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Best Approximative Properties Of Exposed Faces

N. Prakash, K. V. Sangeetha
Department of Mathematics, National Institute of Technology Tiruchirappalli, Trichy 620015, Tamil Nadu, India
prakashn@nitt.edu, sangeekv@gmail.com

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Let $X = c_0$ or $L_1(\mu)$. We will show that the exposed face of the unit sphere of X is strongly proximinal and the corresponding metric projection map is Hausdorff metric continuous.

Keywords: Proximinal, strongly proximinal, norm attaining functional, exposed face, Hausdorff metric continuity.

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1. Introduction

Let X be a real normed linear space. A closed, convex and extremal subset of the closed unit ball B_X of X is called an exposed face of B_X. It is well known by the famous James' theorem that every closed convex subset of a reflexive Banach space is proximinal. But strong proximinality of every closed convex subset of a Banach space demands a very stronger condition on the Banach space X, namely, reflexivity and the Kadec-Klee property [3]. Recall that a Banach space X is said to have the Kadec-Klee property if the relative weak and norm topologies coincide on the unit sphere S_X of X. In a non-reflexive space, there exists a closed convex set which is not even proximinal. In this paper, we consider some special convex subsets of non-reflexive spaces and their strong proximinality properties. The strong proximinality of the exposed faces of some non-reflexive Banach spaces has been studied recently in few papers ([7], [12]). Let E be an exposed face of the unit ball of the non-reflexive Banach spaces c_0 or $L_1(\mu)$. Then we will prove that the closed convex set E is strongly proximinal and the metric projection onto E is Hausdorff metric continuous.

We will introduce the notation and some definitions to discuss in detail. Throughout the paper, X denotes a real Banach space, $B_X = \{x \in X : \|x\| \leq 1\}$ is the closed unit ball of X, $S_X = \{x \in X : \|x\| = 1\}$ is the unit sphere of X, and X^* is the dual of X.

For x in X and a non-empty subset E of X, $d(x, E)$ denotes the distance of x from E. That is, $d(x, E) = \inf_{y \in E} \|x - y\|$. The set of all best approximations to x from E is denoted by $P_E(x)$ and

$$P_E(x) = \{y \in E : \|x - y\| = d(x, E)\}.$$

If $P_E(x)$ is a non-empty set for each x in X, we say that the subset E is proximinal in X. For $\delta > 0$ we set

$$P_E(x, \delta) = \{z \in E : \|x - z\| < d(x, E) + \delta\}.$$

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We say that a proximinal set E of a normed linear space X is *strongly proximinal* if for each x in X and $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sup\{d(z, P_E(x)) : z \in P_E(x, \delta)\} \; < \; \epsilon.$$

Strong proximinality was introduced in [6] and it plays an important role in applications, since it ensures that the set $P_E(x, \delta)$, which can be described as the set of "almost nearest" elements when δ is small enough, is close to the set of nearest elements of x. However, while every Banach space has plenty of proximinal hyperplanes as shown by the Bishop-Phelps Theorem, there are separable Banach spaces which do not have any strongly proximinal hyperplanes [6]. It can be easily verified that the metric projection onto a strongly proximinal set is upper Hausdorff semi continuous [6]. Strong proximinality has been studied by many authors and some of the interesting results related to this notion can be found in [3], [4], [7], [8], [9], [10], [11], [12], [18].

Let X and Y be normed linear spaces. We say F is a set-valued map from X into Y if F is a map from X into the class of all non-empty subsets of Y and in this case, we write $F : X \rightrightarrows Y$ is a set-valued map. Let x_0 be in X. The set-valued map F is said to be

1. *lower semi-continuous* at x_0 if for each open set $V \subset Y$ with the property $F(x_0) \cap V \neq \emptyset$ there exists a neighborhood $U \subset X$ of x_0 such that $F(x) \cap V \neq \emptyset$ whenever $x \in U$.
2. *lower Hausdorff semi-continuous* at x_0 if for each $\epsilon > 0$ there exists a neighborhood $U \subset X$ of x_0 such that $F(x_0) \subset F(x) + \epsilon B_Y$ whenever $x \in U$.
3. *upper semi-continuous* at x_0 if for each open set $V \subset Y$ such that $F(x_0) \subset V$ there exists a neighborhood $U \subset X$ of x_0 such that $F(x) \subset V$ whenever $x \in U$.
4. *upper Hausdorff semi-continuous* at x_0 if for each $\epsilon > 0$ there exists a neighborhood $U \subset X$ of x_0 such that $F(x) \subset F(x_0) + \epsilon B_Y$ whenever $x \in U$.

The set-valued map F is lower (upper, lower Hausdorff, upper Hausdorff) semi-continuous on X if it is lower (upper, lower Hausdorff, upper Hausdorff) semi-continuous at each point of X and is called *Hausdorff metric continuous* if it is both lower and upper Hausdorff semi-continuous.

We observe that upper semi continuity implies upper Hausdorff semi continuity, while the lower semi continuity is implied by lower Hausdorff semi continuity.

Definition 1.1. A *selection* for the set-valued map F is a map $f : X \longrightarrow Y$ such that $f(x)$ is in $F(x)$, for each x in X. A selection of the set-valued map F which is continuous on X, is called a *continuous selection* of the map F. □

In this paper, we will show the Hausdorff metric continuity of the set-valued metric projection map $P_E : X \rightrightarrows E$ defined by $P_E(x) = \{y \in E : d(x, E) = \|x - y\|\}$. By the well known Michael's selection Theorem [15, 16], it is obvious that if the metric projection onto a closed, convex subset of a Banach space is lower semi continuous then it has a continuous selection.

A *face* of a normed linear space X is a convex, extremal subset of the closed unit ball B_X. Let $f \in X^*$ and $J_X(f) = \{x \in S_X : f(x) = \|f\|\}$.

It is easily verified that if the set $J_X(f)$ is non-empty, then it is a closed, convex and extremal subset of B_X and is called an *exposed face* of B_X.

We denote by $NA(X)$, the set of all norm attaining functionals on X and by $NA_1(X)$ the set $NA(X) \cap S_X$. If $H = \ker f$, then it is well known from [20] and [5] that

$$f \in NA(X) \Leftrightarrow J_X(f) \neq \emptyset \Leftrightarrow H \text{ is proximinal in X.}$$

A subspace Y of a normed linear space X is called ball proximinal if B_Y is a proximinal set. It is easily verified that ball proximinality implies proximinality [1].

Best approximative properties of hyperplanes or subspaces in general, of Banach spaces are closely related to structure of the unit ball and its exposed faces and study of geometric structure of the unit ball in view of this link, is not new ([2], [17] and [14]). It is known from [8] that if a hyperplane H, kernel of a functional f in the dual of X is ball proximinal then the set $J_X(f)$ satisfies a restricted proximinality condition. It is also shown in [12] and [7] that if C is an exposed face of the sequence space l_l or $C(Q)$, the Banach space of real valued continuous functions defined on a compact, Hausdorff space, then C is strongly proximinal and the corresponding metric projection map is Hausdorff metric continuous. In this paper, we will show that the similar results holds true for the exposed faces of the non-reflexive Banach spaces c_0 and $L_1(\mu)$.

2. The space c_0 and strong proximinality of exposed faces

Let $\mathbb{N}$ denote the set of positive natural numbers. We consider the real Banach space $X = c_0$, the space of all sequences of real scalars which converges to zero with sup norm. Then the dual of c_0 is l_1, the space of all sequences $(x_n$)of real scalars with $\|x\|_1 = \sum_{n\in\mathbb{N}} |x_n|$. For z in X^*, we recall that the support of z is the set

$$\Lambda = \{n \in \mathbb{N} : z_n \neq 0\}.$$

Now let $\Lambda^+ = \{n \in \mathbb{N} : z_n > 0\}$ and $\Lambda^- = \{n \in \mathbb{N} : z_n < 0\}$.

Then $\Lambda = \Lambda^+ \cup \Lambda^-$. Let $z = (z_n) \in l_1$ and $\|z\| = 1$. If f_z is the element of the dual of c_0, induced by z then $f_z \in NA_1(c_0)$ if and only if support of z is finite. We denote the exposed face $J_{c_0}(f_z)$ by $J_X(z)$. It is known that

$$J_X(z) \neq \emptyset \Leftrightarrow \Lambda \neq \emptyset \ \ \& \ \ \Lambda \text{ is finite} \tag{1}$$

and in this case it can be easily verified that

$$J_X(z) = \big\{x \in S_X : x_n = 1 \text{ on } \Lambda^+,\ x_n = -1 \text{ on } \Lambda^-\big\}. \tag{2}$$

First we prove that the set $J_X(z)$ is proximinal in X.

Fact 2.1. *The set $J_X(z)$, if non-empty, is proximinal in X.*

Proof. Since $J_X(z)$ is non-empty, (2) holds. Let $x \in X$ and set α as

$$\alpha = \max\ \Big[\max_{n\in\Lambda^+} |x_n - 1|, \max_{n\in\Lambda^-} |x_n + 1|\ \Big]. \tag{3}$$

Then
$$d = d(x, J_X(z)) \geq \alpha. \tag{4}$$

We discuss two cases.

Case (1): $\sup\limits_{n\in N\setminus\Lambda} d(x_n,[-1,1]) \leq \alpha$.

We have $x \in X$ and for $\alpha > 0, \exists\ n_0 \in N$ such that $|x_n| < \alpha$ for all $n \geq n_0$. If $\big\{1, 2..., n_0\big\} \subseteq \Lambda$, we define

$$y_n = \begin{cases} 1 & \text{if } n \in \Lambda^+ \\ -1 & \text{if } n \in \Lambda^- \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|y\|=1$ and $\|x-y\| \leq \alpha \leq d$ and so y becomes a nearest element to x from E. If there exists $\big\{n_1, n_2, ..., n_k\big\}$ such that $n_i \notin \Lambda$ and $n_i \leq n_0,\ 1 \leq i \leq k$ then define

$$y_n = \begin{cases} 1 & \text{if } n \in \Lambda^+ \ \ or \ \ x_{n_i} \geq 1,\ 1 \leq i \leq k \\ -1 & \text{if } n \in \Lambda^- \ \ or \ \ x_{n_i} \leq -1,\ 1 \leq i \leq k \\ x_{n_i} + \min\{\alpha, 1 - x_{n_i}\} & \text{if } 0 \leq x_{n_i} < 1,\ 1 \leq i \leq k \\ x_{n_i} - \min\{\alpha, 1 + x_{n_i}\} & \text{if } -1 < x_{n_i} < 0,\ 1 \leq i \leq k \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

We get y is the best approximation to x from $J_X(z)$ as $\|y\| = 1$ and $\|x-y\| \leq \alpha$.

Case (2): $\beta = \sup\limits_{n\in N\setminus\Lambda} d(x_n,[-1,1]) > \alpha$.

In this case, note that since $J_X(z)$, is a subset of the unit sphere S_X, so that

$$d(x, J_X(z)) \geq \sup_{n\in N\setminus\Lambda} d(x_n,[-1,1]) = \beta. \tag{6}$$

Define $p_n = \min\{x_n, 1+\alpha\}$ and $q_n = \max\{p_n, -1-\alpha\}$. Then $q_n = x_n$ on Λ and $\sup_{n\in N\setminus\Lambda} d(q_n,[-1,1]) \leq \alpha$. By Case(1), there exists $y \in P_{J_X(z)}(q)$. Thus

$$\|q-y\| \ = \ d(q, J_X(z) = \alpha. \tag{7}$$

We now claim that y is a best approximation to x from $J_X(z)$. For this, we would show that $\|x-y\| \leq \beta$. Note that

$$A = \big\{n \in N : x_n \neq q_n\big\} \subseteq \big\{n \in N : x_n > 1+\alpha\big\} \cup \big\{n \in N : x_n < -1-\alpha\big\}. \tag{8}$$

It is enough to show that $|y_n - x_n| \leq \beta$ for all $n \in A$. If $x_n > 1+\alpha$, then $q_n = 1+\alpha$ and if $x_n < -1-\alpha$ then $q_n = -1-\alpha$. Now $\|q-y\| \leq \alpha$ and $\|y\| \leq 1$ implies that $y_n = 1$ if $q_n = 1+\alpha$ and $y_n = -1$ if $q_n = -1-\alpha$. Clearly, in either case we have

$$\|x-y\| = \beta \leq d(x, J_X(z))$$

and y is a best approximation to x from $J_X(z)$. □

We now show that the set $J_X(z)$ is strongly proximinal and the corresponding metric projection map is Hausdorff metric continuous.

Theorem 2.2. *The set $E = J_X(z)$ is strongly proximinal and P_E is Hausdorff metric continuous. In particular, P_E has a continuous selection.*

Proof. Let $x \in X$ and α as given by (3). Then there exists $n_0 \in N$ such that $|x_n| < \alpha$ for all $n \geq n_0$. If $d = d(x, E)$, then $d \geq \alpha$. Recall that $E = J_X(z)$ is given by (2).

By Fact 2.1 above, E is proximinal. Pick any $u \in P_E(x)$. Since E is a subset of the unit sphere, we have $\|u\| = 1$ and $\|x - u\| = d$. Hence

$$\|x\| \leq \|u\| + d = 1 + d. \tag{9}$$

Pick $v \in E$ such that $\|x - v\| < d + \delta$ for some $\delta > 0$. We will construct $\widetilde{v} \in P_E(x)$ such that $\|v - \widetilde{v}\| \leq \delta$. This would prove E is strongly proximinal.

Let
$$y_n = \begin{cases} \max\{-d, \min\{x_n - v_n, d\}\} & \text{if } n \leq n_0 \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Then $y = (y_n) \in c_0$ and for any $n \geq n_0$ we have

$$x_n - v_n \geq d \Rightarrow y_n = d, \quad x_n - v_n \leq -d \Rightarrow y_n = -d$$

and
$$|x_n - v_n| \leq d \Rightarrow y_n = x_n - v_n.$$

Define $\widetilde{v} = x - y$. Then

$$\widetilde{v} = v \qquad \text{on} \qquad \{n \in \mathbb{N} : |x_n - v_n| \leq d\}. \tag{11}$$

If $x_n - v_n > d$ then $y_n = d$ and $\widetilde{v_n} = x_n - y_n \leq 1 + d - d \leq 1$ by (9). Also

$$\widetilde{v_n} = x_n - y_n = v_n + (x_n - v_n) - y_n \geq v_n + d - d = v_n \geq -1.$$

If $x_n - v_n < -d$, then $y_n = -d$ and $\widetilde{v_n} = x_n - y_n \geq -1 - d + d = -1$ by (9). Further in this case,

$$\widetilde{v_n} = x_n - y_n = v_n + (x_n - v_n) - y_n \leq v_n - d + d = v_n \leq 1.$$

Thus
$$|\widetilde{v_n}| \leq 1 \text{ if } |x_n - v_n| > d$$

and
$$|\widetilde{v_n}| = |v_n| \leq 1 \text{ if } |x_n - v_n| \leq d.$$

Hence $\|\widetilde{v}\| \leq 1$.

We have $d = d(x, E) \geq \alpha$ by (4). Since $v \in E$, using (2) and (9) we have

$$\Lambda \subseteq \{n \in N : |x_n - v_n| \leq \alpha \leq d\}.$$

Hence, by (11), $\widetilde{v} = v$ on Λ and therefore, (2) implies that $\widetilde{v} \in E$, since $\|\widetilde{v}\| \leq 1$. Now $x - \widetilde{v} = x - (x - y) = y$ and by (10), $\|y\| \leq d$. So $\widetilde{v} \in P_E(x)$.

We now show that $\|v - \widetilde{v}\| \leq \delta$. If $|x_n - v_n| \leq d$ then $\widetilde{v_n} = v_n$ by (11). Hence

$$|\widetilde{v_n} - v_n| \leq \delta \quad \text{if} \quad |x_n - v_n| \leq d. \tag{12}$$

So we consider the case $|x_n - v_n| > d$. If $x_n - v_n > d$ then by (10), $y_n = d < x_n - v_n$. Now $\widetilde{v_n} = x_n - y_n$ and therefore

$$\widetilde{v_n} - v_n = x_n - y_n - v_n > v_n - v_n = 0.$$

Also since $\|x - v\| < d + \delta$, we have

$$0 < \widetilde{v_n} - v_n = x_n - v_n - y_n \leq d + \delta - d = \delta.$$

If $x_n - v_n < -d$, then $y_n = -d > x_n - v_n$. This implies $x_n - y_n < v_n$ and hence

$$\widetilde{v_n} - v_n = x_n - y_n - v_n < v_n - v_n = 0.$$

Further $$0 > \widetilde{v_n} - v_n = x_n - v_n - y_n \geq -d - \delta + d = -\delta.$$

Thus if $|x_n - v_n| > d$ then either $0 < \widetilde{v_n} - v_n \leq \delta$ or $0 > \widetilde{v_n} - v_n \geq -\delta$. So $|\widetilde{v_n} - v_n| \leq \delta$ if $|x_n - v_n| > d$. This with (12) implies that

$$\|v - \widetilde{v}\| \leq \delta$$

and completes the proof for strong proximinality of E.

We now show that the metric projection P_E is Hausdorff metric continuous. Since E is strongly proximinal, P_E is upper Hausdorff semi continuous. We only need to show that P_E is lower Hausdorff semi continuous. From the above, for any $x \in c_0$ and $\delta > 0$, we have

$$u \in C, \|x - u\| < d(x, E) + \delta \Rightarrow \exists\, \widetilde{u} \in P_E(x) \text{ such that } \|\widetilde{u} - u\| \leq \delta.$$

Let $x \in c_0$ and $u \in P_E(x)$. Let $\epsilon > 0$ be given and V be an ϵ-neighborhood of u. We need to show that there exists $\delta > 0$ such that

$$\|x - y\| < \delta \Rightarrow V \cap P_E(y) \neq \phi. \tag{13}$$

Choose $0 < \delta < \frac{\epsilon}{4}$. Then $\|x - y\| < \delta$ implies

$$|d(x, E) - d(y, E)| \leq \|x - y\| < \delta$$

and we have

$$\|y - u\| \leq \|y - x\| + \|x - u\| < \delta + d(x, E) \leq d(y, E) + 2\delta.$$

Hence there exists $\widetilde{u} \in P_E(y)$ such that $\|u - \widetilde{u}\| \leq 2\delta < \epsilon$ and (13) is satisfied. So for any $\epsilon-$neighborhood V of u there exists a $\delta-$neighborhood U of x such that $V \cap P_E(y) \neq \emptyset$ for every $y \in U$. Thus P_E is lHsc and by the well known Michael selection theorem [15, 16], P_E has a continuous selection. $\square$

3. The space $L_1(\mu)$ and proximinality of exposed faces

Let (T, Σ, μ) be a measure space, where T is a compact Hausdorff space. We consider the real Banach space $L_1(\mu)$, the space of all μ-measurable functions x on T with $\|x\| = \int_T |x(t)| d\mu$.

For a real number α, let

$$sgn\ \alpha = \begin{cases} 1 & \text{if } \alpha > 0 \\ 0 & \text{if } \alpha = 0 \\ -1 & \text{if } \alpha < 0. \end{cases}$$

For a function $x \in L_1(\mu)$, we set $supp(x) = \{t \in T : x(t) \neq 0\}$ and for any subset Λ of T, we set

$$\|x\|_\Lambda = \int_{t\in\Lambda} |x(t)| d\mu.$$

Note that the dual of $L_1(\mu)$ is $L_\infty(\mu)$. Let $f \in L_\infty(\mu)$ and $\|f\|_\infty = 1$. Then f is in $NA_1(L_1(\mu))$ if and only if $\mu\{t \in T : |f(t)| = \|f\|_\infty\} > 0$ [20]. For any element $y \in L_1(\mu)$, we set

$$M_y = \{A \subseteq supp(y) : \mu(A) > 0\}, \quad \text{and}$$

$$\Lambda^+ = \{t \in T : f(t) = \|f\|_\infty\}, \quad \Lambda^- = \{t \in T : f(t) = -\|f\|_\infty\} \quad \text{and} \quad \Lambda = \Lambda^+ \cup \Lambda^-.$$

We now characterize the exposed faces of $L_1(\mu)$.

Fact 3.1. *Let $X = L_1(\mu)$ and f be in $NA_1(X)$, then*

$$J_X(f) = \{y \in L_1(\mu) : f(y) = \|f\|_\infty = 1\} = E = E_1,$$

where

$$E = \left\{ y \in L_1(\mu) \ : \begin{array}{l} \|y\|_1 = 1,\ \mu(supp(y) > 0 \ \textit{and}\ \forall\, A \in M_y \ \textit{we have} \\ A \cap \Lambda \neq \emptyset \ \textit{and}\ \ \exists\, t \in A \ \ni\ sgn\ y(t) = sgn\ f(t) \end{array} \right\}$$

and

$$E_1 = \left\{ y \in L_1(\mu) \ : \begin{array}{l} \|y\|_1 = 1,\ \mu(supp(y)) > 0 \ \textit{and}\ \forall\ A \in M_y \\ \exists\ t \in A \cap \Lambda \ \ni\ sgn\ y(t) = sgn\ f(t) \end{array} \right\}.$$

Proof. First we will show that $J_X(f) = E$. Let $y \in J_X(f)$. Then we have $f(y) = 1$. To prove $y \in E$, we first claim that $\mu(supp(y)) > 0$. Suppose not. Then

$$f(y) = \int_T f(t) y(t) d\mu = 0,$$

which is a contradiction. Hence $\mu(supp(y)) > 0$. Now we will verify that $A \cap \Lambda \neq \emptyset$ for every $A \in M_y$. Suppose not. Then there exists a set $A \in M_y$ such that $A \cap \Lambda = \emptyset$. That is, for every $t \in A$, $|f(t)| < 1$ and we have

$$\begin{aligned} 1 &= \int_T f(t) y(t) d\mu \leq \int_T |f(t)||y(t)| d\mu \\ &= \int_\Lambda |f(t)||y(t)| d\mu + \int_A |f(t)||y(t)| d\mu + \int_{T\setminus A\cup\Lambda} |f(t)||y(t)| d\mu \\ &< \int_\Lambda |y(t)| d\mu + \int_A |y(t)| d\mu + \int_{T\setminus A\cup\Lambda} |y(t)| d\mu = \int_T |y(t)| d\mu = 1 \end{aligned}$$

which gives a contradiction. It remains to prove the last condition that for every $A \in M_y$ there exists $t \in A$ such that $sgn\ y(t) = sgn\ f(t)$. Suppose not. Then we have a set $A \in M_y$ such that for every $t \in A$, $sgn\ y(t) \neq sgn\ f(t)$. Now

$$\begin{aligned} 1 &= \int_T f(t)y(t)d\mu \\ &= \int_{\Lambda\setminus A} |f(t)||y(t)|d\mu - \int_A |f(t)||y(t)|d\mu + \int_{T\setminus A\cup\Lambda} |f(t)||y(t)|d\mu \\ &< \int_\Lambda |y(t)|d\mu + \int_A |y(t)|d\mu + \int_{T\setminus A\cup\Lambda} |y(t)|d\mu = \int_T |y(t)|d\mu = 1, \end{aligned}$$

which is a contradiction. So we have $J_X(f) \subseteq E$. Note that if $y \in J_X(f)$, then

$$\int_{T\setminus\Lambda} |f(t)||y(t)|d\mu = 0 = \int_{T\setminus\Lambda} |y(t)|d\mu \tag{14}$$

Now we will show that $E \subseteq J_X(f)$. Let $y \in E$. Then we have

$$\begin{aligned} f(y) &= \int_T f(t)y(t)d\mu = \int_\Lambda |f(t)||y(t)|d\mu + \int_{T\setminus\Lambda} |f(t)||y(t)|d\mu \\ &= \int_\Lambda |y(t)|d\mu + \int_{T\setminus\Lambda} |y(t)|d\mu = \int_T |y(t)|d\mu = 1 = \|f\|_\infty. \end{aligned}$$

Hence we get $E \subseteq J_X(f)$. This with the above shows that $E = J_X(f)$.

Now we proceed to prove that $E = E_1$. Let $y \in E$. Then $A \cap \Lambda \neq \emptyset$ for every $A \in M_y$. Suppose that $y \notin E_1$. Then there exists a set $A \in M_y$ such that for every $t \in A \cap \Lambda$, $sgn\ y(t) \neq sgn\ f(t)$. But we have $A \cap \Lambda \in M_y$ and there exists $t \in (A \cap \Lambda) \cap \Lambda$ such that $sgn\ y(t) = sgn\ f(t)$, which is a contradiction. Hence we get $E \subseteq E_1$. This proves that $E = E_1$, because the other inclusion is trivial. □

For a fixed x in $L_1(\mu)$, we set

$$S^+ = S_x^+ = \{t \in T : x(t) \geq 0\} \text{ and } S^- = S_x^- = \{t \in T : x(t) < 0\}.$$

Let

$$\Lambda_1 = (\Lambda^+ \cap S^+) \cup (\Lambda^- \cap S^-) \text{ and } \Lambda_2 = (\Lambda^+ \cap S^-) \cup (\Lambda^- \cap S^+). \tag{15}$$

Then $\Lambda = \Lambda_1 \cup \Lambda_2$. Further, for any y in E, it follows from Fact 2.1 that for all $A \in M_y$ there exists $t \in A \cap \Lambda$ such that $sgn\ y(t) = sgn\ f(t)$. Therefore if $t \in A \cap \Lambda_1$ we obtain

$$sgn\ x(t) = sgn\ f(t) = sgn\ y(t) \tag{16}$$

and if $t \in A \cap \Lambda_2$ then $y(t)$ and $f(t)$ are of opposite sign. Thus for y in E, we have

$$\|x - y\|_{\Lambda_1} = \int_{\Lambda_1} |x(t) - y(t)|d\mu = \int_{\Lambda_1} |\ |x(t)| - |y(t)|\ |d\mu \tag{17}$$

and

$$\|x-y\|_{\Lambda_2} = \int_{\Lambda_2} |x(t)-y(t)|d\mu = \int_{\Lambda_2} (|x(t)|+|y(t)|)d\mu = \|x\|_{\Lambda_2} + \|y\|_{\Lambda_2}. \tag{18}$$

Now we are in a position to show that the set $E = J_X(f)$ is proximinal in $L_1(\mu)$. Also, for $x \in L_1(\mu)$, we characterize $P_E(x)$, the set of all best approximations to x from E.

Fact 3.2. *Let $X = L_1(\mu)$, f be in $NA_1(X)$ and $E = J_X(f)$. Pick any $x \in X$. Then the set $P_E(x)$ is non-empty and*

$$P_E(x) = \begin{cases} E, & if\ \Lambda_1 = \emptyset \\ \{y \in E : \|y\|_{\Lambda_2} = 0 \ \ and \ \ \|x-y\|_{\Lambda_1} = \|x\|_{\Lambda_1} - \|y\|_{\Lambda_1}\}, & if\ \|x\|_{\Lambda_1} \geq 1 \\ \{y \in E : \|x-y\|_{\Lambda_1} = \|y\|_{\Lambda_1} - \|x\|_{\Lambda_1}\}, & if\ \|x\|_{\Lambda_1} < 1. \end{cases}$$

Proof. We first assume that $\Lambda_1 = \emptyset$. Then for any $y \in E$, we have $1 = \|y\| = \|y\|_{\Lambda_2}$ and further,

$$\begin{aligned} \|x-y\| &= \|x-y\|_{\Lambda_1} + \|x-y\|_{\Lambda_2} + \|x\|_{T\setminus\Lambda} = \|x-y\|_{\Lambda_2} + \|x\|_{T\setminus\Lambda} \\ &= \|x\|_{\Lambda_2} + \|y\|_{\Lambda_2} + \|x\|_{T\setminus\Lambda} = \|x\|_{\Lambda_2} + \|y\| + \|x\|_{T\setminus\Lambda} = \|x\| + 1. \end{aligned}$$

Hence, in this case,

$$d(x,E) = \|x\| + 1 \quad \text{and} \quad P_E(x) = E. \tag{19}$$

So we now assume that the set Λ_1 is non-empty. First we prove that $P_E(x)$ is non-empty in the following two cases.

Case (i). $\|x\|_{\Lambda_1} \geq 1$. We have

$$\begin{aligned} \|x-y\| &= \|x-y\|_{\Lambda_1} + \|x-y\|_{\Lambda_2} + \|x\|_{T\setminus\Lambda} \\ &\geq \|x\|_{\Lambda_1} - \|y\|_{\Lambda_1} + \|x\|_{\Lambda_2} + \|y\|_{\Lambda_2} + \|x\|_{T\setminus\Lambda} \\ &\geq \|x\|_{\Lambda_1} - 1 + \|x\|_{\Lambda_2} + \|x\|_{T\setminus\Lambda}, \end{aligned} \tag{20}$$

since $\|y\|_{\Lambda_1} \leq 1$.

Case (ii): $\|x\|_{\Lambda_1} < 1$. Now

$$\begin{aligned} \|x-y\| &= \|x-y\|_{\Lambda_1} + \|x-y\|_{\Lambda_2} + \|x\|_{T\setminus\Lambda} \\ &\geq \|y\|_{\Lambda_1} - \|x\|_{\Lambda_1} + \|x\|_{\Lambda_2} + \|y\|_{\Lambda_2} + \|x\|_{T\setminus\Lambda} \\ &= 1 - \|x\|_{\Lambda_1} + \|x\|_{\Lambda_2} + \|x\|_{T\setminus\Lambda} \end{aligned} \tag{21}$$

as $1 = \|y\|_{\Lambda} = \|y\|_{\Lambda_1} + \|y\|_{\Lambda_2}$. It is clear from (20) and (21) that

$$d(x,E) \geq |1 - \|x\|_{\Lambda_1}| + \|x\|_{\Lambda_2} + \|x\|_{T\setminus\Lambda}. \tag{22}$$

Define
$$y(t) = \begin{cases} \dfrac{x(t)}{\|x\|_{\Lambda_1}}, & \text{if } t \in \Lambda_1 \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|y\| = 1$ and using Fact 3.1 and (16) we conclude y is in E. Also,

$$\|x - y\| = |1 - \|x\|_{\Lambda_1}| + \|x\|_{\Lambda_2} + \|x\|_{T\setminus\Lambda}.$$

This together with (22) implies that $d(x, E) = \|x - y\|$ and therefore y is a nearest element to x from E. This proves the proximinality of E. Also it is clear that

$$d(x, E) = |1 - \|x\|_{\Lambda_1}| + \|x\|_{\Lambda_2} + \|x\|_{T\setminus\Lambda} = |1 - \|x\|_{\Lambda_1}| + \alpha, \tag{23}$$

where $\alpha = \|x\|_{\Lambda_2} + \|x\|_{T\setminus\Lambda}$.

We have also shown that $P_E(x)$ is non-empty if Λ_1 is a non-empty set. We now characterize the set $P_E(x)$ in this case.

The proof involves discussion of two cases, as in the case of the previous section of this proof.

Case 1: $\|x\|_{\Lambda_1} \geq 1$.

Pick any $y \in P_E(x)$. Then $\|y\|_{\Lambda_1} \leq 1$ and also using (23), we have

$$\begin{aligned} d(x, E) &= |1 - \|x\|_{\Lambda_1}| + \alpha = \|x - y\| \\ &= \|x - y\|_{\Lambda_1} + \|x\|_{\Lambda_2} + \|y\|_{\Lambda_2} + \|x\|_{T\setminus\Lambda} = \|x - y\|_{\Lambda_1} + \|y\|_{\Lambda_2} + \alpha \\ &\geq \|x\|_{\Lambda_1} - \|y\|_{\Lambda_1} + \|y\|_{\Lambda_2} + \alpha \geq \|x\|_{\Lambda_1} - 1 + \|y\|_{\Lambda_2} + \alpha \\ &\geq |1 - \|x\|_{\Lambda_1}| + \alpha. \end{aligned}$$

So equality holds throughout and therefore

$$\|y\|_{\Lambda_2} = 0 \quad \text{and} \quad \|x - y\|_{\Lambda_1} = \|x\|_{\Lambda_1} - \|y\|_{\Lambda_1}.$$

Conversely, if y is in E and the above two conditions are satisfied then we have $\|y\| = \|y\|_{\Lambda_1} = 1$ and

$$\|x - y\| = \|x - y\|_{\Lambda_1} + \|y\|_{\Lambda_2} + \alpha = \|x\|_{\Lambda_1} - \|y\|_{\Lambda_1} + \alpha = \|x\|_{\Lambda_1} - 1 + \alpha = d(x, E)$$

from (23) and y is in $P_E(x)$.

It is now clear that if $\|x\|_{\Lambda_1} \geq 1$

$$y \in P_E(x) \Leftrightarrow y \in E, \|y\|_{\Lambda_2} = 0 \ \ \textit{and} \ \ \|x - y\|_{\Lambda_1} = \|x\|_{\Lambda_1} - \|y\|_{\Lambda_1}. \tag{24}$$

Case (ii): $\|x\|_{\Lambda_1} < 1$.

In this case, for $y \in P_E(x)$, we have again using (23),

$$d(x, E) = 1 - \|x\|_{\Lambda_1} + \alpha = \|x - y\| = \|x - y\|_{\Lambda_1} + \|y\|_{\Lambda_2} + \alpha.$$

Hence $$\|x - y\|_{\Lambda_1} + \|y\|_{\Lambda_2} = 1 - \|x\|_{\Lambda_1}.$$

Now

$$\begin{aligned} \|x - y\|_{\Lambda_1} + \|y\|_{\Lambda_2} = 1 - \|x\|_{\Lambda_1} \quad &\Leftrightarrow \quad \|x - y\|_{\Lambda_1} + \|x\|_{\Lambda_1} = 1 - \|y\|_{\Lambda_2} \\ &\Leftrightarrow \quad \|x - y\|_{\Lambda_1} + \|x\|_{\Lambda_1} = \|y\|_{\Lambda_1} \\ &\Leftrightarrow \quad \|x - y\|_{\Lambda_1} = \|y\|_{\Lambda_1} - \|x\|_{\Lambda_1}. \end{aligned}$$

Hence $y \in P_E(x)$ implies that $\|x-y\|_{\Lambda_1} = \|y\|_{\Lambda_1} - \|x\|_{\Lambda_1}$.

Conversely, if y is in E, we have $1 = \|y\| = \|y\|_{\Lambda_1} + \|y\|_{\Lambda_2}$. Further, if y is such that $\|x-y\|_{\Lambda_1} = \|y\|_{\Lambda_1} - \|x\|_{\Lambda_1}$, then

$$\begin{aligned}\|x-y\| &= \|x-y\|_{\Lambda_1} + \|y\|_{\Lambda_2} + \alpha = \|y\|_{\Lambda_1} - \|x\|_{\Lambda_1} + \|y\|_{\Lambda_2} + \alpha \\ &= \|y\|_{\Lambda_1} + \|y\|_{\Lambda_2} - \|x\|_{\Lambda_1} + \alpha = 1 - \|x\|_{\Lambda_1} + \alpha = d(x,E).\end{aligned}$$

Hence $y \in P_E(x)$. Thus if $\|x\|_{\Lambda_1} < 1$,

$$y \in P_E(x) \Leftrightarrow y \in E \text{ and } \|x-y\|_{\Lambda_1} = \|y\|_{\Lambda_1} - \|x\|_{\Lambda_1}. \qquad \square \tag{25}$$

4. Strong proximinality and semi continuity of metric projection

We now prove the strong proximinality of the set $J_X(f)$.

Theorem 4.1. *Let $X = L_1(\mu)$, f be in $NA_1(X)$ and $E = J_X(f)$. Pick any x in X. If y is in E and $\|x-y\| < d(x,E) + \delta$ for some $\delta > 0$, then there exists z in $P_E(x)$ such that $\|y-z\| < 2\delta$. In particular, E is strongly proximinal in $L_1(\mu)$.*

Proof. If the set Λ_1, given by (15), is empty, then $P_E(x) = E$ by (19). Hence the theorem is trivially true in this case. So we assume that the set Λ_1 is non-empty.

We have, by (23),

$$\begin{aligned}\|x-y\| &= \|x-y\|_{\Lambda_1} + \|x\|_{\Lambda_2} + \|y\|_{\Lambda_2} + \|x\|_{T\setminus\Lambda} \\ &= \{\|x-y\|_{\Lambda_1} - |1 - \|x\|_{\Lambda_1}|\} + \|y\|_{\Lambda_2} + d(x,E) \\ &< d(x,E) + \delta\end{aligned}$$

which implies
$$\{\|x-y\|_{\Lambda_1} - |1 - \|x\|_{\Lambda_1}|\} + \|y\|_{\Lambda_2} < \delta. \tag{26}$$

We now discuss three cases: $\|x\|_{\Lambda_1} = 1$, $\|x\|_{\Lambda_1} > 1$ *and* $\|x\|_{\Lambda_1} < 1$.
In the following, for $y \in E$, we set

$$L_y = \{t \in \Lambda_1 : |x(t)| > |y(t)|\}, \quad E_y = \{t \in \Lambda_1 : |x(t)| = |y(t)|\}$$

and
$$G_y = \{t \in \Lambda_1 : |y(t)| > |x(t)|\}.$$

Then $L_y \cup E_y \cup G_y = \Lambda_1$.

Case (i): $\|x\|_{\Lambda_1} = 1$.

In this case, by (26), $\|x-y\|_{\Lambda_1} + \|y\|_{\Lambda_2} < \delta$. Set

$$p(t) = \begin{cases} x(t) & \text{if } t \in \Lambda_1 \\ 0 & \text{otherwise.}\end{cases}$$

Then $\|p\| = \|x\|_{\Lambda_1} = 1$. Also, it follows from Fact 3.1 that p is in E. We have $\|p\|_{\Lambda_2} = 0$ and $0 = \|x-p\|_{\Lambda_1} = \|x\|_{\Lambda_1} - 1 = \|x\|_{\Lambda_1} - \|p\|_{\Lambda_1}$. So using (24), we conclude that p is in $P_E(x)$. Further

$$\|y-p\| = \|x-y\|_{\Lambda_1} + \|y\|_{\Lambda_2} < \delta.$$

Taking $z = p$, completes the proof for this case.

Case (ii): $\|x\|_{\Lambda_1} > 1$. Now

$$\begin{aligned}
\|x-y\|_{\Lambda_1} &= \int_{\Lambda_1} |x(t)-y(t)|d\mu = \int_{\Lambda_1} |\ |x(t)|-|y(t)|\ |d\mu \\
&= \int_{L_y\cup E_y} (|x(t)|-|y(t)|)d\mu + \int_{G_y} (|y(t)|-|x(t)|)d\mu \\
&= \int_{L_y\cup E_y\cup G_y} |x(t)|d\mu - \int_{L_y\cup E_y\cup G_y} |y(t)|d\mu - 2\int_{G_y} |x(t)|d\mu + 2\int_{G_y} |y(t)|d\mu \\
&= \|x\|_{\Lambda_1} - \|y\|_{\Lambda_1} - 2\int_{G_y} |x(t)|d\mu + 2\int_{G_y} |y(t)|d\mu
\end{aligned}$$

This together with (26) implies that

$$\|x\|_{\Lambda_1} - \|y\|_{\Lambda_1} + 2[\int_{G_y} (|y(t)|-|x(t)|)\, d\mu] - \{\|x\|_{\Lambda_1} - 1\} + \|y\|_{\Lambda_2} < \delta$$

which, in turn, implies

$$2[\int_{G_y} (|y(t)|-|x(t)|)\, d\mu + \|y\|_{\Lambda_2}] < \delta, \tag{27}$$

as $1 - \|y\|_{\Lambda_1} = \|y\|_{\Lambda_2}$. Define $p \in L_1(\mu)$ by

$$p(t) = \begin{cases} y(t) & \text{on } L_y \\ x(t) & \text{on } E_y \cup G_y \\ 0 & \text{on } T \setminus (L_y \cup E_y \cup G_y) = T \setminus \Lambda_1. \end{cases}$$

Then $\|y-p\| < \delta$ by (27) and so $\|p\| > 1-\delta$. Further, it is clear from the definition of p that $\|p\| \le \|y\| = 1$. Thus $1-\delta < \|p\| \le 1$.

Now, using (16), we get for every $A \in M_p$ there exists $t \in A$ such that

$$sgn\ p(t) = \ sgn\ f(t) = \ sgn\ x(t). \tag{28}$$

This with Fact 3.1 would imply that p is in E if $\|p\| = 1$. Thus if $\|p\| = 1$, since $\|p\|_{\Lambda_2} = 0$ and

$$\begin{aligned}
\|x-p\|_{\Lambda_1} &= \int_{L_y} |x(t)-y(t)|d\mu = \int_{L_y} (\ |x(t)|-|y(t)|\ )d\mu \\
&= \int_{L_y} |x(t)|d\mu + \int_{G_y\cup E_y} |x(t)|d\mu - \int_{G_y\cup E_y} |x(t)|d\mu - \int_{L_y} |y(t)|d\mu \\
&= \|x\|_{\Lambda_1} - \|p\|_{\Lambda_1},
\end{aligned} \tag{29}$$

we can conclude using (24) that $p \in P_E(x)$ and take $z = p$ to complete the proof for this case. So we assume that $\|p\| < 1$.

Let $\|x\|_{\Lambda_1} = 1+\epsilon$. Then $\epsilon > 0$ as $\|x\|_{\Lambda_1} > 1$. Let $\|p\| = 1-\delta_1$. Then $0 < \delta_1 < \delta$. We have $|p(t)| \le |x(t)|$ for all $t \in \Lambda_1$ and so

$$\|x-p\|_{\Lambda_1} = \int_{\Lambda_1} (\ |x(t)| - |p(t)|\ )d\mu = \|x\|_{\Lambda_1} - \|p\|_{\Lambda_1} = \epsilon + \delta_1.$$

Let $\lambda = \dfrac{\epsilon}{\epsilon+\delta_1}$ and define

$$z(t) = \begin{cases} \lambda p(t) + (1-\lambda)x(t), & \text{if } t \in \Lambda_1 \\ 0, & \text{otherwise.} \end{cases}$$

We have, using (28),

$$\begin{aligned} \|z\| = \|z\|_{\Lambda_1} &= \int_{\Lambda_1} |z(t)|d\mu = \int_{\Lambda_1} |\lambda\ p(t) + (1-\lambda)x(t)|d\mu \\ &= \int_{\Lambda_1} (\ \lambda\ |p(t)| + (1-\lambda)|x(t)|\ )d\mu = \lambda\ \|p\|_{\Lambda_1} + (1-\lambda)\|x\|_{\Lambda_1} \\ &= \lambda(1-\delta_1) + (1-\lambda)(1+\epsilon) = \frac{\epsilon}{\epsilon+\delta_1}(1-\delta_1) + \frac{\delta_1}{\epsilon+\delta_1}(1+\epsilon) = 1. \end{aligned}$$

Further,

$$\|z-p\| \le (1-\lambda)\|x-p\|_{\Lambda_1} = \frac{\delta_1}{\epsilon+\delta_1}\ (\epsilon+\delta_1) = \delta_1\ <\ \delta.$$

Now $\|z\| = 1$ and also it follows from (28) that for every $A \in M_z$ there exists $t \in A$ such that $sgn\ z(t) = sgn\ f(t)$. Therefore z is in E by Fact 3.1. Clearly, $\|z\|_{\Lambda_2} = 0$. Further it is easily verified that

$$\|x-z\|_{\Lambda_1} = \lambda\|x-p\|_{\Lambda_1} \quad and \quad \|x\|_{\Lambda_1} - \|z\|_{\Lambda_1} = \lambda\ [\ \|x\|_{\Lambda_1} - \|p\|_{\Lambda_1}\ ].$$

Now, using (24) and (29), we see that z is in $P_E(X)$. Also,

$$\|y-z\| \le \|y-p\| + \|p-z\| < \delta + \delta = 2\delta.$$

This completes the proof for this case.

Case (iii): $\|x\|_{\Lambda_1} < 1$. Now

$$\begin{aligned} \|x-y\|_{\Lambda_1} &= \int_{\Lambda_1} |x(t)-y(t)|d\mu \\ &= \int_{G_y\cup E_y} (|y(t)| - |x(t)|)d\mu + \int_{L_y} (|x(t)| - |y(t)|)d\mu \\ &= \int_{G_y\cup E_y\cup L_y} |y(t)|d\mu - \int_{L_y\cup E_y\cup G_y} |x(t)|d\mu + 2\int_{L_y} (|x(t)| - |y(t)|)d\mu \\ &= \|y\|_{\Lambda_1} - \|x\|_{\Lambda_1} + 2\int_{L_y} (|x(t)| - |y(t)|)d\mu. \end{aligned}$$

This together with (26) we have

$$\|y\|_{\Lambda_1} - \|x\|_{\Lambda_1} + 2\int_{L_y} (|x(t)| - |y(t)|)\, d\mu + \|y\|_{\Lambda_2} - |1 - \|x\|_{\Lambda_1}| < \delta.$$

Since $1 = \|y\| = \|y\|_{\Lambda_1} + \|y\|_{\Lambda_2}$, this in turn implies

$$1 - \|x\|_{\Lambda_1} + 2\int_{L_y} (|x(t)| - |y(t)|) d\mu - (1 - \|x\|_{\Lambda_1}) = \int_{L_y} (|x(t)| - |y(t)|) d\mu < \frac{\delta}{2}. \quad (30)$$

Define $p \in L_1(\mu)$ by $$p(t) = \begin{cases} x(t) & \text{on } L_y \\ y(t) & \text{on } T \setminus L_y. \end{cases}$$

Then, by (30), $\|y - p\| < \delta$. Also it is clear from the definition of p that for every $A \in M_p$ there exists $t \in A$ such that

$$sgn\ p(t) = sgn\ f(t) \quad (31)$$

and it is easy to verify that, $\|p\| \geq \|y\| = 1$.

If $\|p\| = 1$, then $p \in E$ by Fact 3.1. Further, since $|x(t)| = |y(t)| = |p(t)|$ for $t \in E_y$, using the definition of p, we have

$$\begin{aligned} \|x - p\|_{\Lambda_1} &= \int_{\Lambda_1 \setminus L_y} |x(t) - y(t)| d\mu = \int_{G_y} (\ |y(t)| - |x(t)|\ ) d\mu \\ &= \int_{\Lambda_1} (\ |p(t)| - |x(t)|\ ) d\mu = \|p\|_{\Lambda_1} - \|x\|_{\Lambda_1}. \end{aligned} \quad (32)$$

Hence (25) holds and we conclude that $p \in P_E(x)$. We take $z = p$ and complete the proof in this case. Therefore we assume that $\|p\| > 1$. In this case, we discuss two subcases: $\|p\|_{\Lambda_1} > 1$ and $\|p\|_{\Lambda_1} \leq 1$.

Now consider the case $\|p\|_{\Lambda_1} > 1$.

Let $\|p\| = 1 + \delta_2$ and $\|p\|_{\Lambda_1} = 1 + \delta_3$. Note that we have $\|y\| = 1$ and $\|y - p\| < \delta$.

Therefore $$1 \leq \|p\| \leq 1 + \delta.$$

Then $0 < \delta_3 \leq \delta_2 \leq \delta$. Since $\|x\|_{\Lambda_1} < 1$, we can choose $\epsilon > 0$ such that

$$1 - \epsilon = \|x\|_{\Lambda_1} = \int_{L_y} |x(t)| d\mu + \int_{\Lambda_1 \setminus L_y} |x(t)| d\mu.$$

Let $\lambda = \dfrac{\epsilon}{\epsilon + \delta_3}$ and define

$$z(t) = \begin{cases} \lambda p(t) + (1 - \lambda) x(t), & if\ t \in \Lambda_1 \\ 0, & otherwise. \end{cases}$$

Then $\|z\|_{L_y} = \|x\|_{L_y}$. Moreover

$$\begin{aligned}
\int_{\Lambda_1 \setminus L_y} |z(t)| d\mu &= \int_{\Lambda_1 \setminus L_y} |\lambda\ p(t) + (1-\lambda) x(t)| d\mu \\
&= \int_{\Lambda_1 \setminus L_y} (\ \lambda\ |y(t)| + (1-\lambda)|x(t)|\ ) d\mu \\
&= \lambda\ \|y\|_{\Lambda_1 \setminus L_y} + (1-\lambda)\|x\|_{\Lambda_1 \setminus L_y} \\
&= \lambda\ \|p\|_{\Lambda_1 \setminus L_y} + (1-\lambda)\|x\|_{\Lambda_1 \setminus L_y} \\
&= \lambda(1 - \int_{L_y} |x(t)| d\mu + \delta_3) + (1-\lambda)(1 - \int_{L_y} |x(t)| d\mu - \epsilon) \\
&= 1 - \int_{L_y} |x(t)| d\mu
\end{aligned}$$

and this implies $\|z\| = \|z\|_{\Lambda_1} = \|z\|_{L_y} + \|z\|_{\Lambda_1 \setminus L_y} = 1$. Further

$$\begin{aligned}
\|z - p\| &\leq (1-\lambda)\|x - p\|_{\Lambda_1} + \|p\|_{\Lambda_2} \\
&= \frac{\delta_3}{\epsilon + \delta_3}\ (\epsilon + \delta_3) + (1 + \delta_2 - (1 + \delta_3)) = \delta_2\ \leq\ \delta.
\end{aligned}$$

Now $\|z\| = 1$ and using (31), we have for every $A \in M_z$ there exists $t \in A$ such that

$$sgn\ z(t) = sgn\ f(t).$$

This with Fact 3.1 implies $z \in E$.

It is easily verified using (30) that

$$\|z - x\|_{\Lambda_1} = \|z\|_{\Lambda_1} - \|x\|_{\Lambda_1}$$

and using (25), we conclude that $z \in P_E(x)$. Also,

$$\|y - z\| \leq \|y - p\| + \|p - z\| < \delta + \delta = 2\delta,$$

which completes the proof for this case.

Now consider the last case $\|p\|_{\Lambda_1} \leq 1$. Define

$$q(t) = \begin{cases} p(t), & \text{if } t \in \Lambda_1 \\ \dfrac{(1 - \|p\|_{\Lambda_1})}{\|p\|_{\Lambda_2}} p(t), & \text{if } t \in \Lambda_2 \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\|q\| = \int_{\Lambda_1} |p(t)| d\mu + \int_{\Lambda_2} \frac{(1 - \|p\|_{\Lambda_1})}{\|p\|_{\Lambda_2}} |p(t)| d\mu = \|p\|_{\Lambda_1} + (1 - \|p\|_{\Lambda 1}) = 1.$$

Now by (31), we have for every $A \in M_q$ there exists $t \in A$ such that

$$sgn\ q(t) = sgn\ f(t).$$

This with Fact 3.1 implies $q \in E$. It is easily verified that

$$\|q - x\|_{\Lambda_1} = \|q\|_{\Lambda_1} - \|x\|_{\Lambda_1}$$

and using (25), we conclude that $q \in P_E(x)$. Further,

$$\begin{aligned}
\|q-p\| &= \int_{\Lambda_2} |q(t) - p(t)| d\mu \\
&= \int_{\Lambda 2} \left|\, |p(t)| - \frac{(1 - \|p\|_{\Lambda_1})}{\|p\|_{\Lambda_2})} |p(t)|\, \right| d\mu \\
&= \|p\|_{\Lambda_2} - (1 - \|p\|_{\Lambda_1}) = \delta_2. \qquad (33)
\end{aligned}$$

Finally we have

$$\|y - q\| \leq \|y - p\| + \|p - q\| < \delta + \delta_2 \leq 2\delta.$$

We take $z = q$ and this completes the proof of the theorem . □

We now proceed to show that the metric projection P_E is Hausdorff metric continuous.

Fact 4.2. *Let $X = L_1(\mu)$, f be in $NA_1(X)$ and $E = J_X(f)$. Then the metric projection P_E is Hausdorff metric continuous and has a continuous selection.*

Proof. Since E is strongly proximinal, the metric projection P_E is upper Hausdorff semi-continuous and we need to prove only the lower Hausdorff semi-continuity of P_E. Let $x \in L_1(\mu)$ and $\epsilon > 0$ be given. Choose $\delta > 0$ such that $\delta < \frac{\epsilon}{2}$. We now claim that for any $y \in P_E(x)$ and $p \in L_1(\mu)$ with $\|x - p\| < \frac{\delta}{2}$, there exists $z \in P_E(p)$ satisfying $\|y - z\| < \epsilon$. This would clearly imply the lower Hausdorff semi-continuity of the set-valued map P_E at x. Since $\|x - p\| < \frac{\delta}{2}$, we have

$$|d(x, E) - d(p, E)| \leq \|x - p\| < \frac{\delta}{2}.$$

Thus

$$\|p - y\| \leq \|p - x\| + \|x - y\| < \frac{\delta}{2} + d(x, E) < d(p, E) + \delta.$$

Now, using Theorem 4.1, we conclude that there exists $z \in P_E(p)$ such that

$$\|y - z\| < 2\delta < \epsilon.$$

So for any $\epsilon-$neighborhood V of $y \in P_E(x)$ there exists a $\delta-$ neighborhood U of x such that $V \cap P_E(p) \neq \emptyset$ for every $p \in U$. This implies that P_E is lower Hausdorff semi-continuous at x. Hence it follows that the metric projection P_E is Hausdorff metric continuous and by the well known Michael selection theorem [15, 16], P_E has a continuous selection. □

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