \bar{\boldsymbol{\sigma}}) &: B(A\nabla_S(\bar{\boldsymbol{v}}))\,\mathrm{dx} = \int_{\Omega_F} B(A\nabla_S(\bar{\boldsymbol{v}}) - \bar{\boldsymbol{\sigma}}) : A\nabla_S(\bar{\boldsymbol{v}})\,\mathrm{dx}, \end{aligned}$$

where the last equality is due to B being symmetric. Since

$$g(A\nabla_S(\bar{\boldsymbol{v}})) - g(\bar{\boldsymbol{\sigma}}) = \frac{1}{2}\int_{\Omega_F} B(A\nabla_S(\bar{\boldsymbol{v}}) - \bar{\boldsymbol{\sigma}}) : (A\nabla_S(\bar{\boldsymbol{v}}) + \bar{\boldsymbol{\sigma}})\,\mathrm{dx},$$

the above inequality becomes

$$\int_{\Omega_F} B(A\nabla_S(\bar{\boldsymbol{v}}) - \bar{\boldsymbol{\sigma}}) : A\nabla_S(\bar{\boldsymbol{v}})\,\mathrm{dx} \leq \int_{\Omega_F} B(A\nabla_S(\bar{\boldsymbol{v}}) - \bar{\boldsymbol{\sigma}}) : \bar{\boldsymbol{\sigma}}\,\mathrm{dx}. \tag{43}$$

The positive-definiteness of B and (43) imply

$$\alpha\|A\nabla_S(\bar{\boldsymbol{v}}) - \bar{\boldsymbol{\sigma}}\|^2_{\mathbb{L}^2_S(\Omega_F)} \leq \int_{\Omega_F} B(A\nabla_S(\bar{\boldsymbol{v}}) - \bar{\boldsymbol{\sigma}}) : (A\nabla_S(\bar{\boldsymbol{v}}) - \bar{\boldsymbol{\sigma}})\,\mathrm{dx} \leq 0.$$

Hence, $\bar{\boldsymbol{\sigma}} = A\nabla_S(\bar{\boldsymbol{v}})$.

Now, as $\bar{\boldsymbol{\mu}} \in \mathbb{M}^+$, there exists some $\boldsymbol{u} \in \boldsymbol{K}$ such that $\bar{\boldsymbol{\mu}} = \nabla_S(\boldsymbol{u})$. Moreover as $\nabla_S(\boldsymbol{K}) = \mathbb{M}^+$, we have that $\boldsymbol{u}$ minimizes J on $\boldsymbol{K}$. Thus, from the previous arguments, we have that

$$A\nabla_S(\boldsymbol{u}) = \bar{\boldsymbol{\sigma}}.$$

Since ker $\nabla_S = \{\boldsymbol{0}\}$ in $\boldsymbol{V}$, we have that $\bar{\boldsymbol{v}} = \boldsymbol{u}$, so that

$$A\bar{\boldsymbol{\mu}} = A\nabla_S(\bar{\boldsymbol{v}}) = \bar{\boldsymbol{\sigma}}. \qquad \square$$

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