

Functions on a Convex Set which are both ω -Semiconvex and ω -Semiconcave

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Let $G \subset \mathbb{R}^n$ be an open convex set which is either bounded or contains a translation of a convex cone with nonempty interior. It is known that, for every modulus ω , every function on G which is both semiconvex and semiconcave with modulus ω is (globally) $C^{1,\omega}$ -smooth. We show that this result is optimal in the sense that the assumption on G cannot be relaxed. We also present direct short proofs of the above mentioned result and of some its quantitative versions. Our results have immediate consequences concerning (i) a first-order quantitative converse Taylor theorem and (ii) the problem whether $f \in C^{1,\omega}(G)$ whenever f is continuous and smooth in a corresponding sense on all lines. We hope that these consequences are of an independent interest.

Keywords: ω -semiconvex functions, ω -semiconcave functions, $C^{1,\omega}$ -smooth functions, smoothness on all lines, converse Taylor theorem, strongly $\alpha(\cdot)$ -paraconvex functions.

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1. Introduction

First we recall a classical result on smoothness of functions which are both semiconvex and semiconcave (i.e., semiconvex and semiconcave with linear modulus).

Definition 1.1. Let X be a Hilbert space, $G \subset X$ an open convex set and f a function on G . We say that f is *semiconvex*, if there exists $C \geq 0$ and a continuous convex function g on G such that

$$f(x) = g(x) - \frac{C}{2} \cdot \|x\|^2, \quad x \in G.$$

Then we say that f is *semiconvex with constant C* . We say that f is *semiconcave* on G (with constant C) if $-f$ is semiconvex on G (with constant C). $\square$

Let X , G and f be as in Definition 1.1. Then it is very easy to show (see Remark 2.2) that

$$\begin{aligned} & f' \text{ exists and is Lipschitz with constant } C \\ \implies & f \text{ is both semiconvex and semiconcave with constant } C. \end{aligned} \quad (1)$$

Rather surprisingly, also the converse of (1) holds:

$$\begin{aligned} & f \text{ is both semiconvex and semiconcave with constant } C \\ \implies & f' \text{ exists and is Lipschitz with constant } C. \end{aligned} \quad (2)$$

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Some bibliographic notes concerning this well-known interesting nontrivial result are contained in Section 3, where also a short elementary proof is presented.

Consequently, we have that

$$\begin{aligned} f \text{ is both semiconvex and semiconcave with constant } C \\ \iff f' \text{ exists and is Lipschitz with constant } C. \end{aligned} \quad (3)$$

In particular,

$$f \text{ is both semiconvex and semiconcave} \iff f \in C^{1,1}(G). \quad (4)$$

In the present article we study analogues of (2) and (4) for functions which are both ω -semiconvex and ω -semiconcave. We use the following terminology.

Definition 1.2. We denote by $\mathcal{M}$ the set of all $\omega : [0, \infty) \rightarrow [0, \infty)$ which are non-decreasing and satisfy $\lim_{t \rightarrow 0+} \omega(t) = 0$. $\square$

Definition 1.3. Let X be a normed linear space, $G \subset X$ an open convex set and $\omega \in \mathcal{M}$. We say that a continuous function $f : G \rightarrow \mathbb{R}$ is *semiconvex with modulus* ω (or ω -semiconvex for short) if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) + \lambda(1 - \lambda)\|x - y\|\omega(\|x - y\|)$$

for every $x, y \in G$ and $\lambda \in [0, 1]$. We say that $f : G \rightarrow \mathbb{R}$ is ω -semiconcave if $-f$ is ω -semiconvex. $\square$

Remark 1.4. If X is a Hilbert space, it is easy to show (cf. [5, p.30]) that f is semiconvex on $G \subset X$ with constant $C \geq 0$ if and only if f is ω -semiconvex with modulus $\omega(t) = \frac{C}{2}t$, $t \geq 0$. $\square$

Remark 1.5. The notion of ω -semiconvex functions is very close to Rolewicz's notion of *strongly* $\alpha(\cdot)$ -paraconvex functions (which has in [22] a slightly different sense than in [21]). Namely (see e.g. [6, Remark 2.11]) if $\omega \in \mathcal{M}$ and $\alpha(t) = t\omega(t)$, $t \geq 0$, then (by the definition from [22]),

$$f \text{ is } \omega\text{-semiconvex} \implies f \text{ is strongly } \alpha(\cdot)\text{-paraconvex} \implies f \text{ is } (2\omega)\text{-semiconvex}.$$

Most of the results concerning ω -semiconvex functions have obvious consequences for strongly $\alpha(\cdot)$ -paraconvex functions (and vice versa). We will not formulate such consequences of our results. $\square$

Definition 1.6. If G is an open subset of a normed linear space and $\omega \in \mathcal{M}$, then we denote by $C_*^{1,\omega}(G)$ the set of all Fréchet differentiable $f : G \rightarrow \mathbb{R}$ such that f' is uniformly continuous with modulus ω . We set $C^{1,\omega}(G) := \bigcup_{K>0} C_*^{1,K\omega}(G)$. $\square$

Note that so defined $C^{1,\omega}$ -smoothness, which coincides with (global) $C^{1,1}$ -smoothness in the case of a linear ω , is common in the literature. We will sometimes use also (stronger) $C_*^{1,\omega}$ -smoothness (which is somewhere, e.g. in [11] and [12], called “ $C^{1,\omega}$ -smoothness”).

Let us first mention that a (weaker) version of (1) for general moduli is also easy (see Remark 2.2). It asserts that, if X , G and ω are as in Definition 1.3, then

$$f \in C_*^{1,\omega}(G) \implies f \text{ is both } \omega\text{-semiconvex and } \omega\text{-semiconcave on } G. \quad (5)$$

However (if X , G and ω are as in Definition 1.3) the implication

$$f \text{ is both } \omega\text{-semiconvex and } \omega\text{-semiconcave on } G \implies f \in C^{1,\omega}(G) \quad (6)$$

does not hold in general. The validity of (6) (or of its local versions) was studied e.g. in [5, Theorem 3.3.7], [13] and [15] (and essentially already in [20] for $\omega(t) = t^\alpha$, $0 < \alpha \leq 1$). As far as we know, the strongest and the most general versions of (6) follow from versions of the Converse Taylor Theorem proved in [11] and [8]. In particular (see [15, Theorem 2.6 and Remark 2.7]), [8, Chap. 1, Corollary 126] easily implies the following result.

Theorem 1.7. ([15]) *Let X be a normed linear space and $G \subset X$ an open convex set. Suppose that G is either bounded or has the property that there are $a \in X$, $r > 0$ and $\{u_n\}_{n \in \mathbb{N}} \subset X$, $\|u_n\| = n$, such that $B(a + u_n, nr) \subset G$ for each $n \in \mathbb{N}$. Then, for each modulus $\omega \in \mathcal{M}$ and for each function f on G ,*

$$f \text{ is both } \omega\text{-semiconvex and } \omega\text{-semiconcave on } G \implies f \in C^{1,\omega}(G).$$

Remark 1.8. Using (5), we easily see that the assertion of Theorem 1.7 implies the following analogue of (4):

$$f \text{ is both } (K\omega)\text{-semiconvex and } (K\omega)\text{-semiconcave for some } K > 0 \\ \iff f \in C^{1,\omega}(G). \quad (7)$$

For a local version of (7) see [13, Theorem 6.1]. See also Proposition 6.9 for reformulations of this characterization of $C^{1,\omega}$ -smoothness on some convex sets. $\square$

Let us note that an easy compactness argument shows that an unbounded open convex $G \subset \mathbb{R}^n$ satisfies the assumption of Theorem 1.7 if and only if it contains a translation of a convex cone with nonempty interior.

The main result of the present article is the following theorem which shows that, for $X = \mathbb{R}^n$, Theorem 1.7 is optimal in the sense that the assumption on G cannot be relaxed. It follows immediately from Proposition 5.13 (proved in Section 5) via Remark 5.7 (i).

Theorem 1.9. *Let $G \subset \mathbb{R}^n$ ($n \geq 2$) be an unbounded open convex set which does not contain a translation of a convex cone with nonempty interior. Then there exists a concave modulus $\omega \in \mathcal{M}$ with $\lim_{t \rightarrow \infty} \omega(t) = \infty$ and a function f which is both ω -semiconvex and ω -semiconcave on G but $f \notin C^{1,\omega}(G)$.*

Since [8, Chap. 1, Corollary 126] is a rather general result (working with classes $C^{k,\omega}(G)$ for each $k \in \mathbb{N}$), we find useful to present (in Section 4) direct proofs (which partly follow arguments from [8]) of Theorem 1.7 and of some its quantitative versions (sharper than those which directly follow from the proof of [8, Chap. 1, Corollary 126]). Namely, we prove the following result.

Theorem 1.10. *Let X be a normed linear space and $\emptyset \neq G \subset X$ an open convex set. Let $\omega \in \mathcal{M}$ and let f be a function on G which is both ω -semiconvex and ω -semiconcave. Then the following assertions hold.*

- (i) *If G is bounded, then f' is uniformly continuous with modulus $6e_G\omega$, where $e_G := \text{diam } G / \sup\{r : B(a, r) \subset G\}$.*
- (ii) *If $a \in X$, $r > 0$ and $\{u_n\}_{n \in \mathbb{N}} \subset X$ with $\|u_n\| = n$ are such that $B(a + u_n, nr) \subset G$ for each $n \in \mathbb{N}$, then f' is uniformly continuous with modulus $6(1 + 1/r)\omega$.*

Remark 1.11. (i) The proof of [8, Chap. I, Corollary 126] gives the multiplicative constants 15 and 30 instead of our constants 6 and 6. Using the example from Remark 4.4 (for $G = B(0, 2) \subset X$), we easily see that the constant 6 in (i) cannot be replaced by any constant $c < 2$.

(ii) In the special cases when $G = X$ or ω is linear, we can assert that f is $C_*^{1,4\omega}$ -smooth (see Corollary 4.3 and Remark 4.5). $\square$

In the last Section 6 we show that our results have immediate consequences (or reformulations) concerning

- (i) a first-order quantitative converse Taylor theorem and
- (ii) the problem whether $f \in C^{1,\omega}(G)$ whenever f is continuous and $C_*^{1,\omega}$ -smooth on all lines.

We hope that these versions of our new results can be interesting also for readers which are not interested in ω -semiconvex functions.

At the end of Section 6, a natural open question is formulated.

2. Preliminaries

2.1. Basic definitions

In the following, X will be always a real normed linear space and X^* its dual space. The norm and the origin in any normed linear space will be denoted by $\|\cdot\|$ and 0, respectively. By the *derivative* f' of a function f defined on a subset of X we always mean the Fréchet derivative and the notion of C^1 -smoothness has the standard sense.

The symbol $B(x, r)$ ($\overline{B}(x, r)$) denotes the open (closed) ball with centre x and radius r in X . For $A \subset \mathbb{R}^n$, the symbols $\overline{A}$ and $\text{span } A$ denote the *closure* and the *linear span* of A , respectively. By a *cone* in $\mathbb{R}^n$ we mean a set $C \subset \mathbb{R}^n$ such that $\lambda x \in C$ for each $x \in C$ and $\lambda > 0$. A *ray* in $\mathbb{R}^n$ is a set of the form $\rho = \{a + tv : t \geq 0\}$, where $a \in \mathbb{R}^n$ and $0 \neq v \in \mathbb{R}^n$. By the *dimension* $\dim(C)$ of a convex set $C \subset \mathbb{R}^n$ we mean the dimension of the affine hull of C (see [19, p. 12]).

2.2. ω -semiconvexity and ω -semiconcavity

We frequently use the obvious fact that if f is ω -semiconvex (resp. ω -semiconcave) on G and $c > 0$, then the function cf is $(c\omega)$ -semiconvex (resp. $(c\omega)$ -semiconcave).

Further, it follows easily from Definition 1.3 that ω -semiconvexity is equivalent to “ ω -semiconvexity on all lines”. More precisely, the following statement holds.

Lemma 2.1. *Let X be a normed linear space, $G \subset X$ an open convex set, $\omega \in \mathcal{M}$, and let f be a continuous function on G . Then the following conditions are equivalent.*

- (i) f is ω -semiconvex on G .
- (ii) For every $a \in G$ and $v \in X$, $\|v\| = 1$, the function $f_{a,v} : t \mapsto f(a + tv)$ defined on the open interval $\{t \in \mathbb{R} : a + tv \in G\}$ is ω -semiconvex.

Remark 2.2. (i) Using Lemma 2.1, it is easy to see that, to prove implications (1) and (5), it is sufficient to prove them for $X = \mathbb{R}$. The validity of (1) for $X = \mathbb{R}$ is almost obvious. Indeed, the assumption of (1) implies that $g(x) := f(x) + \frac{C}{2}x^2$ is convex on G , since $g'(x) = f'(x) + Cx$ is clearly nondecreasing on the interval G . And the validity of (5) for $X = \mathbb{R}$ immediately follows e.g. from [5, Proposition 2.1.2].

(ii) Using Lemma 2.1, Remark 1.4 and (1), we obtain that if ω is linear, then we can assert in (5) even that f is both $\tilde{\omega}$ -semiconvex and $\tilde{\omega}$ -semiconcave, where $\tilde{\omega} := \frac{1}{2}\omega$. However, for general modulus it is not true even for $X = \mathbb{R}$ and $\tilde{\omega} := C\omega$, where $C < 1$ is any absolute constant. Indeed, if $0 < C < 1$ is given, we choose $p \in (1, 2)$ such that $C < 1/p$ and set $\omega(t) := t^{p-1}$, $t \geq 0$, and $f(x) := x^p/p$, $x \in (0, 1)$. Then $f'(x) = \omega(x)$, $x \in (0, 1)$, and consequently the concavity of ω implies that f is $C_*^{1,\omega}$ -smooth. However (as observed in [14, Remark 2.10 (ii)]), it is not difficult to show that f is not $(C\omega)$ -semiconcave. $\square$

We will need the following fact which is an easy consequence of (5) and Lemma 2.1.

Lemma 2.3. *Let X be a normed linear space, $G \subset X$ an open convex set, $\omega \in \mathcal{M}$ and let f be a Fréchet differentiable function on G . Suppose that*

$$|f'(y)(y - x) - f'(x)(y - x)| \leq \|y - x\|\omega(\|y - x\|), \quad x, y \in G. \quad (8)$$

Then f is both ω -semiconvex and ω -semiconcave on G .

Proof. We will first observe that if $f_{a,v}$ is any function as in Lemma 2.1 (ii), then

$$f_{a,v} \text{ is } C_*^{1,\omega}\text{-smooth.} \quad (9)$$

Indeed, for each $t_1 \neq t_2$ from the domain of $f_{a,v}$ we can apply (8) to $x := a + t_1v$ and $y := a + t_2v$, and obtain

$$\begin{aligned} |(f_{a,v})'(t_2) - (f_{a,v})'(t_1)| &= |f'(a + t_2v)(v) - f'(a + t_1v)(v)| \\ &= \frac{1}{|t_2 - t_1|} |f'(a + t_2v)((t_2 - t_1)v) - f'(a + t_1v)((t_2 - t_1)v)| \leq \omega(|t_2 - t_1|). \end{aligned}$$

Therefore each $f_{a,v}$ is ω -semiconvex by (5) and so f is ω -semiconvex by Lemma 2.1. Using the same argument to $-f$, we obtain that f is also ω -semiconcave. $\square$

An other basic tool for us is the following essentially well-known result.

Lemma 2.4. *Let X be a normed linear space, $G \subset X$ an open convex set and $\omega \in \mathcal{M}$. Let a function f be both ω -semiconvex and ω -semiconcave on G . Then f is Fréchet differentiable on G and*

$$|f(y) - f(x) - f'(x)(y - x)| \leq \|y - x\|\omega(\|y - x\|), \quad x, y \in G. \quad (10)$$

Proof. A proof in the case $X = \mathbb{R}^n$ can be found in [5] (see the proof of [5, (3.23), p. 60]). The proof of the general case is factually contained in the proof of [15, Theorem 2.6]. For the convenience of the reader, we shortly repeat the argument from [15]. It is based on [21, Theorem 3] (cf. also [22, Proposition 2]) which easily

implies (cf. [15] or [6, Corollary 4.5]) that ω -semiconvexity of f implies that for each $x \in G$ there exists $\Phi_x \in X^*$ such that

$$f(x+h) - f(x) - \Phi_x(h) \geq -\|h\|\omega(\|h\|) \quad \text{whenever } x+h \in G.$$

Since $-f$ is also ω -semiconvex, for each $x \in G$ there exists $\Psi_x \in X^*$ such that

$$-f(x+h) + f(x) - \Psi_x(h) \geq -\|h\|\omega(\|h\|) \quad \text{whenever } x+h \in G.$$

Adding these two inequalities, we easily obtain that $\Psi_x = -\Phi_x$ and

$$|f(x+h) - f(x) - \Phi_x(h)| \leq \|h\|\omega(\|h\|) \quad \text{whenever } x+h \in G,$$

which implies that $f'(x) = \Phi_x$ and so (10) holds. $\square$

Lemma 2.4 and Remark 1.4 immediately imply the following well-known result.

Corollary 2.5. *Let X be a Hilbert space, $G \subset X$ an open convex set, and let f be a function which is both semiconvex and semiconcave with constant $C > 0$ on G . Then f is Fréchet differentiable on G and*

$$|f(y) - f(x) - f'(x)(y-x)| \leq \frac{C}{2}\|y-x\|^2, \quad x, y \in G. \quad (11)$$

The following result (which clearly implies that the converse of Corollary 2.5 holds) is also essentially well-known, but we prefer to present a simple proof.

Lemma 2.6. *If X , G , ω are as in Lemma 2.4 and f satisfies (10), then f is both (2ω) -semiconvex and (2ω) -semiconcave. Moreover, if ω is linear, then f is both ω -semiconvex and ω -semiconcave.*

Proof. For each $x \in G$, $y \in G$, we can apply (10) for $x := y$ and $y := x$ and obtain

$$|f(x) - f(y) - f'(y)(x-y)| \leq \|x-y\|\omega(\|x-y\|).$$

Adding this inequality with (10) and using the triangle inequality, we obtain

$$|f'(y)(y-x) - f'(x)(y-x)| \leq 2\|y-x\|\omega(\|y-x\|).$$

Since we have proved that (8) implies (9), we obtain that each function $f_{a,v}$ is $C_*^{1,2\omega}$ -smooth and consequently both (2ω) -semiconvex and (2ω) -semiconcave by (5) (and even both ω -semiconvex and ω -semiconcave by Remark 2.2 (ii) if ω is linear). So the assertions of the lemma follow by Lemma 2.1. $\square$

Remark 2.7. The second assertion of Lemma 2.6 does not hold for general ω . In [16], it is shown that if $\omega(t) = t^{1/2}$, $t \geq 0$, $2^{-1/2} < C < 1$ and $f(x) = -C|x|^{3/2}$, $x \in G := \mathbb{R}$, then (10) holds but f is not ω -semiconvex.

We will need also the following special case of [7, Lemma 2.7].

Lemma 2.8. *Let $\omega \in \mathcal{M}$ and $n \geq k \geq 2$. Let $\emptyset \neq G \subset \mathbb{R}^n$ be an open convex set, and let $L : \mathbb{R}^n \rightarrow \mathbb{R}^k$ be a linear surjection. Let f be a function on $L(G)$ which is both ω -semiconvex and ω -semiconcave on $L(G)$. Then there exists $C > 0$ such that the function $f \circ L$ is both $(C\omega)$ -semiconvex and $(C\omega)$ -semiconcave on G .*

(Note that by well-known facts, $L(G)$ is an open convex set.)

2.3. Recession cones

In the proof of our main result, we will use some properties of recession cones (called sometimes also asymptotic cones) of unbounded convex sets $A \subset \mathbb{R}^n$. We refer to the exposition in [19], however we denote the recession cone of A by $\text{rec}(A)$ (and not 0^+A as in [19]).

Definition 2.9. Let $\emptyset \neq A \subset \mathbb{R}^n$ be a convex set. By the *recession cone* of A we mean the set $\text{rec}(A)$ of all $y \in \mathbb{R}^n$ such that $x + \lambda y \in A$ for each $x \in A$ and $\lambda \geq 0$. $\square$

We will need the following two lemmas on open convex sets which easily follow from results proved in [19] for closed convex sets.

Lemma 2.10. Let $\emptyset \neq G \subset \mathbb{R}^n$ be an open convex set. Then the following assertions hold.

- (i) $\text{rec}(G) = \text{rec}(\overline{G})$.
- (ii) $\text{rec}(G)$ is a closed convex cone in $\mathbb{R}^n$ and $0 \in \text{rec}(G)$.
- (iii) $\text{rec}(G) = \{y \in \mathbb{R}^n : \exists x \in G \forall \lambda \geq 0 : x + \lambda y \in G\}$.
- (iv) If G is unbounded then $\text{rec}(G) \setminus \{0\} \neq \emptyset$.
- (v) $\dim(\text{rec}(G)) = \dim(\text{span}(\text{rec}(G)))$.
- (vi) $\dim(\text{rec}(G)) = n$ if and only if G contains a translation of a cone with nonempty interior.

Proof. Property (i) follows from [19, Corollary 8.3.1] and (ii) from [19, Theorem 8.2] and (i). Theorem [19, Theorem 8.3] and (i) imply (iii). Property (iv) follows from [19, Theorem 8.4] and (i). Property (v) is obvious since $0 \in \text{rec}(G)$. Using (iii) and (v), it is easy to prove (vi). $\square$

The following lemma is an easy consequence of [19, Theorem 9.1].

Lemma 2.11. Let $G \subset \mathbb{R}^n$ be an unbounded open convex set and let $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear surjection with the property $L^{-1}(\{0\}) \cap \text{rec}(G) = \{0\}$. Then we have $\text{rec}(L(G)) = L(\text{rec}(G))$.

Proof. By Lemma 2.10 (i) we have $\text{rec}(G) = \text{rec}(\overline{G})$. So $L^{-1}(\{0\}) \cap \text{rec}(\overline{G}) = \{0\}$ and thus [19, Theorem 9.1] (applied to $C := \overline{G}$) implies that $L(\overline{G})$ is closed and $\text{rec}(L(\overline{G})) = L(\text{rec}(\overline{G}))$. It is an easy well-known fact that $L(G)$ is an open convex set. Continuity of L implies $L(\overline{G}) \subset \overline{L(G)}$ and so $L(\overline{G}) = \overline{L(G)}$ since $L(\overline{G})$ is closed. Therefore $\text{rec}(L(\overline{G})) = \text{rec}(L(G))$ by Lemma 2.10 (i) and the equality $\text{rec}(L(G)) = L(\text{rec}(G))$ follows. $\square$

3. Linear modulus

The present short section is devoted to result (2) which is rewritten below as Theorem 3.1. We do not know the origin of this interesting useful result which is well-known for a long time.

It is stated (for $G = X$) in [17, p. 265](1986), where it is written (without a reference) that the case $G = X = \mathbb{R}^n$ is well-known and it is shown how the general case easily follows from the finite-dimensional case. The case $G = X$ is completely proved in [9](1989). Let us note that a weaker version of Theorem 3.1 which asserts that

$f \in C^{1,1}(G)$ only, is factually proved in [20](1979) (for $G = X$ and Lipschitz f) and *very easily* follows (for $G = X$) from [18, Proposition 10.b](1965) which asserts that if g and h are continuous convex functions such that $g(x) + h(x) = (1/2)\|x\|^2$, $x \in X$, then g' is Lipschitz with constant 1. The equality $G = X$ is vital in [18] and [9].

Several proofs of Theorem 3.1 work with second derivatives of f . For example, [5, Corollary 3.8] and [2, Exercise 3.8, p. 183] use distributional derivatives, and [4, Lemma 2.7] and [1, Theorem 2.3] use Aleksandrov's theorem on a.e. second differentiability of convex functions.

We present below a short elementary proof using the procedure (based on (13)) which will be used also in the proof of Lemma 4.2. Our proof is similar to elementary proofs given in [10, Theorem 1.1] and [3, Corollary 5.1] and we guess that a similar proof is (essentially) contained also in a much older article.

Theorem 3.1. (well-known) *Let X be a Hilbert space, $G \subset X$ an open convex set and let a function f on G be both semiconvex and semiconcave with constant C . Then f is differentiable on G and f' is Lipschitz with constant C .*

Proof. By Corollary 2.5 we know that f' exists on G and (11) holds. First we will show that if $u, v \in G$, $u \neq v$, then

$$\overline{B}\left(\frac{u+v}{2}, \frac{\|v-u\|}{2}\right) \subset G \implies \|f'(v) - f'(u)\| \leq C\|v-u\|. \quad (12)$$

To this end, suppose that the assumption of (12) holds. Without any loss of generality we can suppose that

$$f(u) - f(v) + \frac{1}{2}f'(u)(v-u) - \frac{1}{2}f'(v)(u-v) \leq 0, \quad (13)$$

since otherwise we could work with $\tilde{f} := -f$ instead of f . Now consider an arbitrary $w \in X$, $\|w\| = \|v-u\|$. Then we have $\frac{1}{2}(u+v+w) \in G$ and therefore, using (11) and (13), we obtain

$$\begin{aligned} & f'(v)\left(\frac{1}{2}w\right) - f'(u)\left(\frac{1}{2}w\right) \\ &= f\left(\frac{1}{2}(u+v+w)\right) - f(u) - f'(u)\left(\frac{1}{2}(w+v-u)\right) \\ &\quad + f(u) - f(v) + \frac{1}{2}f'(u)(v-u) - \frac{1}{2}f'(v)(u-v) \\ &\quad - \left(f\left(\frac{1}{2}(u+v+w)\right) - f(v) - f'(v)\left(\frac{1}{2}(w-v+u)\right)\right) \\ &\leq \frac{C}{8}\|w+(v-u)\|^2 + \frac{C}{8}\|w-(v-u)\|^2 \\ &= \frac{C}{4}\|w\|^2 + \frac{C}{4}\|v-u\|^2 = \frac{C}{2}\|v-u\|^2 \end{aligned}$$

and hence

$$\|f'(v) - f'(u)\| = \sup_{w \in X, \|w\|=\|v-u\|} \frac{f'(v)(w) - f'(u)(w)}{\|w\|} \leq C\|v-u\|.$$

Now consider arbitrary $x, y \in G$, $x \neq y$. We can clearly choose $n \in \mathbb{N}$ so large that, for each $1 \leq i \leq n$,

$$\overline{B}\left(\frac{z_{i-1} + z_i}{2}, \frac{\|z_i - z_{i-1}\|}{2}\right) \subset G, \quad \text{where } z_i := x + \frac{i}{n}(y - x), \quad i \in \{0, \dots, n\}.$$

Then (12) implies that $\|f'(z_i) - f'(z_{i-1})\| \leq C\|z_i - z_{i-1}\|$ for each $1 \leq i \leq n$, and consequently

$$\|f'(y) - f'(x)\| \leq \sum_{i=1}^n \|f'(z_i) - f'(z_{i-1})\| \leq \sum_{i=1}^n C\|z_i - z_{i-1}\| = C\|y - x\|. \quad \square$$

4. Results for a general modulus

In this section, we prove Theorem 1.10 and interesting Corollary 4.3. Other more special versions of implication (6) (for $X = \mathbb{R}$ or for special ω) can be found in Remark 4.5 and Remark 5.3.

Remark 4.1. Note that in the proof of Lemma 4.2 (and so also in the proofs of Corollary 4.3 and Theorem 1.10) the assumption that f is both ω -semiconvex and ω -semiconcave is applied via property (10) only. $\square$

Proceeding similarly as in the proof of Theorem 3.1, we obtain the following estimate.

Lemma 4.2. *Let X be a normed linear space, $G \subset X$ an open convex set, $\omega \in \mathcal{M}$, and let f be a function on G which is both ω -semiconvex and ω -semiconcave. If $y, z \in G$, $r > 0$ and $\overline{B}(y, r) \subset G$, then*

$$\|f'(z) - f'(y)\| \leq 2\left(1 + \frac{\|z - y\|}{r}\right)\omega\left(\frac{1}{2}r + \frac{1}{2}\|z - y\|\right). \quad (14)$$

Proof. As in the proof of Theorem 3.1, we may suppose that

$$f(y) - f(z) + \frac{1}{2}f'(y)(z - y) - \frac{1}{2}f'(z)(y - z) \leq 0.$$

Then for every $w \in X$, $\|w\| = r$, we obtain, using $\frac{1}{2}(z + y + w) \in G$ and (10),

$$\begin{aligned} & f'(z)\left(\frac{1}{2}w\right) - f'(y)\left(\frac{1}{2}w\right) \\ &= f\left(\frac{1}{2}(z + y + w)\right) - f(y) - f'(y)\left(\frac{1}{2}(w + z - y)\right) \\ &\quad + f(y) - f(z) + \frac{1}{2}f'(y)(z - y) - \frac{1}{2}f'(z)(y - z) \\ &\quad - \left(f\left(\frac{1}{2}(z + y + w)\right) - f(z) - f'(z)\left(\frac{1}{2}(w - z + y)\right)\right) \\ &\leq \frac{1}{2}\|w + (z - y)\|\omega\left(\frac{1}{2}\|w + (z - y)\|\right) + \frac{1}{2}\|w - (z - y)\|\omega\left(\frac{1}{2}\|w - (z - y)\|\right) \\ &\leq (r + \|z - y\|)\omega\left(\frac{1}{2}(r + \|z - y\|)\right) \end{aligned}$$

and hence
$$\begin{aligned} \|f'(z) - f'(y)\| &= \sup_{w \in X, \|w\|=r} \frac{f'(z)(w) - f'(y)(w)}{\|w\|} \\ &\leq 2 \left(1 + \frac{\|z - y\|}{r}\right) \omega \left(\frac{1}{2}r + \frac{1}{2}\|z - y\|\right). \end{aligned} \quad \square$$

Using (14) for $z \neq y$ and $r := \|z - y\|$, we immediately obtain the following result.

Corollary 4.3. *Let X be a normed linear space and $\omega \in \mathcal{M}$. Suppose that a function f on X is both ω -semiconvex and ω -semiconcave. Then f' is uniformly continuous with modulus 4ω .*

Remark 4.4. Corollary 4.3 is optimal in the sense that the constant 4 cannot be diminished, even for linear modulus. It is factually shown in [3, Example 5], where the authors consider the example of the function $f(x, y) = x^2 - y^2$ on the space $X = (\mathbb{R}^2, \|\cdot\|_\infty)$. They (factually) show that then (10) holds with $G = X$ and $\omega(t) = t$, which implies by Lemma 2.6 that f is both ω -semiconvex and ω -semiconcave on X . Observing that $\|f'((1, 1)) - f'((0, 0))\| \geq 4\|(1, 1)\|_\infty = 4$ (see [3, Example 5]), our assertion follows. $\square$

Using Lemma 4.2, we give also a relatively short proof of Theorem 1.10 formulated in the introduction.

Proof of Theorem 1.10: (i) Suppose that G is bounded.

Let $x, x + h \in G$ and $\varepsilon > 0$. We find $y_0 \in G$ and $r_0 > 0$ such that $\overline{B}(y_0, r_0) \subset G$ and $\text{diam } G / r_0 \leq e_G + \varepsilon$. Set $\lambda := \|h\| / \text{diam } G$, $s := x + \frac{1}{2}h$, $y := \lambda y_0 + (1 - \lambda)s$ and $r := \lambda r_0$. Then

$$\overline{B}(y, r) = \lambda \overline{B}(y_0, r_0) + (1 - \lambda)\{s\} \subset G \quad \text{and} \quad \frac{\|h\|}{e_G + \varepsilon} \leq r \leq \frac{\|h\|}{e_G}.$$

Clearly $\|s - y_0\| \leq \text{diam } G - r_0$ and so

$$\|s - y\| = \|\lambda(s - y_0)\| = \frac{\|h\|}{\text{diam } G} \|s - y_0\| \leq \|h\| \left(1 - \frac{r_0}{\text{diam } G}\right) \leq \|h\| - \frac{1}{e_G + \varepsilon} \|h\|.$$

Consequently, if $z = x$ or $z = x + h$, we have $\|z - y\| \leq \frac{3}{2}\|h\| - \frac{1}{e_G + \varepsilon}\|h\|$. Using this inequality, estimates of r and the obvious inequality $e_G \geq 2$, we obtain

$$2 \left(1 + \frac{\|z - y\|}{r}\right) \leq 2 \left(1 + \frac{3}{2}(e_G + \varepsilon) - 1\right) = 3(e_G + \varepsilon),$$

and
$$\frac{1}{2}r + \frac{1}{2}\|z - y\| \leq \frac{1}{2} \left(\frac{\|h\|}{e_G} + \frac{3}{2}\|h\| - \frac{1}{e_G + \varepsilon}\|h\|\right) < \|h\|.$$

Therefore (14) gives $\|f'(z) - f'(y)\| \leq 3(e_G + \varepsilon)\omega(\|h\|)$ and

$$\|f'(x + h) - f'(x)\| \leq \|f'(x + h) - f'(y)\| + \|f'(x) - f'(y)\| \leq 6(e_G + \varepsilon)\omega(\|h\|)$$

which implies the assertion of (i).

(ii) Suppose that $a \in X$, $r > 0$ and $\{u_n\}_{n \in \mathbb{N}}$ are such that we have $\|u_n\| = n$ and $B(a + u_n, nr) \subset G$ for each $n \in \mathbb{N}$.

If $r > 1$, it is easy to see that $G = X$ and thus Corollary 4.3 implies assertion (ii).

So suppose $0 < r \leq 1$. Let $x, x + h \in G$ and $\varepsilon > 0$. Then there exists $n \in \mathbb{N}$ such that $x, x + h \in B(a, \varepsilon nr)$. Set

$$s := a + \frac{r+1}{2}u_{2n} \quad \text{and} \quad H := G \cap B(s, n(r+1+\varepsilon r)).$$

If $y \in B(a + u_{2n}, 2nr)$, then

$$\|y - s\| \leq \|y - (a + u_{2n})\| + \|s - (a + u_{2n})\| < 2nr + n(1 - r) = n(r + 1).$$

Therefore $B(a + u_{2n}, 2nr) \subset H$ and consequently

$$e_H \leq \frac{\text{diam } H}{2nr} \leq \frac{2n(r+1+\varepsilon r)}{2nr} = 1 + \frac{1}{r} + \varepsilon.$$

If $y \in B(a, \varepsilon nr)$, then

$$\|y - s\| \leq \|y - a\| + \|s - a\| < \varepsilon nr + n(r + 1) = n(r + 1 + \varepsilon r),$$

and consequently $x, x + h \in H$. Hence $\|f'(x + h) - f'(x)\| \leq 6(1 + 1/r + \varepsilon)\omega(\|h\|)$ by case (i), and the assertion of (ii) follows. $\square$

Remark 4.5. Let X be a normed linear space, $G \subset X$ an open convex set, $\omega \in \mathcal{M}$, and let f be both ω -semiconvex and ω -semiconcave on G .

(i) If X is a Hilbert space and ω is linear, then $f \in C_*^{1,2\omega}(G)$.

(ii) If $X = \mathbb{R}$, then $f \in C_*^{1,2\omega}(G)$ too.

(iii) If X is general and ω is linear, then $f \in C_*^{1,4\omega}(G)$.

Indeed, (i) is a reformulation of (2) by Remark 1.4. Easy assertion (ii) follows e.g. from [7, Proposition 2.8 (i)]. To prove (iii), we can proceed similarly as in the second part of the proof of Theorem 3.1, choosing for given x, y the number n so large that the points z_i have the property that $\overline{B}(z_i, \|z_i - z_{i-1}\|) \subset G$, $i = 1, \dots, n$. Then, applying (14) to $y := z_i$, $z := z_{i-1}$ and $r := \|z_i - z_{i-1}\|$, the inequality $\|f'(y) - f'(x)\| \leq 4\omega(\|y - x\|)$ easily follows. $\square$

5. Optimality in finite-dimensional spaces

The main result of the present section is Proposition 5.13 which immediately implies Theorem 1.9.

By Remark 4.5 (iii), we know that if ω is linear, then the implication (6) holds for each open convex subset G of an arbitrary normed linear space. This assertion holds also for some nonlinear $\omega \in \mathcal{M}$ (cf. Remark 5.3 below), but the following example shows that for the most interesting nonlinear moduli this assertion does not hold.

Example 5.1. Let $X = \mathbb{R}^2$ and $G := \mathbb{R} \times (0, 1)$. Let $\omega \in \mathcal{M}$ be such that

$$\liminf_{t \rightarrow 0+} \frac{\omega(t)}{t} > 0 \quad \text{and} \quad \liminf_{t \rightarrow \infty} \frac{\omega(t)}{t} = 0. \quad (15)$$

Then the implication (6) does not hold.

Proof. Set $g(x) := x_1 x_2$ where $x = (x_1, x_2) \in G$. By (15) we can choose $1 > \delta > 0$ and $c > 0$ such that $ct \leq \omega(t)$ for each $t \in (0, \delta]$.

Then the monotonicity of ω implies $c\delta \leq \omega(t)$ for each $t \in (\delta, \infty)$. We will show that

$$g \text{ is both } \left(\frac{2}{c\delta}\omega\right)\text{-semiconvex and } \left(\frac{2}{c\delta}\omega\right)\text{-semiconcave on } G. \quad (16)$$

To this end consider $x = (x_1, x_2) \in G$ and $h = (h_1, h_2) \in \mathbb{R}^2$ with $x + h \in G$. Note that $|h_2| < 1$. We have

$$|g(x+h) - g(x) - g'(x)(h)| = |(x_1+h_1)(x_2+h_2) - x_1x_2 - (x_2h_1 + x_1h_2)| = |h_1h_2|.$$

Now we distinguish two cases. If $\|h\| \leq \delta$, then

$$|g(x+h) - g(x) - g'(x)(h)| = |h_1h_2| \leq \|h\| \cdot \|h\| \leq \|h\| \frac{1}{c}\omega(\|h\|).$$

If $\|h\| > \delta$, we have

$$|g(x+h) - g(x) - g'(x)(h)| = |h_1h_2| \leq |h_1| \leq \|h\| \frac{1}{c\delta}\omega(\|h\|).$$

So Lemma 2.6 implies (16), and consequently the function $f := \frac{c\delta}{2}g$ is both ω -semiconvex and ω -semiconcave on G .

However, we will show that $f \notin C^{1,\omega}(G)$. Suppose to the contrary that f' is uniformly continuous with modulus $D\omega$ on G for some $D > 0$. Then also the function $p(x_1, x_2) := \frac{\partial g}{\partial x_2}(x_1, x_2) = x_1$ is uniformly continuous with modulus $E\omega$ on G for some $E > 0$. By (15) there exists $t_0 > 0$ such that $E\omega(t_0) < t_0$ and therefore

$$t_0 = p\left(t_0, \frac{1}{2}\right) = p\left(t_0, \frac{1}{2}\right) - p\left(0, \frac{1}{2}\right) \leq E\omega(t_0) < t_0,$$

which is a contradiction. $\square$

Remark 5.2. We have proved that if $\omega \in \mathcal{M}$ has properties (15), then there exists a C^1 function f on the strip $\mathbb{R} \times (0, 1)$ which is both ω -semiconvex and ω -semiconcave, and for which f' is not uniformly continuous on the ray $[0, \infty) \times \{\frac{1}{2}\}$ with modulus $D\omega$ for any $D > 0$.

Remark 5.3. (1) If $\liminf_{t \rightarrow 0+} \frac{\omega(t)}{t} = 0$, then (for arbitrary X and G) f is ω -semiconvex (resp. ω -semiconcave) on G if and only if f is continuous and convex (resp. concave) on G (see e.g. [15, p. 3, (2)]). Consequently (6) clearly holds for arbitrary X and G .

(2) If $\liminf_{t \rightarrow \infty} \frac{\omega(t)}{t} > 0$, then (as proved in [16]) (6) holds for $X = \mathbb{R}^2$ and an arbitrary $G \subset X$.

It follows that condition (15) in Example 5.1 cannot be relaxed. $\square$

The proof of Theorem 1.9 for $n = 2$ is based on the constructions of the following two lemmas. (The case of general n needs a nontrivial inductive argument.)

Lemma 5.4. Let $\eta : [0, \infty) \rightarrow [0, \infty)$ be a continuous nondecreasing nonconstant concave function with $\eta(0) = 0$ and $\lim_{t \rightarrow \infty} \eta(t)/t = 0$. Then there exists a concave modulus ω such that

- (i) $\omega(t) \geq \min(1, t)$, $t \in [0, \infty)$,
- (ii) the function $g(t) := \frac{\eta(t) \cdot \log t}{t \cdot \omega(t)}$, $t \in [1, \infty)$, is bounded,
- (iii) $\lim_{t \rightarrow \infty} \omega(t)/\log t = 0$,
- (iv) $\lim_{t \rightarrow \infty} \omega(t) = \infty$.

Proof. First we will construct a concave modulus $\tilde{\omega} \in \mathcal{M}$, for which conditions (i)–(iii) hold. Set $\tilde{\omega}(t) := t$, $t \in [0, 1]$, and

$$\tilde{\omega}(t) := 1 + \frac{1}{\eta(1)} \cdot \int_1^t \frac{\eta(u)}{u^2} du, \quad t \geq 1.$$

Then clearly $\tilde{\omega} \in \mathcal{M}$ and (i) holds for $\tilde{\omega}$.

Note that the function $u \mapsto \eta(u)/u$ is nonincreasing on $[1, \infty)$ by concavity of η . So the function $u \mapsto \eta(u)/u^2$ is nonincreasing on $[1, \infty)$ too. Consequently we obtain that $\tilde{\omega}$ is concave on $[1, \infty)$ and since $\tilde{\omega}'_+(1) = 1$, we obtain that $\tilde{\omega}$ is concave on $[0, \infty)$.

Since $u \mapsto \eta(u)/u$ is nonincreasing on $[1, \infty)$, we obtain that, for $1 < u \leq t$, we have

$$\frac{\eta(u)}{u^2} \geq \frac{\eta(t)}{t} \cdot \frac{1}{u},$$

consequently $\tilde{\omega}(t) \geq \frac{1}{\eta(1)} \cdot \frac{\eta(t)}{t} \cdot \int_1^t \frac{1}{u} du = \frac{1}{\eta(1)} \cdot \frac{\eta(t)}{t} \cdot \log t$, $t \geq 1$, and so (ii)

for $\tilde{\omega}$ follows. Finally, (iii) for $\tilde{\omega}$ follows by L'Hospital's rule, since

$$\lim_{t \rightarrow \infty} \frac{\int_1^t \frac{\eta(u)}{u^2} du}{\log t} = \lim_{t \rightarrow \infty} \frac{\frac{\eta(t)}{t^2}}{\frac{1}{t}} = 0.$$

Now it is easy to see that the modulus

$$\omega(t) := \tilde{\omega}(t) + \sqrt{\log(1+t)}, \quad t \in [0, \infty),$$

has all demanded properties. □

Lemma 5.5. *Let $\eta : [0, \infty) \rightarrow [0, \infty)$ be a nondecreasing nonconstant continuous concave function such that $\eta(0) = 0$ and $\lim_{t \rightarrow \infty} \eta(t)/t = 0$. Set*

$$H := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 1, |x_2| < \eta(x_1)\}$$

and $f(x_1, x_2) := \log(x_1)x_2$, $(x_1, x_2) \in H$.

Then there exists a concave $\omega \in \mathcal{M}$ such that

$$\lim_{t \rightarrow \infty} \omega(t) = \infty, \quad \lim_{t \rightarrow \infty} \frac{\omega(t)}{\log t} = 0, \quad (17)$$

and f is both $(C\omega)$ -semiconvex and $(C\omega)$ -semiconcave for some $C > 0$. (18)

Proof. By Lemma 5.4 there exists a concave $\omega \in \mathcal{M}$ such that (17) and conditions (i), (ii) of Lemma 5.4 hold. So we can find $D > 0$ such that

$$\eta(t) \log(t) \leq Dt\omega(t) \quad \text{and} \quad \eta(t) \leq Dt, \quad t \geq 1. \quad (19)$$

To prove (18), we will use Lemma 2.3. So we consider arbitrary $x = (x_1, x_2) \in H$, $y = (y_1, y_2) \in H$ and set $h = (h_1, h_2) := y - x$. By the symmetry, we can suppose that $h_1 \geq 0$. Now we will estimate from above

$$V := |f'(y)(y - x) - f'(x)(y - x)| = |f'(x + h)(h) - f'(x)(h)|.$$

An elementary computation gives

$$\begin{aligned} V &= \left| \frac{x_2 + h_2}{x_1 + h_1} \cdot h_1 + \log(x_1 + h_1)h_2 - \frac{x_2}{x_1} h_1 - \log(x_1)h_2 \right| \\ &\leq h_1 \left| \frac{x_2 + h_2}{x_1 + h_1} - \frac{x_2}{x_1} \right| + |h_2| (\log(x_1 + h_1) - \log(x_1)) =: A + B. \end{aligned}$$

We have
$$A = h_1 \left| \frac{h_2 x_1 - h_1 x_2}{(x_1 + h_1)x_1} \right| \leq \frac{h_1 |h_2|}{x_1 + h_1} + \frac{(h_1)^2 |x_2|}{(x_1 + h_1)x_1}.$$

Using Lemma 5.4 (i) and $x_1 > 1$, we obtain

$$\frac{h_1 |h_2|}{x_1 + h_1} \leq \min(\|h\|, \|h\|^2) = \|h\| \min(1, \|h\|) \leq \|h\| \omega(\|h\|)$$

and, using also $|x_2| \leq \eta(x_1) \leq Dx_1$,

$$\frac{(h_1)^2 |x_2|}{(x_1 + h_1)x_1} \leq D \frac{(h_1)^2}{x_1 + h_1} \leq D \min(\|h\|, \|h\|^2) \leq D \|h\| \omega(\|h\|),$$

and consequently $A \leq (D + 1) \|h\| \omega(\|h\|)$.

If $h_1 \leq 1$, then $h_1 \leq \omega(h_1)$ by Lemma 5.4 (i), and therefore

$$B = |h_2| \log \left(1 + \frac{h_1}{x_1} \right) \leq |h_2| \frac{h_1}{x_1} \leq |h_2| h_1 \leq |h_2| \omega(h_1) \leq \|h\| \omega(\|h\|).$$

If $1 \leq h_1 \leq x_1$, note that Lemma 5.4 (i) implies $1 \leq \omega(h_1)$ and we obviously get $|h_2| \leq 2\eta(x_1 + h_1)$. Therefore we obtain, using also (19),

$$\begin{aligned} B &= |h_2| \log \left(1 + \frac{h_1}{x_1} \right) \leq 2\eta(x_1 + h_1) \frac{h_1}{x_1} \leq 2 \frac{\eta(2x_1)}{x_1} h_1 \\ &\leq 4Dh_1 \leq 4Dh_1 \omega(h_1) \leq 4D \|h\| \omega(\|h\|). \end{aligned}$$

If $x_1 \leq h_1$, then we obtain, using $|h_2| \leq 2\eta(x_1 + h_1)$, (19) and concavity of ω ,

$$\begin{aligned} B &= |h_2| (\log(x_1 + h_1) - \log(x_1)) \\ &\leq 2\eta(x_1 + h_1) \log(x_1 + h_1) \leq 2D(x_1 + h_1) \omega(x_1 + h_1) \\ &\leq 4Dh_1 \omega(2h_1) \leq 8Dh_1 \omega(h_1) \leq 8D \|h\| \omega(\|h\|). \end{aligned}$$

Using the above estimates, we obtain $V = A + B \leq (9D + 2) \|h\| \omega(\|h\|)$. So Lemma 2.3 implies that (18) holds for $C := 9D + 2$. $\square$

The proof of our main result in general $\mathbb{R}^n$ uses an inductive argument which needs the following auxiliary notion.

Definition 5.6. Let $n \in \mathbb{N}$ and $\omega \in \mathcal{M}$. Then we denote by $\mathcal{G}_n(\omega)$ the set of all open convex sets $G \subset \mathbb{R}^n$ for which there exist a ray $\rho \subset G$ and a function $f \in C^1(G)$ which is both ω -semiconvex and ω -semiconcave and for which there is no real $D > 0$ such that f' is uniformly continuous on ρ with modulus $D\omega$. $\square$

Remark 5.7. (1) If $G \in \mathcal{G}_n(\omega)$, then implication (6) clearly does not hold.
 (2) In Definition 5.6, we can equivalently write the weaker property “ f is both $(C\omega)$ -semiconvex and $(C\omega)$ -semiconcave for some $C > 0$ ”. Indeed, then the function $f^* := (1/C)f$ shows that $G \in \mathcal{G}_n(\omega)$. $\square$

Remark 5.8. Note that Remark 5.2 immediately implies that, if $\omega \in \mathcal{M}$ has properties (15), then $\mathbb{R} \times (0, 1) \in \mathcal{G}_2(\omega)$. $\square$

The following result on further unbounded open convex subsets in $\mathbb{R}^2$ easily follows from Lemma 5.5.

Lemma 5.9. *Let u be a positive nondecreasing concave function on $(1, \infty)$ and let l be a negative nonincreasing convex function on $(1, \infty)$ such that $\lim_{x \rightarrow \infty} u(x)/x = 0$ and $\lim_{x \rightarrow \infty} l(x)/x = 0$. Then there exists a concave modulus $\omega \in \mathcal{M}$ such that $\lim_{t \rightarrow \infty} \omega(t) = \infty$ and*

$$G := \{(x, y) : x > 1, l(x) < y < u(x)\} \in \mathcal{G}_2(\omega). \quad (20)$$

Proof. Set $h(x) := u(x) - l(x)$, $x > 1$, $a := \max(h(2), h'_+(2))$ and $b := 2a - h(2)$. Clearly $a > 0$, $b > 0$. Let $\eta(x) := ax$ for $x \in [0, 2]$ and $\eta(x) := h(x) + b$ for $x \in (2, \infty)$. Then η is clearly a nonnegative nonconstant nondecreasing continuous concave function on $[0, \infty)$ such that $\eta(0) = 0$, $\lim_{x \rightarrow \infty} \eta(x)/x = 0$ and

$$G \subset H := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 1, |x_2| < \eta(x_1)\}.$$

Let $f(x_1, x_2) := \log(x_1)x_2$, $(x_1, x_2) \in H$, and let $\omega \in \mathcal{M}$ be a modulus which corresponds to η and f by Lemma 5.5. Suppose to the contrary $G \notin \mathcal{G}_2(\omega)$. Since $G \subset H$ and (18) implies that f is both $(C\omega)$ -semiconvex and $(C\omega)$ -semiconcave on G for some $C > 0$, using Remark 5.7 (ii) we obtain that f' is uniformly continuous with modulus $D\omega$ on the ray $\rho := [2, \infty) \times \{0\} \subset G$ for some $D > 0$. Consequently

$$\left| \frac{\partial f}{\partial x_2}(t, 0) - \frac{\partial f}{\partial x_2}(2, 0) \right| = |\log t - \log 2| \leq D\omega(t - 2), \quad t > 2,$$

which clearly contradicts (17). $\square$

The following lemma is geometrically obvious.

Lemma 5.10. *Let V be a two-dimensional linear subspace of $\mathbb{R}^n$ and $C \subset V$ a two-dimensional closed convex cone such that $\{v, -v\} \subset C$ for no $v \neq 0$. Then there exists $0 \neq w \in V$ such that $\text{span}(\{w\}) \cap C = \{0\}$.*

Lemma 5.11. *Let $\omega \in \mathcal{M}$ be concave. Let $n \geq k \geq 2$, let $G \subset \mathbb{R}^n$ be an open convex set, and let $L : \mathbb{R}^n \rightarrow \mathbb{R}^k$ be a linear surjection. Suppose that $H := L(G) \in \mathcal{G}_k(\omega)$ and $\text{rec}(H) \subset L(\text{rec}(G))$. Then $G \in \mathcal{G}_n(\omega)$.*

Proof. Since $H \in \mathcal{G}_k(\omega)$, there exist a ray $\rho \subset H$ and a function $f \in C^1(H)$ which is both ω -semiconvex and ω -semiconcave and for which f' is not uniformly continuous on ρ with modulus $D\omega$ for any $D > 0$. Set $\tilde{f}(y) := f(L(y))$ for $y \in G$.

Then $\tilde{f} \in C^1(G)$ and there exists $C > 0$ such that $\tilde{f}$ is both $(C\omega)$ -semiconvex and $(C\omega)$ -semiconcave on G by Lemma 2.8.

We can write $\rho = \{a + tv : t \in [0, \infty)\}$, where $a \in H$ and $v \in \mathbb{R}^k \setminus \{0\}$. Then $v \in \text{rec}(H)$ by Lemma 2.10 (iii) and so, by the assumptions, there exists $\tilde{v} \in \text{rec}(G)$ with $L(\tilde{v}) = v$. Choose $\tilde{a} \in G$ with $L(\tilde{a}) = a$. Then $\tilde{\rho} := \{\tilde{a} + t\tilde{v} : t \in [0, \infty)\} \subset G$. It is easy to see that $L(\tilde{\rho}) = \rho$ and, for $\alpha := \|\tilde{v}\|/\|v\|$,

$$\alpha \cdot \|L(\tilde{x}) - L(\tilde{y})\| = \|\tilde{x} - \tilde{y}\| \quad \text{for each } \tilde{x}, \tilde{y} \in \tilde{\rho}. \quad (21)$$

Now suppose, to the contrary, that $G \notin \mathcal{G}_n(\omega)$. Then by Remark 5.7 (ii) there exists $\tilde{D} > 0$, such that

$$\|(\tilde{f})'(\tilde{x}) - (\tilde{f})'(\tilde{y})\| \leq \tilde{D}\omega(\|\tilde{x} - \tilde{y}\|), \quad \tilde{x}, \tilde{y} \in \tilde{\rho}. \quad (22)$$

Consider the dual mapping $L^* : (\mathbb{R}^k)^* \rightarrow (\mathbb{R}^n)^*$ defined by $L^*(\varphi) := \varphi \circ L$, $\varphi \in (\mathbb{R}^k)^*$. Since L is surjective, L^* is injective and so $L^* : (\mathbb{R}^k)^* \rightarrow L^*((\mathbb{R}^k)^*)$ is a linear isomorphism. So there exists $\beta > 0$ such that

$$\beta\|\varphi \circ L\| \geq \|\varphi\| \quad \text{for each } \varphi \in (\mathbb{R}^k)^*. \quad (23)$$

Now consider arbitrary $x, y \in \rho$ and find $\tilde{x}, \tilde{y} \in \tilde{\rho}$ such that $L(\tilde{x}) = x$, $L(\tilde{y}) = y$. Using (23) for $\varphi := f'(x) - f'(y)$, (22), (21) and the concavity of ω , we obtain

$$\begin{aligned} \|f'(x) - f'(y)\| &\leq \beta\|(f'(x) - f'(y)) \circ L\| = \beta\|(f'(L(\tilde{x})) - f'(L(\tilde{y}))) \circ L\| \\ &= \beta\|(\tilde{f})'(\tilde{x}) - (\tilde{f})'(\tilde{y})\| \leq \beta\tilde{D}\omega(\|\tilde{x} - \tilde{y}\|) \\ &= \beta\tilde{D}\omega(\alpha\|x - y\|) \leq \beta\tilde{D}\max(1, \alpha)\omega(\|x - y\|). \end{aligned}$$

So f' is uniformly continuous on ρ with modulus $D\omega$ for $D := \beta\tilde{D}\max(1, \alpha)$, which is a contradiction. $\square$

As an easy consequence we obtain the following fact.

Lemma 5.12. *Let $\omega \in \mathcal{M}$ be concave. Let $n \geq 2$, let $G \subset \mathbb{R}^n$ be an open convex set, and let $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an affine bijection. Then $G \in \mathcal{G}_n(\omega)$ if and only if $A(G) \in \mathcal{G}_n(\omega)$.*

Proof. If A is linear, then the assertion follows from Lemma 2.11 and Lemma 5.11. If A is a translation, then the proof is obvious. So the lemma clearly follows. $\square$

As already mentioned, Theorem 1.9 immediately follows from the following proposition via Remark 5.7 (i).

Proposition 5.13. *Let $n \geq 2$ and let $G \subset \mathbb{R}^n$ be an unbounded open convex set which contains no translation of a convex cone with nonempty interior. Then there exists a concave modulus $\omega \in \mathcal{M}$ with $\lim_{t \rightarrow \infty} \omega(t) = \infty$ such that $G \in \mathcal{G}_n(\omega)$.*

Proof. We will proceed by induction on n .

(a) Let $n = 2$. Then our assumptions and Lemma 2.10 (iv), (vi) imply that $\dim(\text{rec}(G)) = 1$. On account of Lemma 5.12 we can suppose $(1, 0) \in \text{rec}(G)$. Now we will distinguish two cases.

If also $-(1, 0) \in \text{rec}(G)$, then clearly $G = \mathbb{R} \times H$ for an open convex set $H \subset \mathbb{R}$. Since $\dim(\text{rec}(G)) = 1$, H is a bounded open interval. By Lemma 5.12 we can suppose $H = (0, 1)$ and so $G \in \mathcal{G}_2(\omega)$ e.g. for $\omega(t) = \sqrt{t}$ by Remark 5.8.

If $-(1, 0) \notin \text{rec}(G)$, then we can suppose (by Lemma 5.12) that

$$\{x \in \mathbb{R} : (x, 0) \in G\} = (1, \infty).$$

We can also suppose that a support line to $\overline{G}$ at $(1, 0)$ is parallel to the y -axis. For $x > 1$, set $u(x) := \sup\{y : (x, y) \in G\}$ and $l(x) := \inf\{y : (x, y) \in G\}$. It is easy to show that u is a positive nondecreasing concave function on $(1, \infty)$, l is a negative nonincreasing convex function on $(1, \infty)$ and

$$G = \{(x, y) : x > 1, l(x) < y < u(x)\}.$$

Then $A := \lim_{x \rightarrow \infty} u(x)/x = 0$, since otherwise $(1, A/2) \in \text{rec}(G)$ and thus we have $\dim(\text{rec}(G)) = 2$. Similarly we obtain $\lim_{x \rightarrow \infty} l(x)/x = 0$. So our assertion follows from Lemma 5.9.

(b) Now suppose that $n \geq 3$ is given and that the proposition “holds for $n - 1$ ”. We will distinguish two cases.

(b1) There exists $v \neq 0$ such that $\{v, -v\} \subset \text{rec}(G)$ (i.e., G contains a line).

By Lemma 5.12, we can suppose that $v = (0, \dots, 0, 1)$ and thus there exists an open convex set $H \subset \mathbb{R}^{n-1}$ such that $G = H \times \mathbb{R}$. Then clearly $\dim(\text{rec}(H)) < n - 1$ and so the inductive assumption gives that $H \in \mathcal{G}_{n-1}(\omega)$ for some concave $\omega \in \mathcal{M}$ with $\lim_{t \rightarrow \infty} \omega(t) = \infty$. Let $L(x_1, \dots, x_n) := (x_1, \dots, x_{n-1})$. Then clearly $L(G) = H$ and $\text{rec}(H) \subset L(\text{rec}(G))$. Therefore $G \in \mathcal{G}_n(\omega)$ by Lemma 5.11.

(b2) There is no $v \neq 0$ such that $\{v, -v\} \subset \text{rec}(G)$.

We will show that there exists a linear surjection $L : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ such that

$$L^{-1}(\{0\}) \cap \text{rec}(G) = \{0\} \quad (24)$$

$$\text{and} \quad \dim(L(\text{rec}(G))) < n - 1. \quad (25)$$

If $\dim(\text{rec}(G)) < n - 1$, it is easy (using Lemma 2.10 (v)) to find L satisfying (24) and observe that (25) holds.

In the opposite case (v) and (vi) in Lemma 2.10 imply that $\dim(\text{rec}(G)) = n - 1$ and there exist linearly independent vectors $v_1, \dots, v_{n-1} \in \text{rec}(G)$. Now we apply Lemma 5.10 to the space $V := \text{span}\{v_1, v_2\}$ and the cone $C := V \cap \text{rec}(G)$ (which is closed by Lemma 2.10 (ii)) and obtain a corresponding vector w . Then any linear surjection $L : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ with $L(w) = 0$ clearly satisfies (24). Moreover, $L(w) = 0$ implies that the vectors $L(v_1)$ and $L(v_2)$ are linearly dependent and consequently $\dim(L(\text{rec}(G))) < n - 1$.

So there exists L for which (24) and (25) hold. Using (24) and Lemma 2.11 we obtain $\text{rec}(L(G)) = L(\text{rec}(G))$. So $\dim(\text{rec}(L(G))) < n - 1$ and consequently $L(G) \in \mathcal{G}_{n-1}(\omega)$ for some concave $\omega \in \mathcal{M}$ with $\lim_{t \rightarrow \infty} \omega(t) = \infty$ by the induction assumption. Therefore we obtain $G \in \mathcal{G}_n(\omega)$ by Lemma 5.11. $\square$

6. Consequences and open questions

As already mentioned, our results are very closely related to a first-order quantitative converse Taylor theorem and to the problem whether $f \in C^{1,\omega}(G)$ whenever f is continuous and smooth in a corresponding sense on all lines.

To describe precisely these relations, first recall [8, Chap. 1, Definition 121] in the special case of real-valued functions and $p = 1$ (which deal with approximation of a function by polynomials of degree $p \leq 1$).

Definition 6.1. Let X be a normed linear space and $G \subset X$ an open set. We say that a function $f : G \rightarrow \mathbb{R}$ is UT^1 -smooth on G with modulus $\omega \in \mathcal{M}$ (or shortly UT_ω^1 -smooth) if for each $x \in G$ there exist $b \in \mathbb{R}$ and $x^* \in X^*$ such that

$$|f(x+h) - (b + x^*(h))| \leq \|h\| \omega(\|h\|) \quad \text{whenever } x+h \in G. \quad \square \quad (26)$$

Remark 6.2. Note that if (26) holds then clearly $b = f(x)$ and $x^* = f'(x)$. So f is UT_ω^1 -smooth on G if and only if (10) holds. $\square$

Further, we will use the following terminology.

Definition 6.3. Let X be a normed linear space, $G \subset X$ an open convex set, $\omega \in \mathcal{M}$, and let f be a function on G . We will say that f is $C_*^{1,\omega}$ -smooth on all lines if, for every $a \in G$ and $v \in X$, $\|v\| = 1$, the function $f_{a,v} : t \mapsto f(a + tv)$ defined on the open interval $\{t \in \mathbb{R} : a + tv \in G\}$ is $C_*^{1,\omega}$ -smooth. $\square$

Using this notation, we can reformulate some already mentioned facts as follows.

Proposition 6.4. Let X be a normed linear space, $G \subset X$ an open convex set, $\omega \in \mathcal{M}$, and let f be a continuous function on G . Then the following implications hold.

- (i) If f is both ω -semiconvex and ω -semiconcave, then f is UT_ω^1 -smooth.
- (ii) If f is UT_ω^1 -smooth, then f is both (2ω) -semiconvex and (2ω) -semiconcave.
- (iii) If f is $C_*^{1,\omega}$ -smooth on all lines, then f is both ω -semiconvex and ω -semiconcave.
- (iv) If f is both ω -semiconvex and ω -semiconcave, then f is $C_*^{1,2\omega}$ -smooth on all lines.

Proof. Assertions (i) and (ii) are reformulations of Lemma 2.4 and Lemma 2.6, respectively. Using (5) and Lemma 2.1 we obtain (iii). Lemma 2.1 together with Remark 4.5 (ii) give (iv). $\square$

Remark 6.5. If ω in Proposition 6.4 is linear, then the converse of (i) holds by Lemma 2.6 and the converse of (iv) holds by Remark 2.2 (ii) and Lemma 2.1. $\square$

Using Proposition 6.4, Remark 4.1 and Remark 6.2, we easily see that

Remark 6.6. Theorem 1.10 and Corollary 4.3 remain true if we replace the property “ f is both ω -semiconvex and ω -semiconcave” by the (slightly weaker) property that “ f is UT^1 -smooth function with modulus ω ” or by the (slightly stronger) property that “ f is continuous and $C_*^{1,\omega}$ -smooth on all lines”. $\square$

So we obtain in the very special case $p = 1$, $Y = \mathbb{R}$ (and finite modulus ω) a more precise qualitative version of converse Taylor theorem [8, Chap. 1, Corollary 126] and also of [12, Proposition 12] and [8, Chap. 1, Theorem 127] (which deal with functions which are “ $C_*^{p,\omega}$ -smooth on all lines”).

In particular, we obtain the following version of Corollary 4.3.

Corollary 6.7. *Let X be a normed linear space and $\omega \in \mathcal{M}$. Suppose that a continuous function f on X is $C_*^{1,\omega}$ -smooth on all lines. Then f is $C_*^{1,4\omega}$ -smooth on X .*

Further, Proposition 6.4 shows that our basic result Theorem 1.9 can be reformulated in the following form.

Theorem 6.8. *Let $G \subset \mathbb{R}^n$ ($n \geq 2$) be an unbounded open convex set which does not contain a translation of a convex cone with nonempty interior. Then there exists a concave modulus $\omega \in \mathcal{M}$ with $\lim_{t \rightarrow \infty} \omega(t) = \infty$ and a continuous function f such that*

- (i) f is $C_*^{1,\omega}$ -smooth on all lines,
- (ii) f is UT^1 -smooth function with modulus ω and
- (iii) $f \notin C^{1,\omega}(G)$.

In particular, the assumptions on G of [8, Chap. 1, Corollary 126] cannot be relaxed in the case $X = \mathbb{R}^n$.

Finally observe that, by Proposition 6.4, equivalence (7) can be extended as follows.

Proposition 6.9. *Let X and G be as in Theorem 1.7. Then, for each $\omega \in \mathcal{M}$ and for each function f on G , the following conditions are equivalent.*

- (1) $f \in C^{1,\omega}(G)$.
- (2) f is both $(K\omega)$ -semiconvex and $(K\omega)$ -semiconcave for some $K > 0$.
- (3) f is $UT_{K\omega}^1$ -smooth for some $K > 0$.
- (4) f is continuous and $C_*^{1,K\omega}$ -smooth on all lines for some $K > 0$.

The following natural (cf. Remark 5.3) questions remain open.

Question 6.10. Let $\omega \in \mathcal{M}$, $n \geq 2$ and let $G \subset \mathbb{R}^n$ be an open convex set which contains no translation of a convex cone with nonempty interior. Does there exist a function f on G which is both ω -semiconvex and ω -semiconcave and does not belong to $C^{1,\omega}(G)$, if

- (1) $\liminf_{t \rightarrow 0^+} \frac{\omega(t)}{t} > 0$ and $\liminf_{t \rightarrow \infty} \frac{\omega(t)}{t} = 0$?
- (2) ω is concave, $\liminf_{t \rightarrow 0^+} \frac{\omega(t)}{t} > 0$ and $\liminf_{t \rightarrow \infty} \frac{\omega(t)}{t} = 0$?
- (3) $\omega(t) = t^\alpha$, $t \geq 0$, for some $0 < \alpha < 1$?
- (4) $\omega(t) = \sqrt{t}$, $t \geq 0$?

Remark 6.11. It is proved in [16] that the implication (6) does not hold in the case $\omega(t) = t^\alpha$, $t \geq 0$, $G = \{(x, y) \in \mathbb{R}^2 : x > 0, |y| < x^\beta\}$, $0 < \alpha < 1$ and $0 < \beta < 1$. $\square$

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