Journal of Convex Analysis
Volume 29 (2022), No. 3, 789–794

On the Convexity of the Closure of the Domain/Range of a Maximally Monotone Operator

Maria E. Verona
Department of Mathematics, University of Southern California, Los Angeles, U.S.A.
verona@usc.edu

Andrei Verona
Department of Mathematics, California State University, Los Angeles, U.S.A.
averona10@yahoo.com

Received: July 28, 2021
Accepted: November 1, 2021

We prove that a sufficient condition (due to Simons) for the convexity of the closure of the domain/range of a monotone operator is also necessary when the operator has bounded domain/range and is maximal.

Keywords: Monotone operator, maximally monotone operator, Fitzpatrick function, Simons' conditions, type (FP).

2010 Mathematics Subject Classification: Primary 47H05; secondary 47A65.

1. Introduction

Throughout this note X will denote a non zero Banach space with topological dual X^*. For $(x, x^*) \in X \times X^*$, $\langle x, x^* \rangle$ (or $\langle x^*, x \rangle$) will denote the natural evaluation map. If $T : X \rightrightarrows X^*$ is a (multivalued) operator then

$$G(T) = \{(x, x^*) \in X \times X^* : x^* \in T(x)\}$$

is called its *graph*, $D_T = \{x \in X : T(x) \neq \emptyset\}$ is called its *domain* and

$$R_T = \{x^* \in X^*; \text{ there exists } x \in X \text{ such that } x^* \in T(x)\}$$

is called its *range*. The operator T is called *monotone* if $\langle x^* - y^*, x - y \rangle \geq 0$ for all $(x, x^*), (y, y^*) \in G(T)$; T is called *maximally monotone* if there exists no monotone operator S such that $G(S)$ strictly contains $G(T)$. A pair $(x, x^*) \in X \times X^*$ is called *monotonically related to* the monotone operator T if $\langle x^* - z^*, x - z \rangle \geq 0$ for any $(z, z^*) \in G(T)$. If C is a convex subset of X and $x \in C$ we shall denote by $N_C(x)$ the normal cone to C at x, i.e.

$$N_C(x) = \{x^* \in X^*; \langle x^*, y - x \rangle \leq 0, \text{ for all } y \in C\}.$$

Recall that $N_C(x)$ is the *subdifferential* at x of the indicator function of C.

Throughout this note we shall assume that all Banach spaces are non trivial.

ISSN 0944-6532 / \$ 2.50

2. On the convexity of the closure of the domain of a maximally monotone operator

Recall that the *Fitzpatrick function* $\varphi_T : X \times X^* \to (-\infty, \infty]$ of a monotone operator T is defined by

$$\varphi_T(x, x^*) = \sup\{\langle x^* - z^*, z - x\rangle; (z, z^*) \in G(T)\} + \langle x^*, x\rangle.$$

It is a proper, lower semicontinuous convex function. The following facts follow immediately from the definition:

$$D_T \subset \operatorname{co} D_T \subset \pi_1 \operatorname{dom} \varphi_T, \quad R_T \subset \operatorname{co} R_T \subset \pi_2 \operatorname{dom} \varphi_T. \tag{1}$$

where *co* represents the convex hull and $\pi_1 : X \times X^* \to X$, $\pi_2 : X \times X^* \to X^*$ are the projections.

One of the outstanding open problems in convex analysis is whether the closure of the domain of a maximally monotone operator in an arbitrary Banach space is convex. Rockafellar [5] showed that this is true when the domain of the operator has non-empty interior. A summary of what is known about this problem can be found in [1]. In [6, Theorem 27.5] Simons proved the following result:

Fact 2.1. *Let X be a Banach space and $T\colon X \rightrightarrows X^*$ be a monotone operator satisfying*

$$x \notin \overline{D_T} \quad \textit{implies that} \quad \sup\left\{\frac{\langle z^*, x - z\rangle}{||x - z||}; z^* \in T(z)\right\} = \infty. \tag{2}$$

Then:

(a) $\pi_1 \operatorname{dom} \varphi_T \subset \overline{D_T}$.

(b) $\overline{D_T} = \overline{\operatorname{co} D_T} = \overline{\pi_1(\operatorname{dom} \varphi_T)}$.

(c) $\overline{D_T}$ *is convex.*

Remark. In fact, part (a) is the important one; part (b) follows immediately from it and (1). Obviously (b) implies (c). □

In [1, Theorem 3.6] Borwein and Yao proved the following result:

Fact 2.2. *Let $T : X \rightrightarrows X^*$ be a maximally monotone operator. Then we have*

$$\overline{\operatorname{co} D_T} = \overline{\pi_1(\operatorname{dom} \varphi_T)}.$$

We shall use these results to prove a converse to Simons' result in the case when D_T is bounded and T is maximal.

Theorem 2.3. *Let X be a Banach space and $T : X \rightrightarrows X^*$ be a monotone operator with bounded domain. Consider the following conditions:*

(i) $x \notin \overline{D_T}$ *implies that* $\sup\left\{\frac{\langle z^*, x - z\rangle}{||x - z||}; z^* \in T(z)\right\} = \infty$.

(ii) $\pi_1 \operatorname{dom} \varphi_T \subset \overline{D_T}$.

(iii) $\overline{D_T} = \overline{\operatorname{co} D_T} = \overline{\pi_1 \operatorname{dom} \varphi_T}$.

(iv) $\overline{D_T}$ *is convex.*

Then the first three conditions are equivalent and imply the fourth one. If T is maximally monotone all four conditions are equivalent.

Proof. In view of Fact 2.1, we only need to prove that (iii) implies (i) and, when T is maximally monotone, that (iv) implies (iii). To this end let $x \notin \overline{D_T}$ and assume that

$$\sup\left\{\frac{\langle z^*, x-z\rangle}{||x-z||};\, z^* \in T(z)\right\} < \infty.$$

It follows that there exists $M > 0$ such that $\langle z^*, x-z\rangle \le M\|x-z\|$ for any $(z, z^*) \in G(T)$. Since D_T is bounded,

$$\varphi_T(x, 0^*) = \sup\{\langle z^*, x-z\rangle; (z, z^*) \in G(T)\} \le M \sup_{z\in D_T} \|x-z\| < \infty,$$

hence $x \in \pi_1(\operatorname{dom}\varphi_T)$. If (iii) is true, $x \in \overline{D_T}$ and this contradiction proves that (iii) implies (i).

If T is maximally monotone, from Fact 2.2 it follows that $\overline{\operatorname{co} D_T} = \overline{\pi_1(\operatorname{dom}\varphi_T)}$. Since $\overline{D_T}$ is convex, $\overline{\operatorname{co} D_T} = \overline{D_T}$ and thus (iii) is proved. □

Theorem 2.4. *Let $T : X \rightrightarrows X^*$ be a monotone operator with the following property: there exists a family of closed, convex and bounded sets $\{C_\alpha\}$ such that*

$$\bigcup \operatorname{int}(C_\alpha) = X,\ D_T \cap \operatorname{int}(C_\alpha) \ne \emptyset,$$

and $T + N_{C_\alpha}$ is maximally monotone for each α. Then

(a) *T is maximally monotone.*

(b) *Conditions* (i), (ii), (iii), *and* (iv) *from Theorem* 2.3 *are equivalent.*

Proof. (a) Assume that (x, x^*) is monotonically related with T. Choose α such that $x \in \operatorname{int}(C_\alpha)$ and let $z \in D_T \cap C_\alpha$. Let also $z^* \in T(z)$ and $u^* \in N_{C_\alpha}(z)$. Then

$$\langle x^* - z^* - u^*, x - z\rangle = \langle x^* - z^*, x - z\rangle + \langle u^*, z - x\rangle \ge 0 + 0 = 0$$

which means that (x, x^*) is monotonically related with $T + N_{C_\alpha}$ which is maximally monotone. Thus $x \in D_T \cap C_\alpha$ and $x^* \in T(x) + N_{C_\alpha}(x)$. Since $N_{C_\alpha}(x) = \{0^*\}$, assertion (a) is proved.

(b) Since (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is is enough to prove that (iv) implies (i). So, assume that $\overline{D_T}$ is convex and choose $x \notin \overline{D_T}$. Let α be such that $x \in \operatorname{int}(C_\alpha)$. Then $\overline{D_{T+N_{C_\alpha}}} = \overline{D_T \cap C_\alpha} = \overline{D_T} \cap C_\alpha$ is convex, bounded and $x \notin \overline{D_{T+N_{C_\alpha}}}$. We have

$$\sup\left\{\frac{\langle z^*, x-z\rangle}{||x-z||};\, z\in D_T,\, z^*\in T(z)\right\} \ge \sup\left\{\frac{\langle z^*, x-z\rangle}{||x-z||};\, z\in D_T\cap C_\alpha,\, z^*\in T(z)\right\}$$

$$\ge \sup\left\{\frac{\langle z^*+u^*, x-z\rangle}{||x-z||};\ z\in D_T\cap C_\alpha,\ z^*\in T(z),\ u^*\in N_{C_\alpha}(z)\right\} = \infty$$

the last equality because Theorem 2.3 applies to $T + N_{C_\alpha}$. This proves (b). □

3. On the convexity of the closure of the range of a maximally monotone operator

In [6, Theorem 27.6] Simons also proved a result dual to Fact 2.1.

Fact 3.1. *Let X be a Banach space and $T : X \rightrightarrows X^*$ be a monotone operator satisfying*

$$x^* \notin \overline{R_T} \ \text{ implies that } \ \sup\left\{\frac{\langle x^* - z^*, z\rangle}{\|x^* - z^*\|};\ z^* \in T(z)\right\} = \infty. \tag{3}$$

Then:

(a) $\pi_2 \operatorname{dom} \varphi_T \subset \overline{R_T}$.

(b) $\overline{R_T} = \overline{\operatorname{co} R_T} = \overline{\pi_2(\operatorname{dom} \varphi_T)}$.

(c) $\overline{R_T}$ *is convex.*

In [2, Theorem 4.23] Borwein and Yao proved a result dual to Fact 2.2. Here it is:

Fact 3.2. *Let $T : X \rightrightarrows X^*$ be a maximally monotone operator.*
Then $\overline{\operatorname{co} R_T}^{w^*} = \overline{\pi_2(\operatorname{dom} \varphi_T)}^{w^*}$, *where w^* indicates the weak*-closure.*

Finally, Theorem 2.3 has a dual counterpart which we now state.

Theorem 3.3. *Let X be a Banach space and $T : X \rightrightarrows X^*$ be a monotone operator with bounded range. Consider the following conditions:*

(i) $x^* \notin \overline{R_T}$ *implies that* $\sup\left\{\frac{\langle x^* - z^*, z\rangle}{\|x^* - z^*\|};\ z^* \in T(z)\right\} = \infty$.

(ii) $\pi_2(\operatorname{dom} \varphi_T) \subset \overline{R_T}$.

(iii) $\overline{R_T} = \overline{\operatorname{co} R_T} = \overline{\pi_2(\operatorname{dom} \varphi_T)}$.

(iv) $\overline{R_T}$ *is convex.*

(i′) $x^* \notin \overline{R_T}^{w^*}$ *implies that* $\sup\left\{\frac{\langle x^* - z^*, z\rangle}{\|x^* - z^*\|};\ z^* \in T(z)\right\} = \infty$.

(ii′) $\pi_2(\operatorname{dom} \varphi_T) \subset \overline{R_T}^{w^*}$.

(iii′) $\overline{R_T}^{w^*} = \overline{\operatorname{co} R_T}^{w^*} = \overline{\pi_2(\operatorname{dom} \varphi_T)}^{w^*}$.

(iv′) $\overline{R_T}^{w^*}$ *is convex.*

Then

(1) (i), (ii) *and* (iii) *are equivalent and imply* (iv)*;*

(2) (i′), (ii′) *and* (iii′) *are equivalent and imply* (iv′)*;*

(3) (i) $\Rightarrow$ (i′); (ii) $\Rightarrow$ (ii′); (iv) $\Rightarrow$ (iv′);

(4) *if T is maximally monotone then* (iv′) $\Rightarrow$ (i′) *and thus* (i′), (ii′), (iii′), *and* (iv′) *are equivalent.*

Proof. (1) From Fact 3.1 it follows that (i) $\Rightarrow$ (ii). Clearly (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). Exactly as in the proof of Theorem 2.3, we can show that (iii) $\Rightarrow$ (i).

(2) We begin by showing (by contradiction) that (i′) $\Rightarrow$ (ii′). Assume that there exists $(x, x^*) \in \operatorname{dom} \varphi_T$ but $x^* \notin \overline{R_T}^{w^*}$. Then $x^* \notin \overline{R_T}$ and thus there exists $r > 0$ such that $\|z^* - x^*\| \geq r$ for any $z^* \in R_T$. Since $\varphi_T(x, x^*)$ is finite and R_T is bounded it is easily seen that $\varphi_T(0, x^*)$ is also finite.

We have

$$\sup\left\{\frac{\langle x^*-z^*,z\rangle}{\|x^*-z^*\|};\ z^*\in T(z)\right\}\leq \sup\left\{\frac{\langle x^*-z^*,z\rangle}{r};\ z^*\in T(z)\right\}\leq \varphi_T(0,x^*)/r<\infty,$$

contradicting (i′).

(ii′) and (iii′) are clearly equivalent and, exactly as in the proof of Theorem 2.3, we can show that (iii′) $\Rightarrow$ (i′).

(3) The first two implications are obvious since the norm topology is finer than the w^*-topology. To prove the third one, assume that $\overline{R_T}$ is convex. Then

$$\operatorname{co} R_T \subset \overline{R_T} \subset \overline{R_T}^{w^*}$$

and it follows that

$$\overline{R_T}^{w^*} \subset \overline{\operatorname{co} R_T}^{w^*} \subset \overline{(\overline{R_T})}^{w^*} \subset \overline{(\overline{R_T}^{w*})}^{w^*} = \overline{R_T}^{w^*}.$$

Therefore $\overline{R_T}^{w^*} = \overline{\operatorname{co} R_T}^{w^*}$, showing that $\overline{R_T}^{w^*}$ is convex.

(4) When T is maximally monotone, exactly as in the proof of (iv) $\Rightarrow$ (i) in Theorem 2.3 (but using Fact 3.2 instead of Fact 2.3), we can show that we have (iv′) $\Rightarrow$ (i′). □

Finally, we have

Theorem 3.4. *Let $T: X \rightrightarrows X^*$ be a maximally monotone operator with bounded domain. Then T is of type* (FP) *if and only if it satisfies Simons' condition* (3).

Proof. In the proof of Theorem 43.1 in [6] Simons showed that an operator of type (FP) satisfies condition (3). To prove the converse assertion, notice first that

$$\operatorname{dom}\varphi_T = \pi_1(\operatorname{dom}\varphi_T)\times X^*;\quad \text{in particular}\quad \pi_2(\operatorname{dom}\varphi_T) = X^*. \tag{4}$$

Indeed, let $(x,x^*)\in\operatorname{dom}\varphi_T$, i.e. $\varphi_T(x,x^*)<\infty$ and let $y^*\in X^*$. Then

$$\begin{aligned}\varphi_T(x,y^*) &= \sup\{\langle y^*-z^*,z-x\rangle;(z,z^*)\in G(T)\}+\langle x^*,x\rangle\\ &\leq \sup\{\langle x^*-z^*,z-x\rangle;(z,z^*)\in G(T)\}\\ &\qquad +\sup\{\langle y^*-x^*,z-x\rangle; z\in D_T\}+\langle x^*,x\rangle\\ &\leq \varphi_T(x,x^*)+\|y^*-x^*\|\sup\{\|z-x\|;z\in D_T\}<\infty\end{aligned}$$

and thus $(x,y^*)\in\operatorname{dom}\varphi_T$, proving (4). In view of Fact 3.1, $\overline{R_T}=X^*$. The assertion follows from Theorem 4.8 in [4]. □

References

[1] J. M. Borwein, L. Yao: *Some results on the convexity of the closure of the domain of a maximally monotone operator*, Optimization Letters 8 (2014) 237–246.

[2] J. M. Borwein, L. Yao: *Recent progress on monotone operator theory*, in: *Infinite Products of Operators and Their Applications*, Haifa 2012, S. Reich et al. (eds.), Contemporary Mathematics 636, American Mathematical Society, Providence (2015) 51–81.

[3] S. P. Fitzpatrick: *Representing monotone operators by convex functions*, in: Workshop/Miniconference on Functional Analysis and Optimization, Canberra 1988, Proc. Centre Math. Analysis Vol. 20, Australian National University, Canberra (1988) 59–65.

[4] R. Phelps: *Lectures on Maximal Monotone Operators*, 2nd Summer School on Banach Spaces, Related Areas and Applications, Prague and Paseky, August 15-28 (1993).

[5] R. T. Rockafellar: *Local boundedness of nonlinear, monotone operators*, Michigan Math. J. 16 (1969) 397–407.

[6] S. Simons: *From Hahn-Banach to Monotonicity*, 2nd ed., Lecture Notes in Mathematics 1693, Springer, Berlin (2008).