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On Some Properties of Measurable Functions in Abstract Spaces

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In this short note we present several infinite dimensional theorems which generalize corresponding facts from the finite dimensional differential inclusions theory.

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1. Introduction

In [2] a concept of a generalized solution to an ODE with nonsmooth right hand side is introduced. Namely, the following initial value problem

$$\dot{x} = f(t, x), \quad x(0) = \hat{x} \tag{1}$$

is considered in some domain $D \subset \mathbb{R}^m$, $\quad x \in D$. The function f is defined in $[0, T] \times D$ and measurable with respect to the standard Lebesgue measure. Moreover it is assumed that there exists a function $u \in L^1(0, T)$ such that

$$|f(t, x)| \le u(t)$$

holds for almost all $(t, x) \in [0, T] \times D$.

Filippov defines a generalized solution $x(t)$ to IVP (1) as an absolutely continuous function in $[0, T]$ such that $x(0) = \hat{x}$ and the inclusion

$$\dot{x}(t) \in \bigcap_{r>0} \bigcap_{N \in \mathcal{N}} \operatorname{conv} f\Big(t, B_r\big(x(t)\big) \backslash N\Big) \tag{2}$$

holds for almost all $t \in [0, T]$.

Here conv stands for the *closed convex hull*; $B_r(x) \subset \mathbb{R}^m$ denotes the open ball of radius r centered at a point x; and we use $\mathcal{N}$ to denote a set of all measure-null subsets of $\mathbb{R}^m$.

For any differential equation this definition gives rise to several questions from real analysis.

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The main ones are:

(1) whether the set in the right-hand side of (2) is nonvoid for almost all t?

(2) assume in addition that the function f is continuous in $[0,T]\times D$. Is it true that

$$\bigcap_{r>0}\bigcap_{N\in\mathcal{N}}\operatorname{conv} f\Big(t, B_r\big(x(t)\big)\backslash N\Big)=\{f(t,x)\}\,?$$

The article [2] answers both questions positively and contains a sketch of the corresponding proofs. In particular, if the function f is continuous then the definition of a generalized solution turns into the standard definition of the classical solution to an ODE with a continuous right-hand side.

Though this sketch is completely correct we believe that it would be useful to provide a detailed proof; moreover we get rid of some extra hypotheses and generalize Filippov's assertions to an infinite dimensional case.

Perhaps such a generalization can be of some interest by itself or for the modern studies of the inclusions in infinite dimensional spaces.

2. The main theorems

The Theorems 2.1, 2.7 (see below) generalize Filippov's results which are obtained for the case $X=D,\quad Y=\mathbb{R}^m$.

2.1. The case of a measurable mapping

Let a set X be equipped with a σ-algebra and a measure μ.

Let Y stand for a Hausdorff topological space with a countable base. A mapping $f\colon X\to Y$ is supposed to be measurable with respect to the Borel σ-algebra in Y.

Theorem 2.1. *There exists a measurable set $N_0\subset X,\quad \mu(N_0)=0$ such that*

$$\bigcap_{\mu(N)=0}\overline{f(X\backslash N)}=\overline{f(X\backslash N_0)}. \tag{3}$$

The intersection is taken over all measurable sets of measure zero; the line denotes the closure.

Corollary 2.2. *Assume in addition that Y is a topological vector space then*

$$\bigcap_{\mu(N)=0}\operatorname{conv} f(U\backslash N)=\operatorname{conv} f(U\backslash N_0).$$

Indeed, the inclusion $\displaystyle\bigcap_{\mu(N)=0}\operatorname{conv} f(U\backslash N)\subset\operatorname{conv} f(U\backslash N_0)$ is evident: N_0 is one of the sets N from the left side of the formula.

The inverse inclusion is obtained by means of theorem 2.1 with the help of the formula

$$\operatorname{conv}\bigcap_{\alpha}U_\alpha\subset\bigcap_{\alpha}\operatorname{conv} U_\alpha.$$

Definition 2.3. We shall say that $y \in Y$ is a *bad point* of the mapping f if it has an open neighbourhood U_y such that $\mu(f^{-1}(U_y)) = 0$. We shall call other points $y \in Y$ *good points* of the mapping f.

In the sequel when it does not cause confusion we just say a good point or a bad point. Consequently $y \in Y$ is a good point if it has a base $\mathcal{B}$ of the open neighbourhoods such that

$$\mu(f^{-1}(W)) \neq 0, \quad \forall W \in \mathcal{B}.$$

The notion of a good point has a direct relation to the standard concept of the essential range [5].

Generalizing the definition of the essential range to the infinite dimensional case, we say that the *essential range* of the function f coincides with the set of the good points:

$$\text{ess.im}\, f := \{y \in Y \mid y \text{ is good}\}.$$

A finite dimensional version of the following theorem is well known.

Theorem 2.4. *The set in the left hand side of* (3) *coincides with the essential range of f:*

$$\bigcap_{\mu(N)=0} \overline{f(X\backslash N)} = \text{ess.im}\, f. \tag{4}$$

Remark 2.5. If X is a Hausdorff topological space and μ is a Borel measure, moreover if $X = Y$ and $f = \mathrm{id}_X$, then the set of the good points coincides by definition with the support of the measure μ.

Theorem 2.6. *Let $Q \subset X$ be a measurable set. We use f_Q to denote the following restriction $f_Q = f\,|_Q\colon Q \to Y$. Then there is a measurable set*

$$N_0^Q \subset Q, \quad \mu(N_0^Q) = 0, \quad N_0 \cap Q \subset N_0^Q$$

such that

$$\bigcap_{\mu(N)=0} \overline{f_Q(Q\backslash N)} = \overline{f_Q(Q\backslash N_0^Q)}.$$

2.2. The case of a continuous mapping at a point

Theorem 2.7. *Let X be a topological vector space with the Borel σ-algebra and a measure μ.*

Assume that

(1) *for any open nonvoid set $S \subset X$ one has $\mu(S) > 0$;*

(2) *Y is a locally convex Hausdorff space;*

(3) *a mapping $f : X \to Y$ is continuous at a point $\tilde{x} \in X$.*

Then it follows that

$$\bigcap_{U \in \mathcal{U}} \bigcap_{\mu(N)=0} \operatorname{conv} f(U\backslash N) = \{f(\tilde{x})\},$$

where $\mathcal{U}$ is a base of the open neighbourhoods at the point $\tilde{x}$.

Remark 2.8. (1) The function f in Theorem 2.7 needs not to be measurable.

(2) If in Theorem 2.7 we will replace condition (2) by "Y is a Hausdorff topological space", then the assertion is replaced with

$$\bigcap_{U\in\mathcal{U}}\bigcap_{\mu(N)=0}\overline{f(U\backslash N)}=\{f(\tilde{x})\}.$$

If in addition f satisfies the conditions of theorem 2.1 then this formula implies $f(\tilde{x})\in \mathrm{ess.im}\, f$.

3. Proofs

3.1. Proof of Theorem 2.1

All the points of U_y are evidently bad provided y is bad. Let B denote the set of bad points. Then we have

$$B=\bigcup_{y\in B}U_y. \tag{5}$$

Here U_y are the neighbourhoods from definition 2.3.

By the second Lindelof theorem [1] one can extract a countable subcovering

$$\{U_{y_n}\},\quad \{y_n\}_{n=1}^{\infty}\subset B \text{ such that } B=\bigcup_{n\in\mathbb{N}}U_{y_n}.$$

We claim that $N_0:=f^{-1}(B)$ is the set we are looking for. Indeed,

$$\mu(N_0)=\mu(f^{-1}(B))=\mu\left(f^{-1}\left(\bigcup_{n\in\mathbb{N}}U_{y_n}\right)\right)$$
$$=\mu\left(\bigcup_{n\in\mathbb{N}}f^{-1}(U_{y_n})\right)\le\sum_{n\in\mathbb{N}}\mu(f^{-1}(U_{y_n}))=0.$$

The inclusion
$$\bigcap_{\mu(N)=0}\overline{f(X\backslash N)}\subset\overline{f(X\backslash N_0)}$$

is trivial: N_0 is one of the sets N from the left side of the formula.

Let us check the backward inclusion "$\supset$". Show that $\overline{f(X\backslash N_0)}$ does not contain the bad points. Indeed,

$$f(X\backslash N_0)\cap B=f((X\backslash N_0)\cap f^{-1}(B))=f((X\backslash N_0)\cap N_0)=f(\emptyset)=\emptyset.$$

Here we employ a set theory formula [3]:

$$f(A)\cap B=f\left(A\cap f^{-1}(B)\right).$$

It follows that $f(X\backslash N_0)\subset Y\backslash B$. By formula (5) the set B is open, thus $Y\backslash B$ is closed. We consequently obtain

$$\overline{f(X\backslash N_0)}\subset Y\backslash B. \tag{6}$$

This means that $\overline{f(X\backslash N_0)}$ consists of the good points only.

Observe that if $y \in Y\backslash B$ is a good point then for any measure-null set $N \subset X$ we have

$$y \in \overline{f(X\backslash N)}.$$

Indeed, take a set N, $\mu(N) = 0$. Let y be a good point. This means that any open neighbourhood V_y of the point y is such that

$$\mu\big(f^{-1}(V_y)\big) > 0.$$

Thus there exists a point $x' \in f^{-1}(V_y)\backslash N$. This implies $f(x') \in V_y \cap f(X\backslash N)$.

Consequently for any open neighbourhood V_y the set $V_y \cap f(X\backslash N)$ is nonvoid. Therefore we have $y \in \overline{f(X\backslash N)}$. Since N is an arbitrary measure-null set and y is an arbitrary good point we conclude that the set

$$\bigcap_{\mu(N)=0} \overline{f(X\backslash N)} \tag{7}$$

contains all the good points and thus

$$\overline{f(X\backslash N_0)} \subset \bigcap_{\mu(N)=0} \overline{f(X\backslash N)}.$$

This completes the proof of the theorem.

3.2. Proof of Theorem 2.4

We have proved that the set (7) contains all the good points. Due to formula (6) we see that set (7) coincides with the set of good points. This gives (4) and the theorem is proved.

3.3. Proof of Theorem 2.6

Let B_Q stand for the set of bad points of the mapping f_Q. It is clear that $B \subset B_Q$. Thus one has

$$N_0 \cap Q \subset f^{-1}(B_Q) \cap Q = f_Q^{-1}(B_Q) = N_0^Q$$

and the theorem is proved.

3.4. Proof of Theorem 2.7

First let us check that for any $U \in \mathcal{U}$ and for any measurable N, $\mu(N) = 0$, we have

$$\tilde{x} \in \overline{U\backslash N}. \tag{8}$$

Indeed, assume the converse: $\tilde{x} \notin \overline{U\backslash N}$. Since $\overline{U\backslash N}$ is a closed set there exists an open set $\tilde{U} \in \mathcal{U}$ such that

$$\tilde{U} \cap (\overline{U\backslash N}) = \emptyset$$

and all the more

$$\tilde{U} \cap (U\backslash N) = \emptyset. \tag{9}$$

On the other hand we have $\tilde{U} \cap (U \backslash N) = (\tilde{U} \cap U) \backslash (\tilde{U} \cap N)$.

Since $\tilde{x} \in \tilde{U} \cap U$ the set $\tilde{U} \cap U$ is open and nonvoid. Thus $\mu(\tilde{U} \cap U) > 0$ and also $\mu(\tilde{U} \cap N) = 0$. Therefore

$$\mu\big(\tilde{U} \cap (U \backslash N)\big) > 0.$$

This is a contradiction to (9). Hence the inclusion (8) is proved.

Let us prove the inclusion

$$\bigcap_{U \in \mathcal{U}} \bigcap_{\mu(N)=0} \operatorname{conv} f(U \backslash N) \supset \{f(\tilde{x})\}. \tag{10}$$

From the properties of continuous functions [3] inclusion (8) implies $f(\tilde{x}) \in \overline{f(U \backslash N)}$. Now the following trivial observation

$$\bigcap_{U \in \mathcal{U}} \bigcap_{\mu(N)=0} \operatorname{conv} f(U \backslash N) \supset \bigcap_{U \in \mathcal{U}} \bigcap_{\mu(N)=0} \overline{f(U \backslash N)}$$

implies the inclusion (10).

Let us check the inclusion

$$\bigcap_{U \in \mathcal{U}} \bigcap_{\mu(N)=0} \operatorname{conv} f(U \backslash N) \subset \{f(\tilde{x})\}. \tag{11}$$

The point $f(\tilde{x})$ has a base of convex closed neighbourhoods $\mathcal{V}$ [4].

Thus for any $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $f(U) \subset V$; therefore

$$\operatorname{conv} f(U \backslash N) \subset V.$$

We consequently have

$$\bigcap_{U \in \mathcal{U}} \bigcap_{\mu(N)=0} \operatorname{conv} f(U \backslash N) \subset V.$$

Since V is an arbitrary element of the base $\mathcal{V}$ the inclusion (11) is proved.

This completes the proof of the theorem.

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