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Residuality Properties of Certain Classes of Convex Functions on Normed Linear Spaces

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We study the existence of residual sets of certain classes of convex functions on a normed linear space. Such sets play an important role in many optimization algorithms for which elements of a certain class can be used as good approximants of given convex functions and thus the abundance of such elements is crucial for the aforementioned algorithms.

Keywords: Baire category, convex function, local uniform convexity, lower semi-continuity, normed space, residual set, strict convexity, uniformity.

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1. Introduction and background

Let K be a nonempty convex subset of a normed linear space $(X, \|\cdot\|)$. We denote by $B(r)$ the open ball in $(X, \|\cdot\|)$ of center zero and radius $r > 0$. Let $\mathfrak{M}$ be the set of all convex functions $f : K \to \mathbb{R}$. Denote by $\mathfrak{M}_l$ the subset of all lower semi-continuous functions $f \in \mathfrak{M}$, by $\mathfrak{M}_c$ the subset of all continuous functions $f \in \mathfrak{M}$ and by $\mathfrak{M}_b$ the subset of all functions $f \in \mathfrak{M}$ which are bounded on bounded subsets of K. We equip the set $\mathfrak{M}$ with the topology induced by the uniformity determined by the following basis:

$$E(n) = \left\{(f, g) \in \mathfrak{M} \times \mathfrak{M} : |f(x) - g(x)| < n^{-1} \ \forall x \in K \cap B(n)\right\}, \tag{1}$$

where n is a positive integer. This topology will be denoted by τ. It is not difficult to see that this uniform space is completely metrizable. Clearly, $\mathfrak{M}_l$, $\mathfrak{M}_c$ and $\mathfrak{M}_b$ are closed subsets of $\mathfrak{M}$ with respect to this uniform topology. We provide these subspaces with the relative topologies inherited from $\mathfrak{M}$. This topology will be called the *strong topology* on $\mathfrak{M}_l$ and will be denoted by τ_1. More information on uniformities and uniform spaces can be found, for example, in [5].

We next describe the second topology with which we equip $\mathfrak{M}_l$. Recall that the *epigraph* of a function $f : K \to \mathbb{R}$ is the set

$$\mathrm{epi}(f) = \{(x, t) \in K \times \mathbb{R} : t \geq f(x)\}.$$

Let $\|\cdot\|_\infty$ be the norm defined on $X \times \mathbb{R}$ by $\|(x, t)\|_\infty := \max\{\|x\|, |t|\}$ for each point $(x, t) \in X \times \mathbb{R}$. Denote by $B_{\|\cdot\|_\infty}(r)$ the open ball in $(X \times \mathbb{R}, \|\cdot\|)$ of center zero and radius $r > 0$. Define the distance from $\tilde{x} = (x, t)$ to A by

$$\rho(\tilde{x}, A) := \inf_{(a,s)\in A} \{\|(x, t) - (a, s)\|_\infty\} \tag{2}$$

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for each $\tilde{x} = (x, t) \in X \times \mathbb{R}$ and each nonempty set $A \subset X \times \mathbb{R}$. We define the Attouch-Wets metric d_{AW} on $\mathfrak{M}_l$ by using distances to the epigraphs of f and g in $X \times \mathbb{R}$ as follows:

$$d_{AW}(f, g) := \sum_{n=1}^{\infty} 2^{-n} \min \left\{ 1, \sup_{\tilde{x} \in B_{\|\cdot\|_\infty}(n)} \left| \rho(\tilde{x}, \mathrm{epi}(f)) - \rho(\tilde{x}, \mathrm{epi}(g)) \right| \right\}. \tag{3}$$

Since epi (f) is closed in $K \times \mathbb{R}$ for each $f \in \mathfrak{M}_l$, we see that d_{AW} is indeed a metric on $\mathfrak{M}_l$. We denote by τ_2 the topology induced by the metric d_{AW} on $\mathfrak{M}_l$ and by $B_{AW}(g, r)$ the open ball in $(\mathfrak{M}_l, d_{AW})$ of center $g \in \mathfrak{M}_l$ and radius $r > 0$. This topology will be called the *weak topology* on $\mathfrak{M}_l$. In the sequel we show that τ_2 is indeed weaker than τ_1.

The τ_2 topology can be defined differently. Consider the topology induced by the uniformity determined by the basis

$$F(n) = \left\{ (f, g) \in \mathfrak{M}_l \times \mathfrak{M}_l : \begin{array}{l} |\rho(\tilde{x}, \mathrm{epi}(f)) - \rho(\tilde{x}, \mathrm{epi}(g))| < n^{-1} \\ \text{for each } \tilde{x} \in B_{\|\cdot\|_\infty}(n) \end{array} \right\}, \tag{4}$$

where n is a positive integer. It turns out that this topology is the same as the τ_2 topology, as we show in the following lemma. We refer the reader to [1] for more information concerning Attouch-Wets topologies and metrics.

Lemma 1.1. *The τ_3 topology on $\mathfrak{M}_l$ induced by the uniformity determined by the basis defined in* (4) *coincides with the τ_2 topology.*

Proof. Let the set U be open in τ_3 and let $g \in U$. There exists a positive integer N such that for each $f \in \mathfrak{M}_l$, if $(f, g) \in F(N)$, then $f \in U$. In order to show that U is open in τ_2, it is enough to prove that there exists $\varepsilon > 0$ such that whenever $f \in \mathfrak{M}_l$ satisfies $d_{AW}(f, g) < \varepsilon$, we have $(f, g) \in F(N)$. Set $\varepsilon_0 := \min\{N^{-1}, \sum_{n=N}^{\infty} 2^{-n}\}$ and set $\varepsilon := \varepsilon_0^2$. Assume $f \in \mathfrak{M}_l$ is such that $d_{AW}(f, g) < \varepsilon$. By the definition of ε_0 and by (3), there exists a natural number $N_0 \geq N$ such that

$$\sup_{\tilde{x} \in B_{\|\cdot\|_\infty}(N_0)} |\rho(\tilde{x}, \mathrm{epi}(f)) - \rho(\tilde{x}, \mathrm{epi}(g))| < \varepsilon_0.$$

Therefore for each $\tilde{x_0} \in B_{\|\cdot\|_\infty}(N)$, we have

$$|\rho(\tilde{x_0}, \mathrm{epi}(f)) - \rho(\tilde{x_0}, \mathrm{epi}(g))| < N^{-1},$$

and hence $(f, g) \in F(N)$.

Conversely, assume that U is open in τ_2 and $g \in U$. There exists a number $\varepsilon > 0$ such that $B_{AW}(g, \varepsilon) \subset U$. In order to show that U is open in τ_3, it is enough to prove that there exists a positive integer N such that whenever $f \in \mathfrak{M}_l$ satisfies $(f, g) \in F(N)$, we have $d_{AW}(f, g) < \varepsilon$. Choose a positive integer N such that $N^{-1} < 2^{-1}\varepsilon$ and $\sum_{n=N+1}^{\infty} 2^{-n} < 2^{-1}\varepsilon$. Assume $f \in \mathfrak{M}_l$ is such that $(f, g) \in F(N)$. Using (3), we obtain that $d_{AW}(f, g) < \varepsilon$, as required. $\square$

Recall that a subset Z of a topological space Y is called *residual* if it contains a countable intersection of open and dense subsets of Y. In the case where the space Y is completely pseudo-metrizable, the Baire category theorem guarantees that Z is also dense. In our work we are interested in the residual sets of some classes of convex functions defined on normed spaces. We call elements of such sets *generic elements*. For more applications of such a generic approach, see, for example, [8]. We recall the following well-known result due to Alexandrov and Hausdorff (see Theorem 1.1 in [7] and references therein). It is useful in the quest for residual sets.

Theorem 1.2. *A metrizable space X is completely metrizable if and only if it is a G_δ subset of a complete metric space.*

For a general vector space V with a seminorm p we denote by $B_p(x_0, \varepsilon)$ the open ball in (V, p) of center $x_0 \in V$ and radius $\varepsilon > 0$. Recall that a topological vector space V with the topology T is called a *locally convex space* if there exists a family $\mathscr{P}$ of seminorms on V such that the family of open balls $\{B_p(x_0,) : x_0 \in V, \varepsilon > 0, p \in \mathscr{P}\}$ is a subbasis for T and $\cap_{p\in P} Z_p = \{0\}$, where $Z_p = \{x \in V : p(x) = 0\}$ for each $p \in P$. Note that the condition $\cap_{p\in P} Z_p = \{0\}$ is equivalent to the space V being Hausdorff with respect to the topology T. Clearly, a normed space (as a topological vector space with respect to its norm) is a locally convex space. In the sequel we use the following result (see Theorem 3.9 in [4]).

Theorem 1.3. *Let V be a real locally convex topological vector space, and let A and B be two disjoint closed and convex subsets of V. If either A or B is compact, then A and B are strictly separated, that is, there is $\alpha \in \mathbb{R}$ and a continuous linear functional $\phi : V \to \mathbb{R}$ such that $\phi(a) > \alpha$ for each $a \in A$ and $\phi(b) < \alpha$ for each $b \in B$.*

We also use the following results which bring out the connection between boundedness on bounded sets of a convex function and its uniform continuity on bounded sets. We prove them for the convenience of the reader.

Lemma 1.4. *Assume $f : X \to \mathbb{R}$ is a convex function which is bounded on $B(r)$ for some $r > 0$. For each $0 < r' < r$, set $R = (4 \sup_{w\in B(r)} |f(w)| + 1)/(r - r')$. Then f satisfies a Lipschitz condition on $B(r')$ with a Lipschitz constant R, that is,*

$$|f(x) - f(y)| \le R \|x - y\| \tag{5}$$

for each $x, y \in B(r')$.

Proof. Let $0 < r' < r$ and $x, y \in B(r')$. Without loss of generality we may assume $x \neq y$. By the mean value theorem, there exists $t_0 > 1$ such that

$$\|x + t_0(y - x)\| = 2^{-1}(r + r').$$

Set $z := x + t_0(y - x)$. Then $t_0 = \frac{\|z-x\|}{\|y-x\|} \neq 0$, $y = t_0^{-1} z + \left(1 - t_0^{-1}\right) x$ and by the convexity of f we have

$$f(y) - f(x) \le t_0^{-1}(f(z) - f(x)) = \frac{\|y - x\|}{\|z - x\|}(f(z) - f(x)).$$

By the triangle inequality, $\|z-x\| > 2^{-1}(r-r')$ and therefore

$$f(y)-f(x) \leq \frac{4\sup_{w\in B(r)}|f(w)|}{r-r'}\|y-x\| \leq R\|y-x\|.$$

Since this is true for each $x, y \in B(r')$, the proof is complete. □

Corollary 1.5. *Assume $f : X \to \mathbb{R}$ is a convex function. Then f is bounded on bounded sets if and only if it is uniformly continuous on bounded subsets of X.*

Proof. If f is bounded on bounded sets, then it follows from Lemma 1.4 that f satisfies a Lipschitz condition on bounded sets and therefore is certainly uniformly continuous on bounded sets.

Conversely, assume that K is a nonempty and convex subset of X, and assume that $f : K \to \mathbb{R}$ is a uniformly continuous function on bounded subsets of K. Without loss of generality we may assume $0 \in K$. It is sufficient to show that f is bounded on $K \cap B(r)$ for each $r > 0$. Since f is uniformly continuous on $K \cap B(r)$, there exists $\delta > 0$ such that $|f(x) - f(y)| < 1$ for each $x, y \in K \cap B(r)$ satisfying $\|x - y\| < \delta$. Choose a positive integer n such that $n^{-1}r < \delta$. Then for each $x \in K \cap B(r)$, we have

$$f(x)-f(0) = \sum_{k=1}^{n}\left(f\left(\frac{k}{n}x\right) - f\left(\frac{k-1}{n}x\right)\right).$$

Using the triangle inequality and the uniform continuity of f on $K \cap B(r)$, we obtain that $|f(x)| < |f(0)| + n$ for each $x \in K \cap B(r)$. This is true, in particular, for $K = X$. Note that we have proved this direction for a function f the domain of which is an arbitrary nonempty and convex subset of X. □

Since $\mathfrak{M}_l$ is a closed subset of $\mathfrak{M}$ with respect to the τ topology, it is a countable intersection of open subsets of $\mathfrak{M}$ with respect to this topology, that is, it is a G_δ subset of $\mathfrak{M}$ with respect to this topology, while $\mathfrak{M}$ with this topology is a completely metrizable space. Since $\mathfrak{M}_l$ with the τ_2 topology is a metrizable space, it follows from Theorem 1.2 that $\mathfrak{M}_l$ with the τ_2 topology is completely metrizable. The same argument shows that $\mathfrak{M}_c$ and $\mathfrak{M}_b$ with their relative τ_2 topologies are completely metrizable. Nevertheless, $\mathfrak{M}_l$ is not complete with respect to the metric d_{AW}, since the sequence of constant functions $\{f_n\}_{n=1}^{\infty}$, defined by $f_n(x) = -n$ for each $x \in K$ and each $n = 1, 2, \ldots$, is a Cauchy sequence which does not converge in $(\mathfrak{M}_l, d_{AW})$.

Using Corollary 1.5 and the definitions of the sets $\mathfrak{M}_c, \mathfrak{M}_l$ and $\mathfrak{M}$, we see that $\mathfrak{M}_b \subset \mathfrak{M}_c \subset \mathfrak{M}_l \subset \mathfrak{M}$ in the case where $K = X$. Therefore $d_{AW}|_{\mathfrak{M}_c}$ and $d_{AW}|_{\mathfrak{M}_b}$ are metrics, respectively, on $\mathfrak{M}_c$ and $\mathfrak{M}_b$ in this case. But it is easy to see that if $K \neq X$, then $\mathfrak{M}_b$ is not necessarily a subset of $\mathfrak{M}_c$. Take, for example, $X = \mathbb{R}$ with the Euclidean norm, $K = [-1, 1]$ and define

$$f : K \to \mathbb{R} \quad \text{by} \quad f(x) := \begin{cases} x^2 & \text{if } x \in (-1,1) \\ 2 & \text{otherwise} \end{cases}, \quad \text{for each } x \in K.$$

Clearly, f is convex and bounded on bounded subsets of K, but it is not even lower semi-continuous.

Recall that a function $f \in \mathfrak{M}$ is called *strictly convex* if for each $x, y \in K$ such that $x \neq y$ and each $\lambda \in (0,1)$, we have

$$f(\lambda x + (1-\lambda) y) < \lambda f(x) + (1-\lambda) f(y). \tag{6}$$

It is not difficult to see that this definition is equivalent to the requirement that f is convex and for each $x, y \in K$ such that $x \neq y$, there exists $\lambda \in (0,1)$ satisfying (6), as well as to the requirement that f is convex and $\lambda = 2^{-1}$ satisfies (6) for each $x, y \in K$ such that $x \neq y$. We denote the set of all strictly convex functions $f \in \mathfrak{M}$ by $\mathcal{F}$.

Recall that a function $f \in \mathfrak{M}$ is called *locally uniformly convex* if for each sequence $\{x_n\}_{n=1}^{\infty} \subset K$ and each $x \in K$,

$$\lambda f(x_n) + (1-\lambda) f(x) - f(\lambda x_n + (1-\lambda) x) \underset{n\to\infty}{\longrightarrow} 0 \tag{7}$$

implies $\|x - x_n\| \underset{n\to\infty}{\longrightarrow} 0$ for each $0 < \lambda < 1$. It is not difficult to see that this definition is equivalent to the requirement that $f \in \mathfrak{M}$ and for each $x \in K$, there exists $\lambda \in (0,1)$ such that (7) implies that $\|x - x_n\| \underset{n\to\infty}{\longrightarrow} 0$, as well as to the requirement that $f \in \mathfrak{M}$ and for each $x \in K$, (7) with $\lambda = 2^{-1}$ implies that $\|x - x_n\| \underset{n\to\infty}{\longrightarrow} 0$.

It turns out (see Exercise 5.3.1 in [1]) that it is enough to consider only bounded sequences in the definition of a locally uniformly convex function.

Recall that a normed space $(X, \|\cdot\|)$ is called strictly convex if $\|x+y\| < 2$ whenever $x, y \in X$ are such that $x \neq y$ and $\|x\| = \|y\| = 1$, and locally uniformly convex if $\|x - x_n\| \underset{n\to\infty}{\longrightarrow} 0$ whenever $\|x_n + x\| \underset{n\to\infty}{\longrightarrow} 2$ for each sequence $\{x_n\}_{n=1}^{\infty}$ in the closed unit ball of center zero and each x in this ball. It is not difficult to see that $(X, \|\cdot\|)$ is strictly convex (respectively, locally uniformly convex) if and only if the square of its norm is a strictly convex function (respectively, locally uniformly convex function), as well as that a locally uniformly convex function is strictly convex and in the case where $K = X$ and the dimension of the vector space X is finite, the converse is also true. But in the case where the dimension of X is infinite, a strictly convex function may no longer be locally uniformly convex (see, for instance, the example on pages 229–230 in [6]). We denote the set of all locally uniformly convex functions $f \in \mathfrak{M}$ by $\mathcal{G}$.

2. Main results

It was shown in [3] that under the assumption of the existence of a continuous and strictly convex function $f_* \in \mathfrak{M}_b$, the set $\mathcal{F}$ of all strictly convex functions defined on K is residual in $\mathfrak{M}$ with the τ topology, and that the sets $\mathcal{F} \cap \mathfrak{M}_l$ and $\mathcal{F} \cap \mathfrak{M}_c$ are residual in, respectively, $\mathfrak{M}_l$ and $\mathfrak{M}_c$ with their relative τ topologies. It can be verified that the same result is true for the set $\mathcal{F} \cap \mathfrak{M}_b$ in the space $\mathfrak{M}_b$ with the relative τ topology. In the present paper we prove a stronger result, which provides all these residuality properties for the set $\mathcal{G}$ (which is a subset of $\mathcal{F}$) of all locally uniformly convex functions defined on K.

In the recent work described in [9], residual sets of the space $\Gamma(X)$ are considered, where $\Gamma(X)$ is the set all lower semi-continuous functions $f : X \to \mathbb{R}$ with our τ_2 topology and $(X, \|\cdot\|)$ is a real Banach space. In particular, it is shown there that if $(X, \|\cdot\|)$ is a locally uniformly convex space (respectively, a strictly convex space), then the set of all locally uniformly convex (respectively, strictly convex) functions is residual in $\Gamma(X)$. By using similar techniques, we show that in the case where $K = X$ (where $(X, \|\cdot\|)$ is not necessarily a Banach space), the relative strong topology of $\mathfrak{M}_b$ is the same as its relative weak topology and hence, the sets $\mathcal{F} \cap \mathfrak{M}_b$ and $\mathcal{G} \cap \mathfrak{M}_b$ are residual in $\mathfrak{M}_b$ with both of these topologies. For a finite dimensional vector space X, we know that each convex function $f : X \to \mathbb{R}$ is continuous, and therefore this result can be applied to all of $\mathfrak{M}$, because $\mathfrak{M}_b = \mathfrak{M}$ in this case.

Namely, we have the following results which we establish in Section 4 below.

Theorem 2.1. *Suppose that there exists a strictly convex function $f_* \in \mathfrak{M}_b$. Then the sets $\mathcal{G}$ and $\mathcal{G} \cap \mathfrak{M}_b$ are residual in, respectively, $\mathfrak{M}$ and $\mathfrak{M}_b$ with the relative τ topology. If, in addition, f_* is lower semi-continuous (respectively, continuous), then the set $\mathcal{G} \cap \mathfrak{M}_l$ (respectively, $\mathcal{G} \cap \mathfrak{M}_c$) is residual in $\mathfrak{M}_l$ (respectively, $\mathfrak{M}_c$) with the relative strong topology.*

Remark 2.2. In particular, this result is true for a strictly convex normed linear space X, since the square of its norm is a continuous strictly convex function which is bounded on bounded sets. An important class of such spaces consists of inner product spaces and, in particular, Hilbert spaces.

Theorem 2.3. *In the case where $K = X$, the relative weak topology of $\mathfrak{M}_b$ is the same as the relative strong topology of $\mathfrak{M}_b$. As a result, if $\mathcal{F} \cap \mathfrak{M}_b \neq \emptyset$, then the set $\mathcal{G} \cap \mathfrak{M}_b$ (and therefore $\mathcal{F} \cap \mathfrak{M}_b$) is residual in $\mathfrak{M}_b$ with both of these topologies.*

3. Auxiliary results

Suppose
$$\mathcal{K} = \{K_{m,n} : m, n \geq 1 \text{ are integers}\}$$
is a family of bounded (with respect to one of the product metrics) subsets of $K \times K$, which do not intersect the diagonal $\Delta(K) := \{(x, x) : x \in K\}$ of the set $K \times K$, such that for each positive integer n, we have

$$((K \cap B(n)) \times (K \cap B(n))) \setminus \Delta(K) \subset \cup_{m=1}^{\infty} K_{m,n}.$$

For each pair of positive integers (m, n), denote by $\mathcal{G}_{m,n}$ the set of all $f \in \mathfrak{M}$ for which there exists $\delta > 0$ such that

$$2^{-1} f(x) + 2^{-1} f(y) - f\left(2^{-1}(x + y)\right) < \delta \ \Rightarrow\ \|x - y\| < n^{-1} \tag{8}$$

for each $(x, y) \in K_{m,n}$.

We call a function $f \in \mathfrak{M}$ *locally uniformly convex with respect to the family $\mathcal{K}$* if $f \in \cap_{m=1}^{\infty} \cap_{n=1}^{\infty} \mathcal{G}_{m,n}$. It is not difficult to verify that locally uniformly convex functions with respect to $\mathcal{K}$ are locally uniformly convex functions on K.

Denote by $\mathcal{G}_{\mathcal{K}}$ the set of all functions $f \in \mathfrak{M}$ which are locally uniformly convex with respect to $\mathcal{K}$.

Proposition 3.1. *Suppose that the set $\mathcal{G}_{\mathcal{K}}$ contains a function $f_* \in \mathfrak{M}_b$. Then the sets $\mathcal{G}_{\mathcal{K}}$ and $\mathcal{G}_{\mathcal{K}} \cap \mathfrak{M}_b$ are countable intersections of open and dense subsets of, respectively, $\mathfrak{M}$ and $\mathfrak{M}_b$ with the relative τ topology. If, in addition, f_* is lower semi-continuous (respectively, continuous), then the set $\mathcal{G}_{\mathcal{K}} \cap \mathfrak{M}_l$ (respectively, $\mathcal{G}_{\mathcal{K}} \cap \mathfrak{M}_c$) is a countable intersection of open and dense subsets of $\mathfrak{M}_l$ (respectively, $\mathfrak{M}_c$) with the relative strong topology.*

Proof. We prove the first part of the proposition. The case where f_* is lower semi-continuous (respectively, continuous) is proved similarly. It is clear that

$$\mathcal{G}_{\mathcal{K}} = \cap_{m=1}^{\infty} \cap_{n=1}^{\infty} \mathcal{G}_{m,n},$$

and

$$\mathcal{G}_{\mathcal{K}} \cap \mathfrak{M}_b = \cap_{m=1}^{\infty} \cap_{n=1}^{\infty} \left(\mathcal{G}_{m,n} \cap \mathfrak{M}_b\right).$$

Let (m,n) be an arbitrary pair of positive integers. We claim that the set $\mathcal{G}_{m,n}$ is open with respect to the τ topology (that is, we claim that it belongs to this topology). Indeed, suppose that $f \in \mathcal{G}_{m,n}$. Then there exists $\delta > 0$ such that (8) holds for each $(x,y) \in K_{m,n}$. It is sufficient to show that there exists a positive integer N such that for each $g \in \mathfrak{M}$, if $(g,f) \in E(N)$, then $g \in \mathcal{G}_{m,n}$. Since $K_{m,n}$ is bounded with respect to one of the product metrics, there exists a positive integer N satisfying $N^{-1} < 4^{-1}\delta$ such that $\max\{\|x\|, \|y\|\} < N$ for each $(x,y) \in K_{m,n}$. Suppose $g \in \mathfrak{M}$ is such that $(g,f) \in E(N)$ and let $(x,y) \in K_{m,n}$ satisfy

$$2^{-1}g(x) + 2^{-1}g(y) - g\left(2^{-1}(x+y)\right) < 2^{-1}\delta.$$

Then it follows from (1) that

$$\begin{aligned} 2^{-1}f(x) + 2^{-1}f(y) - f\left(2^{-1}(x+y)\right) &< 2^{-1}g(x) + 2^{-1}g(y) - g\left(2^{-1}(x+y)\right) + 2N^{-1} \\ &< \delta. \end{aligned}$$

By (8), $\|x-y\| < n^{-1}$ and therefore $g \in \mathcal{G}_{m,n}$. Hence the set $\mathcal{G}_{m,n} \cap \mathfrak{M}_b$ is also open in $\mathfrak{M}_b$ with the relative τ topology. In order to complete the proof of the theorem, it is sufficient to show that the sets $\mathcal{G}_{m,n}$ and $\mathcal{G}_{m,n} \cap \mathfrak{M}_b$ are dense subsets of, respectively, $\mathfrak{M}$ and $\mathfrak{M}_b$ with the relative τ topology. To this end, for each $f \in \mathfrak{M}$ (respectively, $f \in \mathfrak{M}_b$) and each $\gamma \in (0,1)$, define

$$(\forall x \in K)\, f_\gamma(x) := f(x) + \gamma f_*(x). \tag{9}$$

It is clear that for each $\gamma \in (0,1)$ and each $f \in \mathfrak{M}$ (respectively, $f \in \mathfrak{M}_b$), we have $f_\gamma \in \mathfrak{M}$ (respectively, $f_\gamma \in \mathfrak{M}_b$). By (1), (9) and since $f_* \in \mathfrak{M}_b$, the sets

$$A = \{f_\gamma : f \in \mathfrak{M}\}, \quad \text{and} \quad A_b = \{f_\gamma : f \in \mathfrak{M}_b\}$$

are dense in, respectively, $\mathfrak{M}$ and $\mathfrak{M}_b$ with the relative τ topology. Since the function f_* is locally uniformly convex with respect to the family $\mathcal{K}$, there exists $\delta > 0$ with

$$2^{-1}f_*(x) + 2^{-1}f_*(y) - f_*\left(2^{-1}(x+y)\right) < \delta \;\Rightarrow \|x-y\| < n^{-1} \tag{10}$$

for each $(x,y) \in K_{m,n}$. Assume that $\gamma \in (0,1)$ and $f \in \mathfrak{M}$ (respectively, $f \in \mathfrak{M}_b$). We claim that $f_\gamma \in \mathcal{G}_{m,n}$ (respectively, $f_\gamma \in \mathcal{G}_{m,n} \cap \mathfrak{M}_b$). Indeed, suppose that $(x,y) \in K_{m,n}$ satisfies

$$2^{-1}f_\gamma(x) + 2^{-1}f_\gamma(y) - f_\gamma\left(2^{-1}(x+y)\right) < \gamma\delta. \tag{11}$$

Since f is a convex function, it follows from (9) and (11) that

$$2^{-1}f_*(x) + 2^{-1}f_*(y) - f_*\left(2^{-1}(x+y)\right) < \delta.$$

Hence by (10), we have $\|x-y\| < n^{-1}$ and therefore we conclude $f_\gamma \in \mathcal{G}_{m,n}$ (respectively, $f_\gamma \in \mathcal{G}_{m,n} \cap \mathfrak{M}_b$), as claimed. As a result, the set $\mathcal{G}_{m,n}$ (respectively, $\mathcal{G}_{m,n} \cap \mathfrak{M}_b$) contains the dense set A (respectively, A_b) in the relative τ topology and therefore it is dense, as asserted. □

Next, we show that the τ_2 topology on $\mathfrak{M}_l$ is indeed weaker than the τ_1 topology on $\mathfrak{M}_l$. To this end, we need the following two technical lemmata.

Lemma 3.2. *Assume that $f, g \in \mathfrak{M}_l$, $(x_0, t_0) \in K \times \mathbb{R}$ and that n is a positive integer such that*

$$\rho((x_0,t_0), \operatorname{epi}(f)) = \inf\{\|(x,t)-(x_0,t_0)\|_\infty : (x,t) \in \operatorname{epi}(f) \cap B_\infty(n)\}, \tag{12}$$

$$\rho((x_0,t_0), \operatorname{epi}(f)) \le \rho((x_0,t_0), \operatorname{epi}(g)) \tag{13}$$

and

$$(f,g) \in E(n). \tag{14}$$

Then

$$\rho((x_0,t_0), \operatorname{epi}(g)) - \rho((x_0,t_0), \operatorname{epi}(f)) \le n^{-1}.$$

Proof. Set $S_1 := \left\{\|(x-x_0, t-t_0)\|_\infty : (x,t) \in \operatorname{epi}(f) \cap B_{\|\cdot\|_\infty}(n)\right\}$

and $S_2 := \{\|(x-x_0, t-t_0)\|_\infty : (x,t) \in \operatorname{epi}(g)\}$.

It is clear that for each $(x,t) \in \operatorname{epi}(f) \cap B_{\|\cdot\|_\infty}(n)$, we have

$$(x, g(x) + t - f(x)) \in \operatorname{epi}(g).$$

By (14), for each element $u \in S_1$, there is $v \in S_2$ such that $|u-v| < n^{-1}$. Combining this with (12) and (13), we see that

$$\rho((x_0,t_0), \operatorname{epi}(g)) - \rho((x_0,t_0), \operatorname{epi}(f)) \le n^{-1}. \qquad \square$$

Lemma 3.3. *Assume that N is a positive integer and that $M > 0$. Then there exists a positive integer $n > N$ such that*

$$\rho((x_0,t_0), \operatorname{epi}(f)) = \inf\left\{\|(x,t)-(x_0,t_0)\|_\infty : (x,t) \in \operatorname{epi}(f) \cap B_{\|\cdot\|_\infty}(n)\right\}$$

for each $f \in \mathfrak{M}_l$ and each $(x_0,t_0) \in B_{\|\cdot\|_\infty}(N)$ satisfying $\rho((x_0,t_0), \operatorname{epi}(f)) < M$.

Proof. Choose a positive integer n such that $n > M + N$. Let $f \in \mathfrak{M}_l$ and $(x_0,t_0) \in B_{\|\cdot\|_\infty}(N)$ be such that $\rho((x_0,t_0), \operatorname{epi}(f)) < M$. Clearly, we have $n > N$ and

$$\rho((x_0,t_0), \operatorname{epi}(f)) \le \inf\left\{\|(x,t)-(x_0,t_0)\|_\infty : (x,t) \in \operatorname{epi}(f) \cap B_{\|\cdot\|_\infty}(n)\right\}. \tag{15}$$

Let $(x,t) \in \operatorname{epi}(f)$. If $(x,t) \in B_{\|\cdot\|_\infty}(n)$, then

$$\|(x,t)-(x_0,t_0)\|_\infty \ge \inf\left\{\|(x,t)-(x_0,t_0)\|_\infty : (x,t) \in \operatorname{epi}(f) \cap B_{\|\cdot\|_\infty}(n)\right\}. \tag{16}$$

Otherwise,

$$\|(x,t)-(x_0,t_0)\|_\infty \ge \|(x,t)\|_\infty - \|(x_0,t_0)\|_\infty \ge n - N > M + N - N = M.$$

Hence, there exists $(x', t') \in \operatorname{epi}(f)$ such that

$$\left\|(x', t') - (x_0, t_0)\right\|_\infty < M < \left\|(x, t) - (x_0, t_0)\right\|_\infty . \tag{17}$$

It follows that

$$\left\|(x', t')\right\|_\infty \leq \left\|(x', t') - (x_0, t_0)\right\|_\infty + \left\|(x_0, t_0)\right\|_\infty < M + N < n.$$

As a result, $(x', t') \in \operatorname{epi}(f) \cap B_{\|\cdot\|_\infty}(n)$ and by (17), inequality (16) holds in this case as well. We obtain

$$\rho\left((x_0, t_0), \operatorname{epi}(f)\right) \geq \inf\left\{\left\|(x, t) - (x_0, t_0)\right\|_\infty : (x, t) \in \operatorname{epi}(f) \cap B_{\|\cdot\|_\infty}(n)\right\}.$$

When combined with (15), this yields the desired conclusion. □

Proposition 3.4. *The τ_2 topology is weaker than the τ_1 topology.*

Proof. Assume that the set U is open in the τ_2 topology. Let $g \in U$ be given. There exists a positive integer N such that for each $f \in \mathfrak{M}_l$, if $(f, g) \in F(N)$, then $f \in U$. In order to show that U is open in τ_1, it is enough to prove that there exists a positive integer n such that whenever $f \in \mathfrak{M}_l$ satisfies $(f, g) \in E(n)$, it follows that $(f, g) \in F(N)$. By the triangle inequality, there exists a number $M > 0$ such that $\rho((x, t), \operatorname{epi}(g)) < M$ for each $(x, t) \in B_{\|\cdot\|_\infty}(N)$. By Lemma 3.3, there exists a positive integer $n > N$ such that

$$\rho\left((x_0, t_0), \operatorname{epi}(h)\right) = \inf\left\{\left\|(x, t) - (x_0, t_0)\right\|_\infty : (x, t) \in \operatorname{epi}(h) \cap B_{\|\cdot\|_\infty}(n)\right\} \tag{18}$$

for each $h \in \mathfrak{M}_l$ and each $(x_0, t_0) \in B_{\|\cdot\|_\infty}(N)$ satisfying $\rho((x_0, t_0), \operatorname{epi}(h)) < M$. In particular,

$$\rho\left((x_0, t_0), \operatorname{epi}(g)\right) = \inf\left\{\left\|(x, t) - (x_0, t_0)\right\|_\infty : (x, t) \in \operatorname{epi}(g) \cap B_{\|\cdot\|_\infty}(n)\right\}$$

for each $(x_0, t_0) \in B_{\|\cdot\|_\infty}(N)$. Assume that $f \in \mathfrak{M}_l$ is such that $(f, g) \in E(n)$ and let $(x_0, t_0) \in B_{\|\cdot\|_\infty}(N)$. If $\rho((x_0, t_0), \operatorname{epi}(g)) \leq \rho((x_0, t_0), \operatorname{epi}(f))$, then by Lemma 3.2,

$$\left|\rho\left((x_0, t_0), \operatorname{epi}(f)\right) - \rho\left((x_0, t_0), \operatorname{epi}(g)\right)\right| \leq n^{-1} < N^{-1}. \tag{19}$$

Otherwise, $\rho((x_0, t_0), \operatorname{epi}(f)) < M$ and by (18),

$$\rho\left((x_0, t_0), \operatorname{epi}(f)\right) = \inf\left\{\left\|(x, t) - (x_0, t_0)\right\|_\infty : (x, t) \in \operatorname{epi}(f) \cap B_{\|\cdot\|_\infty}(n)\right\}.$$

By Lemma 3.2, (19) holds in this case as well. Hence we obtain that $(f, g) \in F(N)$, as required. □

The following two lemmata are slightly stronger variants of Lemma 3.1 and Lemma 3.2 in [9].

Lemma 3.5. *Let N be a positive integer and let $M \geq N$. Assume $f, g \in \mathfrak{M}_l$ are functions which are bounded by M on $K \cap B(N)$ and satisfy a Lipschitz condition with constant $R \geq 1$ on $K \cap B(N+1)$. If $d_{AW}(f, g) < \varepsilon$ for some $0 < \varepsilon < 2^{-M}$, then $\sup_{x \in K \cap B(N)} |f(x) - g(x)| \leq 3 \cdot 2^M R\varepsilon$.*

Proof. Assume $d_{AW}(f,g) < \varepsilon$ for some $0 < \varepsilon < 2^{-M}$ and suppose to the contrary that there exists a point $x_0 \in K \cap B(N)$ such that $|f(x_0) - g(x_0)| > 3 \cdot 2^M R\varepsilon$. Without loss of generality, say $g(x_0) > f(x_0) + 3 \cdot 2^M R\varepsilon$. Then for each $x \in K$ such that $\|x - x_0\| < 2^M \varepsilon$, we have $x \in K \cap B(N+1)$ and it follows that

$$g(x) \geq g(x_0) - 2^M R\varepsilon > f(x_0) + 2^{M+1} R\varepsilon > f(x_0) + 2^M \varepsilon.$$

Therefore we have $\rho\big((x_0, f(x_0)), \mathrm{epi}(g)\big) \geq 2^M \varepsilon$, where ρ is as defined in (2). Since $x_0 \in K \cap B(M)$ and $\|(x_0, f(x_0))\|_\infty < M$, it follows that $d_{AW}(f,g) \geq 2^{-M} 2^M \varepsilon = \varepsilon$. The contradiction we have reached concludes the proof of Lemma 3.5. □

Lemma 3.6. *Let N be a positive integer. Assume $K = X$, $f \in \mathfrak{M}_l$ and $g \in \mathfrak{M}_l$ is a bounded function on $B(N+2)$. Set $M = \max\left\{N, \sup_{x \in B(N+2)} |g(x)|\right\}$. If $d_{AW}(f,g) < 2^{-M-1}$, then $\sup_{x \in B(N)} f(x) \leq M + 1$.*

Proof. Assume $d_{AW}(f,g) < 2^{-M-1}$. Suppose to the contrary that there exists a point $x_0 \in B(N)$ such that $f(x_0) > M + 1$. Let $C = \{x \in X : f(x) \leq M + 1\}$. Since C is closed and convex in $(X, \|\cdot\|)$ and $(X, \|\cdot\|)$ is a real normed space, by Theorem 1.3 there exists a continuous linear functional $\phi \in X^*$ such that $\|\phi\| = 1$ and $\sup_{x \in C} \phi(x) < \phi(x_0)$. Set $d := \phi(x_0) - \sup_{x \in C} \phi(x)$ and choose $x_1 \in B(1)$ such that $\phi(x_1) > 1 - d$. Then $\bar{x} = x_1 + x_0$ belongs to $B(N+1)$ and we have $\phi(\bar{x}) > \sup_{x \in C} \phi(x) + 1$. Let $U = \{x \in X : \|x - \bar{x}\| < 1\}$. Then for each $x \in U$, we have

$$f(x) > M + 1 \text{ and } g(x) \leq M. \tag{20}$$

Set $\tilde{x} = (\bar{x}, g(\bar{x}))$. By the definition of U and (20) we conclude $\rho(\tilde{x}, \mathrm{epi}(f)) \geq 1$, where ρ is as defined in (2). It then follows that $d_{AW}(f,g) \geq 2^{-M-1}$.
This contradiction completes the proof. □

Now we show that the relative τ_1 topology of $\mathfrak{M}_b$ is weaker than the relative τ_2 topology of $\mathfrak{M}_b$ in the case where $K = X$.

Proposition 3.7. *Assume $K = X$. Then the relative τ_1 topology of $\mathfrak{M}_b$ is weaker than the relative τ_2 topology of $\mathfrak{M}_b$.*

Proof. Assume that the set U is open in the relative τ_1 topology of $\mathfrak{M}_b$. Let $g \in U$ be given. Then there exists a positive integer N such that

$$(f,g) \in E(N) \;\Rightarrow\; f \in U \tag{21}$$

for each $f \in \mathfrak{M}_b$. Set $M' := \max\left\{N + 2, \sup_{x \in B(N+4)} |g(x)|\right\}$. By Lemma 3.6, we have

$$d_{AW}(f,g) < 2^{-M'-1} \;\Rightarrow\; \sup_{x \in B(N+2)} f(x) \leq M' + 1 \tag{22}$$

for each $f \in \mathfrak{M}_l$. Therefore by (22) and by Lemma 1.4, there exist

$$M = \max\{M' + 1, N\}$$

and a real number $R \geq 1$ such that, whenever $f \in \mathfrak{M}_l$ satisfies $d_{AW}(f,g) < 2^{-M'-1}$, it follows that f and g are bounded by M on $B(N+2)$ (and therefore on $B(N)$), and satisfy a Lipschitz condition with a Lipschitz constant R on $B(N+1)$. Set

$$\varepsilon := 3^{-1}N^{-1}R^{-1}2^{-M-1}$$

and assume $f \in \mathfrak{M}_b$ is such that $d_{AW}(f,g) < \varepsilon$. It follows from Lemma 3.5 that $\sup_{x\in B(N)} |f(x) - g(x)| \leq 2^{-1}N^{-1}$ and (21) implies that $f \in U$. Therefore $B_{AW}(g,\varepsilon) \subset U$ and so U is open in the relative τ_2 topology of $\mathfrak{M}_b$. □

4. Proofs of the main results

Proof of Theorem 2.1 Define a function $\psi : K \times K \to \mathbb{R}$ by

$$(\forall (x,y) \in K \times K)\, \psi(x,y) := 2^{-1}f_*(x) + 2^{-1}f_*(y) - f_*\left(2^{-1}(x+y)\right).$$

For each pair of positive integers (m,n), set

$$K_{m,n} = \left\{ \begin{array}{l} (x,y) \in \big((K\cap B(n)) \times (K \cap B(n))\big)\backslash\Delta(K) : \\ \psi(x,y) < m^{-1} \;\Rightarrow\; \|x-y\| < n^{-1} \end{array} \right\}.$$

Since f_* is a strictly convex function, for each positive integer n, we have

$$\cup_{m=1}^{\infty} K_{m,n} = \{(x,y) \in ((K\cap B(n)) \times (K\cap B(n)))\backslash\Delta(K)\}.$$

Set $\mathcal{K} := \{K_{m,n} : m, n \text{ are positive integers}\}$. Evidently, $f_* \in \mathfrak{M}_b$ is locally uniformly convex with respect to $\mathcal{K}$. By Proposition 3.1, the set $\mathcal{G}_{\mathcal{K}}$ (respectively, $\mathcal{G}_{\mathcal{K}} \cap \mathfrak{M}_b$) is a countable intersection of open and dense subsets of $\mathfrak{M}$ (respectively, $\mathfrak{M}_b$) with the relative τ topology. If, in addition, f_* is lower semi-continuous (respectively, continuous), then the set $\mathcal{G}_{\mathcal{K}} \cap \mathfrak{M}_l$ (respectively, $\mathcal{G}_{\mathcal{K}} \cap \mathfrak{M}_c$) is a countable intersection of open and dense subsets of $\mathfrak{M}_l$ (respectively, $\mathfrak{M}_c$) with the relative strong topology. Since $\mathcal{G}_{\mathcal{K}} \subset \mathcal{G}$ (respectively, $\mathcal{G}_{\mathcal{K}} \cap \mathfrak{M}_l \subset \mathcal{G} \cap \mathfrak{M}_l$, $\mathcal{G}_{\mathcal{K}} \cap \mathfrak{M}_c \subset \mathcal{G} \cap \mathfrak{M}_c$ and, finally, $\mathcal{G}_{\mathcal{K}} \cap \mathfrak{M}_b \subset \mathcal{G} \cap \mathfrak{M}_b$), the proof is complete. □

Proof of Theorem 2.3 This is an immediate consequence of Proposition 3.4, Proposition 3.7 and Theorem 2.1. □

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References

[1] G. Beer: *Topologies on Closed and Closed Convex Sets*, Kluwer, Dordrecht (1993).

[2] J. M. Borwein, J. D. Vanderwerff: *Convex Functions: Constructions, Characterizations and Counterexamples*, Cambridge University Press, Cambridge (2010).

[3] D. Butnariu, S. Reich, A. J. Zaslavski: *There are many totally convex functions*, J. Convex Analysis 13 (2006) 623–632.

[4] J. B. Conway: *A Course in Functional Analysis*, 2nd ed., Springer, New York (1990).

[5] J. L. Kelley: *General Topology*, D. Van Nostrand, New York (1955).

[6] A. R. Lovaglia: *Locally uniformly convex Banach spaces*, Trans. Amer. Math. Soc. 78 (1955) 225–238.

[7] E. Michael: *A note on completely metrizable spaces*, Proc. Amer. Math. Soc. 96 (1986) 513–522.

[8] S. Reich, A. J. Zaslavski: *Genericity in Nonlinear Analysis*, Springer, New York (2014).

[9] J. Vandewerff: *On the residuality of certain classes of convex functions*, Pure Appl. Funct. Analysis 5 (2020) 791–806.