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Extremal Problems for the Distance Between Elements of Two Sets

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Given two sets A and C in a Banach space, we consider four extremal problems for the distance between two elements, one from A and the other from C. The first problem is to minimize the distance by choosing elements from these sets; the second problem is to maximize it; the third one is the minimax problem; the fourth one is the maximin problem. These problems arise in approximation theory and constrained optimization, they are generalizations of the best approximation problem and the problem of farthest points. In terms of prox-regularity and of the property of being a summand of the ball for sets A and C, we obtain sharp sufficient conditions for each of the problems to have a unique solution and, moreover, to be Tykhonov well-posed. We also develop the calculus of convexity parameters for subsets of a Banach space in connection with the Minkowski sum and difference.

Keywords: Distance, antidistance, metric projection, metric antiprojection, prox-regularity, summand of the ball, weak convexity in the sense of Efimov-Stechkin, convexity parameters, Tykhonov well-posedness, Minkowski sum, Minkowski difference.

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1. Introduction

Let X be a real normed vector space. For a set $A \subset X$ by $\operatorname{int} A$, $\overline{A}$, ∂A and $X \setminus A$ we denote the interior, the closure, the boundary and the complement of A, respectively. We use $\langle p, x \rangle$ to denote the value of the functional $p \in X^*$ at the vector $x \in X$, where X^* is the topological dual of X.

Given two subsets A and C of X, consider the following extremal problems

$$P_{\min\min}: \qquad \min_{c\in C,\ a\in A} \|a-c\|,$$

$$P_{\max\max}: \qquad \max_{c\in C,\ a\in A} \|a-c\|,$$

$$P_{\min\max}: \qquad \min_{a\in A} \max_{c\in C} \|a-c\|,$$

$$P_{\max\min}: \qquad \max_{c\in C} \min_{a\in A} \|a-c\|.$$

Consider also the best approximation problem

$$P_{\min}: \qquad \min_{a\in A} \|x-a\|$$

and the problem of farthest points

$$P_{\max}: \qquad \max_{c\in C} \|x-c\|,$$

where x is a given point from X.

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Remark 1.1. The four problems $P_{\min\min}$ – $P_{\max\min}$ are generalizations of the problems $P_{\min}$ and $P_{\max}$. Indeed, if $C = \{x\}$, then the problems $P_{\min\min}$ and $P_{\min\max}$ coincide with the best approximation problem $P_{\min}$. If $A = \{x\}$ the problems $P_{\max\max}$ and $P_{\max\min}$ coincide with $P_{\max}$. $\square$

The values of problems $P_{\min}$ and $P_{\max}$ are respectively the *distance* from the point x to the subset A

$$d(x, A) = \inf_{a\in A} \|x - a\|.$$

and the *antidistance* from the point $x \in X$ to the subset C

$$f(x, C) = \sup_{c\in C} \|x - c\|.$$

The sets of solutions of problems $P_{\min}$ and $P_{\max}$ are respectively the *metric projection* of the point x on the set A

$$P(x, A) = \{a \in A : \ \|x - a\| = d(x, A)\}$$

and the *metric antiprojection* of the point x on the set C

$$F(x, C) = \{c \in C : \ \|x - c\| = f(x, C)\}.$$

As shown in [2] the antiprojection on a set can be expressed through metric projection on another ("dual") set and vice versa.

Denote the values of the problems $P_{\min\min}$ – $P_{\max\min}$ as $\varrho_{\inf\inf}(A, C)$, $\varrho_{\sup\sup}(A, C)$, $\varrho_{\inf\sup}(A, C)$ and $\varrho_{\sup\inf}(C, A)$, respectively. Observe that

$$\varrho_{\inf\inf}(C, A) = \inf_{c\in C} d(c, A), \qquad \varrho_{\sup\sup}(C, A) = \sup_{c\in C} f(c, A),$$

$$\varrho_{\inf\sup}(A, C) = \inf_{a\in A} f(a, C), \qquad \varrho_{\sup\inf}(C, A) = \sup_{c\in C} d(c, A).$$

For $r \geq 0$ and $c \in X$ we define the closed ball with center c and radius r and the complement to the open ball with the same center and radius as

$$B_r(c) = \{x \in X : \ \|x - c\| \leq r\}, \qquad \widetilde{B}_r(c) = \{x \in X : \ \|x - c\| \geq r\}.$$

For brevity, we denote $B_r = B_r(0)$, $\widetilde{B}_r = \widetilde{B}_r(0)$.

The *modulus of convexity* of X is defined as

$$\delta_X(\varepsilon) = \inf\Big\{1 - \frac{1}{2}\|x + y\| : \ x, y \in X, \ \|x\| \leq 1, \ \|y\| \leq 1, \ \|x - y\| \geq \varepsilon\Big\}, \quad \varepsilon \in [0, 2].$$

The space X is called *uniformly convex* if $\delta_X(\varepsilon) > 0$ for any $\varepsilon \in (0; 2]$. X is called *strictly convex* if for any two distinct points $x, y \in \partial B_1$ one has $\frac{1}{2}(x + y) \in \operatorname{int} B_1$. Note that every uniformly convex space is strictly convex. The space X is called *smooth* if

$$\langle p_1, x\rangle = \langle p_2, x\rangle = \|p_1\| = \|p_2\| = 1 \quad \Rightarrow \quad p_1 = p_2 \qquad \forall x \in \partial B_1 \quad \forall p_1, p_2 \in X^*.$$

It is well known that any Hilbert space and L^p spaces with $p \in (1, +\infty)$ are smooth and uniformly convex.

The rest of the paper is organized as follows. In Section 2 we recall the definitions of prox-regular sets and summands of the ball, which are the main objects of this study. In the same section we also recall known results on extremal problems for the distance, including the sufficient conditions for well-posedness of $P_{\min\min}$. In Section 3 we formulate Theorems 3.1, 3.4, 3.7 which specify sufficient conditions for well-posedness of $P_{\max\max}$, $P_{\min\max}$ and $P_{\max\min}$, respectively. The conditions are formulated in terms of relations between the parameters of convexity of sets A and C. We show by examples that the relations are sharp. In Section 4 we obtain some auxiliary results. Section 5 is devoted to properties of weakly convex sets in the sense of Efimov-Stechkin. In Section 6 we develop the calculus of convexity parameters in connection with the Minkowski operations. This calculus is used in Section 7 to prove Theorems 3.1, 3.4, and 3.7.

2. Known results on extremal problems for the distance

In 1963 Stechkin [30] proved that if A is a closed subset of a uniformly convex Banach space X, then the set of $x \in X$ such that the metric projection $P(x, A)$ is a singleton, is a dense subset of X. In 1966 Edelstein [10] obtained a similar result for the metric antiprojection $F(x, C)$ and a closed bounded set C.

Let be given a set $A \subset X$ and a function $J : A \to \mathbb{R}$. A point a_0 is called a *minimizer* or a *solution* of the problem

$$\min_{a \in A} J(a) \tag{1}$$

if $a_0 \in A$ and $J(a_0) = \inf_{a \in A} J(a)$. A sequence $\{a_k\}$ is called a *minimizing sequence* of problem (1) if $a_k \in A$ for all $k \in \mathbb{N}$ and $J(a_k) \to \inf_{a \in A} J(a)$ as $k \to \infty$. Problem (1) is called *Tykhonov well-posed* if any of its minimizing sequences converges to its minimizer. The notions of a maximizer, a maximizing sequence of the problem $\max_{a \in A} J(a)$ and its Tykhonov well-posedness are defined similarly.

Note that if problem (1) is Tykhonov well-posed, then its solution is unique. The property of Tykhonov well-posedness is important not only in theoretical study but also in numerical algorithms, since this property provides the stability of the algorithms. A comprehensive study of well-posedness in various problems of optimization and optimal control is given in the book by Dontchev and Zolezzi [9].

As noted by Cobzaş [8], Stechkin's results [30] imply that if A is a closed subset of a uniformly convex Banach space X, then the set of $x \in X$ such that $P_{\min}$ is Tykhonov well-posed, is a dense G_δ subset of X. The counterpart of this result for $P_{\max}$ and a closed bounded set C has been obtained by De Blasi and Myjak [6].

If for any $x \in X$ the problem $P_{\min}$ is Tykhonov well-posed, then, obviously, A is an approximative compact Chebyshev set, i.e. for any $x \in X$ the problem $P_{\min}$ has a unique solution and any its minimizing sequence contains a convergent subsequence.

According to Theorem 3 by Efimov and Stechkin [12] any approximative compact Chebyshev set in a uniformly convex smooth Banach space is a convex set. This proves one implication of the following proposition. The other implication of this proposition follows easily from the results of Efimov and Stechkin.

Proposition 2.1. *Let A be a closed subset of a uniformly convex smooth Banach space. Then A is convex iff for any $x \in X$ the problem $P_{\min}$ is Tykhonov well-posed.*

Given $r > 0$, the *r-neighbourhood* and *r-antineighbourhood* of the set $A \subset X$ are defined as

$$U_r(A) = \{x \in X : \ d(x, A) < r\}, \qquad T_r(A) = \{x \in X : \ f(x, A) > r\}. \tag{2}$$

Proposition 2.1 provides a characterization of convexity of a set A in terms of well-posedness of $P_{\min}$ all over X. Further we consider prox-regularity of A and the property of C to be a summand of the ball. These two properties will be characterized in terms of well-posedness of $P_{\min}$ and $P_{\max}$ on $U_r(A)$ and $T_r(C)$, respectively.

The *proximal normal cone* to the set $A \subset X$ at the point $a \in A$ is

$$N(a, A) = \{z \in X | \ \exists t > 0 : \ a \in P(a + tz, A)\}.$$

Given $r > 0$, a set $A \subset X$ is called *uniformly r-prox-regular* if

$$P(a + z, A) = \{a\} \qquad \forall a \in A \quad \forall z \in N(a, A) \cap \operatorname{int} B_r.$$

The notion of prox-regularity for sets was introduced in [28] for subsets of a Hilbert space. This notion is a descendant of the notion of sets with positive reach, introduced by H. Federer [13] for subsets of $\mathbb{R}^n$. Clarke, Stern and Wolenski [7] introduced and studied the *proximally smooth* subsets of a Hilbert space. A set A is proximally smooth if there exists $r > 0$ such that the distance function $d(\cdot, A)$ is continuously differentiable on $U_r(A)$. Poliquin, Rockafellar and Thibault [28] showed that in a Hilbert space proximally smooth sets are exactly uniformly prox-regular sets. The notion of prox-regularity has been extended to Banach spaces by Bernard, Thibault and Zlateva in [4] and to asymmetric seminormed spaces by the author of the current paper in [18]. The properties and applications of prox-regular sets and sets from akin classes were investigated also in [1, 5, 14, 22–28, 31, 33]. In some articles, prox-regularity and its extension to asymmetric seminormed spaces are called weak convexity.

The next characterizations of the uniform prox-regularity in a strictly convex space follow immediately from the definitions.

Proposition 2.2. *Let X be a strictly convex normed vector space. For any $A \subset X$ and $r > 0$ the following assertions are equivalent:*

(i) *A is uniformly r-prox-regular;*

(ii) $a \in P(a + z, A) \qquad \forall a \in A \quad \forall z \in N(a, A) \cap \partial B_r$;

(iii) $d(a + z, A) = r \qquad \forall a \in A \quad \forall z \in N(a, A) \cap \partial B_r$;

(iv) $A \cap \operatorname{int} B_r(a + z) = \emptyset \qquad \forall a \in A \quad \forall z \in N(a, A) \cap \partial B_r$.

We denote by $\mathcal{PR}(r)$ the class of all closed and uniformly r-prox-regular sets $A \subset X$. Note that if X is a strictly convex normed space, then any convex closed set $A \subset X$ belongs to the class $\mathcal{PR}(r)$ for any $r > 0$.

If X is uniformly convex and smooth and $A \in \mathcal{PR}(r)$ for all $r > 0$, then A is convex. The set $\widetilde{B}_r(c)$ is a nonconvex representative of the class $\mathcal{PR}(r)$ in any strictly convex normed space.

Remark 2.3. If $0 < r_1 < r_2$, then $\mathcal{PR}(r_2) \subset \mathcal{PR}(r_1)$.

So, the classes $\mathcal{PR}(r)$ are parametrized with scalar parameter $r > 0$, which is a measure of possible nonconvexity of $A \in \mathcal{PR}(r)$.

Proposition 2.4. *Let X be a uniformly convex Banach space, A be a closed subset of X and $r > 0$. The following assertions are equivalent:*

(i) *$A \in \mathcal{PR}(r)$;*

(ii) *the projection mapping $x \mapsto P(x, A)$ is single-valued and continuous on $U_r(A)$;*

(iii) *for any $x \in U_r(A)$ the problem $P_{\min}$ is Tykhonov well-posed.*

The equivalence of items (i) and (ii) in Proposition 2.4 has been established in [4] for Banach spaces with the moduli of uniform convexity and uniform smoothness of power type. The current version of Proposition 2.4 has been obtained in [18] as a consequence of a more general result for asymmetric seminormed spaces.

Proposition 2.4 shows that the uniform prox-regularity of A should be assumed to obtain Tykhonov well-posedness of problem $P_{\min}$ on $U_r(A)$. In view of Remark 1.1 the uniform prox-regularity of A should be assumed to obtain Tykhonov well-posedness of problems $P_{\min\min}$ and $P_{\min\max}$.

The set $C \subset X$ is called a *summand of the ball B_r*, if there exists a set $C_1 \subset X$ such that $C + C_1 = B_r$, where $C + C_1 = \{c + c_1 : \ c \in C, \ c_1 \in C_1\}$ is the Minkowski sum of C and C_1. We denote by $\mathcal{SB}(r)$ the family of all convex closed summands of the ball B_r.

A set $C \subset X$ is called *r-strongly convex* if there exists $C' \subset X$ such that

$$C = \bigcap_{c \in C'} B_r(c).$$

As shown in [3, 29] if $C \in \mathcal{SB}(r)$, then C is r-strongly convex. The converse statement is valid if X is a Hilbert space, but invalid in general for a Banach space.

Remark 2.5. If $0 < r_1 < r_2$, then $\mathcal{SB}(r_1) \subset \mathcal{SB}(r_2)$.

So, the classes $\mathcal{SB}(r)$ are parametrized with scalar parameter $r > 0$, which is a measure of strong convexity of $C \in \mathcal{SB}(r)$.

In a number of problems we deal simultaneously with two sets, one of which is uniformly prox-regular (or weakly convex) and the other is a summand of the ball (or strongly convex). In such problems weak convexity of one of the sets can be compensated by strong convexity of the other one. For example, the metric projection and the antiprojection can be expressed as follows

$$P(x, A) = A \cap B_{d(x,A)}(x), \qquad F(x, C) = C \cap \widetilde{B}_{f(x,C)}(x).$$

If $A \in \mathcal{PR}(R)$ with $R > d(x, A)$ and $C \in \mathcal{SB}(r)$ with $r < f(x, C)$, then the metric projection and antiprojection are the intersections of the images of two continuous multifunctions, where one of the multifunctions has uniformly R-prox-regular images and the images of the other multifunction are summands of the ball B_r with $r < R$.

So, the continuity of the projection and antiprojection mappings may be derived from the results on the continuity of the intersection of the multifunctions obtained in [19].

The following characterization of the class $\mathcal{SB}(r)$ has been obtained in [17].

Proposition 2.6. *Let X be a uniformly convex Banach space with Fréchet differentiable norm and C be a convex closed subset of X, $r > 0$. The following assertions are equivalent:*

(i) $C \in \mathcal{SB}(r)$*;*

(ii) *the antiprojection mapping $x \mapsto F(x, C)$ is single-valued and continuous on $T_r(C)$;*

(iii) *for any $x \in T_r(C)$ the problem $P_{\max}$ is Tykhonov well-posed.*

Proposition 2.6 and Remark 1.1 imply that the assumption $C \in \mathcal{SB}(r)$ should be made to obtain Tykhonov well-posedness of problems $P_{\max\max}$ and $P_{\max\min}$.

In [25, Theorem 4.1] the following sufficient condition for problem $P_{\min\min}$ to be Tykhonov well-posed has been obtained.

Proposition 2.7. *Let X be a uniformly convex Banach space. Assume $A \in \mathcal{PR}(r_A)$, $C \in \mathcal{SB}(r_C)$, $\varrho_{\inf\inf}(C, A) < r_A - r_C$ and $A \cap \operatorname{int} C = \emptyset$. Then $P_{\min\min}$ is Tykhonov well-posed.*

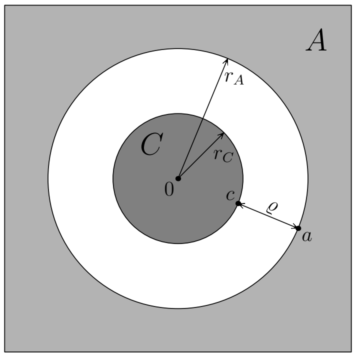
Remark 2.8. The estimate

$$\varrho_{\inf\inf}(C, A) < r_A - r_C$$

in Proposition 2.7 is sharp. Indeed, let $0 < r_C < r_A$. Consider $C = B_{r_C}$ and $A = \widetilde{B}_{r_A}$. Then we have $A \in \mathcal{PR}(r_A)$, $C \in \mathcal{SB}(r_C)$ and

$$\varrho_{\inf\inf}(C, A) = \varrho = r_A - r_C,$$

but it is evident that for any unit vector $e \in X$ the pair $(a, c) = (r_A e, r_C e)$ is a minimizer of $P_{\min\min}$. So, $P_{\min\min}$ is not Tykhonov well-posed.



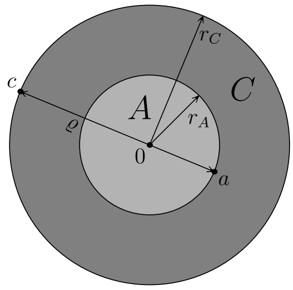
Remark 2.9. The implication (i)⇒(iii) in Proposition 2.4 is a special case of Proposition 2.7, where $C = \{x\}$ is a singleton.

3. Sufficient conditions for well-posedness of $P_{\max\max}$, $P_{\min\max}$ and $P_{\max\min}$

Theorem 3.1. *Let X be a uniformly convex Banach space. Assume $A \in \mathcal{SB}(r_A)$, $C \in \mathcal{SB}(r_C)$, $r_A + r_C < \varrho_{\sup\sup}(C, A)$. Then $P_{\max\max}$ is Tykhonov well-posed.*

The proofs of theorems of this section will be given in Section 7.

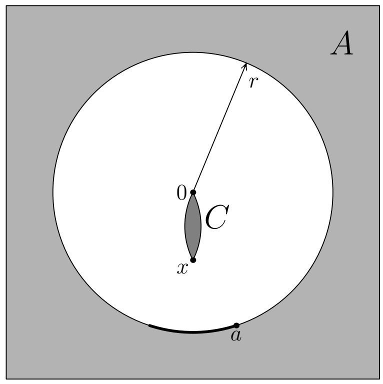
Remark 3.2. The estimate $r_A + r_C < \varrho_{\sup\sup}(C, A)$ in Theorem 3.1 is sharp. Indeed, $A = B_{r_A}$ and $C = B_{r_C}$ satisfy all assumptions of Theorem 3.1 except for the inequality $r_A + r_C < \varrho_{\sup\sup}(C, A)$, which turns into the equalities $r_A + r_C = \varrho = \varrho_{\sup\sup}(C, A)$. For any unit vector $e \in X$ the pair $(a, c) = (r_A e, -r_C e)$ is a maximizer of $P_{\max\max}$. So, $P_{\max\max}$ is not Tykhonov well-posed.



Remark 3.3. The implication (i)⇒(iii) in Proposition 2.6 is a special case of Theorem 3.1, where $A = \{x\}$ is a singleton.

Theorem 3.4. *Let X be a Hilbert space, $A \in \mathcal{PR}(r)$, $C \subset X$ and $\varrho_{\inf\sup}(A, C) < r$. Then $P_{\min\max}$ is Tykhonov well-posed.*

Remark 3.5. The assumption $\varrho_{\inf\sup}(A, C) < r$ in Theorem 3.4 is sharp. Let X be a Hilbert space. Fix any $r > 0$, $R > 0$ and $x \in X$ such that $\|x\| < 2R$. Let $A = \widetilde{B}_r$, $C = \bigcap\limits_{c \in X:\ \{0,x\} \subset B_R(c)} B_R(c)$. All assumptions of Theorem 3.4 are satisfied, except for the inequality $\varrho_{\inf\sup}(A, C) < r$, which turns into equality. Since the minimizer of the problem $\min\limits_{a \in A} f(a, C)$ is not unique, $P_{\min\max}$ is not Tykhonov well-posed.



Remark 3.6. The implication (i)⇒(iii) in Proposition 2.4 is a special case of Theorem 3.4, where $C = \{x\}$ is a singleton.

Theorem 3.7. *Let X be a uniformly convex Banach space, $r^- > 0$ and $r^+ > 0$. Let*

(i) $A \in \mathcal{PR}(r^+)$, $(X \setminus \operatorname{int} A) \in \mathcal{PR}(r^-)$ *and* $A = \overline{\operatorname{int} A}$; $C \in \mathcal{SB}(r_C)$;

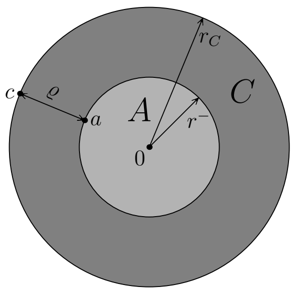
(ii) $r_C < r^- + \varrho_{\sup\inf}(C, A)$;

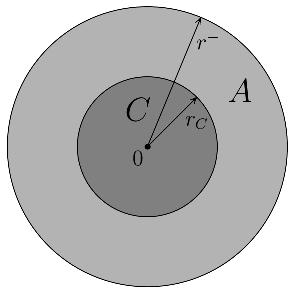
(iii) $0 < \varrho_{\sup\inf}(C, A) < r^+$.

Then $P_{\max\min}$ is Tykhonov well-posed.

Remark 3.8. The estimates $r_C < r^- + \varrho_{\sup\inf}(C, A)$ and $0 < \varrho_{\sup\inf}(C, A) < r^+$ in Theorem 3.7 are sharp. The following three examples show sharpness each of these estimates.

(1) Consider $A = B_{r^-}$, $C = B_{r_C}$ with $r_C > r^- > 0$. All assumptions of Theorem 3.7 except for the inequality $r_C < r^- + \varrho$ with $\varrho = \varrho_{\sup\inf}(C, A)$ are satisfied, but $P_{\max\min}$ is not Tykhonov well-posed.

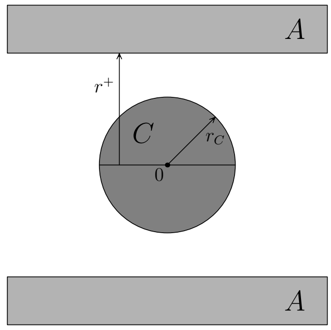




(2) Consider $C = B_{r_C}$, $A = B_{r^-}$ with $r^- > r_C > 0$. Since $C \subset A$ it follows that

$$\varrho_{\sup\inf}(C, A) = \max_{c \in C} d(c, A) = 0.$$

This maximum is attained at any $c \in C$, so $P_{\max\min}$ is not Tykhonov well-posed. All assumptions of Theorem 3.7 except for the inequality $\varrho_{\sup\inf}(C, A) > 0$ are satisfied herein.



3) Fix any positive numbers r_C, r^+, $r^- > r_C$ and a unit functional $p \in X^*$. Consider

$$A = \{x \in X : \ |\langle p, x\rangle| \geq r^+\} \text{ and } C = B_{r_C}.$$

In this example we have

$$\varrho_{\sup\inf}(C, A) = \max_{c \in C} d(c, A) = r^+.$$

The maximum is attained at any $c \in B_{r_C}$ such that $\langle p, c\rangle = 0$. Consequently, $P_{\max\min}$ is not Tykhonov well-posed, while all assumptions of Theorem 3.7 except for the inequality $\varrho_{\sup\inf}(C, A) < r^+$ are satisfied.

Remark 3.9. The implication (i)$\Rightarrow$(iii) in Proposition 2.6 is a limiting case of Theorem 3.7, where $A = B_{r^-}(x)$, $r^- \to 0$.

4. Auxiliary results

Recall that the *Minkowski sum* and *difference* of the sets $A \subset X$ and $C \subset X$ are, respectively,

$$A + C = \{a + c : \ a \in A, \ c \in C\}, \qquad A \overset{*}{-} C = \{x \in X : \ x + C \subset A\}.$$

These definitions immediately imply that

$$A + C = \bigcup_{c \in C} (A + c), \qquad A \overset{*}{-} C = \bigcap_{c \in C} (A - c). \tag{3}$$

Each Minkowski operation can be expressed in terms of another and the complement operation as follows:

$$X \setminus (A \overset{*}{-} C) = (X \setminus A) + (-C), \qquad X \setminus (A + C) = (X \setminus A) \overset{*}{-} (-C). \tag{4}$$

One can derive the following properties of the Minkowski operations immediately from the definitions.

Remark 4.1. For any sets A, C in a normed vector space X one has
(i) $A \subset A + C \overset{*}{-} C$;
(ii) $A \overset{*}{-} C + C \subset A$;
(iii) $\overline{A} + C \subset \overline{A + C}$;
(iv) $\operatorname{int} A + C \subset \operatorname{int}(A + C)$.

Note that for any $r > 0$ and $A \subset X$

$$U_r(A) = A + \operatorname{int} B_r, \qquad T_r(A) = A + \operatorname{int} \widetilde{B}_r. \tag{5}$$

Recall that the *diameter* of a set $A \subset X$ is

$$\operatorname{diam} A = \sup_{x,y\in A} \|x - y\|.$$

The following proposition follows immediately from the definitions.

Proposition 4.2. *Let A be a closed subset of a Banach space X and $J : A \to \mathbb{R}$ be a lower semicontinuous function. Then problem* (1) *is Tykhonov well-posed iff*

$$\lim_{\varepsilon\to+0} \operatorname{diam} \left\{ x \in A : \; J(x) < \inf_{a\in A} J(a) + \varepsilon \right\} = 0.$$

Given two sets $A, C \subset X$ and $\varepsilon > 0$, the *ε-projection* of C onto A is defined as

$$P^{\varepsilon}(C, A) = \{a \in A : \; d(a, C) < \varrho_{\inf\inf}(C, A) + \varepsilon\}. \tag{6}$$

Propositions 4.2 and 2.7 imply the following one.

Proposition 4.3. *Under the assumptions of Proposition* 2.7 *one has*

$$\lim_{\varepsilon\to+0} \operatorname{diam} P^{\varepsilon}(C, A) = \lim_{\varepsilon\to+0} \operatorname{diam} P^{\varepsilon}(A, C) = 0.$$

Lemma 4.4. *Let X be a uniformly convex normed vector space and δ_X be its modulus of convexity. Then for any nonzero vectors $x, y \in X$ one has*

$$\|x + y\| \le \|x\| + \|y\| - 2 \min\{\|x\|, \|y\|\} \cdot \delta_X \left(\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \right).$$

Proof. Denote $a = \frac{x}{\|x\|}$, $b = \frac{y}{\|y\|}$, $\lambda = \|x\|$, $\mu = \|y\|$. Without loss of generality, we assume that $\lambda \ge \mu$. So,

$$\|x + y\| = \|\lambda a + \mu b\| = \|(\lambda - \mu)a + \mu(a + b)\| \le \lambda - \mu + \mu\|a + b\|.$$

The definition of the modulus of convexity implies $\|a + b\| \le 2 - 2\delta_X(\|a - b\|)$. Consequently,

$$\begin{aligned} \|x + y\| &\le \lambda + \mu - 2\mu\delta_X(\|a - b\|) \\ &= \|x\| + \|y\| - 2 \min\{\|x\|, \|y\|\} \cdot \delta_X \left(\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \right). \end{aligned}$$ □

Lemma 4.5. *Let X be a uniformly convex normed vector space and δ_X its modulus of convexity. Consider positive numbers α, β, γ and vectors $x, y \in X$ such that*

$$\|x\| \le \alpha + \gamma, \quad \|y\| \le \beta + \gamma, \quad \|x + y\| \ge \alpha + \beta - \gamma, \quad 4\gamma \le \min\{\alpha, \beta\}.$$

Then
$$\delta_X \left(\frac{1}{2} \left\| \frac{x}{\alpha} - \frac{y}{\beta} \right\| \right) \le \frac{3\gamma}{\min\{\alpha, \beta\}}.$$

Proof. From $\|x\| \ge \|x + y\| - \|y\| \ge \alpha - 2\gamma$ and $\|y\| \ge \|x + y\| - \|x\| \ge \beta - 2\gamma$ it follows that $\min\{\|x\|, \|y\|\} \ge \min\{\alpha - 2\gamma, \beta - 2\gamma\}$.

This and the inequality $\|x\| + \|y\| - \|x+y\| \le 3\gamma$ yield by Lemma 4.4 that

$$2\delta_X\left(\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|\right)\min\{\alpha-2\gamma,\beta-2\gamma\} \le 3\gamma.$$

Since $4\gamma \le \min\{\alpha,\beta\}$ it follows that $2\min\{\alpha-2\gamma,\beta-2\gamma\} \ge \min\{\alpha,\beta\}$. Hence,

$$\delta_X\left(\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|\right) \le \frac{3\gamma}{\min\{\alpha,\beta\}}. \tag{7}$$

Observing that $\left\|\frac{x}{\alpha} - \frac{x}{\|x\|}\right\| = \left|1 - \frac{\|x\|}{\alpha}\right| \le \frac{2\gamma}{\alpha}$, $\left\|\frac{y}{\beta} - \frac{y}{\|y\|}\right\| = \left|1 - \frac{\|y\|}{\beta}\right| \le \frac{2\gamma}{\beta}$, we get

$$\left\|\frac{x}{\alpha} - \frac{y}{\beta}\right\| \le \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| + \frac{4\gamma}{\min\{\alpha,\beta\}}. \tag{8}$$

If $\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \le \frac{4\gamma}{\min\{\alpha,\beta\}}$, then by (8) we have $\left\|\frac{x}{\alpha} - \frac{y}{\beta}\right\| \le \frac{8\gamma}{\min\{\alpha,\beta\}}$. Using the property of the modulus of convexity $\delta_X(2t) \le t$ for all $t \in (0;1]$ we obtain the required inequality. Otherwise, in the case $\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| > \frac{4\gamma}{\min\{\alpha,\beta\}}$ it follows from (8) that $\left\|\frac{x}{\alpha} - \frac{y}{\beta}\right\| \le 2\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|$. From this and (7), due to the monotonicity of the modulus of convexity, the required inequality follows. $\square$

Lemma 4.6. *Let X be a uniformly convex normed vector space and δ_X be its modulus of convexity. Let C be an r-strongly convex subset of X and $\sigma = r\delta_X\left(\frac{\operatorname{diam} C}{2r}\right)$. Then there exists $z \in X$ such that $B_\sigma(z) \subset C \subset B_{2r}(z)$.*

Proof. From the definition of the diameter we conclude that one can find $c_1, c_2 \in C$ such that $\|c_1 - c_2\| \ge \frac{\operatorname{diam} C}{2}$. Define $z = \frac{1}{2}(c_1 + c_2)$. Since C is r-strongly convex there exists $C' \subset X$ such that

$$C = \bigcap_{c \in C'} B_r(c). \tag{9}$$

Fix any $c \in C'$ and denote $x = \frac{1}{r}(c_1 - c)$, $y = \frac{1}{r}(c_2 - c)$, $\varepsilon = \frac{\operatorname{diam} C}{2r}$. Since $\|x\| \le 1$, $\|y\| \le 1$ and $\|x - y\| = \frac{\|c_1 - c_2\|}{r} \ge \varepsilon$ by the definition of the modulus of convexity we obtain $\delta_X(\varepsilon) \le 1 - \frac{1}{2}\|x + y\|$. It implies that $\sigma = r\delta_X(\varepsilon) \le r - \|z - c\|$ and hence

$$B_\sigma(z) \subset B_r(c) \subset B_{2r}(z).$$

Recalling (9), we complete the proof. $\square$

Lemma 4.7. *Let X be a uniformly convex Banach space, $A \in \mathcal{PR}(r)$, $x \in X$, $0 < d(x,A) < r$. Then for any $\varepsilon \in (0, r - d(x,A))$ there exists $x' \in X$ such that $\|x' - x\| = \varepsilon$ and $d(x',A) = d(x,A) + \varepsilon$.*

Proof. According to Proposition 2.4 there exists $a \in P(x,A)$. Denoting $z = \frac{x-a}{\|x-a\|}$ we obtain $z \in N(a,A)$ and $\|z\| = 1$. Fix any $\varepsilon \in (0, r - d(x,A))$ and define $x' = x + \varepsilon z$. Then $\|x' - x\| = \varepsilon$. Since $x' = a + (d(x,A) + \varepsilon)z$ it follows by the definition of the uniform prox-regularity that $a \in P(x',A)$ and hence

$$d(x',A) = \|x' - a\| = d(x,A) + \varepsilon. \qquad \square$$

5. Weakly convex sets in the sense of Efimov-Stechkin

A set $A \subset X$ is called *weakly convex in the sense of Efimov-Stechkin* with radius $r > 0$ if for any $x \in X \setminus A$ there exists $c \in X$ such that $x \in \operatorname{int} B_r(c) \subset X \setminus A$. Weak convexity in the sense of Efimov-Stechkin was introduced in [11], where such sets were called as *a-convex sets*. We denote by $\mathcal{ESWC}(r)$ the class of all sets $A \subset X$ weakly convex in the sense of Efimov-Stechkin with radius r.

One can easily see that any set $A \in \mathcal{ESWC}(r)$ is closed.

Lemma 5.1. *For any convex closed set $A \subset X$ and $r > 0$ one has $A \in \mathcal{ESWC}(r)$.*

Proof. Fix any $x \in X \setminus A$ and $r > 0$. By the Hahn-Banach separation theorem there exists a functional $p \in X^*$ and $\varepsilon > 0$ such that

$$\sup_{a \in A} \langle p, a\rangle + \varepsilon < \langle p, x\rangle.$$

Choose $v \in \operatorname{int} B_r$ such that $\langle p, v\rangle > r\|p\| - \varepsilon$. To show that

$$\operatorname{int} B_r(x + v) \subset X \setminus A \tag{10}$$

fix any $b \in \operatorname{int} B_r(x + v)$. We have

$$\langle p, b\rangle > \langle p, x + v\rangle - r\|p\| > \langle p, x\rangle - \varepsilon > \sup_{a \in A} \langle p, a\rangle$$

and hence $b \notin A$. This proves (10). So, $x \in \operatorname{int} B_r(x + v) \subset X \setminus A$. □

Lemma 5.2. *For any $A \subset X$ and $r > 0$ the following assertion are equivalent:*

(i) $A \in \mathcal{ESWC}(r)$;

(ii) $A + \operatorname{int} B_r \overset{*}{-} \operatorname{int} B_r \subset A$;

(iii) $A + \operatorname{int} B_r \overset{*}{-} \operatorname{int} B_r = A$;

(iv) $\exists A_1 \subset X : \ A = A_1 \overset{*}{-} \operatorname{int} B_r$.

Proof. Observe that we have

$$\begin{aligned} & \exists c \in X : \ x \in \operatorname{int} B_r(c) \subset X \setminus A \\ \Leftrightarrow\ & \exists c \in x + \operatorname{int} B_r : \ c \notin A + \operatorname{int} B_r \\ \Leftrightarrow\ & x + \operatorname{int} B_r \not\subset A + \operatorname{int} B_r \\ \Leftrightarrow\ & x \notin A + \operatorname{int} B_r \overset{*}{-} \operatorname{int} B_r. \end{aligned}$$

Consequently,

$$\begin{aligned} \text{(i)} \quad & \Leftrightarrow \quad (\forall x \in X \setminus A \quad \exists c \in X : \ x \in \operatorname{int} B_r(c) \subset X \setminus A) \\ & \Leftrightarrow \quad (x \notin A + \operatorname{int} B_r \overset{*}{-} \operatorname{int} B_r \quad \forall x \in X \setminus A) \\ & \Leftrightarrow \quad \text{(ii)}. \end{aligned}$$

By Remark 4.1 we have (ii) $\Leftrightarrow$ (iii).

Obviously, (iii) $\Rightarrow$ (iv). To complete the proof it suffices to prove that (iv) $\Rightarrow$ (ii). Now let $A = A_1 \overset{*}{-} \operatorname{int} B_r$ for some $A_1 \subset X$. Then, using Remark 4.1(ii), we get

$$A + \operatorname{int} B_r \overset{*}{-} \operatorname{int} B_r = A_1 \overset{*}{-} \operatorname{int} B_r + \operatorname{int} B_r \overset{*}{-} \operatorname{int} B_r \subset A_1 \overset{*}{-} \operatorname{int} B_r = A.$$

So, (ii) holds true. □

Lemma 5.3. *Let X be a uniformly convex Banach space. Then for any $r>0$ one has*

$$\mathcal{PR}(r) \subset \mathcal{ESWC}(r).$$

Proof. Let $A \in \mathcal{PR}(r)$. According to the definition of the class $\mathcal{ESWC}(r)$ it suffices to show that for any $x \in X \setminus A$ there exists $c \in X$ such that

$$x \in \operatorname{int} B_r(c) \subset X \setminus A. \tag{11}$$

Fix any $x \in X \setminus A$. If $d(x, A) \geq r$, then (11) holds for $c = x$. Now assume that $d(x, A) < r$. According to Proposition 2.4 there exists $a \in P(x, A)$. Denoting $z = \frac{r}{\|x-a\|}(x - a)$, we have $z \in N(a, A) \cap \partial B_r$. In view of r-prox regularity of A we obtain $a \in P(a + z, A)$. Consequently, $d(a + z, A) = \|z\| = r$ and hence $\operatorname{int} B_r(a + z) \subset X \setminus A$. Observing that $\|a + z - x\| = r - \|x - a\| < r$ we conclude, that (11) holds for $c = a + z$. □

The next lemma implies that the inclusion $\mathcal{ESWC}(r_1) \subset \mathcal{PR}(r_2)$ is false for any positive r_1, r_2.

Lemma 5.4. *Let X be a non one-dimensional normed vector space. Then*

$$\bigcap_{r_1>0} \mathcal{ESWC}(r_1) \not\subset \mathcal{PR}(r_2) \quad \forall r_2 > 0.$$

Proof. Let X_2 be some two-dimensional subspace of X. Denote $D = B_1 \cap X_2$, where B_1 is the unit ball of X. According to the Strasziewitcz's theorem there exists a unit functional $p \in X_2^*$, which attains its strict maximum over D at some $e \in D$ (point e is called an exposed point of D). Choose any unit vector $a \in X_2$ such that $\langle p, a\rangle = 0$. For any $t \in [0, 1]$ we define

$$\tau^+(t) = \sup\{\tau > 0 : \|te + \tau a\| \leq 1\}, \quad \tau^-(t) = \sup\{\tau > 0 : \|te - \tau a\| \leq 1\}. \tag{12}$$

Since p attains at e its strict maximum over D and for any $\tau \in \mathbb{R}$

$$\langle p, e + \tau a\rangle = \langle p, e\rangle = \|p\| = 1$$

it follows that $\tau^+(1) = \tau^-(1) = 0$.

Fix any $t_0, t_1, \lambda \in [0, 1]$. Denote $t_\lambda = (1 - \lambda)t_0 + \lambda t_1$, $\tau^+_\lambda = (1 - \lambda)\tau^+(t_0) + \lambda\tau^+(t_1)$. Since $\|t_i e + \tau^+(t_i)a\| \leq 1$ for $i = 0$ and $i = 1$, by the sublinearity of the norm we obtain $\|t_\lambda e + \tau^+_\lambda a\| \leq 1$ and hence $\tau^+(t_\lambda) \geq \tau^+_\lambda$. So, the function τ^+ is concave on

$[0,1]$. Similarly, function τ^- is also concave on $[0,1]$. Therefore, these functions are continuous on $[0,1]$ and, in particular,

$$\lim_{t\to 1-0}\tau^+(t)=\tau^+(1)=0,\qquad \lim_{t\to 1-0}\tau^-(t)=\tau^-(1)=0. \tag{13}$$

Define $A=\{0,r_2a\}$. Since $P\left(\frac{r_2}{2}a,A\right)=A=\{0,r_2a\}$ we have $\frac{r_2}{2}a\in N(0,A)$ and hence $A\notin\mathcal{PR}(r_2)$. Now it suffices to show that

$$A\in\bigcap_{r_1>0}\mathcal{ESWC}(r_1).$$

Fix any $r_1>0$ and $x\in X\setminus A$. We need to prove that there exists $c\in X$ such that

$$x\in\operatorname{int}B_{r_1}(c)\subset X\setminus A. \tag{14}$$

Lemma 5.1 implies that $[0,r_2a]\in\mathcal{ESWC}(r_1)$ and therefore for any $x\in X\setminus[0,r_2a]$ one can find $c\in X$ such that (14) holds true. It remains to consider the case $x\in[0,r_2a]\setminus A$, that is $x=r_2\lambda a$ with some $\lambda\in(0,1)$.

In view of (13) one can find $t\in(0,1)$ such that $\tau^+(t)<\frac{r_2}{r_1}\lambda$ and $\tau^-(t)<\frac{r_2}{r_1}(1-\lambda)$. From (12) it follows that

$$\left\|te+\frac{r_2}{r_1}\lambda a\right\|>1,\qquad \left\|te-\frac{r_2}{r_1}(1-\lambda)a\right\|>1. \tag{15}$$

Denoting $c=x+r_1te$, we obtain $\|c-x\|=r_1t<r_1$, $c=r_2\lambda a+r_1te$. The inequalities (15) can be rewritten as $\|c\|>r_1$ and $\|c-r_2a\|>r_1$, respectively. Thus, (14) holds true. $\square$

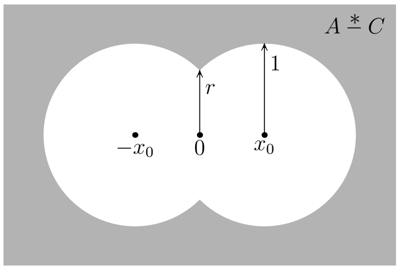
Theorem 5.5. *Let X be a normed vector space and $A\in\mathcal{ESWC}(r)$. Then for any $C\subset X$ one has $A\overset{*}{-}C\in\mathcal{ESWC}(r)$.*

Proof. By Lemma 5.2 there exists $A_1\subset X$ such that $A=A_1\overset{*}{-}\operatorname{int}B_r$. So,

$$A\overset{*}{-}C=A_1\overset{*}{-}\operatorname{int}B_r\overset{*}{-}C=A_1\overset{*}{-}C\overset{*}{-}\operatorname{int}B_r.$$

Using Lemma 5.2 once more, we complete the proof. $\square$

Remark 5.6. In Theorem 5.5 the class $\mathcal{ESWC}(r)$ can not be substituted by $\mathcal{PR}(r)$. Indeed, consider $X=\mathbb{R}^2$, $A=\widetilde{B}_1$ and $C=\{x_0,-x_0\}$, where $0<\|x_0\|<1$. We have $A\in\mathcal{PR}(1)$, but $A\overset{*}{-}C\notin\mathcal{PR}(1)$. Actually, one has only $A\overset{*}{-}C\in\mathcal{PR}(r)$ with $r=\sqrt{1-\|x_0\|^2}$, where $r\to 0$ as $\|x_0\|\to 1$.



Lemma 5.7. *Let X be a uniformly convex Banach space. Let $A\subset X$, $\overline{A}=\overline{\operatorname{int}A}$, $\operatorname{int}A=\operatorname{int}\overline{A}$, $\overline{A}\in\mathcal{ESWC}(r^+)$, $\overline{X\setminus A}\in\mathcal{ESWC}(r^-)$, $a\in\partial A$. Then one can find $v\in\partial B_1$ such that*

$$d(a+r^+v,A)=r^+,\qquad d(a-r^-v,X\setminus A)=r^-.$$

Proof. Since $a \in \overline{A} = \overline{\operatorname{int} A}$ there exists a sequence of $x_k \in \operatorname{int} A$ such that we have $\|x_k - a\| < \frac{1}{k}$ for all $k \in \mathbb{N}$. In view of the assumption $\overline{X \setminus A} \in \mathcal{ESWC}(r^-)$ for any $k \in \mathbb{N}$ one can find $y_k \in X$ such that $x_k \in \operatorname{int} B_{r^-}(y_k) \subset \operatorname{int} A$.

Since $a \notin \operatorname{int} A = \operatorname{int} \overline{A}$ there exists a sequence of $z_n \in X \setminus \overline{A}$ such that $\|z_n - a\| < \frac{1}{n}$ for any $n \in \mathbb{N}$. Due to the assumption $\overline{A} \in \mathcal{ESWC}(r^+)$ for any $n \in \mathbb{N}$ one can find $w_n \in X$ such that $z_n \in \operatorname{int} B_{r^+}(w_n) \subset X \setminus \overline{A}$.

The inclusions $\operatorname{int} B_{r^-}(y_k) \subset \operatorname{int} A$ and $\operatorname{int} B_{r^+}(w_n) \subset X \setminus \overline{A}$ imply that

$$d(y_k, X \setminus A) \geq r^-, \qquad d(w_n, A) \geq r^+. \tag{16}$$

and $\operatorname{int} B_{r^+}(w_n) \bigcap \operatorname{int} B_{r^-}(y_k) = \emptyset$, i.e. $\|w_n - y_k\| \geq r^+ + r^-$.

Applying Lemma 4.5 for $x = x_k - y_k$, $y = w_n - z_n$, $\alpha = r^-$, $\beta = r^+$, $\gamma = \frac{1}{n} + \frac{1}{k}$ we get for sufficiently large n, k

$$\delta_X \left(\frac{1}{2} \left\| \frac{x_k - y_k}{r^-} - \frac{w_n - z_n}{r^+} \right\| \right) \leq \frac{3}{\min\{r^+, r^-\}} \left(\frac{1}{n} + \frac{1}{k} \right).$$

This yields $\left\| \frac{x_k - y_k}{r^-} - \frac{w_n - z_n}{r^+} \right\| \to 0$ as $n, k \to \infty$. Thus, as a consequence of the completeness of X there exists

$$\lim_{k \to \infty} \frac{x_k - y_k}{r^-} = \lim_{n \to \infty} \frac{w_n - z_n}{r^+} = v \in \partial B_1.$$

Consequently, $$\lim_{k \to \infty} y_k = a - r^- v, \quad \lim_{n \to \infty} w_n = a + r^+ v.$$

Passing to the limit in (16) we get $d(a + r^+ v, A) \geq r^+$ and $d(a - r^- v, X \setminus A) \geq r^-$. On the other hand, since $a \in \partial A$, we have

$$d(a + r^+ v, A) \leq \|r^+ v\| = r^+ \ \text{ and } \ d(a - r^- v, X \setminus A) \leq \|r^- v\| = r^-. \qquad \square$$

We say that $A \subset X$ is *weakly convex-concave* with positive constants r^+, r^- and write $A \in \mathcal{WCC}(r^+, r^-)$ if

$$\overline{A} = \overline{\operatorname{int} A}, \quad \operatorname{int} A = \operatorname{int} \overline{A}, \quad \overline{A} \in \mathcal{PR}(r^+), \quad \overline{X \setminus A} \in \mathcal{PR}(r^-).$$

Remark 5.8. For any $A \subset X$ and any positive numbers r^+, r^- one has

$$A \in \mathcal{WCC}(r^+, r^-) \quad \Leftrightarrow \quad (X \setminus A) \in \mathcal{WCC}(r^-, r^+).$$

Theorem 5.9. *Let X be a uniformly convex Banach space, $r^+ > 0$ and $r^- > 0$. Let $A \subset X$, $\overline{A} = \overline{\operatorname{int} A}$ and $\operatorname{int} A = \operatorname{int} \overline{A}$. Then the following assertions are equivalent:*

(i) $A \in \mathcal{WCC}(r^+, r^-)$;

(ii) $\overline{A} \in \mathcal{ESWC}(r^+)$ *and* $\overline{X \setminus A} \in \mathcal{ESWC}(r^-)$.

Proof. The implication (i) $\Rightarrow$ (ii) follows from Lemma 5.3. Prove the converse implication. Let (ii) holds true. Let us prove that $\overline{A} \in \mathcal{PR}(r^+)$. Fix any $a \in A$ and $z \in N(a, A) \cap \partial B_{r^+}$. By the definition of the proximal normal cone there exists $t > 0$ such that $a \in P(a + tz, A)$. Taking into account that $a \in \partial A$ according to Lemma 5.7 one can find $v \in \partial B_1$ such that $d(a + r^+ v, A) = r^+$, $d(a - r^- v, X \setminus A) = r^-$.

Since $\operatorname{int} B_{r^+t}(a+tz) \bigcap \operatorname{int} B_{r^-}(a - r^-v) = \emptyset$ it follows that

$$\|r^-v + tz\| \geq r^- + r^+t = \|r^-v\| + \|tz\|.$$

This by strong convexity of X implies that the vectors r^-v and tz are co-directed, i.e. $r^+v = z$. So, $d(a+z, A) = d(a + r^+v, A) = r^+$ and hence $a \in P(a+z, A)$. According to Proposition 2.2 we get $\overline{A} \in \mathcal{PR}(r^+)$. Similarly, $\overline{X \setminus A} \in \mathcal{PR}(r^-)$. $\square$

In case X is a Hilbert space Theorem 5.9 has been proven in [15, Theorem 7.2].

6. Calculus of convexity parameters in connection with the Minkowski operations

Lemma 6.1. *Let X be a strictly convex normed vector space, $C \in \mathcal{SB}(r)$, $y \in C$, $z \in N(y, C) \cap \partial B_r$. Then $C \subset B_r(y - z)$.*

Proof. By the definition of the proximal normal cone there exists $t > 0$ such that $y \in P(y + tz, C)$.

Since $C \cap \operatorname{int} B_{rt}(y + tz) = \emptyset$ according to the Hahn-Banach separation theorem there exists a unit functional $p \in X^*$ such that

$$\langle p, b\rangle < \langle p, c\rangle \qquad \forall b \in \operatorname{int} B_{rt}(y + tz) \quad \forall c \in C.$$

As $y \in B_{rt}(y + tz) \cap C$ it entails

$$\langle p, b\rangle \leq \langle p, y\rangle \leq \langle p, c\rangle \qquad \forall b \in B_{rt}(y + tz) \quad \forall c \in C. \tag{17}$$

Since $C \in \mathcal{SB}(r)$ there exists $C_1 \subset X$ such that $C + C_1 = B_r$. Therefore $z \in B_r = C + C_1$ and hence one can find $\hat{c} \in C$ and $c_1 \in C_1$ such that $z = \hat{c} + c_1$. By the first inequality of the chain (17) it follows that

$$\langle p, \hat{c} + c_1\rangle = \langle p, z\rangle \leq \langle p, b'\rangle \qquad \forall b' \in B_r. \tag{18}$$

By the second inequality of the chain (17) we have $\langle p, y\rangle \leq \langle p, \hat{c}\rangle$ and in view of (18) we obtain

$$\langle p, y + c_1\rangle \leq \langle p, \hat{c} + c_1\rangle \leq \langle p, b'\rangle \qquad \forall b' \in B_r. \tag{19}$$

Taking into account that $y + c_1 \in B_r$, $\hat{c} + c_1 \in B_r$ and X is a strictly convex space, from (19) we deduce that $y = \hat{c}$. So, $C + c_1 \subset C + C_1 = B_r$, $c_1 = z - \hat{c} = z - y$ and hence $C \subset B_r(-c_1) = B_r(y - z)$. $\square$

Theorem 6.2. *Let X be a uniformly convex Banach space, $A \in \mathcal{PR}(r_A)$, $C \in \mathcal{SB}(r_C)$, $r_A > r_C \geq 0$. Then $A + C \in \mathcal{PR}(r_A - r_C)$.*

Proof. Denote $D = A + C$ and $r_D = r_A - r_C$. Let us show that D is uniformly r_D-prox-regular. Assume that $x \in D$, $z \in N(x, D) \cap \partial B_{r_D}$. According to the definition of the proximal normal cone there exists $t > 0$ such that $x \in P(x + tz, D)$. Since $x \in D = A + C$ one can find $a \in A$ and $c \in C$ such that $x = a + c$. Therefore $a+c \in P(a+c+tz, D)$. Since $a+c \in A+c \subset D$ it follows that $a+c \in P(a+c+tz, A+c)$ and hence $a \in P(a+tz, A)$. Similarly, $c \in P(c+tz, C)$. Thus, $z \in N(a, A) \cap N(c, C)$. Using Proposition 2.2, we get $a \in P\big(a + \frac{r_A}{r_D} z, A\big)$, that is $A \subset \widetilde{B}_{r_A}\big(a + \frac{r_A}{r_D} z\big)$.

Applying Lemma 6.1, we obtain $C \subset B_{r_C}\big(c - \frac{r_C}{r_D} z\big)$. Consequently,

$$D = A + C \subset \widetilde{B}_{r_A}\Big(a + \frac{r_A}{r_D} z\Big) + B_{r_C}\Big(c - \frac{r_C}{r_D} z\Big) = \widetilde{B}_{r_A - r_C}\,(x + z)\,.$$

So, $x \in P(D, x+z)$. It means that D is uniformly r_D-prox-regular.

To complete the proof it suffices to show that D is closed. Fix any $\hat{x} \in \partial D$. Observe that $\varrho_{\inf\inf}(-C, A - \hat{x}) = 0 < r_A - r_C$. If there exists $c_0 \in (\hat{x} - A) \cap \operatorname{int} C$ then $\hat{x} \in A + c_0 \subset A + \operatorname{int} C \subset \operatorname{int} D$, which contradicts the condition $\hat{x} \in \partial D$. So, $(A - \hat{x}) \cap (-\operatorname{int} C) = \emptyset$. According to Proposition 2.7 the problem

$$\min_{c \in C,\ a \in A} \|a - \hat{x} + c\|$$

is Tykhonov well-posed. Hence there exist $\hat{a} \in A$ and $\hat{c} \in C$ such that $\|\hat{a} - \hat{x} + \hat{c}\| = \varrho_{\inf\inf}(-C, A - \hat{x}) = 0$. Consequently, $\hat{x} = \hat{a} + \hat{c} \in A + C = D$. This means that D is closed. $\square$

Remark 6.3. In Theorem 6.2 the class $\mathcal{PR}(r_A)$ can not be substituted by $\mathcal{ESWC}(r_A)$. Indeed, consider $X = \mathbb{R}^2$, $A = \{x_0, -x_0\}$ with $x_0 \in \partial B_1$ and $C = B_2$. We have $C \in \mathcal{SB}(r_C)$ with $r_C = 2$ and $A \in \mathcal{ESWC}(r_A)$ with any $r_A > 0$, in particular, with $r_A = 3$. However, $A + C$ does belong to the class $\mathcal{ESWC}(r_A - r_C)$. Actually, $A + C \notin \mathcal{ESWC}(r)$ for any $r > 0$.

Theorem 6.4. *Let X be a uniformly convex Banach space, $r^- > 0$, $r^+ > 0$, $r_C \in (0, r^+)$. Let $A \in \mathcal{WCC}(r^+, r^-)$, $C \in \mathcal{SB}(r_C)$. Then*

(i) $\overline{\operatorname{int} A + C} = \overline{A + C} = \overline{A} + C$;

(ii) $\operatorname{int}\big(\overline{A} + C\big) = \operatorname{int}(A + C) = \operatorname{int} A + C$;

(iii) $A + C \in \mathcal{WCC}(r^+ - r_C, r^-)$.

Proof. (i). Since $\overline{A} \in \mathcal{PR}(r^+)$ and $C \in \mathcal{SB}(r_C)$ it follows by Theorem 6.2 that $\overline{A} + C \in \mathcal{PR}(r^+ - r_C)$. Consequently, $\overline{A} + C$ is a closed set and hence

$$\overline{\operatorname{int} A + C} \subset \overline{A + C} \subset \overline{A} + C. \tag{20}$$

On the other hand, Remark 4.1(iii) implies that $\overline{\operatorname{int} A} + C \subset \overline{\operatorname{int} A + C}$. Bearing in mind that $\overline{\operatorname{int} A} = \overline{A}$, we get $\overline{A} + C \subset \overline{\operatorname{int} A + C}$. This and (20) yield assertion (i).

Moreover, in view of Lemma 5.3 we get

$$\overline{A + C} = \overline{A} + C \in \mathcal{PR}(r^+ - r_C) \subset \mathcal{ESWC}(r^+ - r_C). \tag{21}$$

(ii). According to Remark 4.1(iv) we have $\operatorname{int} A + C \subset \operatorname{int}(A + C) \subset \operatorname{int}\big(\overline{A} + C\big)$. To prove (ii) we assume the contrary: there exists $x \in \operatorname{int}\big(\overline{A} + C\big)$, $x \notin \operatorname{int} A + C$. Since $x \in \overline{A} + C$ one can find $a \in \overline{A}$ and $c \in C$ such that $x = a + c$. Taking into account that $x \notin \operatorname{int} A + C$ we get $a \in \partial A$. According to Lemma 5.7 there exists $v \in \partial B_1$ such that $d(a + r^+ v, A) = r^+$ and $d(a - r^- v, X \setminus A) = r^-$. Since $x \notin \operatorname{int} A + C$ we have $x - C \subset X \setminus \operatorname{int} A = \overline{X \setminus A}$ and hence $d(a - r^- v, x - C) \geq d(a - r^- v, X \setminus A) = r_-$. On the other hand, $a \in x - C$ and $\|r^- v\| = r^-$. Consequently, $a \in P(a - r^- v, x - C)$ and hence $v \in N(x - a, C)$. Applying Lemma 6.1 we get $C \subset B_{r_C}(x - a - r_C v)$.

From $d(a+r^+v, A) = r^+$ it follows that $\overline{A} \subset \widetilde{B}_{r^+}(a+r^+v)$. So,

$$\overline{A} + C \subset \widetilde{B}_{r^+}(a+r^+v) + B_{r_C}(x-a-r_Cv) = \widetilde{B}_{r^+-r_C}(x-(r^+-r_C)v).$$

Since $x \in \partial \widetilde{B}_{r^+-r_C}(x-(r^+-r_C)v)$ it contradicts $x \in \operatorname{int}\big(\overline{A}+C\big)$.

(iii). Denote $D = A + C$. In view of (i) and (ii) we have

$$\overline{\operatorname{int} D} = \overline{\operatorname{int}(A+C)} \overset{\text{(ii)}}{=} \overline{\operatorname{int} A + C} \overset{\text{(i)}}{=} \overline{A+C} = \overline{D}, \tag{22}$$

$$\operatorname{int} \overline{D} = \operatorname{int}\big(\overline{A+C}\big) \overset{\text{(i)}}{=} \operatorname{int}\big(\overline{A}+C\big) \overset{\text{(ii)}}{=} \operatorname{int}(A+C) = \operatorname{int} D. \tag{23}$$

Using (ii) and (4) we get $X \setminus \operatorname{int} D = X \setminus (\operatorname{int} A + C) = (X \setminus \operatorname{int} A) \overset{*}{-} (-C)$.

Since $A \in \mathcal{WCC}(r^+, r^-)$ it follows that $(X \setminus \operatorname{int} A) \in \mathcal{PR}(r^-) \subset \mathcal{ESWC}(r^-)$ and by Theorem 5.5 we have $(X \setminus \operatorname{int} A) \overset{*}{-} (-C) \subset \mathcal{ESWC}(r^-)$.

Thus, $\overline{X \setminus D} = X \setminus \operatorname{int} D \subset \mathcal{ESWC}(r^-)$. This and (22), (23) by Theorem 5.9 imply that $D \in \mathcal{WCC}(r^+ - r_C, r^-)$. □

Theorem 6.5. *Let X be a uniformly convex Banach space, $r^- > 0$, $r^+ > 0$, $r_C \in (0, r^-)$. Let $A \in \mathcal{WCC}(r^+, r^-)$, $C \in \mathcal{SB}(r_C)$. Then*

(i) $\operatorname{int}\big(\overline{A} \overset{*}{-} C\big) = \operatorname{int}(A \overset{*}{-} C) = \operatorname{int} A \overset{*}{-} C$;

(ii) $\overline{\operatorname{int} A \overset{*}{-} C} = \overline{A \overset{*}{-} C} = \overline{A} \overset{*}{-} C$;

(iii) $A \overset{*}{-} C \in \mathcal{WCC}(r^+, r^- - r_C)$.

Proof. Define $A_1 = X \setminus A$ and $C_1 = -C$. Then, according to Remark 5.8 we have $A_1 \in \mathcal{WCC}(r^-, r^+)$. From the definition of the summand of the ball we obtain $C_1 \in \mathcal{SB}(r_C)$. So, Theorem 6.4 implies that

$$\begin{aligned} &\overline{\operatorname{int} A_1 + C_1} = \overline{A_1 + C_1} = \overline{A_1} + C_1; \\ &\operatorname{int}\big(\overline{A_1} + C_1\big) = \operatorname{int}(A_1 + C_1) = \operatorname{int} A_1 + C_1; \\ &A_1 + C_1 \in \mathcal{WCC}(r^- - r_C, r^+). \end{aligned} \tag{24}$$

In view of (4) we get $X \setminus \operatorname{int}\big(A \overset{*}{-} C\big) = \overline{X \setminus (A \overset{*}{-} C)} = \overline{X \setminus A + (-C)} = \overline{A_1 + C_1}$.

Similarly,

$$\begin{aligned} X \setminus \operatorname{int}\big(\overline{A} \overset{*}{-} C\big) &= \overline{\operatorname{int} A_1 + C_1}, \\ X \setminus (\operatorname{int} A \overset{*}{-} C) &= \overline{A_1} + C_1, \\ X \setminus (A \overset{*}{-} C) &= A_1 + C_1, \\ X \setminus \overline{\operatorname{int} A \overset{*}{-} C} &= \overline{A_1} + C_1, \\ X \setminus \overline{A \overset{*}{-} C} &= \operatorname{int}(A_1 + C_1), \\ X \setminus \big(\overline{A} \overset{*}{-} C\big) &= \operatorname{int} A_1 + C_1. \end{aligned}$$

Substituting the right hand sides of these equations for their left hand sides in (24) and using again Remark 5.8 we complete the proof. □

In the case X is a Hilbert space Theorems 6.4 and 6.5 were proved in [16]. Also for a Hilbert space Theorems 6.4 and 6.5 have been generalized in [20, 21] to nonlinear mappings instead of linear Minkowski operations.

Remark 6.6. Theorem 5.5 and Remark 5.6 show that the Minkowski difference preserves weak convexity in the sense of Efimov-Stechkin but does not preserve prox-regularity. On the other hand, Theorem 6.2 and Remark 6.3 show that the Minkowski sum with a summand of the ball of sufficiently small radius preserves prox-regularity but does not preserve weak convexity in the sense of Efimov-Stechkin. Theorems 6.4 and 6.5 show that both the Minkowski sum and the Minkowski difference with a summand of the ball of sufficiently small radius preserve weak convex-concavity.

Lemma 6.7. *Let X be a uniformly convex Banach space, $r^- > 0$ and $r^+ > 0$. If $A \in \mathcal{WCC}(r^+, r^-)$, $r \in (0, r^+)$, $A_r = A + B_r$, then $A_r \in \mathcal{WCC}(r^+ - r, r^- + r)$.*

Proof. Since $X \setminus \operatorname{int} A \in \mathcal{PR}(r^-) \subset \mathcal{ESWC}(r^-)$ by Lemma 5.2 there exists $D \subset X$ such that $X \setminus \operatorname{int} A = D \overset{*}{-} \operatorname{int} B_{r^-}$. In view of (4) we get $\operatorname{int} A = (X \setminus D) + \operatorname{int} B_{r^-}$.

By Theorem 6.4 we have $\operatorname{int} A_r = \operatorname{int} A + B_r$. Consequently,

$$\operatorname{int} A_r = (X \setminus D) + \operatorname{int} B_{r^-} + B_r = (X \setminus D) + \operatorname{int} B_{r+r^-}$$

and again in view of (4): $X \setminus \operatorname{int} A_r = D \overset{*}{-} \operatorname{int} B_{r+r^-}$.

According to Lemma 5.2 it means that $X \setminus \operatorname{int} A_r \in \mathcal{ESWC}(r^- + r)$.

Applying Theorem 6.4 we conclude that $\overline{A_r} \in \mathcal{PR}(r^+ - r) \subset \mathcal{ESWC}(r^+ - r)$, $\overline{\operatorname{int} A_r} = \overline{A_r}$, $\operatorname{int} \overline{A_r} = \operatorname{int} A_r$. Using Theorem 5.9 we complete the proof. $\square$

7. Proofs of the Theorems 3.1, 3.4, 3.7

Proof of Theorem 3.1. Define $\varrho_0 = \varrho_{\sup\sup}(C, A)$ and then for any $\varepsilon \in (0, \varrho_0)$, $A_\varepsilon = \{a \in A : \ f(a, C) > \varrho_0 - \varepsilon\}$. In view of (2), (5) we have

$$A_\varepsilon = A \cap T_{\varrho_0 - \varepsilon}(C) = A \cap (\operatorname{int} \widetilde{B}_{\varrho_0 - \varepsilon} + C).$$

Since $\operatorname{int} \widetilde{B}_{\varrho_0 - \varepsilon} = \widetilde{B}_{\varrho_0} + \operatorname{int} B_\varepsilon$ it follows that

$$A_\varepsilon = A \cap (G + \operatorname{int} B_\varepsilon), \tag{25}$$

where $G = \widetilde{B}_{\varrho_0} + C$. Taking into account that $\widetilde{B}_{\varrho_0} \in \mathcal{PR}(\varrho_0)$ by Theorem 6.2 we get $G \in \mathcal{PR}(\varrho_0 - r_C)$.

Suppose that there exists $x \in G \cap \operatorname{int} A$. Then one can find $\delta > 0$ such that $B_\delta(x) \subset A$. Since $x \in G = \widetilde{B}_{\varrho_0} + C = \widetilde{B}_{\varrho_0 + \delta} + B_\delta + C$ there exists $x' \in (\widetilde{B}_{\varrho_0 + \delta} + C) \cap B_\delta(x)$. Hence, $x' \in A$ and $\varrho_{\sup\sup}(C, A) \geq f(x', C) \geq \varrho_0 + \delta$, which contradicts the equality $\varrho_0 = \varrho_{\sup\sup}(C, A)$. This contradiction shows that $G \cap \operatorname{int} A = \emptyset$.

By the definition of $\varrho_{\sup\sup}(C, A)$ for any $\delta \in (0, \varrho_0)$ one can find $a^\delta \in A$ and $c^\delta \in C$ such that $\|a^\delta - c^\delta\| > \varrho_0 - \delta$. Since $a^\delta - c^\delta \in \widetilde{B}_{\varrho_0 - \delta} = \widetilde{B}_{\varrho_0} + B_\delta$ it follows that $a^\delta \in \widetilde{B}_{\varrho_0} + B_\delta + C = G + B_\delta$. Consequently, $\varrho_{\inf\inf}(G, A) \leq d(a^\delta, G) \leq \delta$.

Passing to the limit as $\delta \to +0$ we obtain $\varrho_{\inf\inf}(G, A) = 0$.

According to Proposition 4.3 we have

$$\lim_{\varepsilon \to +0} \operatorname{diam} P^{\varepsilon}(G, A) = 0. \tag{26}$$

Observe that

$$P^{\varepsilon}(G, A) \overset{(6)}{=} \{a \in A : \ d(a, G) < \varrho_{\inf\inf}(G, A) + \varepsilon\}$$
$$= \{a \in A : \ d(a, G) < \varepsilon\} \overset{(2)}{=} A \cap U_{\varepsilon}(G) \overset{(5)}{=} A \cap (G + \operatorname{int} B_{\varepsilon}) \overset{(25)}{=} A_{\varepsilon}.$$

This and (26) imply that
$$\lim_{\varepsilon \to +0} \operatorname{diam} A_{\varepsilon} = 0. \tag{27}$$

To complete the proof consider two sequences of elements $a_k \in A$ and $c_k \in C$ such that $\|a_k - c_k\| \to \varrho_0$. Denote $\varepsilon_k = \varrho_0 - \|a_k - c_k\|$. Then $f(a_k, C) \geq \|a_k - c_k\| = \varrho_0 - \varepsilon_k$ and hence $a_k \in A_{\varepsilon_k}$. It follows by (27) that $\{a_k\}$ is a Cauchy sequence and hence converges to some $\hat{a} \in X$. Similarly, $\{c_k\}$ converges to some $\hat{c} \in X$. By the closedness of A and C we have $\hat{a} \in A$, $\hat{c} \in C$. Note that $\|\hat{a} - \hat{c}\| = \varrho_0$. So, the pair $(\hat{a}, \hat{c})$ is a solution of $P_{\max\max}$ and $P_{\max\max}$ is Tykhonov well-posed. □

Proof of Theorem 3.4. For any $t \geq 0$ we define $D_t = B_t \overset{*}{-} (-C)$. Observe that

$$D_t = \bigcap_{\delta > 0} D_{t+\delta} \qquad \forall t \geq 0. \tag{28}$$

Indeed, by the second equation of (3) for any $t \geq 0$ we have

$$D_t = \bigcap_{c \in C} B_t(c) = \bigcap_{c \in C} \bigcap_{\delta > 0} B_{t+\delta}(c) = \bigcap_{\delta > 0} \bigcap_{c \in C} B_{t+\delta}(c) = \bigcap_{\delta > 0} D_{t+\delta}.$$

This proves (28).

By the definition of the antidistance for any $x \in X$ and $t \geq 0$ one has

$$f(x, C) \leq t \iff C \subset B_t(x) \iff x \in D_t. \tag{29}$$

Denote $\varrho_0 = \varrho_{\inf\sup}(A, C)$. Suppose at first that $\varrho_0 = 0$. Then for any $\delta > 0$ there exists $a_\delta \in A$ such that $f(a_\delta, C) < \delta$. Hence $\operatorname{diam} C < 2\delta$ for any $\delta > 0$, i.e. C is a singleton: $C = \{c_0\}$. In this case $P_{\min\max}$ is reduced to $\min_{a \in A} \|a - c_0\|$. Since $d(c_0, A) = 0$ any minimizing sequence of the problem converges to c_0. So, $P_{\min\max}$ is Tykhonov well-posed in this case.

Hereinafter we suppose that $\varrho_0 > 0$. For all $\delta > 0$ we have $\varrho_0 - \delta < f(a, C)$ for all $a \in A$ and there exists $a_\delta \in A$ such that $f(a_\delta, C) < \varrho_0 + \delta$. According to (29) this implies that

$$\forall \delta \in (0, \varrho_0) \qquad A \cap D_{\varrho_0 - \delta} = \emptyset \quad \text{and} \quad A \cap D_{\varrho_0 + \delta} \neq \emptyset. \tag{30}$$

Consider the case $\inf_{\delta > 0} \operatorname{diam} D_{\varrho_0 + \delta} = 0$. In this case by (28) we deduce that D_{ϱ_0} is a singleton $\{x_0\}$ and $x_0 \in A$. In view of (29) any minimizing sequence of $P_{\min\max}$ converges to x_0 and $P_{\min\max}$ is Tykhonov well-posed again.

Now we assume that $\inf_{\delta>0} \operatorname{diam} D_{\varrho_0+\delta} = d > 0$. For any $\delta \in (0, \varrho_0)$ we have $D_{\varrho_0+\delta} = B_{\varrho_0+\delta} \overset{*}{-} (-C) = B_{2\varrho_0} \overset{*}{-} (B_{\varrho_0-\delta} + (-C))$ and hence $D_{\varrho_0+\delta}$ is R-strongly convex with $R = 2\varrho_0$. According to Lemma 4.6 for any $\delta \in (0, \varrho_0)$ there exists $z_\delta \in X$ such that

$$B_\sigma(z_\delta) \subset D_{\varrho_0+\delta} \subset B_{2R}(z_\delta),$$

where $\sigma = R\delta_X\left(\frac{d}{2R}\right)$. Denote $\delta_0 = \min\{\varrho_0, \sigma\}$. In view of [19, Lemma 7.1] for any $\delta \in (0, \delta_0)$ and for $\varepsilon(\delta) = \frac{2R\delta}{\sigma}$ we have $D_{\varrho_0+\delta} \subset D_{\varrho_0+\delta} \overset{*}{-} B_\delta + B_{\varepsilon(\delta)}$. Taking into account that $D_{\varrho_0+\delta} \overset{*}{-} B_\delta = D_{\varrho_0}$ we obtain

$$D_{\varrho_0+\delta} \subset D_{\varrho_0} + B_{\varepsilon(\delta)} \qquad \forall \delta \in (0, \delta_0). \tag{31}$$

If there is $a \in A \cap \operatorname{int} D_{\varrho_0}$, then one can find $\delta \in (0, \varrho_0)$ such that $B_\delta(a) \subset D_{\varrho_0}$ and hence $a \in D_{\varrho_0} \overset{*}{-} B_\delta = D_{\varrho_0-\delta}$. This contradicts the first equation in (30).

Consequently, $$A \cap \operatorname{int} D_{\varrho_0} = \emptyset.$$

By (31) and the second equation in (30) it follows that

$$A \cap (D_{\varrho_0} + B_{\varepsilon(\delta)}) \neq \emptyset \qquad \forall \delta \in (0, \delta_0).$$

Hence, $\varrho_{\inf\inf}(D_{\varrho_0}, A) \leq \varepsilon(\delta)$ for all $\delta \in (0, \delta_0)$. Passing to the limit as $\delta \to +0$, we obtain

$$\varrho_{\inf\inf}(D_{\varrho_0}, A) = 0.$$

Since D_{ϱ_0} is ϱ_0-strongly convex and X is a Hilbert space it follows by [3, Theorem 2.6] that $D_{\varrho_0} \in \mathcal{SB}(\varrho_0)$. Applying Proposition 4.3 we get

$$\lim_{\varepsilon \to +0} \operatorname{diam} P^\varepsilon(D_{\varrho_0}, A) = 0. \tag{32}$$

Let $\{a_k\}$ be a minimizing sequence of $P_{\min\max}$. Denote $\delta_k = f(a_k, C) - \varrho_0$. Then $\delta_k \to 0$ and by (29) we have $a_k \in D_{\varrho_0+\delta_k}$. Using (31) for sufficiently large k we obtain $a_k \in D_{\varrho_0} + B_{\varepsilon(\delta_k)} \subset U_{2\varepsilon(\delta_k)}(D_{\varrho_0})$. Hence $a_k \in P^{2\varepsilon(\delta_k)}(D_{\varrho_0}, A)$ and according to (32) $\{a_k\}$ is a Cauchy sequence. Consequently, $\{a_k\}$ converges to a minimizer of $P_{\min\max}$. $\square$

Proof of Theorem 3.7. Observe that $A \in \mathcal{WCC}(r^+, r^-)$. Denote $\varrho_0 = \varrho_{\sup\inf}(C, A)$, $Y = X \setminus (\operatorname{int} A + B_{\varrho_0})$. According to Lemma 6.7 we have

$$(\operatorname{int} A + B_{\varrho_0}) \in \mathcal{WCC}(r^+ - \varrho_0, r^- + \varrho_0).$$

Using Remark 5.8, we get $Y \in \mathcal{WCC}(r^- + \varrho_0, r^+ - \varrho_0)$.

Suppose that there exists $c_0 \in Y \cap \operatorname{int} C$. Since $c_0 \in Y = X \setminus (\operatorname{int} A + B_{\varrho_0})$ it follows that $d(c_0, \operatorname{int} A) \geq \varrho_0$. Taking into account that $A = \overline{\operatorname{int} A}$ we obtain $d(c_0, A) \geq \varrho_0$. On the other hand, $c_0 \in C$ and hence $d(c_0, A) \leq \sup_{c\in C} d(c, A) = \varrho_0$.

So, $d(c_0, A) = \varrho_0$. By Lemma 4.7 there exists $c' \in X$ sufficiently close to $c_0 \in \operatorname{int} C$ so that $c' \in C$ and $d(c', A) > d(c_0, A) = \varrho_0$. Thus, $\varrho_0 = \sup_{c\in C} d(c, A) \geq d(c', A) > \varrho_0$. This contradiction shows that $Y \cap \operatorname{int} C = \emptyset$.

Since $\varrho_0 = \sup_{c\in C} d(c, A)$ there exists a sequence of points $c_k \in C$ such that $d(c_k, A) \to \varrho_0$. By Lemma 4.7 there exists a sequence $c'_k \in X$ such that $d(c'_k, A) > \varrho_0$ and $c_k - c'_k \to 0$. Hence $c_k \in Y$ and

$$\varrho_{\inf\inf}(Y, C) \leq \lim_{k\to\infty} \|c_k - c'_k\| = 0 < r^- + \varrho_0 - r_C.$$

By Proposition 4.3:
$$\lim_{\varepsilon \to +0} \operatorname{diam} P^{\varepsilon}(C, Y) = 0. \tag{33}$$

Let $\{c_k\}$ be a maximizing sequence of $P_{\max\min}$. Denote $\varepsilon_k = \varrho_0 - d(c_k, A)$. Then $\varepsilon_k \to 0$ and for sufficiently large k one has $c_k \notin A + \operatorname{int} B_{\varrho_0 - \varepsilon_k} = A + \operatorname{int} B_{\varrho_0} \overset{*}{-} B_{\varepsilon_k}$. This means that $c_k \in Y + B_{\varepsilon_k}$. So, $c_k \in P^{\varepsilon_k}(C, Y)$ and according to (33) $\{c_k\}$ is a Cauchy sequence. Consequently, $\{c_k\}$ converges to a maximizer of $P_{\max\min}$. □

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