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Boundedly Polyhedral Sets and F-Simplices

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Generalizing the concept of Choquet simplex, we study a new class of convex solids K in $\mathbb{R}^n$ which satisfy the following condition: all n-dimensional intersections of the form $K \cap (x + K)$, $x \in \mathbb{R}^n$, belong to at most finitely many homothety classes of convex solids. Our description of this class uses new results on boundedly polyhedral sets.

Keywords: Convex set, Choquet simplex, homothety class, polyhedron.

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1. Introduction

This paper deals with two related topics of convex geometry and convex analysis in finite dimensions: boundedly polyhedral sets and Choquet simplices. Following Klee [6], we say that a closed convex set K in the n-dimensional Euclidean space $\mathbb{R}^n$ is *boundedly polyhedral* provided its intersection with every polytope is a polytope. Boundedly polyhedral sets also are called quasipolyhedra in [1] and generalized polyhedra in [5].

Given a closed convex set $K \subset \mathbb{R}^n$ and a point $u \in K$, we say that K is *polyhedral* at u if there is an n-dimensional polytope $P \subset \mathbb{R}^n$ such that $u \in \operatorname{int} P$ and $K \cap P$ is a polytope. Similarly, K is called *conic* at u provided there is an open ball $U_\rho(u) \subset \mathbb{R}^n$ and a cone $C \subset \mathbb{R}^n$ with apex u such that $K \cap U_\rho(u) = C \cap U_\rho(u)$. In the latter case, C is the cone with apex u generated by K and denoted $C_u(K)$ (see Section 2 for definitions and notation). As proved in [6] (see also [1] and [5]), one has $(a) \Rightarrow (b) \Rightarrow (c)$, where (a), (b), and (c) denote the following conditions:

(a) K is polyhedral at u,

(b) K is conic at u,

(c) the cone $C_u(K)$ is closed.

Furthermore, K is boundedly polyhedral if and only if any of the above conditions (a), (b), and (c) holds for all points $u \in K$ (see [6]).

Our Theorems 3.1, 3.4, and 3.6 deal with related characterizations of local conicity and polyhedrality of convex sets that involve extreme points. In particular, Theorem 3.6 refines the above assertion from [6], stating that a line-free closed convex set $K \subset \mathbb{R}^n$ is boundedly polyhedral if and only if the set $\operatorname{ext} K$ of its extreme points is discrete and K is polyhedral at every point $u \in \operatorname{ext} K$.

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These new theorems allow us to describe a new class of convex solids $K \subset \mathbb{R}^n$ which naturally extends the family of Choquet simplices (see Theorem 4.1, Corollary 5.1, and Theorem 5.3).

We recall (see Choquet [2]), that a convex set S in a vector space E is a *Choquet simplex* provided the family of all homothetic copies of S is closed under nonempty intersections:

$$(x + \lambda S) \cap (y + \mu S) = z + \gamma S, \quad x, y, z \in E, \ \lambda, \mu, \gamma \geq 0.$$

It is known (see [12] for details) that a convex set $K \subset \mathbb{R}^n$ is a Choquet simplex if and only if every nonempty intersection of K and a translate of K is a homothetic copy of K (possibly, degenerated into a point):

$$K \cap (x + K) = z + \gamma K, \quad x, z \in \mathbb{R}^n, \ \gamma \geq 0. \tag{1}$$

In what follows, a *convex solid* is a closed convex sets of dimension n in $\mathbb{R}^n$ which contains no lines, and a *convex body* is a compact convex solid. The results of Rogers and Shephard [9] and Gruber [4] provide the following characterizations of Choquet simplices: (i) a convex body $K \subset \mathbb{R}^n$ is a Choquet simplex if and only if it is a usual n-dimensional simplex, (ii) an unbounded convex solid $K \subset \mathbb{R}^n$ is a Choquet simplex if and only if it is an n-dimensional simplicial cone.

The assumption $K \cap (x + K) \neq \varnothing$ in (1) implies that every *nonempty* intersection $K \cap (x + K)$ of dimension $n - 1$ or less (if exists) should be a singleton, and this argument essentially determines the "simplicial" character of K. In this regard, Fourneau [3] considered *quasi-simplices* which are defined as convex solids $K \subset \mathbb{R}^n$ satisfying the following condition: all *n-dimensional* intersections $K \cap (x + K)$, $x \in \mathbb{R}^n$, are homothetic to K.

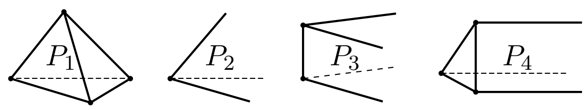

Figure 1.1: Types of generalized 3-simplices in $\mathbb{R}^3$.

As proved in [3], quasi-simplices in $\mathbb{R}^n$ are precisely the generalized simplices in the sense of Rockafellar [8, p. 154]. Let us recall that a convex solid $K \subset \mathbb{R}^n$ is a *generalized simplex* if it can be expressed as the direct vector sum of an r-dimensional simplex and an $(n - r)$-dimensional simplicial cone:

$$K = \operatorname{conv}\{u_0, u_1, \ldots, u_r\} + \sum_{i=r+1}^{n} [o, u_i - u_0\rangle, \quad 0 \leq r \leq n,$$

where $\{u_0, u_1, \ldots, u_n\}$ is an affinely independent set of points, and $[o, u_i - u_0\rangle$ means the halfline through $u_i - u_0$ originated at o. Clearly, usual n-dimensional simplices and n-dimensional simplicial cones are particular cases of generalized simplices. For instance, Figure 1.1 describes all types of generalized simplices in $\mathbb{R}^3$, and the sets Q_1–Q_3 in Figure 1.2 give all types of generalized simplices in $\mathbb{R}^2$.

There are various generalizations of Choquet simplices in $\mathbb{R}^n$ concerning n-dimensional polytopes $K \subset \mathbb{R}^n$ such that the numbers of vertices, edges, or facets of n-dimensional intersections $K \cap (x + K)$, $x \in \mathbb{R}^n$, are constant. These conditions provide characteristic properties of primitive polytopes and direct products of simplices (see [12] for their description and references). We consider below another type of generalization of Choquet simplices which deals with homothety classes of convex solids.

Since the relation of homothety is an equivalence on the family of convex solids of $\mathbb{R}^n$, the latter can be viewed as the union of pairwise disjoint *homothety classes*, each consisting of all pairwise homothetic convex solids. (We recall that nonempty sets X and Y in $\mathbb{R}^n$ are called homothetic provided $X = z + \gamma Y$ for suitable $z \in \mathbb{R}^n$ and $\gamma > 0$.) In these terms, a convex solid $K \subset \mathbb{R}^n$ is a generalized simplex if and only if all n-dimensional intersections $K \cap (x + K)$, $x \in \mathbb{R}^n$, belong to a unique homothety class (generated by K).

The following definition further extends the concept of Choquet simplex.

Definition 1.1. A convex solid $K \subset \mathbb{R}^n$ is called an *F-simplex* provided all n-dimensional intersections $K \cap (x + K)$, $x \in \mathbb{R}^n$, belong to at most finitely many homothety classes of convex solids.

For instance, the convex sets Q_1–Q_4 depicted in Figure 1.2 give all possible types of F-simplices in $\mathbb{R}^2$, and the convex sets P_1–P_7 in Figures 1.1 and 1.3 describe all types of F-simplices in $\mathbb{R}^3$.

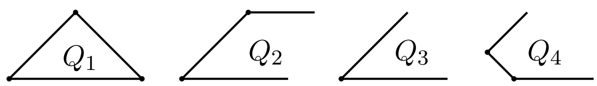

Figure 1.2: Types of F-simplices in $\mathbb{R}^2$.

The description of bounded F-simplices is known from [10]: a convex body $K \subset \mathbb{R}^n$ is an F-simplex if and only if it is a usual n-dimensional simplex. We generalize this result to the case of arbitrary F-simplices (possibly, unbounded). Theorem 4.1 shows that every F-simplex in $\mathbb{R}^n$ is a polyhedron expressible as the sum of a simplex and a polyhedral cone. Corollary 5.1 and Theorem 5.3 give a geometric classification of F-simplices in dimensions two and three. These results may suggest that the problem below has an affirmative answer.

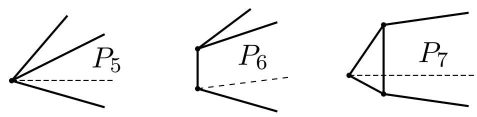

Figure 1.3: Additional types of F-simplices in $\mathbb{R}^3$.

Problem 1.2. *Is it true that a convex solid $K \subset \mathbb{R}^n$ is an F-simplex if and only if all n-dimensional intersections $K \cap (x + K)$, $x \in \mathbb{R}^n$, belong to at most n homothety classes?*

A variation of the condition (1) provides a characteristic property of simplicial cones (see [12]): a convex solid $K \subset \mathbb{R}^n$ is a simplicial cone if and only if all n-dimensional intersections $K\cap(x+K)$, $x \in \mathbb{R}^n$, are translates of K. Theorem 6.3 below refines this result by considering pairs of convex solids in the spirit of [11] and [13]. Namely, we prove that convex solids K_1 and K_2 in $\mathbb{R}^n$ are translates of the same simplicial cone if and only if there is a finite family $\mathcal{F}$ of convex solids such that every n-dimensional intersection $K_1 \cap (x + K_2)$, $x\in\mathbb{R}^n$, is a translate of a suitable set from $\mathcal{F}$.

2. Preliminaries

This section contains necessary definitions, notation, and results on convex sets in the n-dimensional Euclidean space $\mathbb{R}^n$ (see, e. g., [8] and [14] for details). The elements of $\mathbb{R}^n$ are called vectors (or points), and $\|x\|$ means the norm of a vector $x \in \mathbb{R}^n$. Also, $u\cdot v$ will mean the dot product of vectors u and v. In what follows, o stands for the zero vector of $\mathbb{R}^n$. Given distinct points $u, v \in \mathbb{R}^n$, we denote by $[u, v]$ and (u, v) the closed segment and the open segment with endpoints u and v. Also, the sets

$$[u, v\rangle = \{(1 - \lambda)u + \lambda v : \lambda \geq 0\}, \quad (u, v\rangle = \{(1 - \lambda)u + \lambda v : \lambda > 0\}$$

are called, respectively, the closed and open halfline through v with endpoint u. Clearly, $[u, v\rangle = [u, w\rangle$ and $(u, v\rangle = (u, w\rangle$ whenever $w \in (u, v\rangle$. Similarly, $\langle u, v\rangle$ stands for the line through distinct points $u, v \in \mathbb{R}^n$.

By an r-dimensional plane L in $\mathbb{R}^n$, where $0 \leq r \leq n$, we mean a translate of a suitable r-dimensional subspace S of $\mathbb{R}^n$: $L = c + S$, where $c \in \mathbb{R}^n$.

A *hyperplane* is a plane of dimension $n - 1$; it can be described as

$$H = \{x \in \mathbb{R}^n : x\cdot e = \gamma\}, \quad e \neq o, \quad \gamma \in \mathbb{R}. \tag{2}$$

Consequently, a hyperplane of the form (2) is a translate of the hypersubspace

$$S = \{x \in \mathbb{R}^n : x\cdot e = 0\}, \quad e \neq o. \tag{3}$$

Every hyperplane of the form (2) determines a pair of opposite closed halfspaces

$$V_1 = \{x \in \mathbb{R}^n : x\cdot e \leq \gamma\} \text{ and } V_2 = \{x \in \mathbb{R}^n : x\cdot e \geq \gamma\}, \tag{4}$$

and a pair of opposite open halfspaces

$$W_1 = \{x \in \mathbb{R}^n : x\cdot e < \gamma\} \text{ and } W_2 = \{x \in \mathbb{R}^n : x\cdot e > \gamma\}. \tag{5}$$

We say that a closed halfspace V externally (internally) supports a nonempty set X provided $X \cap \operatorname{int} V = \varnothing$ $(X \subset V)$ such that the boundary hyperplane of V supports X.

As usual, $\operatorname{cl} X$ and $\operatorname{int} X$ stand for the closure and interior of a set X. The convex hull and affine span of X, are denoted by $\operatorname{conv} X$ and $\operatorname{aff} X$, respectively, and $\dim X$ is defined as the dimension of the plane $\operatorname{aff} X$. The closed and open balls of radius $\rho > 0$ centered at a point $u \in \mathbb{R}^n$ are denoted by $B_\rho(u)$ and $U_\rho(X)$, respectively.

In what follows, K stands for a *nonempty* convex set in $\mathbb{R}^n$. The relative interior of K is denoted $\operatorname{rint} K$. Both sets $\operatorname{cl} K$ and $\operatorname{rint} K$ are convex. Furthermore, we have $\operatorname{cl} K = \operatorname{cl}(\operatorname{rint} K)$ and $\operatorname{rint} K = \operatorname{rint}(\operatorname{cl} K)$. If $u \in \operatorname{rint} K$ and $v \in K$, then there is a point $w \in K$ such that $u \in (v, w)$. For any point $u \in \operatorname{cl} K$ and an open ball

$U_\rho(u) \subset \mathbb{R}^n$, one has $\operatorname{rint} K \cap U_\rho(u) \neq \varnothing$. If K_1 and K_2 are convex sets in $\mathbb{R}^n$ satisfying the condition $\operatorname{rint} K_1 \cap \operatorname{rint} K_2 \neq \varnothing$, then

$$\operatorname{cl}(K_1 \cap K_2) = \operatorname{cl} K_1 \cap \operatorname{cl} K_2, \quad \operatorname{rint}(K_1 \cap K_2) = \operatorname{rint} K_1 \cap \operatorname{rint} K_2. \tag{6}$$

The relative boundary $\operatorname{rbd} K$ of K is defined by $\operatorname{rbd} K = \operatorname{cl} K \setminus \operatorname{rint} K$. It is known that $\operatorname{rbd} K \neq \varnothing$ if and only if K is not a plane. For any points $u, v \in \operatorname{rbd} K$, either $(u, v) \subset \operatorname{rbd} K$ or $(u, v) \subset \operatorname{rint} K$. If $u \in \operatorname{rint} K$ and $v \in \operatorname{aff} K \setminus K$, then the segment $[u, v]$ meets $\operatorname{rbd} K$ at a single point.

An *extreme face* of K is a convex subset F of K such that for any points $x, y \in K$ and a scalar $\lambda \in (0, 1)$ the inclusion $(1 - \lambda)x + \lambda y \in F$ is possible only if $x, y \in F$. For a given point $u \in K$, the intersection of all extreme faces of K containing u, denoted $F_u(K)$, is called the extreme face of K generated by u. If $x \in \operatorname{rbd} K$, then, $F_u(K) \subset \operatorname{rbd} K$. Zero-dimensional extreme faces of K are called *extreme points* of K and their union is denoted $\operatorname{ext} K$. If F is an extreme face of K, then $\operatorname{ext} F \subset \operatorname{ext} K$. A convex set K is called *line-free* provided it contains no lines. Any line-free closed convex set in $\mathbb{R}^n$ can be expressed as the convex hull of its extreme points and extreme halflines:

$$K = \operatorname{conv}(\operatorname{ext} K \cup \operatorname{extr} K). \tag{7}$$

An *exposed face* of a convex set K is the intersection of K with any hyperplane that supports K. Zero-dimensional exposed faces are called *exposed points* and their union is denoted $\exp K$. Any exposed face of K is its extreme face. In particular, $\exp K \subset \operatorname{ext} K$. Furthermore, $\operatorname{ext} K \subset \operatorname{cl}(\exp K)$.

By *convex cone* with apex u we mean a convex set $C \subset \mathbb{R}^n$ such that $u + \lambda(x - u) \in C$ for all $\lambda \geq 0$ and $x \in C$ (obviously, this definition implies that $u \in C$, although a stronger condition $\lambda > 0$ can be beneficial; see, e. g., [7]). For any point $x \in C \setminus \{u\}$, one has $[u, x\rangle \subset \operatorname{rbd} C$ if $x \in \operatorname{rbd} C$, and $(u, x\rangle \subset \operatorname{rint} C$ if $x \in \operatorname{rint} C$. Furthermore, $x + C \subset C$ whenever $x \in C$.

Given a convex set $K \subset \mathbb{R}^n$ and a point $u \in \mathbb{R}^n$, the generated cone with apex u is defined by

$$C_u(K) = \{u + \lambda(x - u) : \lambda \geq 0,\ x \in K\}.$$

If $u \in \operatorname{rbd} K$, then

$$\operatorname{rint} C_u(K) = C_u(\operatorname{rint} K) \setminus \{u\}. \tag{8}$$

A *polyhedron* is the intersection of finitely many closed halfspaces. Bounded polyhedra are called *polytopes*. A line-free closed convex set in $\mathbb{R}^n$ is a polyhedron if and only if it has finitely many extreme points and extreme halflines. It is known that the families of extreme and exposed faces of a polyhedron coincide (the same is true for the case of bounded faces of boundedly polyhedral sets).

3. Local conicity and polyhedrality

Theorem 3.1. *Given a closed convex set $K \subset \mathbb{R}^n$ and a point $u \in \operatorname{rbd} K$, one has $(a) \Leftrightarrow (b) \Rightarrow (c)$, where (a), (b), and (c) stand for the following conditions:*

(a) *K is conic at u,*

(b) *there is an open ball $U_\rho(u) \subset \mathbb{R}^n$ such that $[u, z] \subset \operatorname{rbd} K$ whenever we have $z \in \operatorname{rbd} K \cap U_\rho(u)$,*

(c) *$u \notin \operatorname{cl}(\operatorname{ext} K \setminus \{u\})$.*

Proof. $(a) \Rightarrow (b)$ The case $\dim K = 1$ is obvious: K is either a segment of the form $[u, v]$, or a closed halfline with endpoint u. In either case, there is an open ball $U_\rho(u) \subset \mathbb{R}^n$ of sufficiently small radius $\rho > 0$ such that $\operatorname{rbd} K \cap U_\rho(u) = \{u\}$; whence condition (b) trivially holds. Suppose that $\dim K \geq 2$. Consequently, $\operatorname{rbd} K \neq \{u\}$. Choose an open ball $U_\rho(u) \subset \mathbb{R}^n$ satisfying the condition $K \cap U_\rho(u) = C_u(K) \cap U_\rho(u)$. Then

$$\operatorname{rbd} K \cap U_\rho(u) = \operatorname{rbd} C_u(K) \cap U_\rho(u).$$

Choose any point $z \in \operatorname{rbd} K \cap U_\rho(u)$ distinct from u. Then $[u, z] \subset [u, z\rangle \subset \operatorname{rbd} C_u(K)$ and $\|u - z\| < \rho$. Hence

$$[u, z] \subset \operatorname{rbd} C_u(K) \cap U_\rho(u) = \operatorname{rbd} K \cap U_\rho(u) \subset \operatorname{rbd} K.$$

$(b) \Rightarrow (a)$ Let $U_\rho(u) \subset \mathbb{R}^n$ be an open ball such that $[u, z] \subset \operatorname{rbd} K$ whenever $z \in \operatorname{rbd} K \cap U_\rho(u)$. We assert that

$$K \cap U_\rho(u) = \operatorname{cl} C_u(K) \cap U_\rho(u). \tag{9}$$

Since the inclusion $K \cap U_\rho(u) \subset \operatorname{cl} C_u(K) \cap U_\rho(u)$ is obvious, it suffices to prove the opposite one. First, we are going to prove the inclusion

$$\operatorname{rint} C_u(K) \cap U_\rho(u) \subset K \cap U_\rho(u). \tag{10}$$

Assume, for contradiction, the existence of a point

$$x \in (\operatorname{rint} C_u(K) \cap U_\rho(u)) \setminus (K \cap U_\rho(u)) = (\operatorname{rint} C_u(K) \setminus K) \cap U_\rho(u).$$

Due to (8), there is a point $v \in \operatorname{rint} K \setminus \{u\}$ such that $x = u + \mu(v - u)$ for a suitable $\mu > 0$. The inclusions $v \in \operatorname{rint} K$ and $x \in C_u(K) \setminus K \subset \operatorname{aff} K \setminus K$ imply the existence of a point $z \in \operatorname{rbd} K \cap [v, x]$. Since $v \in [u, z]$, one has $(u, z) \subset \operatorname{rint} K$. On the other hand, because $\|u - z\| < \|u - x\| < \rho$, condition (b) gives $v \in [u, z] \subset \operatorname{rbd} K$, contrary to the choice of v. Summing up, inclusion (10) holds. Therefore, according to (6),

$$\begin{aligned}\operatorname{cl} C_u(K) \cap B_\rho(u) &= \operatorname{cl}(\operatorname{rint} C_u(K)) \cap \operatorname{cl}(U_\rho(u)) \\ &= \operatorname{cl}(\operatorname{rint} C_u(K) \cap U_\rho(u)) \subset \operatorname{cl}(K \cap U_\rho(u)) = K \cap B_\rho(u).\end{aligned}$$

Hence

$$\begin{aligned}\operatorname{cl} C_u(K) \cap U_\rho(u) &= (\operatorname{cl} C_u(K) \cap B_\rho(u)) \cap U_\rho(u) \\ &\subset (K \cap B_\rho(u)) \cap U_\rho(u) = K \cap U_\rho(u).\end{aligned}$$

Summing up, the equality (9) holds. Since $\operatorname{cl} C_u(K)$ is a cone with apex u, (9) shows that K is conic at u.

$(b) \Rightarrow (c)$ Let $U_\rho(u) \subset \mathbb{R}^n$ be an open ball such that $[u, z] \subset \operatorname{rbd} K$ whenever $z \in \operatorname{rbd} K \cap U_\rho(u)$. Assume for a moment that $u \in \operatorname{cl}(\operatorname{ext} K \setminus \{u\})$. Then there is a point

$$v \in \operatorname{ext} K \cap U_\rho(u) \setminus \{u\} \subset \operatorname{rbd} K \cap U_\rho(u) \setminus \{u\}.$$

Combining (6) and (9), one has

$$\begin{aligned}\operatorname{rbd} K \cap U_\rho(u) &= (K \setminus \operatorname{rint} K) \cap U_\rho(u) = (K \cap U_\rho(u)) \setminus (\operatorname{rint} K \cap U_\rho(u)) \\ &= (K \cap U_\rho(u)) \setminus \operatorname{rint}(K \cap U_\rho(u)) \\ &= (\operatorname{cl} C_u(K) \cap U_\rho(u)) \setminus \operatorname{rint}(\operatorname{cl} C_u(K) \cap U_\rho(u)) \\ &= (\operatorname{cl} C_u(K) \setminus \operatorname{rint}(\operatorname{cl} C_u(K)) \cap U_\rho(u) = \operatorname{rbd} C_u(K) \cap U_\rho(u).\end{aligned}$$

Hence $v \in \operatorname{rbd} C_u(K)$, which implies the inclusion $[u, v\rangle \subset \operatorname{rbd} C_u(K)$. Consequently,

$$[u, v\rangle \cap U_\rho(u) \subset \operatorname{rbd} C_u(K) \cap U_\rho(u) = \operatorname{rbd} K \cap U_\rho(u) \subset K \cap U_\rho(u).$$

Choose a point $w \in ([u, v\rangle \setminus [u, v]) \cap U_\rho(u)$. Obviously $v \in (u, w) \subset K$, and the latter inclusion implies that $v \notin \operatorname{ext} K$, a contradiction with the choice of v. □

Remark 3.2. If $n \geq 3$, then, generally, $(c) \not\Rightarrow (a)$ in the notation of Theorem 3.1. Indeed, let $K \subset \mathbb{R}^3$ be the bounded circular cylinder given by

$$K = \{(x, y, z) : x^2 + y^2 \leq 1, 0 \leq z \leq 1\}.$$

The boundary point $u = (1, 0, 0.5)$ of K is at distance 0.5 from $\operatorname{ext} K$, while K is not conic at u.

Lemma 3.3. *If a halfline h with endpoint u is contained in an open halfspace $W = \{x \in \mathbb{R}^n : x{\cdot}e < \gamma\}$, then $v{\cdot}e \leq u{\cdot}e$ for every point $v \in h$.*

Proof. Assume for a moment the existence of a point $v \in h$ such that $v{\cdot}e > u{\cdot}e$.

Let $$\lambda = (\gamma - u{\cdot}e)/(v{\cdot}e - u{\cdot}e) \quad \text{and} \quad z = (1 - \lambda)u + \lambda v.$$

Then $\lambda > 0$ and $z \in h \subset W$, which gives $z{\cdot}e < \gamma$. On the other hand,

$$z{\cdot}e = ((1 - \lambda)u + \lambda v){\cdot}e = u{\cdot}e + \lambda(v{\cdot}e - u{\cdot}e) = u{\cdot}e + (\gamma - u{\cdot}e) = \gamma,$$

a contradiction. □

Theorem 3.4. *Let $K \subset \mathbb{R}^n$ be a line-free closed convex set. For a point $u \in \operatorname{ext} K$, the following conditions are equivalent:*

- (a) *K is conic at u,*
- (b) *there is an open ball $U_\rho(u) \subset \mathbb{R}^n$ such that $[u, z] \subset \operatorname{rbd} K$ whenever we have $z \in \operatorname{rbd} K \cap U_\rho(u)$,*
- (c) *u is an isolated point in $\operatorname{ext} K$.*

Proof. In view of Theorem 3.1, it suffices to show that $(c) \Rightarrow (b)$. Since the set $\exp K$ is dense in $\operatorname{ext} K$ and u is isolated in $\operatorname{ext} K$, we conclude that $u \in \exp K$. Choose a hyperplane $H \subset \mathbb{R}^n$ with the property $H \cap K = \{u\}$. Express H as $H = \{x \in \mathbb{R}^n : x{\cdot}e = \gamma\}$, where $\gamma = u{\cdot}e$. Suppose that K lies in the halfspace $V = \{x \in \mathbb{R}^n : x{\cdot}e \leq \gamma\}$.

Assume, for contradiction, the existence of points $z_1, z_2, \ldots$ in $\operatorname{rbd} K \setminus \{u\}$ which converge to u such that $[u, z_i] \not\subset \operatorname{rbd} K$ for all $i \geq 1$. Then we have $z_i{\cdot}e < \gamma$ and $\lim_{i\to\infty} z_i{\cdot}e = \gamma$. Let F_i be the extreme face of K generated by z_i. Because we have $z_i \in \operatorname{rbd} K$ we conclude $F_i \subset \operatorname{rbd} K$. Clearly, $u \notin F_i$, since otherwise $[u, z_i] \subset F_i \subset \operatorname{rbd} K$. As a closed convex subset of K, the set F_i is convex and closed. Also, it is line-free as a subset of K. Hence F_i has extreme points.

We assert the existence of a point $v_i \in \operatorname{ext} F_i$ such that $z_i \cdot e \le v_i \cdot e$. Indeed, assume for a moment that $v \cdot e < z_i \cdot e$ for every extreme point v of F_i. Under this assumption, we will show that F_i is contained in the open halfspace

$$W_i = \{x \in \mathbb{R}^n : x \cdot e < \gamma_i\}, \quad \text{where} \quad \gamma_i = z_i \cdot e.$$

Indeed, F_i can be expressed as $F_i = \operatorname{conv}(\operatorname{ext} F_i \cup \operatorname{extr} F_i)$, where, possibly, we have $\operatorname{extr} F_i = \varnothing$. By the assumption, $\operatorname{ext} F_i \subset W_i$. Since $\operatorname{extr} F_i$, if nonempty, is the union of extreme halflines of F_i, each of them originated at an extreme point of F_i and contained in V, Lemma 3.3 implies that $\operatorname{extr} F_i \subset W_i$. Hence $\operatorname{ext} F_i \cup \operatorname{extr} F_i \subset W_i$, and the convexity of W_i gives

$$F_i = \operatorname{conv}(\operatorname{ext} F_i \cup \operatorname{extr} F_i) \subset W_i.$$

In particular, $z_i \in F_i \subset W_i$, contrary to $z_i \cdot e = \gamma_i$.

So, there are points $v_i \in \operatorname{ext} F_i$ satisfying the condition $z_i \cdot e \le v_i \cdot e$, $i \ge 1$. Clearly, $v_i \ne u$ and $v_i \in \operatorname{ext} K$ for all $i \ge 1$. Furthermore, $\lim_{i \to \infty} v_i \cdot e = \gamma$. Combining a compactness argument and the condition $H \cap K = \{u\}$, we finally obtain that $u = \lim_{i \to \infty} v_i \in \operatorname{cl}(\operatorname{ext} K \setminus \{u\})$, in contradiction with the choice of u. □

We recall that a set $X \subset \mathbb{R}^n$ is *discrete* if every its point is isolated in X, that is, for every point $z \in X$ there is a neighborhood $U_\rho(z) \subset \mathbb{R}^n$ that contains no other points of X. We will need the following lemma.

Lemma 3.5. ([6], Corollary 5.7) *If $K \subset \mathbb{R}^n$ is a closed convex set and $K = \operatorname{conv} S$ for a suitable subset S of K, then K is boundedly polyhedral if and only if K is polyhedral at every point of S.* □

Theorem 3.6. *For a line-free closed convex set $K \subset \mathbb{R}^n$, the following conditions are equivalent:*

(a) *K is boundedly polyhedral,*

(b) *K is polyhedral at every point $u \in \operatorname{ext} K$.*

Proof. $(a) \Rightarrow (b)$ Since a boundedly polyhedral set is polyhedral at every its point, it remains to show that the set $\operatorname{ext} K$ is discrete. For this purpose, choose any point $u \in \operatorname{ext} K$. Then K is locally conic at u, and Theorem 3.4 shows that u is an isolated point of $\operatorname{ext} K$.

$(b) \Rightarrow (a)$ A combination of the equality (7) and Lemma 3.5 implies that K is boundedly polyhedral if and only if it is polyhedral at every point of $\operatorname{ext} K \cup \operatorname{extr} K$. Due to condition (b), it suffices to prove that the set K is polyhedral at every point $u \in \operatorname{extr} K \setminus \operatorname{ext} K$. Denote by h_0 the extreme halfline of K which contains u. Let v be the endpoint of h_0, and let a point $z_0 \in h_0$ be such that $u \in (v, z_0)$. Since $v \in \operatorname{ext} K$, the set K is polyhedral at v. Consequently, the generated cone $C_v(K)$ is polyhedral.

Denote by $h_0, h_1, \ldots, h_m$ all extreme halflines of $C_v(K)$. By the polyhedrality of $C_u(K)$, these halflines are exposed in $C_u(K)$. In particular, h_0 is an exposed halfline of K. Choose an open ball $U_\rho(v) \subset \mathbb{R}^n$ such that $K \cap U_\rho(v) = C_v(K) \cap U_\rho(v)$.

Denote by I_i the semi-open segment $h_i \cap U_\rho(v)$, $1 \le i \le m$.

Now, choose a hyperplane $H \subset \mathbb{R}^n$ satisfying the condition $K \cap H = h_0$. Let L be an $(n-2)$-dimensional plane in H satisfying the condition $L \cap h_0 = \{z_0\}$. Since the point v is isolated in $\operatorname{ext} K$, a continuity argument shows that H can be slightly rotated about L such that a new position of H meets every segment I_i at some point z_i, $1 \le i \le m$. Clearly, the set $Q = \operatorname{conv}\{v, z_0, z_1, \dots, z_m\}$ is an n-dimensional polytope. Furthermore, $Q \subset K$ due to the inclusion $\{v, z_0, z_1, \dots, z_m\} \subset K$.

Let V is the closed halfspace determined by H and containing v. Then $h_i \cap V = [v, z_i]$, $0 \le i \le m$, which gives the inclusion $Q \subset C_v(K) \cap V$. Hence

$$\begin{aligned} K \cap V &\subset C_v(K) \cap V = \operatorname{conv}(h_0 \cup h_1 \cup \dots h_m) \cap V \\ &= \operatorname{conv}([v, z_0] \cup [v, z_1] \cup \dots \cup [v, z_m]) = \operatorname{conv}\{v, z_0, z_1, \dots, z_m\} \\ &= Q \subset K \cap V. \end{aligned}$$

Thus $K \cap V = Q$, implying that $K \cap V$ is an n-dimensional polytope. Finally, choose an n-dimensional polytope $P \subset V$ such that $Q \subset P$ and $u \in \operatorname{int} P$. Consequently, $u \in (v, z_0) \subset \operatorname{int} P$. So, K is polyhedral at u, as desired. □

4. Polyhedrality of F-simplices

Theorem 4.1. *Every F-simplex in $\mathbb{R}^n$ is a polyhedron, which can be expressed as the sum of a simplex and a polyhedral cone.*

We divide the proof of Theorem 4.1 into a sequence of lemmas.

Lemma 4.2. *Let $K \subset \mathbb{R}^n$ be a convex set, and let L be a nonempty convex subset of K. If a vector $x \in \mathbb{R}^n$ satisfies the condition $\operatorname{rint} L \cap (x + \operatorname{rint} L) \neq \varnothing$, then $\operatorname{rint} K \cap (x + \operatorname{rint} K) \neq \varnothing$.*

Proof. Assume, for contradiction, that $\operatorname{rint} K \cap (x + \operatorname{rint} K) = \varnothing$. Then there is a hyperplane $H \subset \mathbb{R}^n$ properly separating K and $x + K$ (so that at least one of K and $x + K$ is not in H). The segment $[o, x]$ is not parallel to H, since otherwise both K and $x + K$ would be contained in the same closed halfspace determined by H. Denote by V_1 and V_2 the closed halfspaces determined by H and containing K and $x + K$, respectively. Consequently,

$$L \cap (x + L) \subset K \cap (x + K) \subset V_1 \cap V_2 = H. \tag{11}$$

Choose a point $z \in \operatorname{rint} L \cap (x + \operatorname{rint} L)$. The inclusion $z \in \operatorname{rint} L$ gives

$$z + x \in x + \operatorname{rint} L = \operatorname{rint}(x + L).$$

Similarly, the inclusion $z \in x + \operatorname{rint} L$ results in $z - x \in \operatorname{rint} L$. Since $z \in \operatorname{rint} L$, there is a point $u \in (z, z + x) \cap L$ such that $z \in (u, z - x)$. Similarly, the inclusion $z \in \operatorname{rint}(x + L)$ implies the existence of a point $v \in (z, z - x) \cap (x + L)$ such that $z \in (z + x, v)$. Therefore, $z \in (u, v) \subset L \cap (x + L) \subset H$, according to (11). On the other hand, (u, v) is contained in the line through $z - x$ and $z + x$, and thus $(u, v) \cap H = \{z\}$. The obtained contradiction proves the lemma. □

Lemma 4.3. *Let closed convex sets K_1 and K_2 in $\mathbb{R}^n$ be internally supported by a closed halfspace $V \subset \mathbb{R}^n$ with boundary hyperplane H. If K_1 and K_2 are homothetic, then the sets $H \cap K_1$ and $H \cap K_2$ are homothetic.*

Proof. Choosing any vector $u \in H$ and replacing the sets K_1, K_2, and H with $K_1 - u$, $K_2 - u$, and $H - u$, we may assume that H is a subspace. Since K_1 and K_2 are homothetic, we can write that $K_1 = a + \mu K_2$ for a suitable vector $a \in \mathbb{R}^n$ and a scalar $\mu > 0$. Without loss of generality, we may assume that $\mu \geq 1$ (otherwise renumber K_1 and K_2 in the equality $K_2 = \mu^{-1}a + \mu^{-1}K_1$).

We assert that $a \in H$. Indeed, if $a \in \mathbb{R}^n \setminus V$, then K_1 should be contained in $\operatorname{int} V$ and thus cannot be supported by H. Similarly, if $a \in \operatorname{int} V$, then $K_1 \not\subset V$, and again cannot be supported by H. Hence $a \in H$. Since $H = a + \mu H$, we obtain

$$H \cap K_1 = (a + \mu H) \cap (a + \mu K_2) = a + \mu(H \cap K_2),$$

which shows that the sets $H \cap K_1$ and $H \cap K_2$ are homothetic. □

Given points $u, v \in \operatorname{bd} K$, we say that the segment $[u, v]$ is a *chord* of K provided $[u, v] = K \cap \langle u, v\rangle$. Furthermore, a chord $[u, v]$ of K will be called *special* provided at least one of the points u, v belongs to $\exp K$.

Lemma 4.4. *Let $K \subset \mathbb{R}^n$ be an F-simplex. If there is a vector $x \in \mathbb{R}^n$ such that $\operatorname{int}(x + K)$ contains a special chord $[u, v]$ of K, then $K \subset x + K$.*

Proof. Assume, for contradiction, the existence of a vector $x \in \mathbb{R}^n$ such that the set $\operatorname{int}(x + K)$ contains a special chord $[u, v]$ of K, with $u \in \exp K$, while $K \not\subset x + K$. Choose a point $w \in K \setminus (x + K)$. By a continuity argument, there is a scalar $\delta > 0$ such that $[u, v]$ is in the interior of the convex solid $\varepsilon(w - u) + x + K$ and $w \notin \varepsilon(w - u) + x + K$ for all $0 < \varepsilon < \delta$. Put

$$M(\varepsilon) = K \cap (\varepsilon(w - u) + x + K), \quad 0 < \varepsilon < \delta.$$

Denote by V a closed halfspace of $\mathbb{R}^n$ with the property $V \cap K = \{u\}$. By the above argument, $V \cap M(\varepsilon) = \{u\}$ for all $0 < \varepsilon < \delta$. Hence u is an exposed point of every $M(\varepsilon)$. Let $[u, w(\varepsilon)] = [u, w] \cap M(\varepsilon)$.

By the assumption, the convex solids $M(\varepsilon)$, $0 < \varepsilon < \delta$, belong to at most finitely many homothety classes. Hence there are distinct scalars $\varepsilon_1, \varepsilon_2$ such that $M(\varepsilon_1)$ and $M(\varepsilon_2)$ are homothetic. Clearly, $M(\varepsilon_1) = u + \lambda(M(\varepsilon_2) - u)$, for a suitable $\lambda > 0$. By the choice of δ, we have $[u, v] \cap M(\varepsilon) = [u, v]$ for all $0 < \varepsilon < \delta$. Consequently, $M(\varepsilon_1) = M(\varepsilon_2)$. Therefore, $w(\varepsilon_1) = w(\varepsilon_2)$. On the other hand,

$$\|u - w(\varepsilon)\| = \|u - w(0)\| + \varepsilon\|u - w\|, \quad 0 < \varepsilon < \delta,$$

which implies that the length of $[u, w(\varepsilon)]$ monotonically decreases as $\varepsilon \to 0$. The obtained contradiction shows that $K \subset x + K$. □

Lemma 4.5. *If $K \subset \mathbb{R}^n$ is an F-simplex, then the set $\exp K$ has no accumulation points. Consequently, the set $\operatorname{ext} K$ is discrete.*

Proof. Assume, for contradiction, that $\exp K$ has an accumulation point, say u. Let $u_1, u_2, \ldots$ be a sequence of points from $\exp K \setminus \{u\}$ converging to u. Choose a point $x \in \operatorname{int} K$. Then $u \in \operatorname{int}(u - x + K)$. Hence there is an index r such that $u_i, u_j \in \operatorname{int}(u - x + K)$ for all $i, j \geq r$. Clearly, $[u_r, u_{r+1}]$ is a chord of K, and Lemma 4.4 gives the inclusion $K \subset u - x + K$.

Let $x_1, x_2, \ldots$ be a sequence of points in $\operatorname{int} K$ converging to a point $w \in \exp K \setminus \{u\}$. Because each set $\operatorname{int}(u - x_i + K)$ contains u, we obtain that $K \subset u - x_i + K$, $i \geq 1$. By a continuity argument, $u \in K \subset u - w + K$. Let V be a closed halfspace such that $V \cap K = \{w\}$. Then

$$(u - w + V) \cap (u - w + K) = u - w + V \cap K = u - w + \{w\} = \{u\}.$$

Thus $$u \in (u - w + V) \cap K \subset (u - w + V) \cap (u - w + K) = \{u\},$$

implying that $(u - w + V) \cap K = \{u\}$. Finally, since $u - w + V$ is a translate of V and both these halfspaces externally support K, we obtain that $u - w + V = V$. Consequently,

$$\{w\} = V \cap K = (u - w + V) \cap K = \{u\},$$

in contradiction with the assumption $w \neq u$.

Finally, since $\operatorname{ext} K \subset \operatorname{cl}(\exp K) = \exp K$, the set $\operatorname{ext} K$ has no accumulation point. $\square$

Lemma 4.6. *If $K \subset \mathbb{R}^n$ is an F-simplex, then K is polyhedral at every point $u \in \operatorname{ext} K$.*

Proof. Since the assertion is obvious if K is bounded, we may assume that K is unbounded. Choose a point $u \in \operatorname{ext} K$. Translating K on $-u$, we may assume that $u = o$. According to Lemma 4.5, o is an isolated point of $\operatorname{ext} K$, and Theorem 3.4 shows that K is conic at o. Let $C = C_o(K)$. Choose a scalar $\rho > 0$ such that $K \cap U_\rho(o) = C \cap U_\rho(o)$. Because the cone C is closed and line-free, there is a hyperplane $H \subset \mathbb{R}^n$ such that the $(n-1)$-dimensional convex set $B = K \cap H$ is a compact base for C and is contained in $U_\rho(o)$. Consequently, if $V \subset \mathbb{R}^n$ is a closed halfspace determined by H and containing o, then $K \cap V = C \cap V$.

Assume, for contradiction, that K is not polyhedral at o. Then the cone C is not polyhedral and thus has infinitely many exposed halflines, say $h_1, h_2, \ldots$. By a compactness argument, we may assume that the sequence $h_1, h_2, \ldots$ converges to a halfline $h_0 \subset \operatorname{bd} C$ with endpoint o, which is distinct from every h_i, $i \geq 1$. Every halfline h_i meets B along a single point z_i, which belongs to $\exp B$, $i \geq 1$. Clearly, the sequence $z_1, z_2, \ldots$ converges to the point $z_0 \in h_0 \cap \operatorname{rbd} B$. Let

$$z_0' = z_0/2, \quad w_i = (z_0 - z_i)/2, \quad C_i = w_i + C, \quad K_i = w_i + K, \quad i \geq 1.$$

Clearly, $K_i \cap V = C_i \cap V$. Since $z_0' \in C$, one has $z_0' + C \subset C$. Similarly, because the halfline $w_i + h_i \subset w_i + C$ meets h_0 at z_0', every cone C_i contains $z_0' + C$. Hence

$$z_0' + C \subset C \cap C_i, \quad i \geq 1.$$

Consider the family of closed convex sets $M_i = K \cap K_i$, $i \geq 1$. First, we observe that all sets M_i are convex solids. Indeed, denote by V' a translate of V such that z_0' belongs to the boundary hyperplane of V'. Obviously,

$$K \cap V' = C \cap V' \quad \text{and} \quad K_i \cap V' = C_i \cap V'.$$

Then
$$\begin{aligned} z_0' + (K \cap V') &= z_0' + (C \cap V') = (z_0' + C) \cap (z_0' + V') \\ &\subset (C \cap C_i) \cap V = (C \cap V) \cap (C_i \cap V) \\ &= (K \cap V) \cap (K_i \cap V) \subset K \cap K_i = M_i. \end{aligned}$$

Because the set $z_0' + (K \cap V')$ is n-dimensional, all M_i are convex solids.

The halfline $w_i + h_i$ is an exposed face of C_i. Hence the 1-dimensional set $(w_i + h_i) \cap M_i$ is an exposed face of M_i and has z_0' as an endpoint. Since no two halflines $h_1, h_2, \dots$ are parallel and all points $w_1, w_2, \dots$ are pairwise distinct, we conclude that no two of the convex solids $M_1, M_2, \dots$ can be homothetic. The latter is in contradiction with the assumption on K. □

A combination of Theorem 3.6 and Lemmas 4.5 and 4.6 implies the following corollary.

Corollary 4.7. *Every F-simplex in $\mathbb{R}^n$ is boundedly polyhedral.* □

Lemma 4.8. *If K is an F-simplex in $\mathbb{R}^n$, then every exposed face P of K also is an F-simplex in the space $L = \operatorname{aff} P$.*

Proof. Let P be an exposed face of K. Put $r = \dim P$. Since the cases $r = 0$ and $r = n$ are trivial, we may assume that $1 \leq r \leq n-1$. Then the plane $L = \operatorname{aff} P$ is r-dimensional and $P = K \cap L$. Denote by $\mathcal{K}$ the family of all translates $x + K$, $x \in \mathbb{R}^n$, whose exposed faces $x + P$ are contained in L and the sets $P \cap (x + P)$ have dimension r. Lemma 4.2 shows that $\dim (K \cap (x + K)) = n$ for any choice of $x + P \in \mathcal{K}$. Clearly, the r-dimensional set $P \cap (x + P)$ is an exposed face of $K \cap (x + K)$ such that

$$\operatorname{aff} (P \cap (x + P)) = L \quad \text{and} \quad L \cap ((K \cap (x + K)) = P \cap (x + P).$$

Denote by H a hyperplane supporting K such that $H \cap K = P$. Then $L \subset H$ and $x + L \subset H$ for every set $x + K \in \mathcal{K}$. Thus $H \cap (x + K) = x + L$. By the assumption, the n-dimensional intersections $K \cap (x + K)$ belong to at most finitely many homothety classes. Consequently, Lemma 4.3 implies that their exposed faces $P \cap (x + P)$ belong to at most finitely many homothety classes. In other words, P is an F-simplex in the space $L = \operatorname{aff} P$. □

Lemma 4.8 and the results from [10] imply the following corollary and the fact that bounded extreme faces of the F-simplex K are also its exposed faces.

Corollary 4.9. *If $K \subset \mathbb{R}^n$ is an F-simplex, then every bounded extreme face of K is a usual simplex.* □

Lemma 4.10. *If a boundedly polyhedral set $K \subset \mathbb{R}^n$ has the largest bounded extreme face, then K is a polyhedron.*

Proof. Let P be the largest bounded extreme face of K. Then every vertex of K is contained in P. Since K is boundedly polyhedral, there are finitely many extreme halflines of K originated at P. Furthermore, there are no other extreme halflines of K, since otherwise the endpoints of these halflines can be connected to P by polygonal lines containing vertices of K located outside P. Since K is the convex hull P and of all extreme halflines of K, we conclude that K is a polyhedron. $\square$

Lemma 4.11. *If $K \subset \mathbb{R}^n$ is an F-simplex, then K has the largest bounded extreme face.*

Proof. The assertion is trivial provided K is bounded (in this case K is the largest extreme face of K). Suppose that K is unbounded. According to Corollary 4.7, K is boundedly polyhedral. Since K is line-free, it has bounded faces (for instance, any vertex of K is its 0-dimensional extreme face). Choose a maximal under inclusion bounded extreme face, say P, of K. To show that P is the largest bounded extreme face of K, it suffices prove that every vertex of K belongs to P.

Case 1: $\dim P = n - 1$. By Corollary 4.9, P is an $(n-1)$-dimensional simplex. Denote by H the hyperplane containing P. Let u be a point in $\operatorname{rint} P$ and h be an open halfline with endpoint u entirely contained in $\operatorname{int} K$. Assume, for contradiction, the existence of a vertex v of K which does not belong to P. Let H' the hyperplane through v parallel to H. Clearly, H' meets h at some point w and the open segment (v, w) is contained in $\operatorname{int} K$.

Choose a scalar $\delta > 0$ so small that the intersections $P \cap (\varepsilon(w - v) + u + P)$ are $(n-1)$-dimensional whenever $0 < \varepsilon < \delta$. By Lemma 4.2, all sets

$$M(\varepsilon) = K \cap (\varepsilon(w - v) + u + K), \quad 0 < \varepsilon < \delta, \tag{12}$$

are n-dimensional. Let

$$u(\varepsilon) = \varepsilon(w - v) + u, \quad v(\varepsilon) = \varepsilon(w - v) + v, \quad 0 < \varepsilon < \delta.$$

Clearly, the closed segment $[u(\varepsilon), v(\varepsilon)]$ is a special chord of $M(\varepsilon)$.

Since K is an F-simplex and the family (12) is infinite, there is a pair of distinct scalars ε_1 and ε_2 such that $M(\varepsilon_1)$ and $M(\varepsilon_2)$ are homothetic. Suppose that $\varepsilon_1 < \varepsilon_2$. A simple geometric argument shows that every set $H \cap M(\varepsilon)$ is an $(n-1)$-dimensional simplex and that $H \cap M(\varepsilon_1)$ is larger than $H \cap M(\varepsilon_2)$. Hence $M(\varepsilon_1)$ is larger than $M(\varepsilon_2)$. Consequently, the chord $[u(\varepsilon_1), v(\varepsilon_1)]$ is longer than the parallel chord $[u(\varepsilon_2), v(\varepsilon_2)]$. On the other hand, both points $v(\varepsilon_1)$ and $v(\varepsilon_2)$ are at the same distance from H. The obtained contradiction shows that v cannot be outside P.

Case 2: $\dim P = 0$. Then P is a vertex of K, say u. Assume for a moment that K contains another vertex, say v. Choosing an n-dimensional polytope $Q \subset \mathbb{R}^n$ whose interior contains the set $\{u, v\}$ and considering the polytope $K \cap Q$, we conclude that u and v can be joined by a polygonal line consisting of edges of K. So, there is a segment edge of K, contrary to the choice of P. Hence u is the only vertex of K.

Case 3: $1 \le \dim P \le n - 1$. This case is considered by induction on n. Since the case $n = 1$ is obvious, we may assume that $n \ge 2$. If $n = 2$, then $\dim P = 1$. According to Case 1 above, P is the largest bounded face of K.

Let $n \geq 3$. Denote by L the plane aff P, and let $r = \dim L$. Since P is also an exposed face of K, there is a hyperplane $H \subset \mathbb{R}^n$ satisfying the condition $H \cap K = P$.

Denote by S the subspace parallel to L. Using the polyhedrality of K in a neighborhood of P, one can find $n-r$ hyperplanes, say $G_1, \dots, G_{n-r}$, which contain P and support K along exposed facets of K. Put $Q_i = K \cap G_i$, $1 \leq i \leq n-r$. By Lemma 4.8, every set Q_i is an F-simplex. Clearly, P is the maximal under inclusion bounded face of Q_i. The inductive assumption implies that P is the largest bounded extreme face of Q_i. Clearly, Q_i is the vector sum of P and a polyhedral cone.

Choose a point $u \in \operatorname{rint} P$ and a closed halfline $h_i \subset Q_i$ with endpoint u, where $1 \leq i \leq n-r$. Consequently, the sum $T_i = P + (h_i - u)$ is contained in $\operatorname{rint} Q_i$.

Let $T = \operatorname{conv}(T_1 \cup \dots \cup T_{n-r})$. The above argument shows that $\dim T = n$ and the intersection $(z+S) \cap T$ is r-dimensional whenever $z \in T$. Then, as easy to see, the intersection $(z+S) \cap K$ is r-dimensional whenever $z \in K$.

Now, assume, for contradiction, the existence of a vertex of K which does not belong to P. Since this vertex can be joined with P by a polygonal line consisting of edges of K, we conclude the existence of vertices $v \in P$ and $w \in K \setminus P$ such that the segment $[v, w]$ is an edge of K.

Consider the family of convex solids

$$M(\varepsilon) = K \cap (\varepsilon(u-v) + K), \quad 0 < \varepsilon < 1. \tag{13}$$

Let $$v(\varepsilon) = \varepsilon(u-v) + v, \quad w(\varepsilon) = \varepsilon(u-v) + w, \quad 0 < \varepsilon < 1.$$

By the above argument, $w(\varepsilon) \in w + S \subset K$ and $[v(\varepsilon), w(\varepsilon)]$ is a special chord of $M(\varepsilon)$.

Since K is an F-simplex and the family (13) is infinite, there is a pair of distinct scalars ε_1 and ε_2 such that $M(\varepsilon_1)$ and $M(\varepsilon_2)$ are directly homothetic. Suppose that $\varepsilon_1 < \varepsilon_2$. Then the 1-dimensional face $[u, v(\varepsilon_1)]$ of $M(\varepsilon_1)$ is longer than the respective 1-dimensional face $[u, v(\varepsilon_2)]$ of $M(\varepsilon_2)$. Hence $M(\varepsilon_1)$ is larger than $M(\varepsilon_2)$. Consequently, the chord $[v(\varepsilon_1), w(\varepsilon_1)]$ should be longer than the parallel chord $[v(\varepsilon_2), w(\varepsilon_2)]$. On the other hand, both points $w(\varepsilon_1)$ and $w(\varepsilon_2)$ are at the same distance from H. The obtained contradiction shows that no vertex of K can be outside P. Thus P is the largest bounded face of K. $\square$

The corollary below, which follows from Lemmas 4.10 and 4.11, concludes the proof of Theorem 4.1.

Corollary 4.12. *If $K \subset \mathbb{R}^n$ is an F-simplex, then K is a polyhedron, which can be expressed as the vector sum of a simplex and a polyhedral cone.* $\square$

5. F-simplices in $\mathbb{R}^2$ and $\mathbb{R}^3$

Corollary 4.12 immediately implies the following description of F-simplices in $\mathbb{R}^2$.

Corollary 5.1. *For a convex solid $K \subset \mathbb{R}^2$, the following conditions are equivalent:*

(*a*) *K is an F-simplex,*

(*b*) *K is described by one of the sets in Figure* 1.2,

(*c*) *all two-dimensional intersections $K \cap (x+K)$, $x \in \mathbb{R}^2$, belong to at most two homothety classes.* $\square$

We need the following lemma for a description of F-simplices in $\mathbb{R}^3$. If $K \subset \mathbb{R}^3$ is a polyhedron and x is a vertex of K, then, in a standard way, the degree of x in K, denoted $\deg_K(x)$, is the number of edges of K containing x.

Lemma 5.2. *Let $K \subset \mathbb{R}^3$ be an unbounded F-simplex, and let P be the largest bounded face of K. The following assertions hold:*

(a) *if P is a triangle or a segment, then $\deg_K(x) = 3$ for every vertex $x \in P$,*

(b) *if P is a singleton, say u, then $3 \le \deg_K(u) \le 4$.*

Proof. If x is a vertex of P, then the inequality $\deg_K(x) \ge 3$ is obvious. So, it remains to show that $\deg_K(x) \le 3$ if P is a triangle or a segment, and $\deg_K(x) \le 4$ if P is a singleton.

(a) Let P be a triangle. Assume, for contradiction, that $\deg_K(u) \ge 4$ for a vertex u of P. Then K has distinct extreme halflines $[u, v\rangle$ and $[u, w\rangle$ that belong to a common 2-dimensional face, say G, of K. Denote by L the plane containing P, and let L' be a plane parallel to P which cuts $\operatorname{rint} G$ along a segment $[x, z]$.

Choose a scalar $\delta > 0$ such that for any $\varepsilon \in (0, \delta)$, both sets

$$Q(\varepsilon) = P \cap (\varepsilon(x - z) + P) \quad \text{and} \quad R(\varepsilon) = G \cap (\varepsilon(x - z) + G)$$

are 2-dimensional. Consider the family of convex solids

$$M(\varepsilon) = K \cap (\varepsilon(x - z) + K), \quad 0 < \varepsilon < \delta. \tag{14}$$

Clearly, $Q(\varepsilon)$ and $R(\varepsilon)$ are faces of $M(\varepsilon)$. Furthermore, the face $Q(\varepsilon)$ is a triangle and the face $R(\varepsilon)$ is a cone.

Since K is an F-simplex and the family (14) is infinite, there is a pair of distinct scalars ε_1 and ε_2 such that $M(\varepsilon_1)$ and $M(\varepsilon_2)$ are directly homothetic. Suppose that $\varepsilon_1 < \varepsilon_2$. Then $Q(\varepsilon_1)$ is larger than $Q(\varepsilon_2)$, which implies that $M(\varepsilon_1)$ should be larger than $M(\varepsilon_2)$. On the other hand, the distance from L to the apex of $R(\varepsilon_1)$ is smaller than the distance from L to the apex of $R(\varepsilon_2)$. The obtained contradiction shows that $\deg_K(x) \le 3$ for every vertex x of P.

Now, suppose that P is a segment, say $[u, v]$. Denote by G_1 and G_2 the faces of K which contain $[u, v]$. By Corollary 5.1, G_1 and G_2 have the shape Q_4 from Figure 1.2. Denote by h_1 the halfline with endpoint u which is a side of G_1. Assume, for contradiction, that $\deg_K u \ge 4$. Then, K has at least two conic faces, say F_1 and F_2, with common apex u. We may assume that h_1 is a common edge of G_1 and F_1, and that F_2 is adjacent to F_1 along a halfline h_2.

Denote by L a plane satisfying the condition $K \cap L = [u, v]$, and let L' be a plane parallel to L and cutting $\operatorname{int} K$. Clearly, $K \cap L'$ is a convex polygon with a pair of opposite sides parallel to $[u, v]$. Denote by $[p, q]$ the side of $K \cap L'$ contained in the face F_1 such that $p \in h_1$ and $q \in h_2$.

An obvious geometric argument shows the existence of a scalar $\delta > 0$ such that every convex solid

$$M(\varepsilon) = K \cap (\varepsilon(q - p) + K), \quad 0 < \varepsilon < \delta,$$

has a nontrivial edge $[u(\varepsilon), v(\varepsilon)]$ which is the intersection of faces $\varepsilon(q-p)+G_1$ and G_2.

Therefore, $[u(\varepsilon), v(\varepsilon)]$ is parallel to $[u, v]$. Clearly, the face $\varepsilon(q-p)+F_1$ of the solid $\varepsilon(q-p)+K$ cuts off F_2 a triangle $\Delta(u, r(\varepsilon), s(\varepsilon))$, where $r(\varepsilon) \in h_2$.

Now, if the scalar ε continuously tends to 0, then the segments $[u(\varepsilon), v(\varepsilon)]$ and $[r(\varepsilon), s(\varepsilon)]$ continuously tend to $[u, v]$ and $\{v\}$, respectively. This argument shows that the family $M(\varepsilon)$, $0 < \varepsilon < \delta$, contains infinitely many pairwise nonhomothetic solids, contrary to the assumption on K. Hence $\deg_K(u) = \deg_K(v) \leq 3$.

(b) Let $P = \{u\}$. Then K is a cone with apex u. Assume, for contradiction, that $\deg_K(u) \geq 5$. Then K has at list five extreme halflines, say $h_1, \dots, h_r$, $r \geq 5$. We may suppose that these halflines are numbered according to a certain orientation of bypass of $\operatorname{bd} K$ around u. Denote by F_i the face of K bounded by h_i and h_{i+1}, $1 \leq i \leq r-1$, and let F_0 be the face of K bounded by h_r and h_1. Denote by G_0 the plane containing F_0. Choose a point $v \in \mathbb{R}^3$ such that the following conditions are satisfied:

(i) the solid $K' = (v-u)+K$ overlaps with K,

(ii) the section Q of K by the plane $G_0' = (v-u)+G_0$ is contained in the face $F_0' = (v-u)+F_0$ of K'.

The section Q is an unbounded polygonal region with two boundary halflines $l_1 \subset F_1$ and $l_r \subset F_{r-1}$, which are parallel to h_1 and h_r, respectively, and with $r-2$ vertices, say $u_2, \dots, u_{r-1}$, such that $u_i \in h_i$, $2 \leq i \leq r-1$.

Let L be a plane satisfying the condition $K \cap L = \{u\}$, and let L' be a plane parallel to L and cutting $\operatorname{int} K$. Clearly, $K \cap L'$ is a convex r-gon. Denote by $[p, q]$ the side of $K \cap L'$ contained in the face F_0 such that $p \in h_1$ and $q \in h_r$.

Continuously translating K' in the direction of $p-q$, we obtain a new solid K'' such that $Q \subset K''$ and l_2 is contained in the respective face F_1'' of K''. There is a scalar $\delta > 0$ so that the face $F_1(\varepsilon) = \varepsilon(p-q)+F_1''$ of the convex solid $K(\varepsilon) = \varepsilon(p-q)+K''$ meets the open segment (u_2, u_3) at a point $u_2(\varepsilon)$. Let

$$M(\varepsilon) = K \cap (\varepsilon(p-q)+K''), \quad 0 < \varepsilon < \delta.$$

Since the length of the edge $[u_2(\varepsilon), u_3]$ of $M(\varepsilon)$ continuously decreases as $\varepsilon \to \delta$ while $[u_3, u_4]$ is an edge of every solid $M(\varepsilon)$, we conclude that the family $M(\varepsilon)$, $0 < \varepsilon < \delta$, contains infinitely many pairwise non-homothetic members, contrary to the assumption on K. Hence $\deg_K(u) \leq 4$, as desired. □

Theorem 5.3. *For a convex solid $K \subset \mathbb{R}^3$, the following conditions are equivalent:*

(a) *K is an F-simplex,*

(b) *K is described by one of the sets in Figures* 1.1 *and* 1.3*,*

(c) *all 3-dimensional intersections $K \cap (x+K)$, $x \in \mathbb{R}^3$, belong to at most three homothety classes.*

Proof. $(a) \Rightarrow (b)$ This part immediately follows from Lemma 5.2.

$(b) \Rightarrow (c)$ If K has one of the shapes P_1–P_4, then K is a generalized simplex and every convex solid $M(x) = K \cap (x+K)$, $x \in \mathbb{R}^3$, is homothetic to K. If K has one of the shapes P_5 or P_6, then $M(x)$ is either a translate of P_5 or a homothetic copy of a convex solid of one of the shapes P_6, where the edge of P_6 may have two distinct directions. Finally, if K has the shape P_7, then $M(x)$ is either a translate of P_2 or a homothetic copy of P_7.

$(c) \Rightarrow (a)$ This part is trivial. $\square$

6. Translate classes

Similarly to homothety classes, one may consider the family of convex solids in $\mathbb{R}^n$ as the disjoint union of *translate classes*, each consisting of all translates of a given convex solid. The main result of this section is given by Theorem 6.3.

Lemma 6.1. *If $K \subset \mathbb{R}^n$ is a line-free convex set, then $e = o$ is the only vector in $\mathbb{R}^n$ satisfying the condition $e + K = K$.*

Proof. Assume for a moment the existence of a vector $e \neq o$ such that $e + K = K$. By induction on $r \geq 1$, we obtain the equalities $\pm re + K = K$. Choose a point $u \in K$. Then $\pm re + u \in K$ for all $r \geq 1$. By the convexity of K, the whole line $l = \text{conv}\,\{\pm re + u : r \geq 1\}$ is contained in K. The latter contradicts the assumption on K. $\square$

Lemma 6.2. *Let line-free closed convex sets K_1 and K_2 in $\mathbb{R}^n$, which are translates of each other, be internally supported by a closed halfspace $V \subset \mathbb{R}^n$ with boundary hyperplane H. Then $K_1 = z + K_2$ if and only if $H \cap K_1 = z + H \cap K_2$.*

Proof. Since the "only if" part follows from Lemma 4.3, it remains to show that $K_1 = z + K_2$ provided $H \cap K_1 = z + H \cap K_2$. By the hypothesis, $K_1 = u + K_2$ for a suitable vector $u \in \mathbb{R}^n$. The above argument gives $H \cap K_1 = u + H \cap K_2$. So,

$$H \cap K_1 = z + H \cap K_2 = (z - u) + H \cap K_1$$

Since the set $H \cap K_1$ is line-free, Lemma 6.1 shows that $z - u = o$. Hence we have $K_1 = z + K_2$. $\square$

Theorem 6.3. *For line-free convex solids K_1 and K_2 in $\mathbb{R}^n$, the following conditions are equivalent:*

(a) *both K_1 and K_2 are translates of an n-dimensional simplicial cone,*

(b) *all n-dimensional intersections $K_1 \cap (x + K_2)$, $x \in \mathbb{R}^n$, are translates of a convex solid $K \subset \mathbb{R}^n$,*

(c) *all n-dimensional intersections $K_1 \cap (x + K_2)$, $x \in \mathbb{R}^n$, belong to at most finitely many translate classes of convex solids in $\mathbb{R}^n$.*

Proof. $(a) \Rightarrow (b)$ Let K_1 and K_2 be translates of an n-dimensional simplicial cone $\Gamma \subset \mathbb{R}^n$, that is, $K_1 = u + \Gamma$ and $K_2 = v + \Gamma$ for suitable vectors $u, v \in \mathbb{R}^n$. Choose nonzero vectors $e_1, \ldots, e_n$ along the edges of Γ. Then $\{e_1, \ldots, e_n\}$ is a basis for $\mathbb{R}^n$ and Γ is described as

$$\Gamma = \{(\xi_1, \ldots, \xi_n) : \xi_1 \geq 0, \ldots, \xi_n \geq 0\}.$$

Let $u = (a_1, \ldots, a_n)$ and $v = (b_1, \ldots, b_n)$. Then

$$K_1 = \{(\xi_1, \ldots, \xi_n) : \xi_1 \geq a_1, \ldots, \xi_n \geq a_n\},$$

$$K_2 = \{(\xi_1, \ldots, \xi_n) : \xi_1 \geq b_1, \ldots, \xi_n \geq b_n\}.$$

Given a vector $x = (c_1, \dots, c_n)$, let $f_i = \max\{a_i, b_i + c_i\}$, $1 \le i \le n$, and let $w = (f_1, \dots, f_n)$. Then

$$K_1 \cap (x + K_2) = \{(\xi_1, \dots, \xi_n) : \xi_1 \ge f_1, \dots, \xi_n \ge f_n\} = w + \varGamma.$$

Since the implication $(b) \Rightarrow (c)$ trivially holds, it remains to show that $(c) \Rightarrow (a)$. We divide our argument into a sequence of statements.

(S1) *If there is a vector $x \in \mathbb{R}^n$ such that* $\operatorname{int}(x + K_2)$ *contains an exposed point of* K_1*, then* $K_1 \subset \operatorname{int}(x + K_2)$.

Indeed, choose a point $z \in \exp K_1$ which belongs to $\operatorname{int}(x + K_2)$. Denote by V a closed halfspace of $\mathbb{R}^n$ internally supporting K_1 such that $K_1 \cap \operatorname{bd} V = \{z\}$. Assume, for contradiction, that $K_1 \not\subset \operatorname{int}(x + K_2)$ and choose a point $u \in K_1 \setminus \operatorname{int}(x + K_2)$. Clearly, there is a scalar $\delta > 0$ such that $z \in \operatorname{int}(x + \varepsilon(z - u) + K_2)$ for all $0 < \varepsilon < \delta$. Let $M(\varepsilon) = K_1 \cap (x + \varepsilon(z - u) + K_2)$ and $[v(\varepsilon), z] = [u, z] \cap (x + \varepsilon(z - u) + K_2)$. Because the family of convex solids $M(\varepsilon)$, $0 < \varepsilon < \delta$, is infinite, condition (c) implies the existence of distinct scalars ε_1 and ε_2 such that $M(\varepsilon_1)$ is a translate of $M(\varepsilon_2)$. We may suppose that $\varepsilon_1 < \varepsilon_2$. Since both solids $M(\varepsilon_1)$ and $M(\varepsilon_2)$ are internally supported by V such that

$$M(\varepsilon_1) \cap \operatorname{bd} V = M(\varepsilon_2) \cap \operatorname{bd} V = \{z\},$$

Lemma 6.2 implies that $M(\varepsilon_1) = M(\varepsilon_2)$. Hence the segments $[v(\varepsilon_1), z]$ and $[v(\varepsilon_2), z]$ should coincide. On the other hand, $v(\varepsilon_2) \in (v(\varepsilon_1), z)$, which gives

$$\|v(\varepsilon_1) - z\| = \|v(\varepsilon_1) - v(\varepsilon_2)\| + \|v(\varepsilon_2) - z\| > \|v(\varepsilon_2) - z\|,$$

a contradiction. Summing up, $K_1 \subset \operatorname{int}(x + K_2)$.

(S2) *Both solids K_1 and K_2 are cones.*

Indeed, assume for a moment that K_1 is not a cone. Then K_1 has distinct exposed points z_1 and z_2. Choose a point $u \in \operatorname{int} K_2$ and denote by l the line through u parallel to the segment $[z_1, z_2]$. Since K_2 is line-free, the intersection $M = l \cap \operatorname{int} K_2$ is either an open line segment or an open halfline. Hence a suitable translate $x + M$ contains precisely one of the points z_1 and z_2. The obvious equality

$$[z_1, z_2] \cap (x + \operatorname{int} K_2) = [z_1, z_2] \cap (x + M)$$

implies that $x + \operatorname{int} K_2$ contains precisely one of the points z_1, z_2, in contradiction with (S1).

In what follows, denote by z_1 and z_2, respectively, the apices of the cones K_1 and K_2.

(S3) $K_2 = z_2 - z_1 + K_1$.

Indeed, choose a vector $x \in \mathbb{R}^n$ such that $z_1 \subset \operatorname{int}(x + K_2)$. Then $K_1 \subset \operatorname{int}(x + K_2)$, due to (S1). By the convexity of $x + K_2$, the nontrivial segment $[z_1, x + z_2]$ is contained in $x + K_2$. Consequently, $(z_1, x + z_2) \subset \operatorname{int}(x + K_2)$. If $u \in (z_1, x + z_2) \subset \operatorname{int}(x + K_2)$, then u is the exposed point of $(u - z_1) + K_1$ and, by (S1), one has

$$(u - z_1) + K_1 \subset \operatorname{int}(x + K_2).$$

Tending u towards $x + z_2$, we obtain that $x + z_2 - z_1 + K_1 \subset x + K_2$. Hence $z_2 - z_1 + K_1 \subset K_2$. Similarly, $z_1 - z_2 + K_2 \subset K_1$. Summing up, $K_2 = z_2 - z_1 + K_1$.

Let $K = K_1 - z_1 = K_2 - z_2$. Obviously, K is a closed convex cone with apex o satisfying the following condition:

(c') all n-dimensional intersections $K \cap (x + K)$, $x \in \mathbb{R}^n$, belong to at most finitely many translate classes of convex solids in $\mathbb{R}^n$.

Since translate classes are subfamilies of homothety classes, Theorem 4.1 implies that the cone K is polyhedral.

(S4) *K is a simplicial cone.*

We prove this statement by induction on n. Because the cases $n \leq 2$ are trivial, we may assume that $n \geq 3$. Suppose that the inductive assumption holds for all $r \leq n - 1$, and let an n-dimensional polyhedral cone $K \subset \mathbb{R}^n$ satisfy condition (c'). Choose a facet F of K and denote by V the closed halfspace internally supporting K such that the $(n-1)$-dimensional subspace $H = \operatorname{bd} V$ contains F. Consider the family $\mathcal{F}$ of translates $x+K$, $x \in \mathbb{R}^n$, internally supported by V such that the facets $x + F$ are contained in H and satisfy the condition $\operatorname{rint} F \cap \operatorname{rint}(x + F) \neq \varnothing$. By Lemma 4.2, $\operatorname{int} K \cap \operatorname{int}(x + K) \neq \varnothing$ whenever $x + K \in \mathcal{F}$. A combination of Lemma 6.2 and condition (c') implies that the family of all $(n-1)$-dimensional polyhedra $F \cap (x+F)$, where $x+K \in \mathcal{F}$, belongs to at most finitely many translate classes of $(n-1)$-dimensional solids in the $(n-1)$-dimensional space H. By the inductive hypothesis, F is a simplicial cone.

Now, assume for a moment that K is not a simplicial cone. Then K has at least two exposed halflines, say h_1 and h_2, which are not contained in H. Choose a vector $v \in \operatorname{int} V$ such that the facet $v + F$ of $v + K$ contains the section of K by the hyperplane $H' = v + H$. Clearly, H' meets the halflines h_1 and h_2 at some points z_1 and z_2, respectively. Furthermore, z_1 and z_2 are exposed points of $(v + K) \cap V'$. Consider the convex solids

$$M(\varepsilon) = K \cap (\varepsilon v + K), \quad 0 < \varepsilon < 1.$$

Then εz_1 and εz_2 are exposed points of $M(\varepsilon)$. Since the distance $\|\varepsilon z_1 - \varepsilon z_2\|$ monotonically tends to 0 as $\varepsilon \to 0$, the family $M(\varepsilon)$ contains infinitely many sets which are not translates of each other. The latter is in contradiction with condition (c'). Summing up, K is a simplicial cone. □

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