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Metrization of Idempotent Convex Compacta

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Using a convenient subbase on the second hyperspace of a compactum with the Vietoris topology, we prove that the mapping that takes each closed non-empty subset A of an I-convex compactum X to its closed idempotent convex hull is continuous. This implies that each neighborhood of the diagonal $\Delta_X \subset X \times X$ contains an idempotent convex neighborhood. The main result is the theorem that the topology on an idempotent convex compactum X is determined by a family of idempotent convex pseudometrics (with one idempotent convex metric if X is metrizable).

Keywords: I-convex compactum, idempotent semimodule, locally convex space, Vietoris topology, metrization

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1. Introduction

This paper is a continuation of [14, 15], in that it also deals with idempotent generalizations of convexity, but with less algebra, without category theory, with only real-valued coefficients instead of elements of completely distributive lattices, i.e., closer to classical functional analysis.

For a reader who is new to so called idempotent or tropical mathematics, [16] can be a bit humorous introduction. On the other hand, [1] or [11] convince that this theory have found numerous "real life" applications. Each paper on idempotent mathematics contains an advocacy against suspicions that it is rather funny than practical. We omit this phase and pass to the problem itself, as it was said by C. Horvath [8], "with barely no justification as to why one should be interested in max-plus convexity". In fact applications to approximation, selections, and fixed points theorems considered in his paper present such a justification.

Idempotent functional analysis, although quite elaborated nowadays [12], considers objects of classical analysis, like powers of the real line or spaces of real-valued continuous functions, from the idempotent point of view. On the contrary, in "usual" analysis, classes of objects are *defined* and then filled with new examples. We believe that idempotent convex sets with topologies should be defined not only in $\mathbb{R}^\tau$, $C(X, \mathbb{R})$ or even in a vector lattice, although these cases will remain the most important, cf. [4].

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In this article we attempt to develop a general approach to idempotent analogue of convex compactum. In particular, the topology on a convex compactum in a locally convex vector topological space can be determined with a metric or with a saturated family of pseudometrics that play well with the convex structure. We show that a similar statement is valid for the class of I-convex compacta, which all are locally convex.

2. Idempotent convexity

Definition 2.1. A commutative *idempotent semiring with a unit* [1] $(S, \oplus, *)$ is a set S with binary operations $\oplus$, $*$ such that for all $\alpha, \beta, \gamma \in S$:

(1) $\alpha \oplus \beta = \beta \oplus \alpha$;

(2) $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$;

(3) $\alpha \oplus \alpha = \alpha$;

(4) there is $0 \in S$ such that $\alpha \oplus 0 = \alpha$ for all α;

(5) $\alpha * \beta = \beta * \alpha$;

(6) $\alpha * (\beta * \gamma) = (\alpha * \beta) * \gamma$;

(7) there is $1 \in S$ such that $\alpha * 1 = \alpha$ for all α;

(8) $\alpha * (\beta \oplus \gamma) = (\alpha * \beta) \oplus (\alpha * \gamma)$.

Informally speaking, a commutative idempotent semiring is a commutative ring without additive inverses, which are impossible for $|S| > 1$ due to idempotent property of "$\oplus$". We consider $*$ a multiplication, hence it has higher precedence over an addition "$\oplus$". In the sequel we consider only commutative semirings, thus dropping the term "commutative".

Observe that the three first properties imply that S is an upper semilattice with $\alpha \oplus \beta = \sup\{\alpha, \beta\}$. The partial order is $\alpha \le \beta \iff \alpha \oplus \beta = \beta$. Taking into account other properties, we obtain that 0 and 1 are the bottom and the top elements resp.

The two most famous idempotent semirings are $\big([-\infty, +\infty), \max, +\big)$ with the operations, which can be considered as limit case of "deformed" field operations [12] on $(\mathbb{R}, +, \cdot)$, and $\big([-\infty, +\infty], \max, \min\big)$. The latter is isomorphic to $(I, \max, \min)$, where we denote $I = [0, 1]$.

The idempotent analogue of the notion of vector space is idempotent semimodule.

Definition 2.2. A triple $(X, \bar{\oplus}, \bar{*})$ is called an $(S, \oplus, *)$-*semimodule* for an idempotent semiring $(S, \oplus, *)$ (or an S-*semimodule* for short, if $*$ is known) if X is a set with operations $\bar{\oplus} : X \times X \to X$, $\bar{*} : S \times X \to X$ such that for all $x, y, z \in X$, $\alpha, \beta \in S$:

(1) $x \mathbin{\bar{\oplus}} y = y \mathbin{\bar{\oplus}} x$;

(2) $(x \mathbin{\bar{\oplus}} y) \mathbin{\bar{\oplus}} z = x \mathbin{\bar{\oplus}} (y \mathbin{\bar{\oplus}} z)$;

(3) there is a unique element $\bar{0} \in X$ such that $x \mathbin{\bar{\oplus}} \bar{0} = x$ for all x;

(4) $\alpha \mathbin{\bar{*}} (x \mathbin{\bar{\oplus}} y) = (\alpha \mathbin{\bar{*}} x) \mathbin{\bar{\oplus}} (\alpha \mathbin{\bar{*}} y)$, $(\alpha \oplus \beta) \mathbin{\bar{*}} x = (\alpha \mathbin{\bar{*}} x) \mathbin{\bar{\oplus}} (\beta \mathbin{\bar{*}} x)$;

(5) $(\alpha * \beta) \mathbin{\bar{*}} x = \alpha \mathbin{\bar{*}} (\beta \mathbin{\bar{*}} x)$;

(6) $1 \mathbin{\bar{*}} x = x$;

(7) $0 \mathbin{\bar{*}} x = \bar{0}$.

In the two latter properties 0 and 1 are the bottom and the top elements of S. Hence $x \bar{\oplus} x = 1 \bar{*} x \bar{\oplus} 1 \bar{*} x = (1 \oplus 1) \bar{*} x = 1 \bar{*} x = x$, and such an X is an upper semilattice with $x \bar{\oplus} y = \sup\{x, y\}$ as well, and the respective partial order is

$$x \leq y \iff x \bar{\oplus} y = y.$$

A power S^τ with the coordinatewise operations is clearly an example of an S-semimodule. A less obvious one of $([0, +\infty], \max, \cdot)$-semimodule is an Archimedean vector lattice L with unit u if $\alpha \otimes x = \inf\{\alpha u, x\}$ [8].

An S-convex combination of a finite number of elements in an S-semimodule X is defined in a straightforward way, namely for $x_1, x_2, \ldots, x_n \in X$ and coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n \in S$ such that $\alpha_1 \oplus \alpha_2 \oplus \ldots \oplus \alpha_n = 1$, it is the expression

$$\alpha_1 \bar{*} x_1 \bar{\oplus} \alpha_2 \bar{*} x_2 \bar{\oplus} \ldots \bar{\oplus} \alpha_n \bar{*} x_n.$$

Mostly we denote it simply with $\alpha_1 x_1 \bar{\oplus} \alpha_2 x_2 \bar{\oplus} \ldots \bar{\oplus} \alpha_n x_n$.

Definition 2.3. A subset A of an S-semimodule X is called *S-convex* (or idempotent convex), if it contains all S-convex combinations of its elements.

As $1x_1 \oplus 1x_2 \oplus \ldots \oplus 1x_n = x_1 \oplus x_2 \oplus \ldots \oplus x_n$ is a particular case of S-convex combination, any S-convex subset of an S-semimodule is its upper subsemilattice.

If S is linearly ordered, then for S-convexity of a subset A of an S-semimodule X it is sufficient that for all points $x_1, x_2 \in A$ and all $\alpha_1, \alpha_2 \in S$ such that $\max\{\alpha_1, \alpha_2\} = 1$, the S-convex combination $\alpha_1 x_1 \bar{\oplus} \alpha_2 x_2$ is in A as well.

In particular, it is the case for idempotent semirings $(I, \max, *)$, i.e., $\alpha \oplus \beta = \max\{\alpha, \beta\}$, which we consider in our paper. Then "$*$" should be an arbitrary Łukasiewicz t-norm [10], which we also require to be continuous. The obvious examples are the previously mentioned $* = \min$ and the usual multiplication $* = \cdot$. See [13] for nice pictures and detailed description of segments, convex sets, hyperplanes, halfspaces, etc, in finite powers of $([-\infty, +\infty], \max, \min) \cong (I, \max, \min)$.

I-convex sets are in many aspects similar to the "usual" convex sets, but the analogy is not complete, e.g., all compact finite-dimensional convex sets are homeomorphic to Euclidean cubes, but the set

$$A = \{(x_1, x_2, x_3) \in I^3 \mid \text{at least two of } x_1, x_2, x_3 \text{ are equal to } 1\}$$

is compact, $(I, \max, \cdot)$-convex and topologically is a simple triod. To verify $(I, \max, \cdot)$-convexity, consider n points $x^k = (x_1^k, x_2^k, x_3^k) \in A$, $1 \leq k \leq n$, and coefficients $\alpha_1, \ldots, \alpha_n \in I$ such that $\max\{\alpha_1, \ldots, \alpha_n\} = 1$. Then some α_i is equal to 1, hence

$$\begin{aligned} x = (x_1, x_2, x_3) &= \alpha_1 \bar{*} x^1 \bar{\oplus} \ldots \alpha_{i-1} \bar{*} x^{i-1} \bar{\oplus} 1 \bar{*} x^i \bar{\oplus} \alpha_{i+1} \bar{*} x^{i+1} \bar{\oplus} \ldots \bar{\oplus} \alpha_n \bar{*} x_n \\ &\geq 1 \bar{*} x^i = x^i = (x_1^i, x_2^i, x_3^i), \end{aligned}$$

therefore at least two of x_1, x_2, x_3 should also be equal to 1.

Nevertheless, the hyperspaces of idempotent convex compact nonempty subsets in finite powers of real line and in $C(X, \mathbb{R})$ for compact X have properties similar to ones of the hyperspaces of "traditionally" convex compact nonempty subsets of these spaces [2].

3. I-convex compacta

As said above, from now on $I = (I, \oplus, *)$ is an idempotent semiring, $\oplus = \max$, and $* : I \times I \to I$ is continuous. Each I-semimodule $(X, \bar{\oplus}, \bar{*})$ is an upper semilattice. Recall that a topological semilattice is called *Lawson* if in each point it has a base consisting of subsemilattices. Fundamental Theorem on Compact Semilattices [6] implies that any compact Hausdorff Lawson upper semilattice is complete and each its filtered subset has an infimum.

Definition 3.1. An I-semimodule $(M, \bar{\oplus}, \bar{*})$ with a compact Hausdorff topology such that M is a Lawson upper semilattice and the multiplication $\bar{*} : I \times M \to M$ is continuous, is called a *compact Hausdorff Lawson I-semimodule.*

Such an M has a bottom element $\bar{0}$, therefore is a complete lattice.

Definition 3.2. A closed I-convex subset of a compact Hausdorff Lawson I-semimodule with the induced operation of I-convex combination of finite number of points is called an *I-convex compactum.*

It is straightforward to show that an I-convex compactum is a complete upper subsemilattice of a compact Hausdorff Lawson semilattice, hence is a compact Hausdorff Lawson semilattice as well, therefore each its filtered subset has an infimum.

Example 3.3. 1. The power I^τ with the product topology is compact for any cardinality τ and is a Lawson upper semilattice with the coordinatewise partial order (and even Lawson lower semilattice, which is defined analogously). If $\bar{\oplus} : I^\tau \times I^\tau \to I^\tau$ and $\bar{*} : I \times I^\tau \to I^\tau$ are also defined coordinatewise, then $(I^\tau, \bar{\oplus}, \bar{*})$ is a compact Hausdorff Lawson I-semimodule.

The subset $V = \{(c_i)_{i\in\tau} \mid \text{ at least one of } x_i \text{ is } \geq \frac{1}{2}\}$ is I-convex but not compact for an infinite τ. If τ is finite, then V is closed, hence compact, therefore is an I-convex compactum.

2. Choose a metric compactum K and let

$$M = \{f : K \to [0, \operatorname{diam} K] \mid f \text{ is non-expanding}\}.$$

By the Arzelà-Ascoli Theorem M is compact in the topology induced by the uniform norm. Define "$\bar{\oplus}$" to be argumentwise maximum, and "$\bar{*}$" to be argumentwise multiplication of functions by numbers in I. Then M is a compact Hausdorff Lawson I-semimodule. Its subset $V = \{f \in M \mid f(x) = \operatorname{diam} K \text{ for some } x \in K\}$ is not a compact Hausdorff Lawson I-semimodule, but is an I-convex compactum.

3. For a convex compact set K in a locally convex vector topological space L fix a point $x_0 \in K$ and put $M = \{A \subset K \mid A \ni x_0 \text{ is convex and closed}\}$. If the Vietoris topology, which will be defined further in this paper, is considered on M, then M is compact and Hausdorff. For $A, B \in M$ and $\alpha \in I$, define $A \oplus B$ to be the closed convex hull of $A \cup B$, and $\alpha \bar{*} A = \alpha(A - x_0) + x_0 = \{\alpha x + (1 - \alpha)x_0 \mid x \in A\}$. Then M is a compact Hausdorff Lawson I-semimodule. For a fixed subset $K_0 \subset K$, the subset $M_0 = \{A \in M \mid A \supset K_0\}$ is an I-convex compactum.

4. Principal examples of infinite-dimensional compact Hausdorff Lawson I-semimodules are spaces of possibility measures [14], i.e., of upper semicontinuous functions that take closed subsets of some compactum K to non-negative numbers, in particular, $\emptyset$ to 0, and $A \cup B$ to the greater of the values for A and for B. If we require also that K is taken to 1, then the subspaces of *normalized* possibility measures are obtained, which are I-convex compacta.

We can observe that in a compact Hausdorff Lawson I-semimodule the elements can be multiplied by numbers in I so that the multiplication is continuous and agrees with the partial order: if $x \le y$, then $\alpha \bar{*} x \le \alpha \bar{*} y$. Compact Hausdorff Lawson I-semimodules can be defined in a purely algebraic manner, cf. [1, 14], without even mentioning any topology, and the topology is completely determined with the order.

It is also possible to equivalently define I-convex compactum "internally" as a compactum X with a continuous ternary operation $ic : X \times I \times X \to X$ (for the sake of brevity we write $x \mathbin{\bar\oplus} \alpha y$ instead of $ic(x, \alpha, y)$) such that for all $x, y, z \in X$, $\alpha, \beta \in I$:

(1) $x \mathbin{\bar\oplus} \alpha x = x$;

(2) $(x \mathbin{\bar\oplus} \alpha y) \mathbin{\bar\oplus} \beta z = (x \mathbin{\bar\oplus} \beta z) \mathbin{\bar\oplus} \alpha y$;

(3) $x \mathbin{\bar\oplus} \alpha(y \mathbin{\bar\oplus} \beta z) = (x \mathbin{\bar\oplus} \alpha y) \mathbin{\bar\oplus} (\alpha * \beta) z$;

(4) $x \mathbin{\bar\oplus} 1y = y \mathbin{\bar\oplus} 1x$;

(5) $x \mathbin{\bar\oplus} 0y = x$;

(6) for each neighborhood U of x there is a neighborhood $V \ni x$, $V \subset U$, such that $y \mathbin{\bar\oplus} 1z \in V$ for all $y, z \in V$.

It is then possible to define the I-convex combination of finitely many points as

$$1x_0 \mathbin{\bar\oplus} \alpha_1 x_1 \mathbin{\bar\oplus} \alpha_2 x_2 \mathbin{\bar\oplus} \ldots \mathbin{\bar\oplus} \alpha_n x_n \tag{1}$$

$$= (\ldots((x_0 \mathbin{\bar\oplus} \alpha_1 x_1) \mathbin{\bar\oplus} \alpha_2 x_2) \mathbin{\bar\oplus} \ldots \mathbin{\bar\oplus} \alpha_{n-1} x_{n-1}) \mathbin{\bar\oplus} \alpha_n x_n. \tag{2}$$

Such a compactum embeds into a compact Hausdorff Lawson I-semimodule [15]. Properties (1)–(6) have a corollary of crucial importance:

(6+) for each neighborhood U of x there is a neighborhood $V \ni x$, $V \subset U$, such that $y \mathbin{\bar\oplus} \alpha z \in V$ for all $y, z \in V$, $\alpha \in I$.

Then $V \subset X$ is I-convex, hence all I-convex compacta, in particular, all compact Hausdorff Lawson I-semimodules, are *locally convex*, i.e., have local bases consisting of I-convex sets.

The intersection of a family of I-convex sets is I-convex as well, therefore for any subset $A \subset X$ there is the least I-convex superset of A, which is denoted with $\operatorname{conv} A$ and called its I-convex hull. Clearly $\operatorname{conv} A$ consists of all I-convex combinations of the elements of A.

It is an advantage of I-convex compacta over convex compacta that we can define well the I-convex combination of infinitely, and even uncountably many points. We do this through finite I-convex combinations. Coefficients $\alpha_i \in I$, $i \in \mathcal{I}$, by definition are such that $\sup_{i \in \mathcal{I}} \alpha_i = 1$. If $\max_{i \in \mathcal{I}} \alpha_i = \alpha_j = 1$ happens, i.e. some α_j attains the maximal possible value 1, put

$$\mathop{\bar\bigoplus}_{i \in \mathcal{I}} \alpha_i x_i = \sup\{x_j \mathbin{\bar\oplus} \alpha_i x_i \mid i \in \mathcal{I}\}. \tag{3}$$

Observe that all the pairwise combinations $x_j \mathbin{\bar\oplus} \alpha_i x_i$, $i \in \mathcal{I}$, are well defined, hence we can calculate their least upper bound in the complete upper semilattice X irregard of the cardinality of the index set $\mathcal{I}$.

If $\sup_{i\in\mathcal{I}}\alpha_i = 1$, but maximum is not attained, then we cannot get to the result from below, hence the definition is a little more cumbersome. We cover a (possibly infinite) index set $\mathcal{I}$ with a finite number n of its subsets $\mathcal{I}_1$, $\mathcal{I}_2$, ..., $\mathcal{I}_n$, then by associativity of suprema

$$\sup_{i\in\mathcal{I}}\alpha_i = \sup_{i\in\mathcal{I}_1}\alpha_i \oplus \sup_{i\in\mathcal{I}_2}\alpha_i \oplus \ldots \oplus \sup_{i\in\mathcal{I}_n}\alpha_i,$$

therefore at least one of $\sup_{i\in\mathcal{I}_1}\alpha_i, \sup_{i\in\mathcal{I}_2}\alpha_i, \ldots, \sup_{i\in\mathcal{I}_n}\alpha_i$ equals 1. The *finite* I-convex combination

$$\sup_{i\in\mathcal{I}_1}\alpha_i \mathbin{\bar{*}} \sup_{i\in\mathcal{I}_1}x_i \mathbin{\bar{\oplus}} \sup_{i\in\mathcal{I}_2}\alpha_i \mathbin{\bar{*}} \sup_{i\in\mathcal{I}_2}x_i \mathbin{\bar{\oplus}} \ldots \mathbin{\bar{\oplus}} \sup_{i\in\mathcal{I}_n}\alpha_i \mathbin{\bar{*}} \sup_{i\in\mathcal{I}_n}x_i$$

is then well defined, and is non-increasing with respect to refinement of a covering $\{\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n\}$. Thus we define

$$\bar{\bigoplus_{i\in\mathcal{I}}}\alpha_i x_i = \inf\left\{\begin{array}{l}\sup\limits_{i\in\mathcal{I}_1}\alpha_i \mathbin{\bar{*}} \sup\limits_{i\in\mathcal{I}_1}x_i \mathbin{\bar{\oplus}} \sup\limits_{i\in\mathcal{I}_2}\alpha_i \mathbin{\bar{*}} \sup\limits_{i\in\mathcal{I}_2}x_i \mathbin{\bar{\oplus}} \ldots \mathbin{\bar{\oplus}} \sup\limits_{i\in\mathcal{I}_n}\alpha_i \mathbin{\bar{*}} \sup\limits_{i\in\mathcal{I}_n}x_i \mid \\ n\in\mathbb{N}, \quad \mathcal{I} = \mathcal{I}_1\cup\mathcal{I}_2\cup\ldots\cup\mathcal{I}_n\end{array}\right\}, \tag{4}$$

and the latter infimum exists as the greatest lower bound of a filtered set.

Let $\mathcal{A} = \{(x_i, \alpha_i) \mid i \in \mathcal{I}\} \subset X \times I$ contain pairs of elements of an I-convex compactum X and the respective coefficients. Then for the sake of brevity we (rather informally) define $\bar{\bigoplus}_{i\in\mathcal{I}}\alpha_i x_i = \bar{\bigoplus}\mathcal{A}$. It can be easily shown that $\bar{\bigoplus}(\mathrm{Cl}\,\mathcal{A}) = \bar{\bigoplus}\mathcal{A}$ for all $\mathcal{A} \subset X \times I$ such that $\sup(\mathrm{pr}_2(\mathcal{A})) = 1$, but $\mathrm{pr}_2(\mathrm{Cl}\,\mathcal{A}) \ni 1$. Thus taking closure allows to use the simpler formula for infinite I-convex combination.

4. Closed idempotent convex hull

Similarly to I-convex hull, the intersection of all closed I-convex supersets of A is the least such superset $\overline{\mathrm{conv}}\,A$, which is called the closed I-convex hull of A.

To describe it, we need some hyperspaces. For a compactum K the set of all non-empty closed (hence compact) subset of K is denoted $\exp K$. It is endowed with the Vietoris topology determined by a base of all sets of the form

$$\langle V_1, V_2, ..., V_n\rangle = \{A \in \exp X \mid A \subset V_1 \cup V_2 \cup ... \cup V_n,\ A \cap V_i \neq \emptyset,\ i \in \{1, \ldots, n\}\}, \tag{5}$$

for $n \in \mathbb{N}$ and $V_1, V_2, \ldots, V_n$ open in X. The set $\exp K$ with this topology is called the hyperspace (of compact subsets) of K. It is a compactum as well, hence we can consider $\exp^2 K = \exp(\exp K)$ etc.

For each closed $A \subset X$ denote $\exp_1(A\times I)$ the set of all closed subsets of $A\times I \subset X\times I$ that contain at least one pair of the form $(x, 1)$, $x \in A$. Then

$$\exp_1(A \times I) \underset{\mathrm{cl}}{\subset} \exp_1(X \times I) \underset{\mathrm{cl}}{\subset} \exp(X \times I)$$

with respect to the Vietoris topology.

It was proved by Hlushak [7] that the defined above mapping $\bar{\bigoplus} : \exp_1(X \times I) \to X$ that takes each $\mathcal{A} \in \exp_1(X \times I)$ to the I-convex combination $\bar{\bigoplus}\mathcal{A}$ is continuous. Therefore $\bar{\bigoplus}\big(\exp_1(A \times I)\big) \subset X$ is closed and I-convex.

Moreover, if $x_i \in A$, $\alpha_i \in I$ for all $i \in \mathcal{I}$, and $\alpha_j = 1$:

$$\bar{\bigoplus_{i\in\mathcal{I}}}\alpha_i x_i = \sup\{x_j \mathbin{\bar{\oplus}} \alpha_{i_1}x_{i_1} \mathbin{\bar{\oplus}} \ldots \mathbin{\bar{\oplus}} \alpha_{i_k}x_{i_k} \mid k \in \mathbb{N}, \{i_1, \ldots, i_k\} \in \mathcal{I}\}. \tag{6}$$

The latter set of finite I-convex combinations of elements of A is directed, hence form a net that converges to $\bar{\oplus}_{i\in\mathcal{I}_1}\alpha_i x_i$. It implies that an infinite I-convex combination of elements A is a closure point of $\operatorname{conv} A$, hence belongs to any closed I-convex superset of A. Thus:

Proposition 4.1. *For a closed non-empty subset $A \subset X$, the image $\bar{\oplus}\big(\exp_1(A\times I)\big)$ is the closed I-convex hull of A.*

Now we are going to show that $\overline{\operatorname{conv}}\, A$ depends continuously on A. To prove this, we need an auxilliary statement. It is well known that, for a compactum X, the collection of all the sets of the form

$$\langle V\rangle = \{A \in \exp X \mid A \subset V\}, \quad \langle X, V\rangle = \{A \in \exp X \mid A\cap V \neq \emptyset\}, \tag{7}$$

where $V \underset{\mathrm{op}}{\subset} X$ is a subbase of the Vietoris topology.

We need as simple subbase for $\exp^2 X$ as possible. For a family of open sets $\mathcal{P}$ to be a subbase of a compactum X, it is sufficient that for all $x, y \in X$ there exist $U, V \in \mathcal{P}$ such that $x \in U$, $y \in V$, $U \cap V = \emptyset$.

Proposition 4.2. *For a compactum X, the family $\mathcal{P}$ of all sets of the following forms is a subbase of* $\exp^2 X$*:*

$$\big\langle \exp X, \langle V_1, V_2, \dots, V_n\rangle\big\rangle, \quad \big\langle \langle X, V_0\rangle \cup \langle V_1\rangle \cup \langle V_2\rangle \cup \ldots \cup \langle V_n\rangle\big\rangle, \tag{8}$$

where $n \in \mathbb{N}$, $V_1, V_2, \dots, V_n$ run through all open sets of X, $V_0 = V_1 \cap V_2 \cap \ldots \cap V_n$.

Proof. Show that the sets of the above forms can "separate" arbitrary points of the compactum $\exp^2 X$.

Assume $\mathcal{F}, \mathcal{G} \in \exp^2 X, \mathcal{F} \neq \mathcal{G}$, e.g., there is $G \in \mathcal{G}$ such that $G \notin \mathcal{F}$. The family $\mathcal{F}$ is closed in $\exp X$, hence $\exp X \setminus \mathcal{F}$ is open, $X \setminus \mathcal{F}$ contains G with some its neighborhood $\langle U_1, U_2, \dots, U_n\rangle$, U_i are open in X, i.e., $G \in \bigcup\limits_{i=1}^{n} U_i$, $G \cap U_i \neq \emptyset$ for all $i \in \{1, 2, \dots, n\}$. By Shrinking Lemma there is an open cover $\{V_1, V_2, \dots, V_n\}$ of G such that $V_i \subset \operatorname{Cl} V_i \subset U_i$, $V_i \cap G \neq \emptyset$.

Then $G \in \langle V_1, V_2, \dots, V_n\rangle \subset \langle \operatorname{Cl} V_1, \operatorname{Cl} V_2, \dots, \operatorname{Cl} V_n\rangle \subset \langle U_1, U_2, \dots, U_n\rangle$. Therefore $\mathcal{F} \cap \langle \operatorname{Cl} V_1, \operatorname{Cl} V_2, ..., \operatorname{Cl} V_n\rangle = \emptyset$, i.e., for all $F \in \mathcal{F}$ either $F \not\subset \operatorname{Cl} V_1 \cup \operatorname{Cl} V_2 \cup ... \cup \operatorname{Cl} V_n$, or there is $i \in \{1, 2, \dots, n\}$ such that $F \cap \operatorname{Cl} V_i = \emptyset$.

Define $W_i = X \setminus \operatorname{Cl} V_i$, $W_0 = W_1 \cap W_2 \cap \ldots \cap W_n$, then all $F \in \mathcal{F}$ either are subsets of some W_i, or have non-empty intersection with $X \setminus (\operatorname{Cl} V_1 \cup \operatorname{Cl} V_2 \cup \ldots \cup \operatorname{Cl} V_n) = W_0$.

We obtain $\qquad \mathcal{F} \in \big\langle \langle X, W_0\rangle \cup \langle W_1\rangle \cup \langle W_2\rangle \cup \ldots \cup \langle W_n\rangle\big\rangle = \mathcal{U} \in \mathcal{P}.$

On the other hand, $\mathcal{G} \in \big\langle \exp X, \langle V_1, V_2, \dots, V_n\rangle\big\rangle = \mathcal{V} \in \mathcal{P}$, and it is straightforward to verify that $\mathcal{U} \cap \mathcal{V} = \emptyset$. □

Proposition 4.3. *The mapping $k\colon \exp X \to \exp^2(X \times I)$ that takes each $A \underset{\mathrm{cl}}{\subset} X$, $A \neq \emptyset$, to $\exp_1(A \times I) \underset{\mathrm{cl}}{\subset} \exp(X \times I)$, is continuous.*

Proof. It is sufficient to show that the preimages of elements of the constructed above subbase of $\exp^2 X$ are open.

(1) Let, for $V_1, V_2, \ldots, V_n$ open sets in $X \times I$,

$$\begin{aligned} \mathscr{U} &= \big\langle \exp(X \times I), \langle V_1, V_2, \ldots, V_n\rangle\big\rangle \\ &= \big\{\mathfrak{F} \in \exp^2(X \times I) \mid \mathfrak{F} \cap \langle V_1, V_2, \ldots, V_n\rangle \neq \emptyset\big\} \\ &= \big\{\mathfrak{F} \in \exp^2(X \times I) \mid \exists F \in \mathfrak{F} : F \subset V_1 \cup V_2 \cup \ldots \cup V_n, \\ &\qquad\qquad F \cap V_i \neq \emptyset \text{ for all } i = 1, 2, \ldots, n\big\}, \end{aligned}$$

Denote for $i = 1, 2, \ldots, n$

$$U_i = \mathrm{pr}_1(V_i), \quad U_i' = \{x \in X \mid (x, 1) \in V_i\} = \mathrm{pr}_1(V_i \cap (X \times \{1\})).$$

The projection is an open mapping, hence all U_i and U_i' are open. It is obvious that for $\mathscr{U} \cap \exp_1(X \times I) \neq \emptyset$ at least one of the U_i' must be non-empty.
We show that the open set

$$\mathscr{V} = \langle X, U_1', U_2, \ldots, U_n\rangle \cup \langle X, U_1, U_2', \ldots, U_n\rangle \cup \ldots \cup \langle X, U_1, U_2, \ldots, U_n'\rangle$$

is the preimage of $\mathscr{U}$ under k.

Assume $A \in \mathscr{V}$, then $A \in \langle X, U_1, U_2, \ldots, U_{i-1}, U_i', U_{i+1}, \ldots, U_n\rangle$ for some i. There is $y_i \in U_i' \cap A$, i.e., $(y_i, 1) \in V_i$. Let $\alpha_i = 1$ and choose for all $k \neq i$ a point $y_k \in U_k \cap A$ and a number $\alpha_k \in I$ such that $(y_k, \alpha_k) \in V_k$.

Then $F = \big\{(y_l, \alpha_l) \mid l = 1, 2, \ldots, n\big\} \in \exp_1(A \times I) \cap \langle V_1, V_2, \ldots, V_n\rangle$, therefore $k(A) = \exp_1(A \times I) \in \mathscr{U}$.

Let $B \in \exp X$ be such that $k(B) = \exp_1(B \times I) \in \mathscr{U}$. Then there is $F \underset{\mathrm{cl}}{\subset} (B \times I)$ and $(y, 1) \in F$ such that $F \subset V_1 \cup V_2 \cup \ldots \cup V_n$, $F \cap V_i \neq \emptyset$ for all $i = 1, 2, \ldots, n$. The point $(y, 1)$ is in some V_i, hence $y \in U_i'$. There are $(y_k, \alpha_k) \in F \cap V_k$ for all $k \neq i$, hence $y_k \in \mathrm{pr}_1(F) \cap \mathrm{pr}_1(V_k) \subset B \cap U_k$, and $B \in \langle U_1, \ldots, U_{i-1}, U_i', U_{i+1}, \ldots, U_n\rangle \subset \mathscr{V}$, which completes the proof that $\mathcal{V} = k^{-1}(\mathscr{U})$.

(2) Define $\qquad \mathscr{U} = \big\langle \langle X \times I, V_0\rangle \cup \langle V_1\rangle \cup \langle V_2\rangle \cup \ldots \cup \langle V_n\rangle\big\rangle,$
where $V_0 = V_1 \cap V_2 \cap \ldots \cap V_n$, $V_1, V_2, \ldots, V_n$ are open in $X \times I$. We show that the preimage of $\mathscr{U}$ under k is equal to

$$k^{-1}(\mathscr{U}) = \mathscr{V} = \langle V_0'\rangle \cup \langle V_1'\rangle \cup \langle V_2'\rangle \cup \ldots \cup \langle V_n'\rangle,$$

where $V_0' = \{x \in X \mid (x, 1) \in V_0\}$, $V_i' = \{x \in X \mid \{x\} \times I \subset V_i\} = X \setminus \mathrm{pr}_1\big((X \times I) \setminus V_i\big)$ for all $i \in \{1, 2, \ldots, n\}$, hence $V_i' \underset{\mathrm{op}}{\subset} X$.

Assume $A \in \mathscr{V}$, then $A \in \langle V_i'\rangle$, i.e., $A \subset V_i'$ for some $i \in \{0, 1, 2, \ldots, n\}$. If $A \subset V_0'$, then for all $F \in \exp_1(A \times I)$ and $(y, 1) \in F$ we have $y \in V_0'$, therefore $(y, 1) \in V_0$, and $k(A) \in \big\langle \langle X \times I, V_0\rangle\big\rangle \subset \mathscr{U}$.

Similarly, if $A \subset V_i'$ for $1 \leq i \leq n$, then for all $(y, \alpha) \in F \in \exp_1(A \times I)$ we have $(y, \alpha) \in \{y\} \times I \subset V_i$, hence $F \subset V_i$, and $k(A) \in \big\langle \langle V_i\rangle\big\rangle \subset \mathscr{U}$. We obtain that $k(A) \in \mathscr{U}$ for all $A \in \mathscr{V}$.

Now let $B \notin \mathscr{V}$, i.e., $B \not\subset V_i'$ for all $i = 0, 1, 2, \ldots, n$. Then there are $x_i \in B \setminus V_i'$, in particular, $(x_0, 1) \notin V_0$, $\{x_1\} \times I \not\subset V_1$, $\{x_2\} \times I \not\subset V_2$, ..., $\{x_n\} \times I \not\subset V_n$, and we can choose $\alpha_1, \alpha_2, \ldots, \alpha_n \in I$ such that $(x_1, \alpha_1) \notin V_1$, $(x_2, \alpha_2) \notin V_2$, ..., $(x_n, \alpha_n) \notin V_n$. The set $F = \{(x_0, 1), (x_1, \alpha_1), (x_2, \alpha_2), \ldots, (x_n, \alpha_n)\} \subset B \times I$ is then in $k(B) \setminus \big(\langle X \times I, V_0\rangle \cup \langle V_1\rangle \cup \langle V_2\rangle \cup \ldots \cup \langle V_n\rangle\big)$, therefore $k(B) \notin \mathscr{U}$.

We have shown that the preimages of the subbase elements under k are open, thus k is continuous. $\square$

In fact k is a function from $\exp X$ into $\exp(\exp_1(X \times I))$. The continuous mapping $\bar{\oplus} : \exp_1(X \times I) \to X$ induces the continuous mapping

$$\exp \bar{\oplus} : \exp(\exp_1(X \times I)) \to \exp X, \text{ namely } \exp \bar{\oplus}(\mathrm{A}) = \{\bar{\oplus}(\mathcal{A}) \mid \mathcal{A} \in \mathrm{A}\}.$$

Then the composition $\exp \bar{\oplus} \circ k : \exp X \to \exp X$ is continuous and by Proposition 4.1 takes each non-empty $A \underset{\mathrm{cl}}{\subset} X$ to its closed I-convex hull.

Thus we arrive at the main result of the section:

Theorem 4.4. *For an I-convex compactum X, the mapping $\overline{\mathrm{conv}}$ that takes each non-empty closed set to its closed I-convex hull, is continuous with respect to Vietoris topology.*

5. Local convexity and I-convex (pseudo-)metrics

Recall that a real topological vector space L is called locally convex if it has a base in zero that consists of convex sets. This property can conveniently be expressed by saying that the topology on L is the initial topology determined with a family of seminorms $(\|\ldots\|_i)_{i\in\mathcal{I}}$. In fact each seminorm $\|\ldots\|_i$ determines the translation invariant pseudometric $d_i(x, y) = \|x - y\|_i$, and all balls for these pseudometrics form a subbase of the topology (a base if the family of seminorms is saturated).

We cannot use this approach because a compact Hausdorff Lawson I-semimodule lacks subtraction, and the question "How much do these elements differ?" is meaningless.

If the topology on L is determined with a translation invariant (pseudo-)metric d, then for the local convexity of L it is necessary and sufficient that for all $\varepsilon > 0$ there is $\delta > 0$ such that, for all $n \in \mathbb{N}$, $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in L$, $\lambda_1, \lambda_2, \ldots, \lambda_n \in I$, $\lambda_1 + \lambda_2 + \ldots + \lambda_n = 1$, the inequalities $d(x_1, y_1) \le \delta$, $d(x_2, y_2) \le \delta$, ..., $d(x_n, y_n) \le \delta$ imply

$$d(\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n, \lambda_1 y_1 + \lambda_2 y_2 + \ldots + \lambda_n y_n) \le \varepsilon. \tag{9}$$

All the balls for d are convex (hence L is locally convex) if and only if $d(x_1, y_1) \le \varepsilon$, $d(x_2, y_2) \le \varepsilon$, ..., $d(x_n, y_n) \le \varepsilon$ for all $\varepsilon > 0$ imply

$$d(\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n, \lambda_1 y_1 + \lambda_2 y_2 + \ldots + \lambda_n y_n) \le \varepsilon, \tag{10}$$

and it is sufficient to verify the latter equality for $n = 2$.

Most "real life" examples of compact Hausdorff Lawson I-semimodules, and, hence, of I-convex compacta are endowed with (pseudo-)metrics with the similar property.

Definition 5.1. A continuous (pseudo-)metric d on an I-convex compactum is called I-convex if, for all $x_1, y_1, x_2, y_2 \in X$, $\lambda_1, \lambda_2 \in I$, $\lambda_1 \oplus \lambda_2 = 1$, the inequalities $d(x_1, y_1) \le \varepsilon$, $d(x_2, y_2) \le \varepsilon$ imply

$$d(\lambda_1 x_1 \bar{\oplus} \lambda_2 x_2, \lambda_1 y_1 \bar{\oplus} \lambda_2 y_2) \le \varepsilon. \tag{11}$$

If d is I-convex, then, for all $n \in \mathbb{N}$, $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in X$, $\lambda_1, \lambda_2, \ldots, \lambda_n \in I$, $\lambda_1 \oplus \lambda_2 \oplus \ldots \oplus \lambda_n = 1$, the inequalities $d(x_1, y_1) \le \varepsilon$, $d(x_2, y_2) \le \varepsilon$, ..., $d(x_n, y_n) \le \varepsilon$

imply $$d(\lambda_1 x_1 \bar{\oplus} \lambda_2 x_2 \bar{\oplus} \ldots \bar{\oplus} \lambda_n x_n, \lambda_1 y_1 \bar{\oplus} \lambda_2 y_2 \bar{\oplus} \ldots \bar{\oplus} \lambda_n y_n) \le \varepsilon. \tag{12}$$

Consider, e.g., $I = (I, \max, \cdot)$, $X = I^n$,

$$d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \max\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\}.$$

It is straightforward to verify that d is an admissible I-convex metric on X.

We already know that all I-convex compacta are locally convex, hence one can expect that such "nice" pseudometrics always exist. The most natural way to construct them is via the unique admissible uniformities on compacta. For our pseudometrics to be I-convex, we need I-convex neighborhoods of diagonals.

Thus the main question of this section is the following: "Does each neighborhood of the diagonal $\Delta_X \subset X \times X$ for an I-convex compactum contain an I-convex neighborhood?" The answer is clearly positive for I^τ, but for arbitrary X we use the previously proved continuity of closed I-convex hull w.r.t. the Vietoris topology.

Theorem 5.2. *For each neighborhood $U \supset \Delta_X$ in $X \times X$, there in a closed neighborhood $V \supset \Delta_X$ such that $\overline{\mathrm{conv}}\, V \subset U$.*

Proof. Consider the family (V_α) of all *closed* neighborhoods of Δ_X. It is filtered, hence can be considered a net, which converges w.r.t. the Vietoris topology to the diagonal. The mapping $\overline{\mathrm{conv}} : \exp(X \times X) \to \exp(X \times X)$ is continuous, for the open neighborhood $\mathrm{Int}\, U \supset \Delta X$ we have $\overline{\mathrm{conv}}(\Delta_X) = \Delta_X \in \langle \mathrm{Int}\, U \rangle$, therefore there is $V = V_\alpha$ such that $\overline{\mathrm{conv}}(V) \in \langle \mathrm{Int}\, U \rangle$, hence $\overline{\mathrm{conv}}\, V \subset U$. □

6. Existence of I-convex pseudometric

Now we are ready to construct an I-convex pseudometric on an I-convex compactum X. We use uniform structures as defined in [5], Chapter VIII. For binary relations U, V on X, i.e., subsets $U, V \subset X \times X$ the *composition* is defined as

$$U \circ V = \{(x, z) \in X \times X \mid \text{ there is } y \in X \text{ such that } (x, y) \in U,\ (y, z) \in V\}. \tag{13}$$

The *inverse* relation to $U \subset X \times X$ is $U^{-1} = \{(y, x) \in X \times X \mid (x, y) \in U\}$.

Recall that a *uniformity* on a set X is a family $\mathcal{U}$ of subsets of the cartesian product $X \times X$ such that:

(1) each $U \in \mathcal{U}$ contains Δ_X;
(2) if $U \in \mathcal{U}$ and $U \subset V \subset X \times X$, then $V \in \mathcal{U}$;
(3) if $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$;
(4) for each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$;
(5) $\bigcap\{U \mid U \in \mathcal{U}\} = \Delta_X$.

Elements of a uniformity are called *entourages*. The above conditions imply that for all $U \in \mathcal{U}$ there is a *symmetric* entourage $V \in \mathcal{U}$ (i.e., $V = V^{-1}$) such that $V \subset U$.

We say that a uniformity $\mathcal{U}$ on a topological space (X, τ) is *admissible* if a point $x \in X$ is an interior point in a set $A \subset X$ if and only if there is entourage $U \in \mathcal{U}$ such that $xU = \{y \in X \mid (x, y) \in U\} \subset A$.

It is known that for a compactum X there exists a unique admissible uniformity, which consists of all subsets $U \subset X \times X$ such that $\Delta_X \subset \operatorname{Int} U$. From now on this uniformity on an I-convex compactum is considered.

Assume that a (not strictly) decreasing sequence $(V_n)_{n\in\mathbb{N}}$ of symmetric closed entourages of the diagonal $\Delta_X \subset X \times X$ is given such that $V_n \supset V_{n+1} \circ V_{n+1}$ for all $n \in \mathbb{N}$.

By the above we can choose a decreasing sequence (W_r^n), of symmetric closed I-convex entourages of the diagonal, which we conveniently index with the pairs of a number $n \in \mathbb{N}$ and a number $r \in D_n^+ = \{\frac{k}{2^n}, 0 < k \leq 2^n\}$. We require that

$$\begin{aligned} X \times X &\supset W_1^1 \supset W_{1/2}^1 \supset W_1^2 \circ W_1^2 \\ &\supset W_1^2 \supset W_{3/4}^2 \supset W_{1/2}^2 \supset W_{1/4}^2 \supset W_1^3 \circ W_1^3 \\ &\dots \\ &\supset W_1^n \supset W_{\frac{2^n-1}{2^n}}^n \supset \dots \supset W_{\frac{2}{2^n}}^n \supset W_{\frac{1}{2^n}}^n \supset W_1^{n+1} \circ W_1^{n+1} \\ &\dots, \end{aligned} \tag{14}$$

$W_r^n \subset V_n$ for all $n \in \mathbb{N}$, and property (*), which will be formulated later, must hold. Each number $r \in D_n^+ \setminus \{1\}$ can be uniquely represented as a sum $\frac{1}{2^{n_1}} + \frac{1}{2^{n_2}} + \dots + \frac{1}{2^{n_k}}$, where $1 \leq n_1 < n_2 < \dots < n_k \leq n$. We take this number to the (not necessarily symmetric) entourage of the diagonal

$$W_r = W^{n_1}_{\frac{1}{2^{n_1}}} \circ W^{n_2}_{\frac{1}{2^{n_1}}+\frac{1}{2^{n_2}}} \circ W^{n_3}_{\frac{1}{2^{n_1}}+\frac{1}{2^{n_2}}+\frac{1}{2^{n_3}}} \circ \dots \circ W^{n_k}_{\frac{1}{2^{n_1}}+\dots+\frac{1}{2^{n_k}}}. \tag{15}$$

It is easy to observe that for $r = \frac{1}{2^{m+1}} + \frac{1}{2^{m+2}} + \dots + \frac{1}{2^{m+l}} < \frac{1}{2^m}$ the inclusion $W_r \subset W^m_{\frac{1}{2^m}} = W_{\frac{1}{2^m}}$ is valid, hence we have $W_r \subset W_q$ for all dyadic rational proper fractions $r < q$. Note that each entourage W_r is I-closed in $X \times X$, which follows from I-convexity of all $W^{n_i}_{\frac{1}{2^{n_1}}+\dots+\frac{1}{2^{n_i}}}$.

Now we can formulate the last requirement (*): for all $p < q$, $p, q \in D_n^+$, such that $q - p \in D_{n-1}^+$, the inclusion $W_p^n \circ W_{q-p} \subset W_{q-p} \circ W_q^n$ should hold.

The required sequence of entourages can be constructed due to Theorem 5.2 and the two following lemmata:

Lemma 6.1. *If U, V are closed entourages of the diagonal in a compactum $X \times X$, then $U \subset \operatorname{Int}(U \circ V)$.*

Lemma 6.2. *If V, W are closed entourages of the diagonal in a compactum $X \times X$ such that $V \subset \operatorname{Int} W$, and (U_α) is a filtered family of closed entourages of the diagonal such that $\bigcap_\alpha U_\alpha = \Delta_X$, then $U_\alpha \circ V \subset W$ for some α.*

Then for the chosen entourages:

Proposition 6.3. *If $p, r > 0$ are dyadic rationals and $p+r < 1$, then $W_p \circ W_r \subset W_{p+r}$.*

Proof. Let $p = \frac{1}{2^{n_1}} + \frac{1}{2^{n_2}} + \dots + \frac{1}{2^{n_k}}$, where $1 \leq n_1 < n_2 < \dots < n_k$, and, similarly,

$$r = \frac{1}{2^{m_1}} + \frac{1}{2^{m_2}} + \dots + \frac{1}{2^{m_l}},\ 1 \leq m_1 < m_2 < \dots < m_l.$$

In the composition

$$
\begin{aligned}
W_p \circ W_r = W^{n_1}_{\frac{1}{2^{n_1}}} \circ W^{n_2}_{\frac{1}{2^{n_1}}+\frac{1}{2^{n_2}}} \circ \cdots \circ W^{n_k}_{\frac{1}{2^{n_1}}+\cdots+\frac{1}{2^{n_k}}} \circ \\
\circ W^{m_1}_{\frac{1}{2^{m_1}}} \circ W^{m_2}_{\frac{1}{2^{m_1}}+\frac{1}{2^{m_2}}} \cdots \circ W^{m_l}_{\frac{1}{2^{m_1}}+\cdots+\frac{1}{2^{m_l}}}.
\end{aligned}
\tag{16}
$$

we transfer the "factor" $W^{n_k}_p$ to the right side along all "factors" with the powers $m_1 < m_2 < \cdots < m_i$ that are less than n_k. Define $p' = \frac{1}{2^{n_1}} + \cdots + \frac{1}{2^{n_{k-1}}} = p - \frac{1}{2^{n_k}}$, $q = \frac{1}{2^{m_1}} + \frac{1}{2^{m_2}} + \cdots + \frac{1}{2^{m_i}}$. Due to (*)

$$
W^{n_k}_p \circ W_q \subset W_q \circ W^{n_k}_{p+q}, \quad p+q = p' + q + \frac{1}{2^{n_k}},
\tag{17}
$$

thus

$$
\begin{aligned}
W_p \circ W_r &\subset W_{p'} \circ W_q \circ W^{n_k}_{p+q} \circ W^{m_{i+1}}_{\frac{1}{2^{m_1}}+\cdots+\frac{1}{2^{m_{i+1}}}} \circ \cdots \circ W^{m_l}_{\frac{1}{2^{m_1}}+\cdots+\frac{1}{2^{m_l}}} \\
&= W_{p'} \circ W_q \circ W^{n_k}_{p'+q+\frac{1}{2^{n_k}}} \circ W^{m_{i+1}}_{q+\frac{1}{2^{m_{i+1}}}} \circ \cdots \circ W^{m_l}_{q+\frac{1}{2^{m_{i+1}}}+\cdots+\frac{1}{2^{m_l}}}.
\end{aligned}
\tag{18}
$$

Taking into account

$$
\begin{aligned}
&W^{m_{i+1}}_{q+\frac{1}{2^{m_{i+1}}}} \subset W^{m_{i+1}}_{p'+q+\frac{1}{2^{n_k}}+\frac{1}{2^{m_{i+1}}}}, \\
&\ldots \\
&W^{m_l}_{q+\frac{1}{2^{m_{i+1}}}+\cdots+\frac{1}{2^{m_l}}} \subset W^{m_l}_{p'+q+\frac{1}{2^{n_k}}+\frac{1}{2^{m_{i+1}}}+\cdots+\frac{1}{2^{m_l}}} = W^{m_l}_{p'+r'},
\end{aligned}
\tag{19}
$$

where $r' = r + \frac{1}{2^{n_k}}$ (hence $p' + r' = p + r$), we obtain

$$
W_p \circ W_r \subset W_{p'} \circ W_q \circ W^{n_k}_{p'+q+\frac{1}{2^{n_k}}} \circ W^{m_{i+1}}_{p'+q+\frac{1}{2^{n_k}}+\frac{1}{2^{m_{i+1}}}} \circ \cdots \circ W^{m_l}_{p+r}.
\tag{20}
$$

We leave unchanged the "tail" of the composition that starts with $W^{n_k}_{p'+q+\frac{1}{2^{n_k}}}$, and similarly transfer the last "factor" from $W_{p'}$ into W_q, then the composition $W_{p'} \circ W_q$ and therefore the total composition increases or is unchanged, etc, until the two increasing sequences $n_1 < n_2 < \cdots < n_k$ and $m_1 < m_2 < \cdots < m_l$ merge into one non-decreasing sequence $N_1 \le N_2 \le \cdots \le N_{k+l}$ such that

$$
W_p \circ W_r \subset W^{N_1}_{\frac{1}{2^{N_1}}} \circ W^{N_2}_{\frac{1}{2^{N_1}}+\frac{1}{2^{N_2}}} \circ \cdots \circ W^{N_{k+l}}_{\underbrace{\frac{1}{2^{N_1}}+\cdots+\frac{1}{2^{N_{k+l}}}}_{=p+r}}.
\tag{21}
$$

Unfortunately in the sequence $N_1 \le N_2 \le \cdots \le N_{k+l}$ elements are not necessarily unique. Hence, moving backwards from the end, we replace the last pair of identical elements $N_i = N_{i+1} = n$ with one element $n-1$, resulting in replacement of the terms $\frac{1}{2^n} + \frac{1}{2^n}$ in the total sum with $\frac{1}{2^{n-1}}$, i.e., the sum does not change and is equal to $p + r$. For the composition

$$
W^{N_i}_{\frac{1}{2^{N_1}}+\cdots+\frac{1}{2^{N_i}}} \circ W^{N_{i+1}}_{\frac{1}{2^{N_1}}+\cdots+\frac{1}{2^{N_i}}+\frac{1}{2^{N_{i+1}}}} \subset W^{n-1}_{\frac{1}{2^{n-1}}} \subset W^{n-1}_{\frac{1}{2^{N_1}}+\cdots+\frac{1}{2^{N_i}}+\frac{1}{2^{N_{i+1}}}},
\tag{22}
$$

therefore the total composition increases or is unchanged. Then we search for the previous two or three identical elements, and join the pair, if two elements are found, or the first two of three, etc, until all elements of the sequence are distinct. Clearly the final composition will then be equal to W_{p+r}, which implies the required inclusion $W_p \circ W_r \subset W_{p+r}$. $\square$

What follows is standard in the spirit of Kakutani's Theorem on metrization of topological groups [9]. For all $x, y \in X$ define $d(x,y) = \inf\{r \mid (x,y) \in W_r \wedge (y,x) \in W_r\}$ assuming $\inf \varnothing = 1$. Then d is a continuous pseudometric on X with the required property: if $d(x,x') \leq \varepsilon$, $d(y,y') \leq \varepsilon$, then for all $\lambda, \mu \in I$, $\lambda \oplus \mu = 1$ the inequality $d(\lambda \bar{*} x \bar{\oplus} \mu \bar{*} y, \lambda \bar{*} x' \bar{\oplus} \mu \bar{*} y') \leq \varepsilon$ is valid as well. If the given family (V_n) of entourages of the diagonal separates the points of the compactum X, i.e., is a countable base [5] of the unique admissible uniformity on X, then d is an admissible metric on X.

For any continuous pseudometric $\rho : X \times X \to I$ we can put

$$V_i = \{(x,y) \in X \times X \mid \rho(x,y) \leq 2^{-i}\},\ i = 1, 2, \ldots.$$

Then $d(x,y) \geq \rho(x,y)$ for all x, y and the constructed above (pseudo-)metric d.

Obviously, by multiplying by a constant factor we can obtain a pseudometric with any range, not necessarily $[0,1]$, hence:

Proposition 6.4. *For each continuous pseudometric ρ on an I-convex compactum X there is a continuous I-convex pseudometric d on X such that $d(x,y) \geq \rho(x,y)$ for all $x, y \in X$.*

Thus we arrive at the main result of the paper:

Theorem 6.5. *The topology on an I-convex compactum is determined with a family of I-convex pseudometrics. If the compactum is metrizable, then there is an admissible I-convex metric.*

7. Conclusions and future work

So complicated construction for an I-convex pseudometric was proposed because of noncommutativity of composition of entourages and lack of subtraction in idempotent semimodules. Compactness of underlying spaces was extensively used. It has been said already that it is possible to define continuous idempotent semimodules in a purely algebraic fashion, where order continuity in the sense of domain theory is involved, and Scott and Lawson topologies arise from order relation, cf. [1]. Metrization of such objects without compactness would require quite new techniques.

Two questions naturally arise concerning the presented above results.

Consider again the semiring $I = (I, \max, \cdot)$ and the semimodule $X = I^n$ with the metric

$$d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \max\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\}.$$

It satisfies even stronger inequality

$$d(\lambda_1 \bar{*} x_1 \bar{\oplus} \lambda_2 \bar{*} x_2,\ \lambda_1 \bar{*} y_1 \bar{\oplus} \lambda_2 \bar{*} y_2) \leq \lambda_1 * d(x_1, y_1) \oplus \lambda_2 * d(x_2, y_2), \tag{23}$$

which, in turn, implies

$$\begin{aligned} &d(\lambda_1 \bar{*} x_1 \bar{\oplus} \lambda_2 \bar{*} x_2 \bar{\oplus} \ldots \bar{\oplus} \lambda_n \bar{*} x_n,\ \lambda_1 \bar{*} y_1 \bar{\oplus} \lambda_2 \bar{*} y_2 \bar{\oplus} \ldots \bar{\oplus} \lambda_n \bar{*} y_n) \\ &\quad \leq \lambda_1 * d(x_1, y_1) \oplus \lambda_2 * d(x_2, y_2) \bar{\oplus} \ldots \bar{\oplus} \lambda_n * d(x_n, y_n). \end{aligned} \tag{24}$$

It is unlikely that such metrics exist for all I-convex compacta, but now we neither have a counterexample nor are able to prove existence.

The answer is positive if another hypothesis is true. Examples of idempotent semimodules usually can be embedded into powers of the respective semiring, or, equivalently, there are separating families of continuous affine mappings (i.e., mappings that preserve idempotent convex combinations) into the semiring. It is unknown whether this is possible for all I-convex compacta.

Last but not least, we are going to verify if the above results can be extended to non-associative settings [3], to which our attention was attracted by the anonymous referee. We also express our deep gratitude to him for careful reading and numerous helpful advices and corrections.

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