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Finite Convergence of Locally Proper Circumcentered Methods

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In view of the great performance of circumcentered isometry methods for solving the best approximation problem, in this work we further investigate the locally proper circumcenter mapping and circumcentered method. Various examples of locally proper circumcenter mapping are presented and studied. Inspired by some results on the circumcentered-reflection method by R. Arefidamghani, R. Behling, Y. Bello-Cruz, A. N. Iusem and L. R. Santos [*The block-wise circumcentered-reflection method*, Comp. Optimization Appl. 76/3 (2019) 675–699; *The circumcentered-reflection method achieves better rates than alternating projections*, ibid. 79/2 (2021) 507–530; *On the circumcentered-reflection method for the convex feasibility problem*, Num. Algorithms 86/4 (2021) 1475–1494], we provide sufficient conditions for one-step convergence of circumcentered isometry methods for finding the best approximation point onto the intersection of fixed point sets of related isometries. In addition, we elaborate the performance of circumcentered reflection methods induced by reflectors associated with hyperplanes and halfspaces for finding the best approximation point onto (or a point in) the intersection of hyperplanes and halfspaces.

Keywords: Best approximation problem, feasibility problem, circumcenter mapping, circumcentered method, properness, halfspace, hyperplane, finite convergence.

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1. Introduction

Throughout this paper, we assume that $\mathcal{H}$ is a real Hilbert space, with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Set $\mathbb{N} := \{0, 1, 2, \ldots\}$, let $m \in \mathbb{N} \smallsetminus \{0\}$, and denote by $\mathrm{I} := \{1, \ldots, m\}$.

Let $C_1, \ldots, C_m$ be closed and convex subsets of $\mathcal{H}$ with $\cap_{i=1}^{m} C_i \neq \varnothing$. Let $x \in \mathcal{H}$. The *best approximation problem* is to find the projection $\mathrm{P}_{\cap_{i=1}^{m} C_i} x$, and the *feasibility problem* is to find a point in $\cap_{i=1}^{m} C_i$.

Denote by $\mathcal{P}(\mathcal{H})$ the set of nonempty subsets of $\mathcal{H}$ containing finitely many elements. The *circumcenter operator* $CC \colon \mathcal{P}(\mathcal{H}) \to \mathcal{H} \cup \{\varnothing\}$ maps every $K \in \mathcal{P}(\mathcal{H})$ to the *circumcenter $CC(K)$ of K*, where $CC(K)$ is either the empty set or the unique point $CC(K)$ such that $CC(K) \in \operatorname{aff}(K)$ and $CC(K)$ is equidistant from all points in K.

Let $\mathcal{K}$ and $\mathcal{D}$ be nonempty subsets of $\mathcal{H}$ with $\mathcal{D} \subseteq \mathcal{K}$. Let $(\forall i \in \mathrm{I})$ $T_i : \mathcal{K} \to \mathcal{H}$. Set $\mathcal{S} := \{T_1, \ldots, T_m\}$ and$(\forall x \in \mathcal{K})$ $\mathcal{S}(x) := \{T_1 x, \ldots, T_m x\}$. The *circumcenter mapping $CC_{\mathcal{S}}$ induced by $\mathcal{S}$* is defined as $(\forall x \in \mathcal{K})$ $CC_{\mathcal{S}}(x) = CC(\mathcal{S}(x))$.

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We say that $CC_{\mathcal{S}}$ is *proper over* $\mathcal{D}$, if $(\forall x \in \mathcal{D})$ $CC_{\mathcal{S}}x \in \mathcal{D}$. In particular, if $\mathcal{D} = \mathcal{H}$, then we omit the phrase "over $\mathcal{H}$"; if $\mathcal{D} \subsetneqq \mathcal{H}$, we also say that $CC_{\mathcal{S}}$ is *locally proper*. Assume that $CC_{\mathcal{S}}$ is proper over $\mathcal{D}$. Let $x \in \mathcal{D}$. The *circumcentered method induced by* $\mathcal{S}$ generates the sequence $(CC^k_{\mathcal{S}}x)_{k\in\mathbb{N}}$ of iterations. When $\mathcal{D} \neq \mathcal{H}$, we also say the related circumcentered method is *locally proper*. Particularly, if all $T_1, \ldots, T_m$ are isometries (resp. reflectors), then we call the method *circumcentered isometry method* (*resp. circumcentered reflection method*) and CIM (resp. CRM) for short.

For every $i \in \mathrm{I}$, let U_i be an affine subspace and let R_{U_i} be the reflector associated with U_i. In [9], Behling, Bello Cruz and Santos defined the circumcentered Douglas-Rachford method (C-DRM), which is the first circumcentered method in the literature and is actually the circumcentered method induced by $S := \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}\mathrm{R}_{U_1}\}$. Then they introduced the circumcentered-reflection method which is the circumcentered method induced by $S := \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}\mathrm{R}_{U_1}, \cdots, \mathrm{R}_{U_m}\cdots\mathrm{R}_{U_2}\mathrm{R}_{U_1}\}$. Motivated by the C-DRM, Bauschke, Ouyang and Wang introduced the circumcenter mapping and circumcentered method defined above and showed the global properness of circumcenter mappings induced by sets of isometries in [5].

Results on circumcentered methods accelerating the well-known Douglas-Rachford method and method of alternating projections can be found in [1], [7], and [9]. Note that circumcenter mappings induced by sets of operators that are not isometric are generally not globally proper, which implies that the related circumcentered methods may not be well-defined on the whole space. But clearly if a circumcenter mapping is proper over a nonempty set, say $\mathcal{D}$, then the related circumcentered method is well-defined with initial points in $\mathcal{D}$. Considering the great performance of CIMs for solving best approximation problems, naturally one may want to know how will the locally proper circumcentered methods perform when solving best approximation or feasibility problems?

In this work, *our goal is to study locally proper circumcenter mappings and to explore the finite convergence of circumcentered methods (not necessarily proper over* $\mathcal{H}$*) for solving the best approximation and feasibility problems.* The main results in this work are the following:

R1: Theorem 3.5 presents an explicit formula for the circumcenter of three points satisfying an equidistant condition, which plays a critical role in proving the one-step convergence of the CIM with initial point from a certain set in Theorem 4.1.

R2: Theorem 4.7 states sufficient conditions for the one-step convergence of circumcenter methods induced by $\mathcal{S}$ being

$$\{\mathrm{Id}, T_1, T_2, \ldots, T_m\} \text{ or } \{\mathrm{Id}, T_1, T_2T_1, \ldots, T_m\cdots T_2T_1\}$$

where $(\forall i \in \mathrm{I})$ T_i is isometric. Moreover, Propositions 4.8 and 4.9 illustrate that the sets of initial points required in Theorem 4.7 are dense in $\mathcal{H}$ and specify clear procedures to find practical initial points.

R3: Theorem 5.14 illustrates that we can always find CRMs induced by sets of appropriate reflectors for finding the best approximation point onto or feasibility point in the intersection of hyperplanes and halfspaces in at most three steps.

The organization of the rest of the work is the following. Some preliminary results are presented in Section 2. In particular, Theorem 2.9 generalizes Bauschke et al. [5, Theorem 4.7] from circumcenter mapping to general operator and specifies sufficient conditions for the weak convergence of the related iteration sequence. In Section 3, we deduce a more practical formula for the circumcenter of three points satisfying a equidistant condition. We also verify that some results on circumcenter mappings with global properness assumption in [5] and [7] actually hold with only local properness assumption (see Fact 3.8 and Proposition 3.9). In Section 4, sufficient conditions for the one-step convergence of CIMs are exhibited in Theorems 4.1 and 4.7. In addition, Propositions 4.8 and 4.9 confirm that the conditions required in Theorem 4.7 are not strong and state clear steps to find desired initial points. In fact, Theorem 4.1(iii), Theorem 4.7(ii), and Proposition 4.4(ii) are generalizations of [12, Lemma 2], [11, Lemma 3], and [12, Lemma 3], respectively. The detailed relations between our results and the related known results are elaborated in Section 4. Last but not least, the finite convergence of circumcentered reflection methods induced by sets of reflectors associated with hyperplanes and halfspaces is explored in Section 5.

We now turn to the notation used in this work. Let C be a nonempty subset of $\mathcal{H}$. The *orthogonal complement* of C is the set $C^{\perp} := \{y \in \mathcal{H} \mid (\forall x \in C)\ \langle x, y\rangle = 0\}$. C is an *affine subspace* of $\mathcal{H}$ if $C \neq \varnothing$ and $(\forall \rho \in \mathbb{R})\ \rho C + (1-\rho)C = C$. The intersection of all affine subspaces of $\mathcal{H}$ containing C is denoted by $\operatorname{aff} C$ and called the *affine hull* of C. An affine subspace U is said to be *parallel* to an affine subspace M if $U = M + a$ for some $a \in \mathcal{H}$. Every affine subspace U is parallel to a unique linear subspace $\operatorname{par} U$, which is given by $(\forall y \in U)\ \operatorname{par} U := U - y = U - U$.

Suppose that C is a nonempty closed convex subset of $\mathcal{H}$. The *projector* (or *projection operator*) onto C is the operator, denoted by P_C, that maps every point in $\mathcal{H}$ to its unique projection onto C. $\mathrm{R}_C := 2\mathrm{P}_C - \mathrm{Id}$ is the *reflector* associated with C. Let $x \in \mathcal{H}$ and let $\epsilon \in \mathbb{R}_+$. Then $\ker x := \{y \in \mathcal{H} : \langle y, x\rangle = 0\}$ and $\mathbf{B}[x;\epsilon] := \{y \in \mathcal{H} : \|y - x\| \leq \epsilon\}$. For two subsets A and B of $\mathcal{H}$ we define $\operatorname{dist}(A,B) := \inf\{\|a-b\| : a \in A \text{ and } b \in B\}$.

Let $\mathcal{D}$ be a nonempty subset of $\mathcal{H}$, and let $T : \mathcal{D} \to \mathcal{H}$. T is said to be *isometric* or an *isometry*, if $(\forall x, y \in \mathcal{D})\ \|T(x) - T(y)\| = \|x - y\|$. The *set of fixed points of the operator* T is denoted by $\operatorname{Fix} T$, i.e., $\operatorname{Fix} T := \{x \in \mathcal{D} \ :\ Tx = x\}$. The *range of* T is defined as $\operatorname{ran} T := \{Tx \ :\ x \in \mathcal{H}\}$; moreover, $\overline{\operatorname{ran}}\, T$ is the closure of $\operatorname{ran} T$. Denote by $\mathcal{B}(\mathcal{H},\mathcal{H}) := \{T : \mathcal{H} \to \mathcal{H} \ :\ T \text{ is bounded and linear}\}$. For every $T \in \mathcal{B}(\mathcal{H},\mathcal{H})$, the *operator norm* $\|T\|$ of T is defined by $\|T\| := \sup_{\|x\|\leq 1}\|Tx\|$. A sequence $(x_k)_{k\in\mathbb{N}}$ in $\mathcal{H}$ *converges weakly* to a point $x \in \mathcal{H}$ if, for every $u \in \mathcal{H}$, $\langle x_k, u\rangle \to \langle x, u\rangle$; in symbols, $x_k \rightharpoonup x$.

For other notation not explicitly defined here, we refer the reader to [2].

2. Auxiliary results

In this section, we collect some results to be used subsequently.

Projections

Fact 2.1. [15, Fact 1.8] *Let A and B be two nonempty closed convex subsets of $\mathcal{H}$. Let $A \subseteq B$ and $x \in \mathcal{H}$. Then $\mathrm{P}_B x \in A$ if and only if $\mathrm{P}_B x = \mathrm{P}_A x$.*

Fact 2.2. [2, Proposition 3.19] *Let C be a nonempty closed convex subset of $\mathcal{H}$ and let $x \in \mathcal{H}$. Set $D := z + C$, where $z \in \mathcal{H}$. Then $\mathrm{P}_D x = z + \mathrm{P}_C(x - z)$.*

Fact 2.3. [2, Example 29.18] *Let $u \in \mathcal{H} \smallsetminus \{0\}$, let $\eta \in \mathbb{R}$, and set*

$$H := \{x \in \mathcal{H} \ : \ \langle x, u\rangle = \eta\}.$$

Then $(\forall x \in \mathcal{H})$ $\mathrm{P}_H x = x + \frac{\eta - \langle x,u\rangle}{\|u\|^2} u$.

Fact 2.4. [16, Remark 2.9] *Let $u \in \mathcal{H} \smallsetminus \{0\}$ and $\eta \in \mathbb{R}$. Set*

$$W := \{x \in \mathcal{H} \ : \ \langle x, u\rangle \leq \eta\} \text{ and } H := \{x \in \mathcal{H} \ : \ \langle x, u\rangle = \eta\}.$$

Then $W \neq \varnothing$, $W^c \neq \varnothing$, and $(\forall x \in W^c)$ $\mathrm{P}_W x = x + \frac{\eta - \langle x,u\rangle}{\|u\|^2} u = \mathrm{P}_H x$.

Lemma 2.5. *Let $u \in \mathcal{H} \smallsetminus \{0\}$ and let $\eta \in \mathbb{R}$. Set $W := \{x \in \mathcal{H} \ : \ \langle x, u\rangle \leq \eta\}$ and $H := \{x \in \mathcal{H} \ : \ \langle x, u\rangle = \eta\}$. Let $x \in \mathcal{H} \smallsetminus W$. Then $\mathrm{R}_H x = \mathrm{R}_W x \in \operatorname{int} W$.*

Proof. According to Fact 2.4,

$$\mathrm{R}_W x = \mathrm{R}_H x = 2\,\mathrm{P}_H(x) - x = x + 2\frac{\eta - \langle x, u\rangle}{\|u\|^2} u.$$

Clearly, $x \notin W$ means $\langle x, u\rangle - \eta > 0$. Hence,

$$\begin{aligned}\langle \mathrm{R}_W x, u\rangle - \eta &= \left\langle x + 2\frac{\eta - \langle x, u\rangle}{\|u\|^2} u, u\right\rangle - \eta = \langle x, u\rangle - \eta + 2\frac{\eta - \langle x, u\rangle}{\|u\|^2}\langle u, u\rangle \\ &= \eta - \langle x, u\rangle < 0,\end{aligned}$$

which implies that $\mathrm{R}_H x = \mathrm{R}_W x \in \operatorname{int} W$. $\square$

Sufficient conditions for weak convergence

Definition 2.6. [2, Definition 5.1] Let C be a nonempty subset of $\mathcal{H}$ and let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{H}$. Then $(x_k)_{k\in\mathbb{N}}$ is *Fejér monotone with respect to C* if

$$(\forall x \in C)(\forall k \in \mathbb{N}) \quad \|x_{k+1} - x\| \leq \|x_k - x\|.$$

Definition 2.7. [2, Definition 4.26] Let $\mathcal{D}$ be a nonempty weakly sequentially closed subset of $\mathcal{H}$, let $T : \mathcal{D} \to \mathcal{H}$, and let $u \in \mathcal{H}$. Then T is *demiclosed at u* if, for every sequence $(x_k)_{k\in\mathbb{N}}$ in $\mathcal{D}$ and every $x \in \mathcal{D}$ such that $x_k \rightharpoonup x$ and $Tx_k \to u$, we have $Tx = u$. In addition, T is *demiclosed* if it is demiclosed at every point in $\mathcal{D}$.

Definition 2.8. [2, Definition 4.1(iv)] and [13, Definition 2.2] Let $\mathcal{D}$ be a nonempty subset of $\mathcal{H}$ and let $\rho \in \mathbb{R}_+$. We say $T : \mathcal{D} \to \mathcal{H}$ is *ρ-strongly quasinonexpansive* (ρ-SQNE), if

$$(\forall x \in \mathcal{D})(\forall z \in \operatorname{Fix} T) \quad \|T(x) - z\|^2 + \rho\|T(x) - x\|^2 \leq \|x - z\|^2. \tag{1}$$

In particular, if $\rho = 0$ and $\rho = 1$ in (1), then T is called *quasinonexpansive* and *firmly quasinonexpansive*, respectively.

Theorem 2.9. *Let $\mathcal{D}$ be a nonempty weakly sequentially closed subset of $\mathcal{H}$, let $T : \mathcal{D} \to \mathcal{D}$ and let $\rho \in \mathbb{R}_{++}$. Assume that T is ρ-SQNE, that $\operatorname{Fix} T \neq \varnothing$, and that $\operatorname{Id} - T$ is demiclosed at 0. Let $x \in \mathcal{D}$. Then $(T^k x)_{k\in\mathbb{N}}$ weakly converges to one point in $\operatorname{Fix} T$. Moreover, if $\operatorname{Fix} T$ is a closed affine subspace of $\mathcal{H}$, then $T^k x \rightharpoonup \mathrm{P}_{\operatorname{Fix} T} x$, and $(\forall k \in \mathbb{N})$ $\mathrm{P}_{\operatorname{Fix} T} T^k x = \mathrm{P}_{\operatorname{Fix} T} x$.*

Proof. Note that, by Definition 2.8, T is ρ-SQNE implies that T is quasinonexpansive. Moreover, because $\operatorname{Fix} T \neq \varnothing$, using [2, Example 5.3], we know that $(T^k x)_{k\in\mathbb{N}}$ is Fejér monotone with respect to $\operatorname{Fix} T$. Now, in view of [2, Theorem 5.5 and Proposition 5.9], it remains to show that very weak sequential cluster point of $(T^k x)_{k\in\mathbb{N}}$ belongs to $\operatorname{Fix} T$.

Let $y \in \mathcal{H}$ and let $(T^{k_i}x)_{i\in\mathbb{N}}$ be a subsequence of $(T^k x)_{k\in\mathbb{N}}$ such that $T^{k_i}x \rightharpoonup y$. Let $z \in \operatorname{Fix} T$. According to [2, Proposition 5.4], $(T^k x)_{k\in\mathbb{N}}$ is Fejér monotone with respect to $\operatorname{Fix} T$ implies that the limit $L_z := \lim_{k\to\infty}\|T^k(x) - z\|$ exists. Because $T : \mathcal{D} \to \mathcal{D}$ is ρ-SQNE, we have that

$$(\forall i \in \mathbb{N}) \quad \rho\|T(T^{k_i}x) - T^{k_i}x\|^2 \leq \|T^{k_i}(x) - z\|^2 - \|T(T^{k_i}x) - z\|^2. \tag{2}$$

Summing over i from 0 to infinity in both sides of (2), we obtain that

$$\sum_{i=0}^{\infty}\|T(T^{k_i}x) - T^{k_i}x\|^2 \leq \frac{1}{\rho}\left(\|T^{k_0}(x) - z\|^2 - L_z\right),$$

which implies that $T(T^{k_i}x) - T^{k_i}x \to 0$. Furthermore, using the assumption, $\operatorname{Id} - T$ is demiclosed at 0, and Definition 2.7, we see that

$$\left.\begin{array}{l} T^{k_i}x \rightharpoonup y \\ T(T^{k_i}x) - T^{k_i}x \to 0 \end{array}\right\} \;\Rightarrow\; (\operatorname{Id} - T)y = 0, \text{ i.e., } y \in \operatorname{Fix} T.$$

Altogether, the proof is complete. $\square$

Remark 2.10. Let $\mathcal{D}$ be a nonempty weakly sequentially closed subset of $\mathcal{H}$ and let $T : \mathcal{D} \to \mathcal{D}$ with $\operatorname{Fix} T \neq \varnothing$. According to [2, Definition 4.1 and Theorem 4.27], T is 1-SQNE implies that $\operatorname{Id} - T$ is demiclosed. Hence, Theorem 2.9 implies that if T is ρ-SQNE with $\rho \geq 1$, then $(T^k x)_{k\in\mathbb{N}}$ weakly converges to one point in $\operatorname{Fix} T$.

3. Circumcenter mapping with local properness

In this section, we investigate the locally proper circumcenter mapping.

Recall that $\mathcal{P}(\mathcal{H})$ is the set of all nonempty subsets of $\mathcal{H}$ containing *finitely many* elements, and that $\mathrm{I} := \{1, 2, \ldots, m\}$.

Circumcenters

According to [4, Proposition 3.3], for every K in $\mathcal{P}(\mathcal{H})$, there is at most one point $p \in \operatorname{aff}(K)$ such that $\{\|p-x\| \mid x \in K\}$ is a singleton. Hence, the following Definition 3.1 is well-defined.

Definition 3.1. (Circumcenter operator, [4, Definition 3.4])
The *circumcenter operator* is

$$CC\colon \mathcal{P}(\mathcal{H}) \to \mathcal{H}\cup\{\varnothing\}\colon K \mapsto \begin{cases} p, & \text{if } p\in\operatorname{aff}(K) \text{ and } \{\|p-x\| \mid x\in K\} \text{ is a singleton;} \\ \varnothing, & \text{otherwise.} \end{cases}$$

In particular, when $CC(K)\in\mathcal{H}$, that is, $CC(K) \neq \varnothing$, we say that the circumcenter of K exists and we call $CC(K)$ the *circumcenter of* K.

Fact 3.2. [4, Theorem 8.4] *Suppose that $S := \{x, y, z\} \subseteq \mathcal{H}$ and denote the cardinality of S by l. Then exactly one of the following cases occurs:*

(i) $l = 1$ *and* $CC(S) = x$.

(ii) $l = 2$, *say* $S = \{u, v\}$, *where* $u, v \in S$ *and* $u \neq v$, *and* $CC(S) = \frac{u+v}{2}$.

(iii) $l = 3$ *and exactly one of the following two cases occurs:*

(a) x, y, z *are affinely independent; equivalently,* $\|y-x\|\|z-x\| > \langle y-x, z-x\rangle$, *and*

$$CC(S) = \frac{\|y-z\|^2\langle x-z, x-y\rangle x + \|x-z\|^2\langle y-z, y-x\rangle y + \|x-y\|^2\langle z-x, z-y\rangle z}{2(\|y-x\|^2\|z-x\|^2 - \langle y-x, z-x\rangle^2)}.$$

(b) x, y, z *are affinely dependent; equivalently,* $\|y-x\|\|z-x\| = \langle y-x, z-x\rangle$, *and* $CC(S) = \varnothing$.

Lemma 3.3. *Let* $\{p, x, y\} \subseteq \mathcal{H}$. *Then the following statements are equivalent:*

(i) $\|p-x\| = \|p-y\|$.

(ii) $\langle p-x, y-x\rangle = \frac{1}{2}\|y-x\|^2$.

(iii) $\langle p - \frac{x+y}{2}, \frac{x+y}{2} - x\rangle = 0$.

(iv) $\langle p - \frac{x+y}{2}, y-x\rangle = 0$.

(v) $\langle x-p, y-p\rangle = \|p-x\|^2 - \frac{1}{2}\|y-x\|^2$.

Proof. By [4, Proposition 3.1], we know that (i)$\Leftrightarrow$(ii). Because $y - x = 2\left(\frac{x+y}{2} - x\right)$, we have that

$$\begin{aligned}
\langle p-x, y-x\rangle = \frac{1}{2}\|y-x\|^2 &\Leftrightarrow 2\Big\langle p-x, \frac{x+y}{2} - x\Big\rangle = 2\Big\|\frac{x+y}{2} - x\Big\|^2 \\
&\Leftrightarrow \Big\langle p - \frac{x+y}{2} + \frac{x+y}{2} - x, \frac{x+y}{2} - x\Big\rangle = \Big\|\frac{x+y}{2} - x\Big\|^2 \\
&\Leftrightarrow \Big\langle p - \frac{x+y}{2}, \frac{x+y}{2} - x\Big\rangle = 0 \\
&\Leftrightarrow \Big\langle p - \frac{x+y}{2}, y-x\Big\rangle = 0,
\end{aligned}$$

which shows the equivalence of (ii), (iii) and (iv). On the other hand,

$$\begin{aligned}
\langle p-x, y-x\rangle = \frac{1}{2}\|y-x\|^2 &\Leftrightarrow \langle p-x, y-p+p-x\rangle = \frac{1}{2}\|y-x\|^2 \\
&\Leftrightarrow \langle x-p, y-p\rangle = \|p-x\|^2 - \frac{1}{2}\|y-x\|^2,
\end{aligned}$$

which necessitates that (ii)$\Leftrightarrow$(v). Altogether, all of the equivalences hold. $\square$

Lemma 3.4. *Let* $\{p, x, y\} \subseteq \mathcal{H}$. *Assume that* $\|p-x\| = \|p-y\|$. *Then the following statements hold:*

(i) $\|y-x\| = 2\|x-p\|$ *if and only if* $p = \frac{x+y}{2}$.

(ii) $\|y-x\| < 2\|x-p\|$ *if and only if* $p \neq \frac{x+y}{2}$.

Proof. (i): If $p = \frac{x+y}{2}$, then clearly $2\|x-p\| = 2\left\|x - \frac{x+y}{2}\right\| = \|y-x\|$.

Assume that $\|y-x\| = 2\|x-p\|$. Because $\|p-x\| = \|p-y\|$, we have

$$\|y-x\| = 2\|x-p\| \Leftrightarrow \|y-p\|^2 + 2\langle y-p, p-x\rangle + \|p-x\|^2 = 4\|x-p\|^2 \tag{3a}$$
$$\Leftrightarrow \langle y-p, p-x\rangle = \|x-p\|^2 \tag{3b}$$
$$\Leftrightarrow \langle y-p+x-p, p-x\rangle = 0 \tag{3c}$$
$$\Leftrightarrow \Big\langle \frac{x+y}{2} - p, p-x\Big\rangle = 0. \tag{3d}$$

Note that $\|p-x\| = \|p-y\|$ implies that $\|y-x\| = 2\|x-p\| \Leftrightarrow \|x-y\| = 2\|y-p\|$. Combine this with the calculation of (3) to deduce that

$$\Big\langle \frac{y+x}{2} - p, p-y\Big\rangle = 0. \tag{4}$$

Now (3d) and (4) add up to

$$\Big\langle \frac{y+x}{2} - p, 2p-x-y\Big\rangle = 0 \Leftrightarrow \Big\langle \frac{y+x}{2} - p, p - \frac{y+x}{2}\Big\rangle = 0 \Leftrightarrow p = \frac{x+y}{2}.$$

Altogether, (i) holds.

(ii): Using the assumption $\|p-x\| = \|p-y\|$ and the triangle inequality, we know that $\|y-x\| \leq \|y-p\| + \|p-x\| = 2\|x-p\|$. Then (ii) follows directly from (i). $\square$

Theorem 3.5. *Let $\{x,y,z\} \subseteq \mathcal{H}$. Assume that $\|x-y\| = \|x-z\|$. Then*

$$CC(\{x,y,z\}) = \begin{cases} x, & \text{if } x = y; \\ \varnothing, & \text{if } x \neq y \text{ and } x = \frac{y+z}{2}; \\ x + \alpha(y-x) + \alpha(z-x), & \\ \qquad \text{where } \alpha := \left(4 - \frac{\|y-z\|^2}{\|y-x\|^2}\right)^{-1} \in \left[\frac{1}{4}, +\infty\right[, & \text{otherwise.} \end{cases}$$

Proof. If $x = y$, then $\|x-y\| = \|x-z\|$ implies that $x = y = z$, and that $CC(\{x,y,z\}) = x$.

Suppose that $x \neq y$ and $x = \frac{y+z}{2}$. Then clearly, $y \neq z$ and $\operatorname{aff}\{x,y,z\} = \operatorname{aff}\{y,z\}$. Assume to the contrary that $CC(\{x,y,z\}) \neq \varnothing$, say $q := CC(\{x,y,z\})$. Then [4, Proposition 3.1(i)⇔(iii)] yields $q = \frac{1}{2}(y+z)$, and so

$$\|x-q\| = \|\frac{1}{2}(y+z) - \frac{1}{2}(y+z)\| = 0 \neq \frac{1}{2}\|y-z\| = \|y-q\| = \|z-q\|$$

which contradicts Definition 3.1. Hence, $CC(\{x,y,z\}) = \varnothing$.

Assume that $x \neq y$ and $x \neq \frac{y+z}{2}$. Applying Lemma 3.4(ii) with replacing p, x, and y by x, y, and z, respectively, we obtain that $\|z-y\|^2 < 4\|y-x\|^2$, which implies that $4 - \frac{\|y-z\|^2}{\|y-x\|^2} \in \left]0, 4\right]$. Hence, $\alpha := \left(4 - \frac{\|y-z\|^2}{\|y-x\|^2}\right)^{-1} \in \left[\frac{1}{4}, +\infty\right[$ is well-defined. Set

$$p := x + \alpha(y-x) + \alpha(z-x). \tag{5}$$

Because, clearly, $p \in \operatorname{aff}\{x, y, z\}$, in order to prove $CC(\{x, y, z\}) = p$, it suffices to prove that

$$\|p - x\| = \|p - y\| = \|p - z\|. \tag{6}$$

Applying Lemma 3.3(i)⇔(v) with p, x, and y substituted by x, y, and z respectively, we have that

$$\|x - y\| = \|x - z\| \Leftrightarrow \langle y - x, z - x\rangle = \|x - y\|^2 - \tfrac{1}{2}\|z - y\|^2. \tag{7}$$

Now,

$$\begin{aligned}
\|p - x\| = \|p - y\| &\Leftrightarrow \langle p - x, y - x\rangle = \tfrac{1}{2}\|y - x\|^2 \quad \text{(by Lemma 3.3(i)}\Leftrightarrow\text{(ii))} && \text{(8a)}\\
&\overset{(5)}{\Leftrightarrow} \alpha\langle z - x, y - x\rangle = (\tfrac{1}{2} - \alpha)\|y - x\|^2 && \text{(8b)}\\
&\overset{(7)}{\Leftrightarrow} -\tfrac{\alpha}{2}\|z - y\|^2 = (\tfrac{1}{2} - 2\alpha)\|y - x\|^2 && \text{(8c)}\\
&\Leftrightarrow \frac{\|z - y\|^2}{\|y - x\|^2} = \tfrac{2(2\alpha - \frac{1}{2})}{\alpha} = 4 - \frac{1}{\alpha} && \text{(8d)}\\
&\Leftrightarrow \alpha = \left(4 - \frac{\|y - z\|^2}{\|y - x\|^2}\right)^{-1}. && \text{(8e)}
\end{aligned}$$

Moreover, repeat similar calculation in (8) for $\|p - x\| = \|p - z\|$ and use $\|y - x\| = \|z - x\|$ to obtain that

$$\|p - x\| = \|p - z\| \Leftrightarrow \alpha = \left(4 - \frac{\|y - z\|^2}{\|y - x\|^2}\right)^{-1}. \tag{9}$$

Combine (8) and (9) with the definition of α to see that (6) holds. □

Circumcenter mapping

Definition 3.6. (Induced circumcenter mapping, [6, Definition 3.1])
Let $\mathcal{K}$ and $\mathcal{D}$ be nonempty subsets of $\mathcal{H}$ with $\mathcal{D} \subseteq \mathcal{K}$. Let $(\forall i \in \mathrm{I})$ $T_i : \mathcal{K} \to \mathcal{H}$. Set $\mathcal{S} := \{T_1, \ldots, T_m\}$ and $(\forall x \in \mathcal{K})$ $\mathcal{S}(x) := \{T_1 x, \ldots, T_m x\}$. The *circumcenter mapping $CC_{\mathcal{S}}$ induced by $\mathcal{S}$* is

$$CC_{\mathcal{S}} \colon \mathcal{K} \to \mathcal{H} \cup \{\varnothing\} \colon x \mapsto CC(\mathcal{S}(x)),$$

that is, for every $x \in \mathcal{K}$, if the circumcenter of the set $\mathcal{S}(x)$ defined in Definition 3.1 does not exist in $\mathcal{H}$, then $CC_{\mathcal{S}}x = \varnothing$. Otherwise, $CC_{\mathcal{S}}x$ is the unique point satisfying the two conditions below:

(i) $CC_{\mathcal{S}}x \in \operatorname{aff}(\mathcal{S}(x)) = \operatorname{aff}\{T_1 x, \ldots, T_{m-1}x, T_m x\}$, and

(ii) $\|CC_{\mathcal{S}}x - T_1 x\| = \cdots = \|CC_{\mathcal{S}}x - T_{m-1}x\| = \|CC_{\mathcal{S}}x - T_m x\|$.

If $(\forall x \in \mathcal{D})$ $CC_{\mathcal{S}}x \in \mathcal{D}$, then we say $CC_{\mathcal{S}}$ is *proper over* $\mathcal{D}$. In particular, if $\mathcal{D} = \mathcal{H}$, then we omit the phrase "over $\mathcal{H}$" and also say $CC_{\mathcal{S}}$ is "globally proper"; otherwise, we say that $CC_{\mathcal{S}}$ is *locally proper.*

Example 3.7. Suppose that $\mathcal{H} = \mathbb{R}^2$. We define the sets $B_1 := \mathbf{B}[(-1, 0); 1]$ and $B_2 := \mathbf{B}[(1, 0); 1]$. Set $\mathcal{S}_1 := \{\mathrm{P}_{B_1}, \mathrm{P}_{B_2}\}$, $\mathcal{S}_2 := \{\mathrm{Id}, \mathrm{P}_{B_1}, \mathrm{P}_{B_2}\}$, and $\mathcal{D} := \mathbb{R} \cdot (0, 1)$. Then, by Fact 3.2 and [2, Example 3.18], it is easy to see that $CC_{\mathcal{S}_1}$ is both globally proper and proper over $\mathcal{D}$, and that $CC_{\mathcal{S}_2}$ is not globally proper but it is proper over $\mathcal{D}$. □

Let $\mathcal{D}$ be a nonempty subset of $\mathcal{H}$, let $t \in \mathbb{N} \smallsetminus \{0\}$, and let $(\forall i \in \{1, \ldots, t\})$ $F_i : \mathcal{D} \to \mathcal{H}$. Denote by

$$\Omega(F_1, \ldots, F_t) := \left\{ F_{i_r} \cdots F_{i_2} F_{i_1} \;\middle|\; r \in \mathbb{N},\ i_0 := 0, \text{ and } i_1, \ldots, i_r \in \{1, \ldots, t\} \right\} \quad (10)$$

the set consisting of all finite composition of operators from $\{F_1, \ldots, F_t\}$. We use the empty product convention, so for $r = 0$, $F_{i_0} \cdots F_{i_1} = \mathrm{Id}$. Hence, $\mathrm{Id} \in \Omega(F_1, \ldots, F_t)$.

Note that in the original papers of results presented in the following Fact 3.8, we have the assumption that the corresponding $CC_{\mathcal{S}}$ is proper. It is easy to see that the global properness assumption in the original papers is too strong, and that using the related proofs, we actually obtain the following results.

Fact 3.8. *Let $\mathcal{D}$ be a nonempty weakly sequentially closed subset of $\mathcal{H}$. Let $(\forall i \in \mathrm{I})$ $T_i : \mathcal{D} \to \mathcal{H}$. Then the following statements hold:*

(i) [6, Proposition 3.10(iii)] and [5, Proposition 2.29] *Suppose that $\mathcal{S}$ is a subset of $\Omega(T_1, \ldots, T_m)$ such that*

$$\{\mathrm{Id}, T_1, \ldots, T_m\} \subseteq \mathcal{S} \quad \text{or} \quad \{\mathrm{Id}, T_1, T_2T_1, \ldots, T_m \cdots T_2T_1\} \subseteq \mathcal{S}.$$

Then $\operatorname{Fix} CC_{\mathcal{S}} = \cap_{i\in \mathrm{I}} \operatorname{Fix} T_i$.

(ii) [6, Theorem 3.17] *Set $\mathcal{S} := \{T_1, \ldots, T_m\}$. Assume that $(\forall x \in \mathcal{D})$ $CC_{\mathcal{S}}(x) \in \mathcal{D}$, that $(\forall i \in \mathrm{I})$ $\mathrm{Id} - T_i$ is demiclosed at 0, and that for every $(x_k)_{k\in\mathbb{N}}$ in $\mathcal{D}$, $x_k - CC_{\mathcal{S}} x_k \to 0$ implies that $(\forall i \in \mathrm{I})$ $CC_{\mathcal{S}} x_k - T_i x_k \to 0$. Then $\mathrm{Id} - CC_{\mathcal{S}}$ is demiclosed at 0, and $\operatorname{Fix} CC_{\mathcal{S}} = \cap_{i\in \mathrm{I}} \operatorname{Fix} T_i$*

(iii) [6, Theorem 3.20] *Set $\mathcal{S} := \{\mathrm{Id}, T_1, \ldots, T_m\}$. Assume that $(\forall x \in \mathcal{D})$ $CC_{\mathcal{S}}(x) \in \mathcal{D}$ and that $(\forall i \in \mathrm{I})$ T_i is nonexpansive. Then $\mathrm{Id} - CC_{\mathcal{S}}$ is demiclosed at 0.*

(iv) *Suppose that $(\forall i \in \mathrm{I})$ $T_i : \mathcal{D} \to \mathcal{H}$ is isometric with $\cap_{i\in \mathrm{I}} \operatorname{Fix} T_i \neq \varnothing$. Define $\mathcal{S} := \{\mathrm{Id}, T_1, \ldots, T_m\}$. Then the following statements hold:*
   (a) [5, Theorem 3.3] $(\forall x \in \mathcal{D})$ $(\forall z \in \cap_{i\in \mathrm{I}} \operatorname{Fix} T_i)$ $CC_{\mathcal{S}} x = \mathrm{P}_{\operatorname{aff}(\mathcal{S}(x))}(z)$.
   (b) [5, Lemma 3.5] *Let $F : \mathcal{D} \to \mathcal{H}$ satisfy $(\forall x \in \mathcal{D})$ $F(x) \in \operatorname{aff}(\mathcal{S}(x))$. Then* $(\forall x \in \mathcal{D})$ $(\forall z \in \cap_{i\in \mathrm{I}} \operatorname{Fix} T_i)$ $\|z - CC_{\mathcal{S}} x\|^2 + \|CC_{\mathcal{S}} x - Fx\|^2 = \|z - Fx\|^2$.

(v) [5, Proposition 3.8] *Suppose that $(\forall i \in \mathrm{I})$ $T_i : \mathcal{D} \to \mathcal{H}$ is isometric with $\cap_{i\in \mathrm{I}} \operatorname{Fix} T_i \neq \varnothing$, and that $\mathcal{S}$ is a subset of $\Omega(T_1, \ldots, T_m)$ such that*

$$\{\mathrm{Id}, T_1, \ldots, T_m\} \subseteq \mathcal{S} \quad \text{or} \quad \{\mathrm{Id}, T_1, T_2T_1, \ldots, T_m \cdots T_2T_1\} \subseteq \mathcal{S}.$$

Then we have

$$(\forall x \in \mathcal{D})(\forall z \in \operatorname{Fix} CC_{\mathcal{S}}) \quad \|CC_{\mathcal{S}}(x) - z\|^2 + \|CC_{\mathcal{S}}(x) - x\|^2 = \|x - z\|^2.$$

Consequently, $CC_{\mathcal{S}}$ is firmly quasinonexpansive.

The following Proposition 3.9(i) generalizes [5, Theorem 4.7] where the global properness assumption is required.

Proposition 3.9. *Let $\mathcal{D}$ be a nonempty weakly sequentially closed subset of $\mathcal{H}$. Let $(\forall i \in \mathrm{I})$ $T_i : \mathcal{D} \to \mathcal{H}$ be isometric such that $\cap_{i\in \mathrm{I}} \operatorname{Fix} T_i \neq \varnothing$. Suppose $\mathcal{S}$ is $\{\mathrm{Id}, T_1, \ldots, T_m\}$ or $\{\mathrm{Id}, T_1, T_2T_1, \ldots, T_m \cdots T_2T_1\}$. Assume that $(\forall x \in \mathcal{D})$ $CC_{\mathcal{S}} x \in \mathcal{D}$. Let $x \in \mathcal{D}$. Then the following statements hold:*

(i) $(\forall k \in \mathbb{N} \smallsetminus \{0\})$ $\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i}(CC_{\mathcal{S}}^k x) = \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x$, *and* $(CC_{\mathcal{S}}^k x)_{k\in\mathbb{N}}$ *weakly converges to* $\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x$.

(ii) *Assume that* $\limsup_{k\to\infty}\|CC_{\mathcal{S}}^k x\| \leq \|\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x\|$ *or that* $\mathcal{H} = \mathbb{R}^n$. *Then* $(CC_{\mathcal{S}}^k x)_{k\in\mathbb{N}}$ *converges to* $\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x$.

Proof. (i): Because $(\forall i \in \mathrm{I})$ $T_i : \mathcal{H} \to \mathcal{H}$ is isometric such that $\cap_{i\in\mathrm{I}} \operatorname{Fix} T_i \neq \varnothing$, by [5, Lemma 2.24], [7, Proposition 3.4] and Fact 3.8(i), we know that $(\forall k \in \mathrm{I})$ $T_k \cdots T_1$ is isometric and $\operatorname{Fix} CC_{\mathcal{S}} = \cap_{i\in\mathrm{I}} \operatorname{Fix} T_i$ is a closed affine subspace of $\mathcal{H}$. Then using Fact 3.8(iii)&(v), we obtain respectively that $\mathrm{Id} - CC_{\mathcal{S}}$ is demiclosed at 0, and that $CC_{\mathcal{S}}$ is firmly quasinonexpansive. Combine these results with Definition 2.8 and Theorem 2.9 to see that (i) holds.

(ii): The required result follows from (i), [2, Lemma 2.51] and the well-known fact that in $\mathbb{R}^n$ weak convergence is equivalent to strong convergence. ☐

Lemma 3.10. *Let* $(\forall i \in \mathrm{I})$ $G_i : \mathcal{H} \to \mathcal{H}$ *with* $\cap_{i\in\mathrm{I}} \operatorname{Fix} G_i \neq \varnothing$ *and let* $z \in \mathcal{H}$. *Define* $(\forall i \in \mathrm{I})$ $(\forall x \in \mathcal{H})$ $F_i x := G_i(x+z) - z$. *Set* $\mathcal{S} := \{G_1, \ldots, G_m\}$ *and* $\mathcal{S}_F := \{F_1, \ldots, F_m\}$. *Let* C *be a nonempty subset of* $\mathcal{H}$, *and let* $k \in \mathbb{N}$. *Then the following statements hold:*

(i) $(\forall i \in \mathrm{I})$ $\operatorname{Fix} F_i = \operatorname{Fix} G_i - z$, *and* $\cap_{i\in\mathrm{I}} \operatorname{Fix} F_i = \cap_{i\in\mathrm{I}} \operatorname{Fix} G_i - z$. *Moreover, if* $(\forall i \in \mathrm{I})$ G_i *is affine and* $z \in \cap_{i\in\mathrm{I}} \operatorname{Fix} G_i$, *then* $(\forall i \in \mathrm{I})$ F_i *is linear.*

(ii) $(\forall x \in \mathcal{H})$ $\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} G_i} x = z + \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} F_i}(x-z)$ *and* $CC_{\mathcal{S}}^k x = z + CC_{\mathcal{S}_F}^k(x-z)$.

(iii) $(\forall x \in C)$ $CC_{\mathcal{S}}^k x = \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} G_i} x \iff (\forall x \in C - z)$ $CC_{\mathcal{S}_F}^k x = \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} F_i} x$. *Consequently,* $(\forall x \in \cap_{i\in\mathrm{I}} \operatorname{Fix} G_i)$ $CC_{\mathcal{S}}^k x = \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} G_i} x$ *if and only if we have* $(\forall x \in \cap_{i\in\mathrm{I}} \operatorname{Fix} F_i)$ $CC_{\mathcal{S}_F}^k x = \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} F_i} x$.

(iv) *Then* $(\forall x \in C)$ $CC_{\mathcal{S}}^k x \in C$ *if and only if* $(\forall x \in C - z)$ $CC_{\mathcal{S}_F}^k x \in C - z$. *Consequently,* $(\forall x \in \cap_{i\in\mathrm{I}} \operatorname{Fix} G_i)$ $CC_{\mathcal{S}}^k x \in \cap_{i\in\mathrm{I}} \operatorname{Fix} G_i$ *if and only if we have* $(\forall x \in \cap_{i\in\mathrm{I}} \operatorname{Fix} F_i)$ $CC_{\mathcal{S}_F}^k x \in \cap_{i\in\mathrm{I}} \operatorname{Fix} F_i$.

Proof. (i): This follows from [7, Lemma 3.8(ii)&(iii)].

(ii): This is from [7, Lemma 4.8(ii)&(iv)].

(iii)&(iv): The desired results are clear from (i) and (ii) above. ☐

According to [6, Theorem 4.3 and Corollary 5.2], the $CC_{\mathcal{S}}$ in Proposition 3.11 below is globally proper. The following result states that this $CC_{\mathcal{S}}$ is also locally proper.

Proposition 3.11. *Let* $(\forall i \in \mathrm{I})$ $u_i \in \mathcal{H} \setminus \{0\}$ *and* $\eta_i \in \mathbb{R}$.
Set $(\forall i \in \mathrm{I})$ $H_i := \{x \in \mathcal{H} : \langle x, u_i\rangle = \eta_i\}$, *and*

$$C := \{x \in \mathcal{H} : (\forall i \in \mathrm{I})\ \|x - \mathrm{P}_{H_i} x\| = \|x - \mathrm{P}_{H_1} x\|\}.$$

Assume that $\cap_{i\in\mathrm{I}} H_i \neq \varnothing$. *Set* $\mathcal{S} := \{\mathrm{Id}, \mathrm{P}_{H_1}, \ldots, \mathrm{P}_{H_m}\}$. *Then* $(\forall x \in C)$ $CC_{\mathcal{S}}(x) \in C$.

Proof. Take $z \in \cap_{i\in\mathrm{I}} H_i$. Set $(\forall i \in \mathrm{I})$ $L_i := H_i - z$, i.e., $L_i := \{x \in \mathcal{H} : \langle x, u_i\rangle = 0\}$. Now, $(\forall i \in \mathrm{I})$ L_i is a linear subspaces of $\mathcal{H}$, P_{L_i} is linear, and by Fact 2.2, $(\forall x \in \mathcal{H})$ $\mathrm{P}_{H_i}(x) = \mathrm{P}_{z+L_i}(x) = z + \mathrm{P}_{L_i}(x-z)$, which implies that

$$\begin{aligned} C - z &= \{y \in \mathcal{H} \ :\ y + z \in C\} \\ &= \{y \in \mathcal{H} \ :\ (\forall i \in \mathrm{I}) \quad \|y + z - \mathrm{P}_{H_i}(y+z)\| = \|y + z - \mathrm{P}_{H_1}(y+z)\|\} \\ &= \{y \in \mathcal{H} \ :\ (\forall i \in \mathrm{I}) \quad \|y - \mathrm{P}_{L_i}(y)\| = \|y - \mathrm{P}_{L_1}(y)\|\}. \end{aligned}$$

Set $\mathcal{S}_L := \{\mathrm{Id}, \mathrm{P}_{L_1}, \ldots, \mathrm{P}_{L_m}\}$. Moreover, by Lemma 3.10(iv), we know that $(\forall x \in C)$ $CC_{\mathcal{S}}x \in C$ if and only if $(\forall x \in C - z)$ $CC_{\mathcal{S}_L}x \in C - z$. Therefore, without loss of generality, we assume that $(\forall i \in \mathrm{I})$ $\eta_i = 0$, that is, $(\forall i \in \mathrm{I})$ H_i is a linear subspace of $\mathcal{H}$ and P_{H_i} is linear.

Let $x \in C$. If there exists $j \in \mathrm{I}$ such that $x \in H_j$, then by definition of C, $x \in C$ implies that $x \in \cap_{i \in \mathrm{I}} H_i$ and so $CC_{\mathcal{S}}(x) = x \in C$. Suppose that $(\forall i \in \mathrm{I})$ $x \notin H_i$.

According to [6, Corollary 5.2], $CC_{\mathcal{S}}(x) \in \mathcal{H}$ satisfies $CC_{\mathcal{S}}(x) \in \mathrm{aff}\{x, \mathrm{P}_{H_1} x, \ldots, \mathrm{P}_{H_m} x\}$ and $(\forall i \in \mathrm{I})$ $\|CC_{\mathcal{S}}(x) - x\| = \|CC_{\mathcal{S}}(x) - \mathrm{P}_{H_i} x\|$. Let $i \in \mathrm{I}$. In consequence of Lemma 3.3(i)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv), we conclude

$$\|CC_{\mathcal{S}}(x) - x\| = \|CC_{\mathcal{S}}(x) - \mathrm{P}_{H_i} x\| \tag{11a}$$

$$\Leftrightarrow \left\langle CC_{\mathcal{S}}(x) - \frac{x + \mathrm{P}_{H_i} x}{2}, \frac{x + \mathrm{P}_{H_i} x}{2} - \mathrm{P}_{H_i} x \right\rangle = 0 \tag{11b}$$

$$\Leftrightarrow \left\langle CC_{\mathcal{S}}(x) - \frac{x + \mathrm{P}_{H_i} x}{2}, x - \mathrm{P}_{H_i} x \right\rangle = 0. \tag{11c}$$

Note that $\eta_i = 0$ implies $H_i = \ker u_i$ and $H_i^{\perp} = \mathrm{span}\,\{u_i\}$. Because $x \notin H_i$, by [2, Corollary 3.24(v)], we know that $x - \mathrm{P}_{H_i} x = \mathrm{P}_{H_i^{\perp}} x \in \mathrm{span}\,\{u_i\} \smallsetminus \{0\}$. Hence, due to (11c), $CC_{\mathcal{S}}(x) - \frac{x + \mathrm{P}_{H_i} x}{2} \in (\mathrm{span}\,\{x - \mathrm{P}_{H_i} x\})^{\perp} = (\mathrm{span}\,\{u_i\})^{\perp} = H_i$, which, by [14, Theorem 5.13(1)&(2)], implies that

$$CC_{\mathcal{S}}(x) - \frac{x + \mathrm{P}_{H_i} x}{2} = \mathrm{P}_{H_i}\left(CC_{\mathcal{S}}(x) - \frac{x + \mathrm{P}_{H_i} x}{2}\right) = \mathrm{P}_{H_i} CC_{\mathcal{S}}(x) - \mathrm{P}_{H_i} x. \tag{12}$$

Using [6, Proposition 2.10(iii)] and (11b), we obtain, respectively, that

$$\|CC_{\mathcal{S}}(x) - \mathrm{P}_{H_i} x\|^2 = \|CC_{\mathcal{S}}(x) - \mathrm{P}_{H_i} CC_{\mathcal{S}}(x)\|^2 + \|\mathrm{P}_{H_i} CC_{\mathcal{S}}(x) - \mathrm{P}_{H_i} x\|^2;$$
$$\|CC_{\mathcal{S}}(x) - \mathrm{P}_{H_i} x\|^2 = \left\|CC_{\mathcal{S}}(x) - \frac{x + \mathrm{P}_{H_i} x}{2}\right\|^2 + \left\|\frac{x + \mathrm{P}_{H_i} x}{2} - \mathrm{P}_{H_i} x\right\|^2.$$

Combine these two identities with (12) to see that

$$\|CC_{\mathcal{S}}(x) - \mathrm{P}_{H_i} CC_{\mathcal{S}}(x)\| = \left\|\frac{x + \mathrm{P}_{H_i} x}{2} - \mathrm{P}_{H_i} x\right\| = \frac{1}{2}\|x - \mathrm{P}_{H_i} x\|.$$

Therefore, $(\forall x \in C)$ $(\forall i \in \mathrm{I})$

$$\|CC_{\mathcal{S}}(x) - \mathrm{P}_{H_i} CC_{\mathcal{S}}(x)\| = \frac{1}{2}\|x - \mathrm{P}_{H_i} x\| = \frac{1}{2}\|x - \mathrm{P}_{H_1} x\| = \|CC_{\mathcal{S}}(x) - \mathrm{P}_{H_1} CC_{\mathcal{S}}(x)\|,$$

which verifies that $CC_{\mathcal{S}}(x) \in C$. $\square$

4. One-step convergence of circumcentered methods

Assume that the circumcenter mapping $CC_{\mathcal{S}}$ is proper over a nonempty subset $\mathcal{D}$ of $\mathcal{H}$. Recall that the related circumcentered method generates the sequence $(CC_{\mathcal{S}}^k x)_{k \in \mathbb{N}}$ of iterations, and that when $\mathcal{D} \neq \mathcal{H}$, we say the related circumcentered method is *locally proper*. In this section, we show one-step convergence of circumcentered methods in Theorem 4.1(iii), Corollary 4.2 and Theorem 4.7 below.

Circumcentered methods with initial points from a certain set

Theorem 4.1(iii) is motivated by and a generalization of [12, Lemma 2]. In fact, Corollary 4.2 below reduces to [12, Lemma 2] when $\mathcal{H} = \mathbb{R}^n$.

Theorem 4.1. *Assume that $T_1 : \mathcal{H} \to \mathcal{H}$ and $T_2 : \mathcal{H} \to \mathcal{H}$ are isometries such that* $\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2 \neq \varnothing$. *Set $\mathcal{S} := \{\mathrm{Id}, T_1, T_2T_1\}$. Then the following statements hold:*

(i) $(\forall x \in \mathcal{H})$ $(\forall z \in \operatorname{Fix} T_1 \cap \operatorname{Fix} T_2)$ $CC_{\mathcal{S}}x = \mathrm{P}_{\operatorname{aff}(\mathcal{S}(x))}(z) \in \mathcal{H}$ *and* $(\forall k \in \mathbb{N})$ $\mathrm{P}_{\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2} CC^k_{\mathcal{S}}x = \mathrm{P}_{\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2} x$. *Moreover, for every $x \in \operatorname{Fix} T_2$,*

$$CC_{\mathcal{S}}(x) = \begin{cases} x & \text{if } x = T_1x; \\ x + \alpha(T_1x - x) + \alpha(T_2T_1x - x), & \\ \qquad \text{where } \alpha := \left(4 - \frac{\|T_2T_1x - T_1x\|^2}{\|T_1x - x\|^2}\right)^{-1} \in \left[\frac{1}{4}, \infty\right[& \text{otherwise.} \end{cases}$$

(ii) *Assume that $T_1(\operatorname{Fix} T_2) \subseteq \operatorname{Fix} T_2^2$ (e.g., when $\operatorname{ran} T_1 \subseteq \operatorname{Fix} T_2^2$ or $T_2^2 = \mathrm{Id}$). Then* $(\forall x \in \operatorname{Fix} T_2)$ $CC_{\mathcal{S}}(x) \in \operatorname{Fix} T_2$.

(iii) *Assume that $T_1(\operatorname{Fix} T_2) \subseteq \operatorname{Fix} T_2^2$, $\operatorname{Fix} T_2 \subseteq \operatorname{Fix} T_1^2$, and $\dim(\operatorname{par}(\operatorname{Fix} T_1)^{\perp}) = 1$. Then* $(\forall x \in \operatorname{Fix} T_2)$ $CC_{\mathcal{S}}(x) = \mathrm{P}_{\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2} x$.

Proof. (i): Note that compositions of isometries are isometries and that

$$\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2 = \operatorname{Fix} T_1 \cap \operatorname{Fix} T_2T_1.$$

Then Fact 3.8(iv)(a) and [5, Theorem 5.7] imply that

$$(\forall x \in \mathcal{H})\ (\forall z \in \operatorname{Fix} T_1 \cap \operatorname{Fix} T_2)\ CC_{\mathcal{S}}x = \mathrm{P}_{\operatorname{aff}(\mathcal{S}(x))}(z) \in \mathcal{H}$$

and $(\forall k \in \mathbb{N})$ $\mathrm{P}_{\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2} CC^k_{\mathcal{S}}x = \mathrm{P}_{\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2} x$. Combine this with Theorem 3.5 to see that

$$\{x \in \mathcal{H} : \|x - T_1x\| = \|x - T_2T_1x\|\} \cap \left\{x \in \mathcal{H} : x \neq T_1x \text{ and } x = \frac{T_1x + T_2T_1x}{2}\right\} = \varnothing.$$

In addition, inasmuch as T_2 is isometric,

$$(\forall x \in \mathcal{H}) \quad x \in \operatorname{Fix} T_2 \Leftrightarrow x = T_2x \Rightarrow \|x - T_2T_1x\| = \|T_2x - T_2T_1x\| = \|x - T_1x\|,$$

which yields $\operatorname{Fix} T_2 \subseteq \{x \in \mathcal{H} : \|x - T_1x\| = \|x - T_2T_1x\|\}$. Therefore, for every $x \in \operatorname{Fix} T_2$ the required formula follows from Theorem 3.5 with x, y and z substituted by x, T_1x and T_2T_1x, respectively.

(ii): According to [7, Proposition 3.4], every isometry must be affine. Then using Lemma 3.10(i)&(iv), without loss of generality, we assume that T_1 and T_2 are linear isometries. Let $x \in \operatorname{Fix} T_2$. In view of (i) above, we have exactly the following two cases:

Case 1: $x = T_1x$. Then $CC_{\mathcal{S}}(x) = x \in \operatorname{Fix} T_2$.

Case 2: $x \neq T_1x$. Then $CC_{\mathcal{S}}(x) = x + \alpha(T_1x - x) + \alpha(T_2T_1x - x)$, where

$$\alpha := \left(4 - \frac{\|T_2T_1x - T_1x\|^2}{\|T_1x - x\|^2}\right)^{-1} \in \left[\frac{1}{4}, \infty\right[.$$

Note that $x \in \operatorname{Fix} T_2$ and $T_1(\operatorname{Fix} T_2) \subseteq \operatorname{Fix} T_2^2$ entail that $T_2T_2T_1x = T_1x$. Combine this with the linearity of T_2 to observe that

$$\begin{aligned} T_2\left(CC_{\mathcal{S}}(x)\right) &= T_2\left(x+\alpha(T_1x-x)+\alpha(T_2T_1x-x)\right)\\ &= x+\alpha(T_2T_1x-x)+\alpha(T_1x-x) = CC_{\mathcal{S}}(x), \end{aligned}$$

which implies that $CC_{\mathcal{S}}(x) \in \operatorname{Fix} T_2$.

(iii): Recall that $\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2 = \operatorname{Fix} T_1 \cap \operatorname{Fix} T_2T_1$. Then similarly to the proof of (ii) above, using Lemma 3.10(i)&(iii), we assume that T_1 and T_2 are linear isometries. Now, $\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2$ is a closed linear subspace of $\mathcal{H}$. Let $x \in \operatorname{Fix} T_2$.

Note that if we can prove that $CC_{\mathcal{S}}x \in \operatorname{Fix} T_1$, then, by (ii), $CC_{\mathcal{S}}x \in \operatorname{Fix} T_1 \cap \operatorname{Fix} T_2$, and so $CC_{\mathcal{S}}x = \mathrm{P}_{\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2} CC_{\mathcal{S}}x \overset{\text{(i)}}{=} \mathrm{P}_{\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2} x$. Therefore, it remains to show that $CC_{\mathcal{S}}x \in \operatorname{Fix} T_1$.

Because T_1 is linear, $x \in \operatorname{Fix} T_2$, and $\operatorname{Fix} T_2 \subseteq \operatorname{Fix} T_1^2$, we have that

$$T_1(\tfrac{T_1x+x}{2}) = \tfrac{x+T_1x}{2}, \text{ that is, } \tfrac{T_1x+x}{2} \in \operatorname{Fix} T_1.$$

If $CC_{\mathcal{S}}(x) - \frac{T_1x+x}{2} = 0$, then $CC_{\mathcal{S}}(x) = \frac{T_1x+x}{2} \in \operatorname{Fix} T_1$. If $T_1x = x$, then by (i), $CC_{\mathcal{S}}(x) = x \in \operatorname{Fix} T_1$. In the rest of the proof, we assume that $T_1x - x \neq 0$ and that $CC_{\mathcal{S}}(x) - \frac{T_1x+x}{2} \neq 0$. Because T_1 is linear and isometric, by [7, Lemma 2.29], $T_1x-x \in \overline{\operatorname{ran}}(T_1-\mathrm{Id}) = \overline{\operatorname{ran}}(\mathrm{Id}-T_1) = (\operatorname{Fix} T_1)^{\perp}$. Combine this with the assumption $\dim(\operatorname{par}(\operatorname{Fix} T_1)^{\perp}) = 1$ and $T_1x - x \neq 0$ to obtain that

$$(\operatorname{Fix} T_1)^{\perp} = \operatorname{span}\{T_1x - x\}. \tag{13}$$

By Definition 3.6, we know that $\|CC_{\mathcal{S}}(x) - x\| = \|CC_{\mathcal{S}}(x) - T_1x\|$.
Because, by Lemma 3.3(i)⇔(iv),

$$\|CC_{\mathcal{S}}(x) - x\| = \|CC_{\mathcal{S}}(x) - T_1x\| \Leftrightarrow \left\langle CC_{\mathcal{S}}(x) - \tfrac{T_1x+x}{2}, T_1x - x \right\rangle = 0,$$

we have that $CC_{\mathcal{S}}(x) - \frac{T_1x+x}{2} \in (\operatorname{span}\{T_1x - x\})^{\perp} \overset{(13)}{=} (\operatorname{Fix} T_1)^{\perp\perp} = \operatorname{Fix} T_1$. Recall that $\frac{T_1x+x}{2} \in \operatorname{Fix} T_1$ and that $\operatorname{Fix} T_1$ is a linear subspace. Hence, we obtain that

$$CC_{\mathcal{S}}(x) \in \tfrac{T_1x+x}{2} + \operatorname{Fix} T_1 = \operatorname{Fix} T_1.$$

Therefore, the proof is complete. ☐

Corollary 4.2. *Let $u \in \mathcal{H} \smallsetminus \{0\}$ and $\eta \in \mathbb{R}$.Let $H := \{x \in \mathcal{H} : \langle x, u\rangle = \eta\}$, and let U be an affine subspace of $\mathcal{H}$ such that $H \cap U \neq \varnothing$.*
Assume that $\mathcal{S} := \{\mathrm{Id}, \mathrm{R}_H, \mathrm{R}_U \mathrm{R}_H\}$. Then $(\forall x \in U)$ $CC_{\mathcal{S}}(x) = \mathrm{P}_{H\cap U} x$.

Proof. Because H and U are affine subspaces, by [5, Lemma 2.23(i)], R_H and R_U are isometries. Moreover, the idempotent property of projectors and the definition of reflectors imply that $\mathrm{R}_H^2 = \mathrm{Id}$ and $\mathrm{R}_U^2 = \mathrm{Id}$. Hence, the desired result follows from Theorem 4.1(iii). ☐

The following Lemma 4.3 is necessary in the proof of Proposition 4.4 below. In fact, the result in the following Lemma 4.3 was directly used in the proofs of [12, Lemma 5] and [1, Proposition 5]. For completeness, we provide a clear analytical proof below.

Lemma 4.3. *Let $u \in \mathcal{H} \smallsetminus \{0\}$ and $\eta \in \mathbb{R}$. Let U be a closed affine subspace of $\mathcal{H}$. Define $H := \{x \in \mathcal{H} : \langle x, u\rangle = \eta\}$ and $W := \{x \in \mathcal{H} : \langle x, u\rangle \leq \eta\}$. Assume that $W \cap U \neq \varnothing$ and $U \smallsetminus W \neq \varnothing$. Then $H \cap U \neq \varnothing$ and $(\forall x \in U \smallsetminus W)$ $\mathrm{P}_{H\cap U} x = \mathrm{P}_{W\cap U} x$.*

Proof. Let $x \in U \smallsetminus W$, and let $y \in W \cap U$. Define $f : \mathbb{R} \to \mathbb{R}$ by $(\forall t \in \mathbb{R})$ $f(t) := \langle tx + (1-t)y, u\rangle - \eta$. Because $y \in W$ and $x \notin W$, we have that $f(0) = \langle y, u\rangle - \eta \leq 0$ and $f(1) = \langle x, u\rangle - \eta > 0$. So, there exists $\bar{t} \in [0, 1[$ such that $f(\bar{t}) = \langle \bar{t}x + (1-\bar{t})y, u\rangle - \eta = 0$, which implies that, $\bar{t}x + (1-\bar{t})y \in H$. Because $x \in U$, $y \in U$, and U is affine, we know that $\bar{t}x+(1-\bar{t})y \in H \cap U \neq \varnothing$. Combine this with the result that $H \cap U$ is a closed affine subspace, and [6, Proposition 2.10(iii)] to see that

$$\begin{aligned}\|x - \mathrm{P}_{H\cap U} x\|^2 + \|\mathrm{P}_{H\cap U} x - (\bar{t}x + (1-\bar{t})y)\|^2 &= \|x - (\bar{t}x + (1-\bar{t})y)\|^2\\ &= (1-\bar{t})^2\|x-y\|^2 \leq \|x-y\|^2.\end{aligned}$$

Note that $y \in W \cap U$ is arbitrary. Hence, we obtain that

$$(\forall y \in W \cap U) \quad \|x - \mathrm{P}_{H\cap U} x\| \leq \|x - y\|,$$

which, combining with the fact that $\mathrm{P}_{H\cap U} x \in H \cap U \subseteq W \cap U$, implies that $\mathrm{P}_{H\cap U} x = \mathrm{P}_{W\cap U} x$. □

The result $(\forall x \in U)$ $CC_{\mathcal{S}}(x) = \mathrm{P}_{H_x\cap U} x$ in the following Proposition 4.4(ii) reduces to [12, Lemma 3] when $\mathcal{H} = \mathbb{R}^n$. In fact, the idea of proof of Proposition 4.4(ii) is almost the same with that of [12, Lemma 3]. Proposition 4.4(iv) illustrates that the inequality (4) in [12, Lemma 5] is actually an equation. In Proposition 4.4(v), we apply the demiclosedness principle for the circumcenter mapping proved in [6, Theorem 3.2] to generalize [12, Theorem 1] from $\mathbb{R}^n$ to general Hilbert spaces and from convergence to weak convergence.

Note that the reflector associated with general convex set is generally not isometric. Hence, the following result is not a direct corollary of Theorem 4.1.

Proposition 4.4. *Let C be a closed convex set of $\mathcal{H}$ and let U be a closed affine subspace of $\mathcal{H}$ with $C \cap U \neq \varnothing$. Set $\mathcal{S} := \{\mathrm{Id}, \mathrm{R}_C, \mathrm{R}_U \mathrm{R}_C\}$. For every $x \in \mathcal{H}$, define*

$$H_x := \begin{cases}\{y \in \mathcal{H} \ : \ \langle y, x - \mathrm{P}_C x\rangle = \langle \mathrm{P}_C x, x - \mathrm{P}_C x\rangle\} & \text{if } x \notin C;\\ C & \text{if } x \in C,\end{cases}$$

$$W_x := \begin{cases}\{y \in \mathcal{H} \ : \ \langle y, x - \mathrm{P}_C x\rangle \leq \langle \mathrm{P}_C x, x - \mathrm{P}_C x\rangle\} & \text{if } x \notin C;\\ C & \text{if } x \in C.\end{cases}$$

Let $x \in U$. Then the following statements hold:

(i) $H_x \cap U \neq \varnothing$ *and* $\mathrm{P}_{H_x\cap U} x = \mathrm{P}_{W_x\cap U} x$.

(ii) *Set* $\tilde{\mathcal{S}} := \{\mathrm{Id}, \mathrm{R}_{H_x}, R_U \mathrm{R}_{H_x}\}$. *Then* $\mathcal{S}(x) = \tilde{\mathcal{S}}(x)$ *and* $CC_{\mathcal{S}}(x) = CC_{\tilde{\mathcal{S}}}(x) = \mathrm{P}_{H_x\cap U} x$.

(iii) $(\forall z \in C \cap U)$ $CC_{\mathcal{S}}x = \mathrm{P}_{\mathrm{aff}(\mathcal{S}(x))} z$. *Moreover,*

$$CC_{\mathcal{S}}(x) = \begin{cases} x & \text{if } x \in C;\\ x + 2\alpha(\mathrm{P}_U \mathrm{R}_C(x) - x), & \\ \qquad \text{where } \alpha = \left(4 - \frac{\|\mathrm{R}_U \mathrm{R}_C x - \mathrm{R}_C x\|^2}{\|\mathrm{R}_C x - x\|^2}\right)^{-1} \in \left[\frac{1}{4}, +\infty\right[& \text{otherwise.}\end{cases}$$

(iv) $(\forall z \in C \cap U)$ $\|CC_{\mathcal{S}}(x) - x\|^2 + \|CC_{\mathcal{S}}(x) - z\|^2 = \|x - z\|^2$. *Consequently,* $CC_{\mathcal{S}} : U \to U$ *is firmly quasinonexpansive.*

(v) $(\forall k \in \mathbb{N})$ $CC^k_{\mathcal{S}}(x) \in U$. *Moreover, there exists* $\bar{x} \in K \cap U$ *such that* $CC^k_{\mathcal{S}}(x) \rightharpoonup \bar{x}$.

Proof. (i): If $x \in C$, then we have in consequence of the definitions of H_x and W_x that $x \in C \cap U = H_x \cap U \neq \varnothing$ and $x = \mathrm{P}_{C\cap U} x = \mathrm{P}_{H_x\cap U} x = \mathrm{P}_{W_x\cap U} x$. Assume that $x \notin C$. Then by the definition of W_x, $x \notin W_x$. So, $x \in U \smallsetminus W_x$. Note that [2, Theorem 3.16] yields $C \subseteq W_x$, which, combining with $C \cap U \neq \varnothing$, deduces that $W_x \cap U \neq \varnothing$. Hence, applying Lemma 4.3 with W and H replaced by W_x and H_x respectively, we see that $H_x \cap U \neq \varnothing$ and $\mathrm{P}_{H_x\cap U} x = \mathrm{P}_{W_x\cap U} x$.

(ii): If $x \in C$, then $H_x = C$. Therefore,

$$\mathcal{S}(x) = \tilde{\mathcal{S}}(x) = \{x\} \text{ and } CC_{\mathcal{S}}(x) = x = \mathrm{P}_{C\cap U} x = \mathrm{P}_{H_x\cap U} x.$$

Assume $x \notin C$. According to Fact 2.3,

$$\mathrm{P}_{H_x} x = x + \frac{\langle \mathrm{P}_C x, x - \mathrm{P}_C x\rangle - \langle x, x - \mathrm{P}_C x\rangle}{\|x - \mathrm{P}_C x\|^2}(x - \mathrm{P}_C x) = x - (x - \mathrm{P}_C x) = \mathrm{P}_C x,$$

which implies that $\mathrm{R}_C x = \mathrm{R}_{H_x} x$ and $\mathrm{R}_U \mathrm{R}_C x = \mathrm{R}_U \mathrm{R}_{H_x} x$. Hence, $\mathcal{S}(x) = \tilde{\mathcal{S}}(x)$ and, by Definition 3.6,

$$CC_{\mathcal{S}}(x) = CC(\{x, \mathrm{R}_C x, \mathrm{R}_U \mathrm{R}_C x\}) = CC(\{x, \mathrm{R}_{H_x} x, \mathrm{R}_U \mathrm{R}_{H_x} x\}) = CC_{\tilde{\mathcal{S}}}(x).$$

Because $x \in U$, and, by (i), $H_x \cap U \neq \varnothing$, applying Corollary 4.2 with $H = H_x$, we know that

$$CC_{\mathcal{S}}(x) = CC_{\tilde{\mathcal{S}}}(x) = CC(\{x, \mathrm{R}_{H_x} x, \mathrm{R}_U \mathrm{R}_{H_x} x\}) = \mathrm{P}_{H_x\cap U} x.$$

(iii): Let $z \in C \cap U$. If $x \in C$, as we did in (ii), it is easy to see that we have $\operatorname{aff} \mathcal{S}(x) = \operatorname{aff}\{x, \mathrm{R}_C x, \mathrm{R}_U \mathrm{R}_C x\} = \{x\}$ and $CC_{\mathcal{S}}(x) = x = \mathrm{P}_{\operatorname{aff}(\mathcal{S}(x))} z$.

Assume $x \notin C$. Now U is an affine subspace and H_x is a hyperplane, which, by [5, Lemma 2.23(i)], implies that both R_{H_x} and R_U are isometries. By definition of H_x and Fact 2.3, it is easy to see that $\mathrm{R}_C x = \mathrm{R}_{H_x} x$, and that $x \notin C$ implies that $x \notin H_x$ and $x \neq \mathrm{R}_{H_x} x$. Hence, applying Theorem 4.1(i) with T_1 and T_2 replaced by R_{H_x} and R_U, respectively, we obtain

$$CC_{\tilde{\mathcal{S}}}(x) = \mathrm{P}_{\operatorname{aff}(\tilde{\mathcal{S}}(x))} z = x + 2\alpha\left(\frac{\mathrm{R}_C x + \mathrm{R}_U \mathrm{R}_C x}{2} - x\right),$$

where $$\alpha := \left(4 - \frac{\|\mathrm{R}_U \mathrm{R}_C x - \mathrm{R}_C x\|^2}{\|\mathrm{R}_C x - x\|^2}\right)^{-1} \in \left[\frac{1}{4}, +\infty\right[.$$

Combine these results with $\frac{\mathrm{R}_C + \mathrm{R}_U \mathrm{R}_C}{2} = \mathrm{P}_U \mathrm{R}_C$ and (ii) to obtain the required results.

(iv): Apply Fact 3.8(i) with T_1, T_2, and $\mathcal{S}$ replaced by R_C, R_U and $\{\mathrm{Id}, \mathrm{R}_C, \mathrm{R}_U \mathrm{R}_C\}$, respectively, to obtain that $\operatorname{Fix} CC_{\mathcal{S}} = C \cap U$.

Let $z \in C \cap U$. By (iii), $CC_{\mathcal{S}} x = \mathrm{P}_{\operatorname{aff}(\mathcal{S}(x))} z$. Note that the definition of $\mathcal{S}(x)$ yields $x \in \mathcal{S}(x)$, and that $\operatorname{aff}(\mathcal{S}(x))$ is an affine subspace. Hence, by [6, Proposition 2.10(iii)], we have that

$$\|z - \mathrm{P}_{\operatorname{aff}(\mathcal{S}(x))} z\|^2 + \|\mathrm{P}_{\operatorname{aff}(\mathcal{S}(x))} z - x\|^2 = \|z - x\|^2$$
$$\Leftrightarrow \ \|z - CC_{\mathcal{S}} x\|^2 + \|CC_{\mathcal{S}}(x) - x\|^2 = \|z - x\|^2,$$

which, by Definition 2.8, implies that $CC_{\mathcal{S}} : U \to U$ is firmly quasinonexpansive.

(v): Because, by (ii), $CC_{\mathcal{S}} : U \to U$ is well-defined, and R_C and $\mathrm{R}_U \mathrm{R}_C$ are nonexpansive, using Fact 3.8(i)&(iii), we have that $\operatorname{Fix} CC_{\mathcal{S}} = C \cap U \neq \varnothing$ and that $\mathrm{Id} - CC_{\mathcal{S}}$ is demiclosed at 0. Moreover, (iv) shows that $CC_{\mathcal{S}} : U \to U$ is firmly quasinonexpansive. Because, by [2, Theorem 3.34], U is a nonempty closed convex set implies that U is weakly sequentially closed, using Theorem 2.9, we obtain that $(CC_{\mathcal{S}}^k(x))_{k\in\mathbb{N}}$ weakly converges to one point in $C \cap U$. □

One-step convergence of circumcentered isometry methods

Lemma 4.5. *Let $F : \mathcal{H} \to \mathcal{H}$ and $T : \mathcal{H} \to \mathcal{H}$ be linear and continuous. Let $x \in \mathcal{H}$ and let $z \in \mathcal{H}$.*

(i) *Suppose that $\mathcal{H} = \mathbb{R}^n$ and that F is isometric. Assume that $z \notin \operatorname{Fix} T$, and that $Fx \in \operatorname{Fix} T$. Then $(\forall t \in \mathbb{R} \smallsetminus \{0\}) \quad F(x + tF^*z) \notin \operatorname{Fix} T$.*

(ii) *Assume that $Fx \notin \operatorname{Fix} T$. Then there exists $\bar{t} \in \mathbb{R}_{++}$ such that*
$$(\forall t \in [-\bar{t}, \bar{t}]) \quad F(x + tz) \notin \operatorname{Fix} T.$$

Proof. (i): Because $\mathcal{H} = \mathbb{R}^n$ and F is a linear isometry, by [7, Lemma 3.9], we have $FF^* = \mathrm{Id}$. Let $t \in \mathbb{R} \smallsetminus \{0\}$. Then $F(x + tF^*z) = Fx + tFF^*z = Fx + tz$. Because $\operatorname{Fix} T$ is a linear subspace, by assumptions $Fx \in \operatorname{Fix} T$ and $z \notin \operatorname{Fix} T$, and therefore $F(x + tF^*z) = Fx + tz \notin \operatorname{Fix} T$.

(ii): Define $f : \mathbb{R} \to \mathbb{R}$ by $(\forall t \in \mathbb{R})\ f(t) := \operatorname{dist}(F(x + tz), \operatorname{Fix} T)$. Note that F is continuous, that $\operatorname{Fix} T$ is a nonempty closed linear subspace of $\mathcal{H}$, and that the distance function $\operatorname{dist}(\cdot, \operatorname{Fix} T)$ is continuous. Hence, f is continuous. Because $\operatorname{Fix} T$ is closed, $Fx \notin \operatorname{Fix} T$ yields $f(0) = \operatorname{dist}(Fx, \operatorname{Fix} T) > 0$. Therefore there exists $\bar{t} > 0$ such that $(\forall t \in [-\bar{t}, \bar{t}])\ \operatorname{dist}(F(x + tz), \operatorname{Fix} T) = f(t) > 0$, which implies that $F(x + tz) \notin \operatorname{Fix} T$. □

Lemma 4.6. *Let $(\forall i \in \mathrm{I})$ T_i is linear and nonexpansive, and let $x \in \mathcal{H}$. The following statements hold:*

(i) $\operatorname{span}\{T_1x - x, T_2T_1x - T_1x, \ldots, T_m \cdots T_2T_1x - T_{m-1} \cdots T_2T_1x\} \subseteq (\cap_{i=1}^m \operatorname{Fix} T_i)^\perp$ *and* $\operatorname{span}\{T_1x - x, T_2x - x, \ldots, T_mx - x\} \subseteq (\cap_{i=1}^m \operatorname{Fix} T_i)^\perp$.

(ii) *Suppose that $(\forall i \in \mathrm{I})$ $(\operatorname{Fix} T_i)^\perp$ is one-dimensional.*
   (a) *Assume that $(\forall i \in \mathrm{I})$ $x \notin \operatorname{Fix} T_i$. Then*
   $\operatorname{span}\{T_1x - x, T_2x - x, \ldots, T_mx - x\} = (\cap_{i=1}^m \operatorname{Fix} T_i)^\perp$.
   (b) *Assume that $(\forall i \in \mathrm{I})$ $T_{i-1} \cdots T_1x \notin \operatorname{Fix} T_i$.*[1] *Then*
   $\operatorname{span}\{T_1x - x, T_2T_1x - T_1x, \ldots, T_m \cdots T_2T_1x - T_{m-1} \cdots T_2T_1x\}$
   $= (\cap_{i=1}^m \operatorname{Fix} T_i)^\perp$.

Proof. Because $(\forall i \in \mathrm{I})$ T_i is linear and nonexpansive, due to [7, Lemma 2.29],
$$(\forall i \in \mathrm{I}) \quad \overline{\operatorname{ran}}(\mathrm{Id} - T_i) = (\operatorname{Fix} T_i)^\perp. \tag{14}$$

Moreover, due to [14, Theorems 4.6(5)],
$$(\cap_{i\in\mathrm{I}} \operatorname{Fix} T_i)^\perp = \overline{\sum_{i\in\mathrm{I}} (\operatorname{Fix} T_i)^\perp} \overset{(14)}{=} \overline{\sum_{i\in\mathrm{I}} \overline{\operatorname{ran}}(\mathrm{Id} - T_i)}. \tag{15}$$

[1] We use the empty product convention, so if $i = 1$, $T_{i-1} \cdots T_1 = \mathrm{Id}$.

(i): Recall that $(\forall i \in \mathrm{I})$ $T_i - \mathrm{Id}$ is linear. So,

$$\begin{aligned}&\operatorname{span}\{T_1x - x, T_2T_1x - T_1x, \ldots, T_m \cdots T_2T_1x - T_{m-1} \cdots T_2T_1x\}\\ &= \sum_{i\in \mathrm{I}} \operatorname{span}\{T_iT_{i-1}\cdots T_1x - T_{i-1}\cdots T_1x\} \subseteq \sum_{i\in\mathrm{I}} \operatorname{ran}(\mathrm{Id} - T_i) \overset{(15)}{\subseteq} \left(\cap_{i=1}^m \operatorname{Fix} T_i\right)^{\perp},\end{aligned}$$

and

$$\begin{aligned}&\operatorname{span}\{T_1x - x, T_2x - x, \ldots, T_mx - x\}\\ &= \sum_{i\in\mathrm{I}} \operatorname{span}\{T_ix - x\} \subseteq \sum_{i\in\mathrm{I}} \operatorname{ran}(\mathrm{Id} - T_i) \overset{(15)}{\subseteq} \left(\cap_{i=1}^m \operatorname{Fix} T_i\right)^{\perp}.\end{aligned}$$

(ii): Because $(\forall i \in \mathrm{I})$ $(\operatorname{Fix} T_i)^{\perp}$ is one-dimensional, the finite-dimensional linear space $\sum_{i\in\mathrm{I}}(\operatorname{Fix} T_i)^{\perp}$ is closed. Combine this with [14, Theorems 4.6(5)] to see that

$$\left(\cap_{i\in\mathrm{I}} \operatorname{Fix} T_i\right)^{\perp} = \overline{\sum_{i\in\mathrm{I}}(\operatorname{Fix} T_i)^{\perp}} = \sum_{i\in\mathrm{I}}(\operatorname{Fix} T_i)^{\perp}. \tag{16}$$

(ii)(a): Because $(\forall i \in \mathrm{I})$ $x \notin \operatorname{Fix} T_i$ and $(\operatorname{Fix} T_i)^{\perp}$ is one-dimensional, we have that $(\forall i \in \mathrm{I})$ $\operatorname{span}\{T_ix - x\} = \overline{\operatorname{ran}}(\mathrm{Id} - T_i) \overset{(14)}{=} (\operatorname{Fix} T_i)^{\perp}$. Therefore,

$$\begin{aligned}&\operatorname{span}\{T_1x - x, T_2x - x, \ldots, T_mx - x\}\\ &= \sum_{i\in\mathrm{I}} \operatorname{span}\{T_ix - x\} = \sum_{i\in\mathrm{I}} (\operatorname{Fix} T_i)^{\perp} \overset{(16)}{=} \left(\cap_{i\in\mathrm{I}} \operatorname{Fix} T_i\right)^{\perp}.\end{aligned}$$

(ii)(b): Because $(\forall i \in \mathrm{I})$ $T_{i-1}\cdots T_1x \notin \operatorname{Fix} T_i$ and $(\operatorname{Fix} T_i)^{\perp}$ is one-dimensional, we get $(\forall i \in \mathrm{I})$ $\operatorname{span}\{T_iT_{i-1}\cdots T_1x - T_{i-1}\cdots T_1x\} = \overline{\operatorname{ran}}(\mathrm{Id} - T_i) \overset{(14)}{=} (\operatorname{Fix} T_i)^{\perp}$. Hence,

$$\begin{aligned}&\operatorname{span}\{T_1x - x, T_2T_1x - T_1x, \ldots, T_m \cdots T_2T_1x - T_{m-1} \cdots T_2T_1x\}\\ &= \sum_{i\in\mathrm{I}} \operatorname{span}\{T_iT_{i-1}\cdots T_1x - T_{i-1}\cdots T_1x\} = \sum_{i\in\mathrm{I}} (\operatorname{Fix} T_i)^{\perp} \overset{(16)}{=} \left(\cap_{i\in\mathrm{I}} \operatorname{Fix} T_i\right)^{\perp}. \qquad \square\end{aligned}$$

The following result provides sufficient conditions for the circumcentered isometry methods finding the best approximation point in one step. In fact, Theorem 4.7 under the assumption (ii) below and Proposition 4.9 reduce to [11, Lemma 3] and [11, Lemma 4], respectively, when $\mathcal{H} = \mathbb{R}^n$ and $(\forall i \in \{1, \ldots, m\})$ $T_i = \mathrm{R}_{H_i}$ with H_i being a hyperplane.

Theorem 4.7. *Let $T_1, \ldots, T_m$ be isometries from $\mathcal{H}$ to $\mathcal{H}$. Assume that we have $z \in \cap_{i=1}^m \operatorname{Fix} T_i \neq \varnothing$ and that $(\forall i \in \mathrm{I})$ $(\operatorname{Fix} T_i - z)^{\perp}$ is one-dimensional linear subspace of $\mathcal{H}$. Let $x \in \mathcal{H}$. Assume that one of the following items hold.*

(i) $\mathcal{S} := \{\mathrm{Id}, T_1, T_2, \ldots, T_m\}$ *and* $(\forall i \in \mathrm{I})$ $x \notin \operatorname{Fix} T_i$.

(ii) $\mathcal{S} := \{\mathrm{Id}, T_1, T_2T_1, \ldots, T_m \cdots T_2T_1\}$ *and* $(\forall i \in \mathrm{I})$ $T_{i-1}\cdots T_1x \notin \operatorname{Fix} T_i$.

Then $CC_{\mathcal{S}}x = \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x$.

Proof. Because $T_1, \ldots, T_m$ are isometries with $z \in \cap_{i\in\mathrm{I}} \operatorname{Fix} T_i \neq \varnothing$ and compositions of isometries are affine isometries, applying Lemma 3.10(i)&(iii), without loss of generality, we assume that $(\forall i \in \mathrm{I})$ T_i is linear and that $z = 0$.

Assume that (i) or (ii) holds. Denote by $\widehat{\operatorname{aff}}\mathcal{S}(x) := \operatorname{par}\operatorname{aff}\mathcal{S}(x)$. Note that $\mathrm{Id} \in \mathcal{S}$ implies that $x \in \operatorname{aff}\mathcal{S}(x)$. Then $\widehat{\operatorname{aff}}\mathcal{S}(x) = \operatorname{aff}\mathcal{S}(x) - x$.

In view of the assumptions and Lemma 4.6(ii), we have that $\widehat{\operatorname{aff}}\mathcal{S}(x) = (\cap_{i=1}^m \operatorname{Fix} T_i)^\perp$. This combined with [14, Theorems 5.8(6)] implies that

$$(\widehat{\operatorname{aff}}\mathcal{S}(x))^\perp = (\cap_{i=1}^m \operatorname{Fix} T_i)^{\perp\perp} = \cap_{i=1}^m \operatorname{Fix} T_i. \tag{17}$$

Using [5, Proposition 4.2(iii)] and [14, Theorems 5.8(2)], we have that

$$\begin{aligned}\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x - CC_{\mathcal{S}}x &= \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} CC_{\mathcal{S}}x - CC_{\mathcal{S}}x \\ &= -\mathrm{P}_{\left(\cap_{i=1}^m \operatorname{Fix} T_i\right)^\perp}(CC_{\mathcal{S}}(x)) \in (\cap_{i=1}^m \operatorname{Fix} T_i)^\perp . \end{aligned} \tag{18}$$

On the other hand, by Fact 3.8(iv)(a), Fact 2.2 and [14, Theorems 5.8(2)], we obtain

$$\begin{aligned}\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x - CC_{\mathcal{S}}x &= \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x - \mathrm{P}_{\operatorname{aff}\mathcal{S}(x)} \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x \\ &= \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x - \Big(x + \mathrm{P}_{\widehat{\operatorname{aff}}\mathcal{S}(x)}(\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i}(x) - x)\Big) \\ &= \mathrm{P}_{(\widehat{\operatorname{aff}}\mathcal{S}(x))^\perp}(\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i}(x) - x) \in (\widehat{\operatorname{aff}}\mathcal{S}(x))^\perp \stackrel{(17)}{=} \cap_{i=1}^m \operatorname{Fix} T_i,\end{aligned}$$

which, combined with (18), implies

$$\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x - CC_{\mathcal{S}}x \in (\cap_{i=1}^m \operatorname{Fix} T_i)^\perp \cap (\cap_{i=1}^m \operatorname{Fix} T_i) = \{0\},$$

that is, $\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x = CC_{\mathcal{S}}x$. □

In the following Propositions 4.8 and 4.9, we affirm that the sets of points x satisfying the conditions presented in Theorem 4.7 are actually dense in $\mathcal{H}$. Note that in the proof of Proposition 4.9, $\mathcal{H} = \mathbb{R}^n$ is required, while it is not necessary in Proposition 4.8.

Proposition 4.8. *Let* $(\forall i \in \mathrm{I})$ $T_i : \mathcal{H} \to \mathcal{H}$ *be isometric such that* $z \in \cap_{i=1}^m \operatorname{Fix} T_i \neq \varnothing$ *and* $(\forall i \in \mathrm{I})$ $(\operatorname{Fix} T_i - z)^\perp \neq \{0\}$. *Set* $\mathcal{S} := \{\mathrm{Id}, T_1, T_2, \ldots, T_m\}$. *Let* $x \in \mathcal{H}$. *Then for every* $\epsilon \in \mathbb{R}_{++}$, *there exists* $y_x \in \mathbf{B}[x;\epsilon]$ *such that* $\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x = \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} y_x$ *and* $(\forall i \in \mathrm{I})$ $y_x \notin \operatorname{Fix} T_i$. *Consequently, there exists a sequence* $(x_k)_{k\in\mathbb{N}}$ *in* $\mathcal{H}$ *such that* $\lim_{k\to\infty} x_k = x$, $(\forall k \in \mathbb{N})$ $\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x = \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x_k$ *and* $(\forall i \in \{1, \ldots, m\})$ $x_k \notin \operatorname{Fix} T_i$.

Proof. Let $\epsilon \in \mathbb{R}_{++}$. Because $(\forall i \in \mathrm{I})$ $T_i : \mathcal{H} \to \mathcal{H}$ is isometric, using [7, Proposition 3.4], we know that $(\forall i \in \mathrm{I})$ $T_i : \mathcal{H} \to \mathcal{H}$ is affine.

Define $(\forall i \in \mathrm{I})$ $(\forall x \in \mathcal{H})$ $F_i x := T_i(x + z) - z$. Then by Lemma 3.10(i), $(\forall i \in \mathrm{I})$ F_i is linear, $\operatorname{Fix} F_i = \operatorname{Fix} T_i - z$, and $\cap_{i=1}^m \operatorname{Fix} F_i = \cap_{i=1}^m \operatorname{Fix} T_i - z$.

Moreover, bearing Fact 2.2 in mind, we notice that there exists $y_x \in \mathbf{B}[x;\epsilon]$ such that $\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x = \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} y_x$ and $(\forall i \in \mathrm{I})$ $y_x \notin \operatorname{Fix} T_i$ if and only if there exists $z_x := y_x - z \in \mathbf{B}[x - z;\epsilon]$ such that $\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} F_i}(x - z) = \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} F_i}(z_x)$ and $(\forall i \in \mathrm{I})$ $z_x \notin \operatorname{Fix} F_i$. Hence, without loss of generality, we assume that $(\forall i \in \mathrm{I})$ $T_i : \mathcal{H} \to \mathcal{H}$ is linearly isometric and $z = 0$.

Suppose $m=1$. If $x \notin \operatorname{Fix} T_1$, then $y_x = x$. If $x \in \operatorname{Fix} T_1$, then take $w \in (\operatorname{Fix} T_1)^\perp \smallsetminus \{0\}$, $y_x = x + \frac{\epsilon}{\|w\|} w$. Because $w \in (\operatorname{Fix} T_1)^\perp$, $\operatorname{Fix} T_1$ is linear, and $\mathrm{P}_{\operatorname{Fix} T_1}$ is linear, we know that in both cases, $y_x \in \mathbf{B}[x;\epsilon]$, $\mathrm{P}_{\operatorname{Fix} T_1} x = \mathrm{P}_{\operatorname{Fix} T_1} y_x$ and $y_x \notin \operatorname{Fix} T_1$.

In the rest of the proof, we assume that $m \geq 2$. If $(\forall i \in \mathrm{I})$ $x \notin \operatorname{Fix} T_i$, then the proof is done with $y_x = x$. Otherwise, assume that j is the smallest index in $\mathrm{I} := \{1, \ldots, m\}$ such that $x \in \operatorname{Fix} T_j$.

Step 1: Take $z_j \in (\operatorname{Fix} T_j)^\perp \smallsetminus \{0\}$. Because $x \in \operatorname{Fix} T_j$, $z_j \notin \operatorname{Fix} T_j$ and $\operatorname{Fix} T_j$ is a linear subspace, we know that $(\forall t \in \mathbb{R} \smallsetminus \{0\})$ $x + t z_j \notin \operatorname{Fix} T_j$.

In addition, for every $i \in \{1, \ldots, j-1\}$, because $x \notin \operatorname{Fix} T_i$, by applying Lemma 4.5(ii) with $F = \mathrm{Id}, T = T_i$, and $z = z_j$, we have that there exists $\bar{t}_i \in \mathbb{R}_{++}$ such that $(\forall t \in [-\bar{t}_i, \bar{t}_i])$ $x + t z_j \notin \operatorname{Fix} T_i$.

Set $\alpha_j := \min_{1 \leq i \leq j-1} \bar{t}_i$ and take $y_j := x + \min\{\frac{\epsilon}{\|z_j\| m}, \alpha_j\} z_j$. Then we obtain that $(\forall i \in \{1, \ldots, j\})$ $y_j \notin \operatorname{Fix} T_i$, and $\|y_j - x\| \leq \frac{\epsilon}{m}$.

Furthermore, using the linearity of $\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i}$, $z_j \in (\operatorname{Fix} T_j)^\perp \subseteq (\cap_{i=1}^m \operatorname{Fix} T_i)^\perp$, and [14, Theorems 5.8(4)], we have that

$$\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} y_j = \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x + \min\{\frac{\epsilon}{\|z_j\| m}, \alpha_j\} \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} z_j = \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x.$$

Recursive Steps: If $j = m$, we go to Step $m-j+2$. Otherwise, for $k = j, \ldots, m-1$ successively, if $y_k \notin \operatorname{Fix} T_{k+1}$, then take $y_{k+1} = y_k$. Otherwise, we repeat Step 1 by substituting $x = y_k$, $j = k+1$ to obtain y_{k+1} such that $(\forall i \in \{1, \ldots, k+1\})$ $y_{k+1} \notin \operatorname{Fix} T_i$, $\|y_{k+1} - y_k\| \leq \frac{\epsilon}{m}$, and $\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} y_{k+1} = \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} y_k = \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x$.

Step $\mathbf{m-j+2}$: Note that from our Step 1 and Recursive Steps, we got that $(\forall k \in \{j, \ldots, m\})$ y_k satisfying $\|y_k - y_{k-1}\| \leq \frac{\epsilon}{m}$, where $y_{j-1} = x$. Moreover, in our Recursive Steps with $k = m-1$, we obtained y_m such that $(\forall i \in \mathrm{I})$ $y_m \notin \operatorname{Fix} T_i$, and $\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} y_m = \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x$.

Now take $y_x = y_m$. Then $\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} y_x = \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x$ and $(\forall i \in \mathrm{I})$ $y_x \notin \operatorname{Fix} T_i$.

Moreover,
$$\begin{aligned}\|y_x - x\| &= \|y_m - x\| \leq \|y_m - y_j\| + \|y_j - x\| \\ &\leq \|y_j - x\| + \sum_{k=j}^{m-1} \|y_{k+1} - y_k\| \leq \frac{\epsilon}{m} + (m-j)\frac{\epsilon}{m} \leq \epsilon.\end{aligned}$$

Altogether, the proof is complete. □

Proposition 4.9. *Suppose that $\mathcal{H} = \mathbb{R}^n$. Let $(\forall i \in \mathrm{I})$ $T_i : \mathcal{H} \to \mathcal{H}$ be isometric such that $z \in \cap_{i=1}^m \operatorname{Fix} T_i \neq \varnothing$ and $(\forall i \in \mathrm{I})$ $(\operatorname{Fix} T_i - z)^\perp \neq \{0\}$. Set $\mathcal{S} := \{\mathrm{Id}, T_1, T_2T_1, \ldots, T_m \cdots T_2T_1\}$. Let $x \in \mathcal{H}$. Then for every $\epsilon \in \mathbb{R}_{++}$, there exists $y_x \in \mathbf{B}[x;\epsilon]$ such that $\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x = \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} y_x$ and $(\forall i \in \mathrm{I})$ $T_{i-1} \cdots T_1 y_x \notin \operatorname{Fix} T_i$. Consequently, there exists a sequence $(x_k)_{k \in \mathbb{N}}$ in $\mathcal{H}$ such that $\lim_{k \to \infty} x_k = x$, $(\forall k \in \mathbb{N})$ $\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x = \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x_k$, and $(\forall i \in \mathrm{I})$ $T_{i-1} \cdots T_1 x_k \notin \operatorname{Fix} T_i$.*

Proof. Similarly with what we explained in the proof of Proposition 4.8, without loss of generality, we assume that $(\forall i \in \mathrm{I})$ $T_i : \mathcal{H} \to \mathcal{H}$ is a linear isometry and $z = 0$.

If $m=1$, then $\mathcal{S} = \{\mathrm{Id}, T_1\}$. Hence, the required result was proved in Proposition 4.8.

Suppose that $m \geq 2$. If $(\forall i \in \mathrm{I})$ $T_{i-1} \cdots T_1 x \notin \operatorname{Fix} T_i$, then the proof is done with $y_x = x$. Otherwise, assume that j is the smallest index in $\mathrm{I} := \{1, \ldots, m\}$ such that $T_{j-1} \cdots T_1 x \in \operatorname{Fix} T_j$.

Step 1: Take $z_j \in (\operatorname{Fix} T_j)^{\perp} \smallsetminus \{0\}$. Because $\mathcal{H} = \mathbb{R}^n$, and $T_{j-1} \cdots T_1 x \in \operatorname{Fix} T_j$, then applying Lemma 4.5(i) with $F = T_{j-1} \cdots T_1$, $T = T_j$ and $z = z_j$, we have that $(\forall t \in \mathbb{R} \smallsetminus \{0\})$ $T_{j-1} \cdots T_1(x + tT_1^* T_2^* \cdots T_{j-1}^* z_j) = T_{j-1} \cdots T_1 x + t z_j \notin \operatorname{Fix} T_j$.

In addition, since $(\forall i \in \{1, \ldots, j-1\})$ $T_{i-1} \cdots T_1 x \notin \operatorname{Fix} T_i$, thus applying Lemma 4.5(ii) with $F = T_{i-1} \cdots T_1, T = T_i$, and $z = T_1^* T_2^* \cdots T_{j-1}^* z_j$, we have that there exists $\bar{t}_i \in \mathbb{R}_{++}$ such that $(\forall t \in [-\bar{t}_i, \bar{t}_i])$ $T_{i-1} \cdots T_1(x + tT_1^* T_2^* \cdots T_{j-1}^* z_j) \notin \operatorname{Fix} T_i$.

Set $\alpha_j := \min_{1 \leq i \leq j-1} \bar{t}_i$ and take $y_j := x + \min\{\frac{\epsilon}{m\|z_j\|}, \alpha_j\} T_1^* T_2^* \cdots T_{j-1}^* z_j$.

Then we obtain that $(\forall i \in \{1, \ldots, j\})$ $T_{i-1} \cdots T_1 y_j \notin \operatorname{Fix} T_i$, and $\|y_j - x\| \leq \frac{\epsilon}{m}$, since $(\forall i \in \mathrm{I})$ T_i is isometric implies that $(\forall i \in \mathrm{I})$ $\|T_i^*\| = \|T_i\| \leq 1$. Furthermore, because $(\forall i \in \mathrm{I})$ T_i is linearly isometric and $\cap_{k=1}^m \operatorname{Fix} T_k^* \subseteq \operatorname{Fix} T_i^*$, by [3, Lemma 2.1] and [7, Lemma 3.7],

$$\mathrm{P}_{\cap_{t=1}^m \operatorname{Fix} T_t} T_1^* T_2^* \cdots T_{j-1}^* = \mathrm{P}_{\cap_{t=1}^m \operatorname{Fix} T_t^*} T_1^* T_2^* \cdots T_{j-1}^* = \mathrm{P}_{\cap_{t=1}^m \operatorname{Fix} T_t^*} = \mathrm{P}_{\cap_{t=1}^m \operatorname{Fix} T_t}. \tag{19}$$

Employing $\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i}$, $z_j \in (\operatorname{Fix} T_j)^{\perp} \subseteq (\cap_{i=1}^m \operatorname{Fix} T_i)^{\perp}$, and [14, Theorems 5.8(4)], we establish that

$$\begin{aligned} \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} y_j &= \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x + \min\{\frac{\epsilon}{m\|z_j\|}, \alpha_j\} \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} T_1^* T_2^* \cdots T_{j-1}^* z_j \\ &\overset{(19)}{=} \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x + \min\{\frac{\epsilon}{m\|z_j\|}, \alpha_j\} \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} z_j = \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x. \end{aligned}$$

Recursive Steps: If $j = m$, we go to Step $m-j+2$. Otherwise, for $k = j, \ldots, m-1$ successively, if $T_k \cdots T_1 y_k \notin \operatorname{Fix} T_{k+1}$, then take $y_{k+1} = y_k$. Otherwise, we repeat Step 1 by replacing $x = y_k$ and $j = k+1$ to obtain y_{k+1} such that $(\forall i \in \{1, \ldots, k+1\})$ $T_{i-1} \cdots T_1 y_{k+1} \notin \operatorname{Fix} T_i$, $\|y_{k+1} - y_k\| \leq \frac{\epsilon}{m}$, and

$$\mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} y_{k+1} = \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} y_k = \mathrm{P}_{\cap_{i=1}^m \operatorname{Fix} T_i} x.$$

Step $\mathbf{m - j + 2}$: This is almost the same as the Step $m - j + 2$ in the proof of Proposition 4.8. Altogether, the proof is complete. □

5. Finite convergence of CRMs associated with hyperplanes and halfspaces

Throughout this section, for every $i \in \mathrm{I} := \{1, \ldots, m\}$, let u_i be in $\mathcal{H}$, let η_i be in $\mathbb{R}$, and set

$$W_i := \{x \in \mathcal{H} \ : \ \langle x, u_i \rangle \leq \eta_i\} \quad \text{and} \quad H_i := \{x \in \mathcal{H} \ : \ \langle x, u_i \rangle = \eta_i\}. \tag{20}$$

In this section, we investigate the finite convergence of CRMs induced by sets of reflectors associated with halfspaces and hyperplanes for solving the related best approximation or feasibility problems.

According to [16, Theorems 4.17 and 6.5], if u_1 and u_2 are linearly dependent or orthogonal, then $\mathrm{P}_{W_2} \mathrm{P}_{W_1} = \mathrm{P}_{W_1 \cap W_2}$ and $\mathrm{P}_{W_2} \mathrm{P}_{H_1} = \mathrm{P}_{H_1} \mathrm{P}_{W_2} = \mathrm{P}_{H_1 \cap W_2}$; moreover, if u_1 and u_2 are linearly independent, we can always find corresponding sequences of iterations of compositions of projections onto halfspaces and hyperplanes with initial points $x \in \mathcal{H}$ converging linearly to $\mathrm{P}_{W_1 \cap W_2} x$ or $\mathrm{P}_{H_1 \cap W_2} x$.

For completeness we shall explore the performance of CRMs associated with hyperplanes and halfspaces in all cases.

CRMs associated with hyperplanes

Note that Corollary 5.1(ii) was first proved in [11, Lemma 3] when $\mathcal{H} = \mathbb{R}^n$.

Corollary 5.1. *Assume that $\cap_{i=1}^m H_i \neq \varnothing$. Let $x \in \mathcal{H}$. Then the following statements hold.*

(i) *Set $\mathcal{S} := \{\mathrm{Id}, \mathrm{R}_{H_1}, \ldots, \mathrm{R}_{H_m}\}$. If $x \notin \cup_{i=1}^m H_i$, then $CC_{\mathcal{S}} x = \mathrm{P}_{\cap_{i=1}^m H_i} x$.*

(ii) *Set $\mathcal{S} := \{\mathrm{Id}, \mathrm{R}_{H_1}, \mathrm{R}_{H_2}\mathrm{R}_{H_1}, \ldots, \mathrm{R}_{H_m}\cdots\mathrm{R}_{H_2}\mathrm{R}_{H_1}\}$.*
If $(\forall i \in \mathrm{I})\ \ \mathrm{R}_{H_{i-1}}\cdots\mathrm{R}_{H_1} x \notin H_i$, then $CC_{\mathcal{S}} x = \mathrm{P}_{\cap_{i=1}^m H_i} x$.

Proof. By [5, Lemmas 2.37], $(\forall i \in \mathrm{I})$ R_{H_i} is an affine isometry with $\operatorname{Fix}\mathrm{R}_{H_i} = H_i$. Let $z \in \cap_{i\in \mathrm{I}} H_i$. Note that $(\forall i \in \mathrm{I})$

$$\dim\big((\operatorname{Fix}\mathrm{R}_{H_i})^{\perp} - z\big) = \dim\big((H_i - z)^{\perp}\big) = \dim\big((\ker u_i)^{\perp}\big) = \dim\left(\operatorname{span}\{u_i\}\right) = 1.$$

Therefore, the required results follow directly from Theorem 4.7 with $T_1 = \mathrm{R}_{H_1}$ and $T_2 = \mathrm{R}_{H_2}, \ldots, T_m = \mathrm{R}_{H_m}$. ☐

Theorem 5.2 below illustrates that the CRM induced by $\mathcal{S} := \{\mathrm{Id}, \mathrm{R}_{H_1}, \mathrm{R}_{H_2}\}$ generally can not find $\mathrm{P}_{H_1\cap H_2} x$ in finite steps. In particular, if $x \in (H_1 \cap H_2) \cup (H_1^c \cap H_2^c)$, then the CRM converges in one step; otherwise, the classical method of alternating projections becomes a subsequence of the CRM.

Theorem 5.2. *Assume that $H_1 \cap H_2 \neq \varnothing$. Set $\mathcal{S} := \{\mathrm{Id}, \mathrm{R}_{H_1}, \mathrm{R}_{H_2}\}$. Let $x \in \mathcal{H}$. Then exactly one of the following cases occurs:*

(i) *$x \in (H_1 \cap H_2) \cup (H_1^c \cap H_2^c)$. Then $CC_{\mathcal{S}} x = \mathrm{P}_{H_1\cap H_2} x$.*

(ii) *$x \in H_1 \cap H_2^c$. Then $(\forall k \in \mathbb{N})$*
$CC_{\mathcal{S}}^{2k} x = (\mathrm{P}_{H_1}\mathrm{P}_{H_2})^k x$ and $CC_{\mathcal{S}}^{2k+1} x = \mathrm{P}_{H_2}(\mathrm{P}_{H_1}\mathrm{P}_{H_2})^k x$.

(iii) *$x \in H_1^c \cap H_2$. Then $(\forall k \in \mathbb{N})$*
$CC_{\mathcal{S}}^{2k} x = (\mathrm{P}_{H_2}\mathrm{P}_{H_1})^k x$ and $CC_{\mathcal{S}}^{2k+1} x = \mathrm{P}_{H_1}(\mathrm{P}_{H_2}\mathrm{P}_{H_1})^k x$.

Proof. (i): If $x \in H_1 \cap H_2$, then $\mathrm{R}_{H_1} x = x$, $\mathrm{R}_{H_2} x = x$, $\mathrm{P}_{H_1\cap H_2} x = x$, and $\mathcal{S}(x) = \{x, \mathrm{R}_{H_1} x, \mathrm{R}_{H_2} x\} = \{x\}$. Hence, by Fact 3.2(i),

$$CC_{\mathcal{S}} x = CC(\mathcal{S}(x)) = CC(\{x\}) = x = \mathrm{P}_{H_1\cap H_2} x.$$

If $x \in H_1^c \cap H_2^c$, that is, $x \notin H_1 \cup H_2$, then by Corollary 5.1(i), $CC_{\mathcal{S}} x = \mathrm{P}_{H_1\cap H_2} x$.

(ii): Assume that $x \in H_1 \cap H_2^c$. We prove (ii) by induction on k.

Clearly $x \in H_1 \cap H_2^c$ yields $\mathcal{S}(x) = \{x, \mathrm{R}_{H_1} x, \mathrm{R}_{H_2} x\} = \{x, \mathrm{R}_{H_2} x\}$. By Fact 3.2(ii), $CC_{\mathcal{S}} x = CC(\mathcal{S}(x)) = (x + \mathrm{R}_{H_2} x)/2 = \mathrm{P}_{H_2} x$. Hence, (ii) holds for $k = 0$.

Assume that (ii) is true for some $k \in \mathbb{N}$.

Denote by $(\forall t \in \mathbb{N})$ $x^{(t)} := CC_{\mathcal{S}}^t x$. By induction hypothesis,

$$x^{(2k+1)} = CC_{\mathcal{S}}^{2k+1} x = \mathrm{P}_{H_2}(\mathrm{P}_{H_1}\mathrm{P}_{H_2})^k x \in H_2,$$

so $\mathrm{R}_{H_2}(x^{(2k+1)}) = x^{(2k+1)}$ and $\mathcal{S}(x^{(2k+1)}) = \{x^{(2k+1)}, \mathrm{R}_{H_1}(x^{(2k+1)})\}$. Then by Fact 3.2(ii) and by induction hypothesis again,

$$\begin{aligned} CC_{\mathcal{S}}^{2(k+1)} x &= CC(\mathcal{S}(x^{(2k+1)})) = \frac{x^{(2k+1)} + \mathrm{R}_{H_1}(x^{(2k+1)})}{2} \\ &= \mathrm{P}_{H_1}(x^{(2k+1)}) = (\mathrm{P}_{H_1}\mathrm{P}_{H_2})^{k+1} x \in H_1, \end{aligned} \tag{21}$$

which implies that

$$\mathrm{R}_{H_1}(x^{(2(k+1))}) = x^{(2(k+1))} \text{ and } \mathcal{S}(x^{(2(k+1))}) = \{x^{(2(k+1))}, \mathrm{R}_{H_2}(x^{(2(k+1))})\}.$$

Furthermore, using Fact 3.2(ii), we obtain that

$$\begin{aligned} CC_{\mathcal{S}}^{2(k+1)+1}x = CC(\mathcal{S}(x^{(2(k+1))})) &= \frac{x^{(2(k+1))} + \mathrm{R}_{H_2}(x^{(2(k+1))})}{2} \\ &= \mathrm{P}_{H_2}(x^{(2(k+1))}) \overset{(21)}{=} \mathrm{P}_{H_2}(\mathrm{P}_{H_1}\mathrm{P}_{H_2})^{k+1}x. \end{aligned}$$

Hence, the desired results hold for $k+1$ and so (ii) is true by induction.

(iii): Switch H_1 and H_2 in (ii) to obtain the required results. $\square$

In Example 1 of the first arXiv version of [11], the authors showed the idea that the CRM induced by $\mathcal{S} := \{\mathrm{Id}, \mathrm{R}_{H_1}, \mathrm{R}_{H_2}\mathrm{R}_{H_1}\}$ needs at most three steps to find a point in $H_1 \cap H_2$ in $\mathbb{R}^n$. This motivates the following Theorem 5.3. The main idea of the following proof comes from that example as well. Note that the proof of that Example didn't consider the Case 3 below, and that taking advantage of our [5, Proposition 4.4], we actually discover that the CRM solves the best approximation problem in at most 3 steps.

Theorem 5.3. *Assume that $H_1 \cap H_2 \neq \varnothing$. Set $\mathcal{S} := \{\mathrm{Id}, \mathrm{R}_{H_1}, \mathrm{R}_{H_2}\mathrm{R}_{H_1}\}$. Then for every $x \in \mathcal{H}$, there exists $k \in \{1,2,3\}$ such that $CC_{\mathcal{S}}^k x = \mathrm{P}_{H_1\cap H_2} x$, that is, the circumcentered reflection method induced by $\mathcal{S}$ needs at most 3 steps to find the best approximation point $\mathrm{P}_{H_1\cap H_2} x$.*

Proof. Applying [5, Proposition 4.4] and [5, Proposition 4.2(iii)] with substituting $m = 3$, $T_1 = \mathrm{Id}$, $T_2 = \mathrm{R}_{H_1}$, $T_3 = \mathrm{R}_{H_2}\mathrm{R}_{H_1}$ and $W = H_1 \cap H_2$, we obtain, respectively,

$$(\forall x \in \mathcal{H}) \quad CC_{\mathcal{S}}x = \mathrm{P}_{H_1\cap H_2} x \Leftrightarrow CC_{\mathcal{S}}x \in H_1 \cap H_2; \tag{22a}$$

$$(\forall x \in \mathcal{H})(\forall k \in \mathbb{N}) \quad \mathrm{P}_{H_1\cap H_2} CC_{\mathcal{S}}^k x = \mathrm{P}_{H_1\cap H_2} x. \tag{22b}$$

Let $x \in \mathcal{H}$. According to Definition 3.6,

$$CC_{\mathcal{S}}x = CC(\mathcal{S}(x)) = CC(\{x, \mathrm{R}_{H_1} x, \mathrm{R}_{H_2}\mathrm{R}_{H_1} x\}). \tag{23}$$

We have exactly the following four cases.

Case 1: $\mathrm{R}_{H_1} x - x = 0$ and $\mathrm{R}_{H_2}\mathrm{R}_{H_1} x - \mathrm{R}_{H_1} x = 0$. Then, in view of (23) and Fact 3.2(i), $CC_{\mathcal{S}}x = CC(\{x\}) = x = \mathrm{P}_{H_1\cap H_2} x$.

Case 2: $\mathrm{R}_{H_1} x - x \neq 0$ and $\mathrm{R}_{H_2}\mathrm{R}_{H_1} x - \mathrm{R}_{H_1} x \neq 0$, that is, $x \notin \operatorname{Fix}\mathrm{R}_{H_1} = H_1$ and $\mathrm{R}_{H_1} x \notin \operatorname{Fix}\mathrm{R}_{H_2} = H_2$. Then, using Corollary 5.1(ii), we obtain that $CC_{\mathcal{S}}x = \mathrm{P}_{H_1\cap H_2} x$.

Case 3: $\mathrm{R}_{H_1} x - x = 0$ and $\mathrm{R}_{H_2}\mathrm{R}_{H_1} x - \mathrm{R}_{H_1} x \neq 0$. Then $\mathcal{S}(x) = \{\mathrm{R}_{H_1} x, \mathrm{R}_{H_2}\mathrm{R}_{H_1} x\}$, which, combining with (23) and fact 3.2(ii), implies that

$$x^{(1)} := CC_{\mathcal{S}}x = \frac{\mathrm{R}_{H_1} x + \mathrm{R}_{H_2}\mathrm{R}_{H_1} x}{2} = \mathrm{P}_{H_2}\mathrm{R}_{H_1} x \in H_2. \tag{24}$$

Case 3.1: Assume $x^{(1)} \in H_1$. Then by (24), $CC_{\mathcal{S}}x \in H_1 \cap H_2$. Hence, by (22a), $CC_{\mathcal{S}}x = \mathrm{P}_{H_1\cap H_2} x$.

Case 3.2: Assume $x^{(1)} \notin H_1$. If $\mathrm{R}_{H_1}(x^{(1)}) \notin H_2$, then apply Corollary 5.1(ii) with $x = x^{(1)}$ to obtain

$$CC_{\mathcal{S}}^2 x = CC_{\mathcal{S}}(x^{(1)}) = \mathrm{P}_{H_1 \cap H_2}(x^{(1)}) = \mathrm{P}_{H_1 \cap H_2} CC_{\mathcal{S}} x \overset{(22b)}{=} \mathrm{P}_{H_1 \cap H_2} x.$$

Assume that $\mathrm{R}_{H_1}(x^{(1)}) \in H_2$. Then $\mathcal{S}(x^{(1)}) = \{x^{(1)}, \mathrm{R}_{H_1}(x^{(1)})\}$, and, by Definition 3.6 and Fact 3.2(ii),

$$x^{(2)} := CC_{\mathcal{S}}(x^{(1)}) = \frac{x^{(1)} + \mathrm{R}_{H_1}(x^{(1)})}{2} = \mathrm{P}_{H_1}(x^{(1)}) \in H_1. \tag{25}$$

Moreover, because H_2 is an affine subspace, $x^{(1)} \in H_2$ and $\mathrm{R}_{H_1}(x^{(1)}) \in H_2$, we know that $x^{(2)} = CC_{\mathcal{S}}(x^{(1)}) = (x^{(1)} + \mathrm{R}_{H_1}(x^{(1)}))/2 \in H_2$, which, combined with (25), yields that $CC_{\mathcal{S}}(x^{(1)}) = CC_{\mathcal{S}}^2 x = x^{(2)} \in H_1 \cap H_2$. Hence,

$$CC_{\mathcal{S}}^2 x = CC_{\mathcal{S}}(x^{(1)}) \overset{(22a)}{=} \mathrm{P}_{H_1 \cap H_2}(x^{(1)}) = \mathrm{P}_{H_1 \cap H_2} CC_{\mathcal{S}} x \overset{(22b)}{=} \mathrm{P}_{H_1 \cap H_2} x.$$

Case 4: $\mathrm{R}_{H_1} x - x \neq 0$ and $\mathrm{R}_{H_2} \mathrm{R}_{H_1} x - \mathrm{R}_{H_1} x = 0$. Then $\mathcal{S}(x) = \{x, \mathrm{R}_{H_1} x\}$, and, by Fact 3.2(ii), $x^{(1)} := CC_{\mathcal{S}} x = (x + \mathrm{R}_{H_1} x)/2 = \mathrm{P}_{H_1} x \in H_1$. Then $\mathrm{R}_{H_1}(x^{(1)}) = x^{(1)}$ and $\mathcal{S}(x^{(1)}) = \{\mathrm{R}_{H_1}(x^{(1)}), \mathrm{R}_{H_2} \mathrm{R}_{H_1}(x^{(1)})\}$.

If $\mathrm{R}_{H_1}(x^{(1)}) = \mathrm{R}_{H_2} \mathrm{R}_{H_1}(x^{(1)})$, then applying case 1 with $x = x^{(1)}$, we derive

$$CC_{\mathcal{S}}^2 x = CC_{\mathcal{S}}(x^{(1)}) = \mathrm{P}_{H_1 \cap H_2}(x^{(1)}) = \mathrm{P}_{H_1 \cap H_2}(CC_{\mathcal{S}} x) \overset{(22b)}{=} \mathrm{P}_{H_1 \cap H_2} x.$$

If $\mathrm{R}_{H_1}(x^{(1)}) \neq \mathrm{R}_{H_2} \mathrm{R}_{H_1}(x^{(1)})$, then employing Case 3 with $x = x^{(1)}$, we obtain that

$$\text{either} \qquad CC_{\mathcal{S}}^2 x = CC_{\mathcal{S}}(x^{(1)}) = \mathrm{P}_{H_1 \cap H_2}(x^{(1)}) \overset{(22b)}{=} \mathrm{P}_{H_1 \cap H_2} x$$

$$\text{or} \qquad CC_{\mathcal{S}}^3 x = CC_{\mathcal{S}}^2(x^{(1)}) = \mathrm{P}_{H_1 \cap H_2}(x^{(1)}) \overset{(22b)}{=} \mathrm{P}_{H_1 \cap H_2} x.$$

Altogether, the proof is complete. □

CRMs associated with halfspaces and hyperplanes

Proposition 5.4. *Suppose that $u_1 \neq 0$ and $u_2 \neq 0$, and that $\langle u_1, u_2 \rangle \geq 0$. Set $\mathcal{S} := \{\mathrm{Id}, \mathrm{R}_{W_1}, \mathrm{R}_{W_2}\}$. Then $(\forall x \in W_1 \cup W_2)$, $CC_{\mathcal{S}} x = \mathrm{P}_{W_1 \cap W_2} x$.*

Proof. Let $x \in W_1 \cup W_2$. Then we have exactly the following three cases:

Case 1: $x \in W_1 \cap W_2$. Then by Fact 3.2(i), clearly

$$CC_{\mathcal{S}} x = CC(\mathcal{S}(x)) = x = \mathrm{P}_{W_1 \cap W_2} x.$$

Case 2: $x \in W_1 \cap W_2^c$. Then we have $\mathrm{R}_{W_1} x = x$ and, by Fact 2.4, $\mathrm{P}_{W_2} x = \mathrm{P}_{H_2} x$ and $\mathrm{R}_{W_2} x = \mathrm{R}_{H_2} x$. Hence, $\mathcal{S}(x) = \{x, \mathrm{R}_{W_1} x, \mathrm{R}_{W_2} x\} = \{x, \mathrm{R}_{H_2} x\}$, and by Fact 3.2(ii)

$$CC_{\mathcal{S}} x = CC(\mathcal{S}(x)) = CC(\{x, \mathrm{R}_{W_2} x\}) = \frac{x + \mathrm{R}_{W_2} x}{2} = \mathrm{P}_{W_2} x.$$

Using definitions of W_1, W_2 in (20) and $x \in W_1 \cap W_2^c$, we know that $\langle x, u_1 \rangle \leq \eta_1$ and $\langle x, u_2 \rangle > \eta_2$. Combine this with the assumption $\langle u_1, u_2 \rangle \geq 0$ to obtain that

$$\langle x, u_1 \rangle + \frac{\eta_2 - \langle x, u_2 \rangle}{\|u_2\|^2} \langle u_2, u_1 \rangle \leq \eta_1. \tag{26}$$

In view of Fact 2.3,

$$\langle \mathrm{P}_{H_2} x, u_1 \rangle = \left\langle x + \frac{\eta_2 - \langle x, u_2 \rangle}{\|u_2\|^2} u_2, u_1 \right\rangle = \langle x, u_1 \rangle + \frac{\eta_2 - \langle x, u_2 \rangle}{\|u_2\|^2} \langle u_2, u_1 \rangle \overset{(26)}{\leq} \eta_1, \tag{27}$$

which implies that $\mathrm{P}_{H_2} x \in W_1$. Hence, $\mathrm{P}_{W_2} x = \mathrm{P}_{H_2} x \in W_1 \cap H_2 \subseteq W_1 \cap W_2$. Apply Fact 2.1 with $A = W_1 \cap W_2$ and $B = W_2$ to obtain that $CC_{\mathcal{S}} x = \mathrm{P}_{W_2} x = \mathrm{P}_{W_1 \cap W_2} x$.

Case 3: $x \in W_1^c \cap W_2$. Swap W_1 and W_2 in Case 2 above to obtain the required result. $\square$

u_1 and u_2 are linearly dependent

Lemma 5.5. *Assume that $u_1 = 0$ or $u_2 = 0$. The following statements hold:*

(i) *Assume that $H_1 \cap H_2 \neq \varnothing$.*
Set $\mathcal{S}_1 := \{\mathrm{Id}, \mathrm{R}_{H_1}, \mathrm{R}_{H_2}\}$ and $\mathcal{S}_2 := \{\mathrm{Id}, \mathrm{R}_{H_1}, \mathrm{R}_{H_2} \mathrm{R}_{H_1}\}$.
Then $(\forall x \in \mathcal{H})$ $CC_{\mathcal{S}_1} x = CC_{\mathcal{S}_2} x = \mathrm{P}_{H_1 \cap H_2} x$.

(ii) *Assume that $W_1 \cap W_2 \neq \varnothing$.*
Set $\mathcal{S}_1 := \{\mathrm{Id}, \mathrm{R}_{W_1}, \mathrm{R}_{W_2}\}$ and $\mathcal{S}_2 := \{\mathrm{Id}, \mathrm{R}_{W_1}, \mathrm{R}_{W_2} \mathrm{R}_{W_1}\}$.
Then $(\forall x \in \mathcal{H})$ $CC_{\mathcal{S}_1} x = CC_{\mathcal{S}_2} x = \mathrm{P}_{W_1 \cap W_2} x$.

(iii) *Assume that $H_1 \cap W_2 \neq \varnothing$. Set $\mathcal{S}_1 := \{\mathrm{Id}, \mathrm{R}_{H_1}, \mathrm{R}_{W_2}\}$,*
$\mathcal{S}_2 := \{\mathrm{Id}, \mathrm{R}_{H_1}, \mathrm{R}_{W_2} \mathrm{R}_{H_1}\}$ and $\mathcal{S}_3 := \{\mathrm{Id}, \mathrm{R}_{W_2}, \mathrm{R}_{H_1} \mathrm{R}_{W_2}\}$.
Then $(\forall x \in \mathcal{H})$ $CC_{\mathcal{S}_1} x = CC_{\mathcal{S}_2} x = CC_{\mathcal{S}_3} x = \mathrm{P}_{H_1 \cap W_2} x$.

Proof. (i): Note that $H_1 \cap H_2 \neq \varnothing$ implies that for every $i \in \{1, 2\}$, if $u_i = 0$, then $H_i = \mathcal{H}$ and $\mathrm{R}_{H_i} = \mathrm{Id}$. Without loss of generality, assume $u_1 = 0$. Then $H_1 = \mathcal{H}$ and, by Fact 3.2(ii), $(\forall x \in \mathcal{H})$

$$CC_{\mathcal{S}_1} x = CC_{\mathcal{S}_2} x = CC(\{x, \mathrm{R}_{H_2} x\}) = \frac{x + \mathrm{R}_{H_2} x}{2} = \mathrm{P}_{H_2} x = \mathrm{P}_{H_1 \cap H_2} x.$$

Hence, (i) holds.

The proofs for (ii) and (iii) are similar to the proof of (i) and are omitted. $\square$

The following results are necessary for the proofs of the Propositions 5.7 and 5.8. The long and mechanical proof of Lemma 5.6 is left for the Appendix.

Lemma 5.6. *Assume that $u_1 \neq 0$ and $u_2 \neq 0$, and that u_1 and u_2 are linearly dependent. The following statements hold:*

(i) *$\langle u_1, u_2 \rangle \neq 0$. Moreover, if $\langle u_1, u_2 \rangle > 0$, then $u_2 = \frac{\|u_2\|}{\|u_1\|} u_1$ and $\langle u_1, u_2 \rangle = \|u_1\| \|u_2\|$; if $\langle u_1, u_2 \rangle < 0$, then $u_2 = -\frac{\|u_2\|}{\|u_1\|} u_1$ and $\langle u_1, u_2 \rangle = -\|u_1\| \|u_2\|$.*

(ii) *Assume that $\langle u_1, u_2 \rangle > 0$. Then $\frac{\eta_1}{\|u_1\|} = \frac{\eta_2}{\|u_2\|} \Leftrightarrow W_1 = W_2 \Leftrightarrow H_1 = H_2$. Moreover, we have $\frac{\eta_1}{\|u_1\|} \neq \frac{\eta_2}{\|u_2\|} \Leftrightarrow H_1 \cap H_2 = \varnothing$, $\frac{\eta_1}{\|u_1\|} < \frac{\eta_2}{\|u_2\|} \Leftrightarrow W_1 \subsetneqq W_2$, and $\frac{\eta_1}{\|u_1\|} > \frac{\eta_2}{\|u_2\|} \Leftrightarrow W_2 \subsetneqq W_1$.*

(iii) *Assume that $\langle u_1, u_2 \rangle < 0$. Then $\frac{\eta_1}{\|u_1\|} = -\frac{\eta_2}{\|u_2\|} \Leftrightarrow H_1 = H_2$. Furthermore $\frac{\eta_1}{\|u_1\|} \neq -\frac{\eta_2}{\|u_2\|} \Leftrightarrow H_1 \cap H_2 = \varnothing$. Moreover, $W_1 \cap W_2 \neq \varnothing \Leftrightarrow \eta_1 \|u_2\| + \eta_2 \|u_1\| \geq 0$. In addition, if $W_1 \cap W_2 \neq \varnothing$, then $\operatorname{int} W_1 \cup W_2 = W_1 \cup \operatorname{int} W_2 = \mathcal{H}$, $H_1 \subseteq W_2$ and $H_2 \subseteq W_1$.*

(iv) *Assume that $\langle u_1, u_2 \rangle > 0$ and $\frac{\eta_1}{\|u_1\|} \geq \frac{\eta_2}{\|u_2\|}$. Then $\|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2} x\| = \frac{\eta_1}{\|u_1\|} - \frac{\eta_2}{\|u_2\|}$ $(\forall x \in \mathcal{H})$. Moreover we have $\mathrm{R}_{W_1} x \in W_2 \Leftrightarrow \|x - \mathrm{P}_{W_1} x\| \geq \|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2} x\|$ $(\forall x \in \mathcal{H} \smallsetminus W_1)$.*

(v) *Assume that* $\langle u_1, u_2\rangle < 0$ *and* $W_1 \cap W_2 \neq \varnothing$.
Then $(\forall x \in \mathcal{H})$ $\|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2} x\| = \frac{\eta_1}{\|u_1\|} + \frac{\eta_2}{\|u_2\|}$.
Moreover, $(\forall x \in \mathcal{H} \smallsetminus W_1)$ $\mathrm{R}_{W_1} x \notin W_2 \Leftrightarrow \|x - \mathrm{P}_{W_1} x\| > \|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2} x\|$.

(vi) *Assume that* $\langle u_1, u_2\rangle < 0$, $W_1 \cap W_2 \neq \varnothing$ *and* $x \in W_1 \cap W_2^c$.
Then $\mathrm{P}_{W_2} x = \mathrm{P}_{W_1 \cap H_2} x = \mathrm{P}_{W_1 \cap W_2} x$.

(vii) *Assume that* $\langle u_1, u_2\rangle < 0$ *and* $W_1 \cap W_2 \neq \varnothing$.
Let $x \in W_2$ *with* $\|x - \mathrm{P}_{H_1} x\| > \|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2} x\|$. *Then*
  (a) $x \notin W_1$.
  (b) $\mathrm{R}_{W_1} x = \mathrm{R}_{H_1} x \notin W_2$.
  (c) $\|\mathrm{R}_{W_2} \mathrm{R}_{W_1} x - \mathrm{R}_{W_1} x\| = 2(\|x - \mathrm{P}_{H_1} x\| - \|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2} x\|) \leq 2\|x - \mathrm{P}_{H_1} x\|$ $= \|x - \mathrm{R}_{W_1} x\|$.

(viii) *Let* $x \in \mathcal{H}$. *Then* $x, \mathrm{R}_{H_1} x, \mathrm{R}_{H_2} x$ *are affinely dependent;* $x, \mathrm{R}_{H_1} x, \mathrm{R}_{H_2} \mathrm{R}_{H_1} x$ *are affinely dependent;* $x, \mathrm{R}_{W_1} x, \mathrm{R}_{W_2} x$ *are affinely dependent; and* $x, \mathrm{R}_{W_1} x, \mathrm{R}_{W_2} \mathrm{R}_{W_1} x$ *are affinely dependent.*

Proposition 5.7. *Assume that* $u_1 \neq 0$ *and* $u_2 \neq 0$ *and that* u_1 *and* u_2 *are linearly dependent. Assume that* $W_1 \cap W_2 \neq \varnothing$. *Set* $\mathcal{S} := \{\mathrm{Id}, \mathrm{R}_{W_1}, \mathrm{R}_{W_2}\}$. *Then exactly one of the following cases occurs:*

Case 1: $\langle u_1, u_2\rangle > 0$. *Then if* $\frac{\eta_1}{\|u_1\|} = \frac{\eta_2}{\|u_2\|}$, *then* $(\forall x \in \mathcal{H})$ $CC_{\mathcal{S}}x = \mathrm{P}_{W_1 \cap W_2} x$*; otherwise,* $(\forall x \in W_1 \cup W_2)$ $CC_{\mathcal{S}}x = \mathrm{P}_{W_1 \cap W_2} x$*, and* $(\forall x \in \mathcal{H} \smallsetminus (W_1 \cup W_2))$ $CC_{\mathcal{S}}x = \varnothing$.

Case 2: $\langle u_1, u_2\rangle < 0$. *Then* $W_1 \cup W_2 = \mathcal{H}$ *and* $(\forall x \in \mathcal{H})$ $CC_{\mathcal{S}}x = \mathrm{P}_{W_1 \cap W_2} x$.

Consequently, $(\forall x \in W_1 \cup W_2)$ $CC_{\mathcal{S}}x = \mathrm{P}_{W_1 \cap W_2} x$.

Proof. The assumptions and Lemma 5.6(i) imply either $\langle u_1, u_2\rangle > 0$ or $\langle u_1, u_2\rangle < 0$.

Case 1: Assume that $\langle u_1, u_2\rangle > 0$. If $\frac{\eta_1}{\|u_1\|} = \frac{\eta_2}{\|u_2\|}$, then by Lemma 5.6(ii) we get $W_1 = W_2 = W_1 \cap W_2$ and $\mathcal{S} = \{\mathrm{Id}, \mathrm{R}_{W_1}\}$. Hence, by Fact 3.2(ii),

$$CC_{\mathcal{S}}x = CC(\{x, \mathrm{R}_{W_1} x\}) = \frac{x + \mathrm{R}_{W_1} x}{2} = \mathrm{P}_{W_1} x = \mathrm{P}_{W_1 \cap W_2} x.$$

Assume $\frac{\eta_1}{\|u_1\|} \neq \frac{\eta_2}{\|u_2\|}$. If $x \in W_1 \cup W_2$, then Proposition 5.4 yields $CC_{\mathcal{S}}x = \mathrm{P}_{W_1 \cap W_2} x$. Suppose that $x \in \mathcal{H} \smallsetminus (W_1 \cup W_2)$. By Lemma 2.5 we get $\mathrm{R}_{W_1} x = \mathrm{R}_{H_1} x \in \operatorname{int} W_1$ and $\mathrm{R}_{W_2} x = \mathrm{R}_{H_2} x \in \operatorname{int} W_2$. Note that, by Lemma 5.6(ii), $H_1 \cap H_2 = \varnothing$. Then clearly $\mathrm{P}_{H_1} x \neq \mathrm{P}_{H_2} x$. Now we have

$$x \neq \mathrm{R}_{W_1} x, x \neq \mathrm{R}_{W_2} x, \text{ and } \mathrm{R}_{W_1} x = 2\,\mathrm{P}_{H_1} x - x \neq 2\,\mathrm{P}_{H_2} x - x = \mathrm{R}_{W_2} x,$$

which, combined with Lemma 5.6(viii) and Fact 3.2(iii)(b), implies $CC_{\mathcal{S}}x = \varnothing$.

Case 2: Assume that $\langle u_1, u_2\rangle < 0$. Then Lemma 5.6(iii) implies that $\mathcal{H} = W_1 \cup W_2$. Hence, we have exactly the following three cases:

Case 2.1: $x \in W_1 \cap W_2$. Then $\mathcal{S}(x) = \{x\}$, and following Fact 3.2(i) we obtain $CC_{\mathcal{S}}x = x = \mathrm{P}_{W_1 \cap W_2} x$.

Case 2.2: $x \in W_1 \cap W_2^c$. Then $\mathcal{S}(x) = \{x, \mathrm{R}_{W_2} x\}$, and by Fact 3.2(ii) and Lemma 5.6(vi), $CC_{\mathcal{S}}x = CC(\{x, \mathrm{R}_{W_2} x\}) = \mathrm{P}_{W_2} x = \mathrm{P}_{W_1 \cap W_2} x$.

Case 2.3: $x \in W_2 \cap W_1^c$. This proof is similar to the proof of Case 2.

Altogether, the proof is complete. ☐

Propositions 5.7 and 5.8 suggest that if u_1 and u_2 are linearly dependent, then the CRM induced by $\mathcal{S} := \{\mathrm{Id}, \mathrm{R}_{W_1}, \mathrm{R}_{W_2}\mathrm{R}_{W_1}\}$ performs worse than the CRM induced by $\mathcal{S} := \{\mathrm{Id}, \mathrm{R}_{W_1}, \mathrm{R}_{W_2}\}$.

Proposition 5.8. *Assume that $u_1 \neq 0$ and $u_2 \neq 0$, and that u_1 and u_2 are linearly dependent. Set $\mathcal{S} := \{\mathrm{Id}, \mathrm{R}_{W_1}, \mathrm{R}_{W_2}\mathrm{R}_{W_1}\}$. Assume that $W_1 \cap W_2 \neq \varnothing$. Then exactly one of the following cases occurs:*

Case 1: $\langle u_1, u_2\rangle > 0$ *and* $\frac{\eta_1}{\|u_1\|} \leq \frac{\eta_2}{\|u_2\|}$. *Then* $(\forall x \in \mathcal{H})$ $CC_{\mathcal{S}}x = \mathrm{P}_{W_1} x = \mathrm{P}_{W_1\cap W_2} x$.

Case 2: $\langle u_1, u_2\rangle > 0$ *and* $\frac{\eta_1}{\|u_1\|} > \frac{\eta_2}{\|u_2\|}$. *Then*

if $x \in W_1$, *then* $CC_{\mathcal{S}}x = \mathrm{P}_{W_2} x = \mathrm{P}_{W_1\cap W_2} x$;

if $x \notin W_1$ *and* $\|x - \mathrm{P}_{W_1} x\| \geq \|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2} x\|$, *then* $CC_{\mathcal{S}}x = \mathrm{P}_{W_1} x = \mathrm{P}_{H_1} x \notin W_1 \cap W_2$; *and*

if $x \notin W_1$ *and* $\|x - \mathrm{P}_{W_1} x\| < \|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2} x\|$, *then* $CC_{\mathcal{S}}x = \varnothing$.

Case 3: $\langle u_1, u_2\rangle < 0$. *Then*

if $x \in W_1$, *then* $CC_{\mathcal{S}}x = \mathrm{P}_{W_1\cap W_2} x$;

if $x \notin W_1$ *and* $\|x - \mathrm{P}_{W_1} x\| \leq \|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2} x\|$, *then* $CC_{\mathcal{S}}x = \mathrm{P}_{W_1} x = \mathrm{P}_{W_1\cap W_2} x$;

if $x \notin W_1$, $\|x - \mathrm{P}_{W_1} x\| > \|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2} x\|$, *and* $\frac{\eta_1}{\|u_1\|} = -\frac{\eta_2}{\|u_2\|}$, *then* $CC_{\mathcal{S}}x = \mathrm{P}_{W_1\cap W_2} x$;

if $x \notin W_1$, $\|x - \mathrm{P}_{W_1} x\| > \|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2} x\|$, *and* $\frac{\eta_1}{\|u_1\|} \neq -\frac{\eta_2}{\|u_2\|}$, *then* $CC_{\mathcal{S}}x = \varnothing$.

Consequently, $(\forall x \in W_1)$ $CC_{\mathcal{S}}x = \mathrm{P}_{W_1\cap W_2} x$.

Proof. Case 1: Assume that $\langle u_1, u_2\rangle > 0$ and $\frac{\eta_1}{\|u_1\|} \leq \frac{\eta_2}{\|u_2\|}$. Then by Lemma 5.6(ii) and Lemma 2.5, $W_1 \subseteq W_2$ and $(\forall x \in \mathcal{H})$ $\mathrm{R}_{W_2}\mathrm{R}_{W_1} x = \mathrm{R}_{W_1} x$. Hence, by Fact 3.2(ii), $(\forall x \in \mathcal{H})$ $CC_{\mathcal{S}}x = \frac{x+\mathrm{R}_{W_1} x}{2} = \mathrm{P}_{W_1} x = \mathrm{P}_{W_1\cap W_2} x$.

Case 2: Assume that $\langle u_1, u_2\rangle > 0$ and $\frac{\eta_1}{\|u_1\|} > \frac{\eta_2}{\|u_2\|}$. Then by Lemma 5.6(ii), $W_2 \subsetneqq W_1$ and so $W_2 = W_1 \cap W_2$. If $x \in W_1$, then $x = \mathrm{R}_{W_1} x$ and $\mathcal{S}(x) = \{x, \mathrm{R}_{W_2} x\}$. Hence, by Fact 3.2(ii), $CC_{\mathcal{S}}x = CC(\{x, \mathrm{R}_{W_2} x\}) = \frac{x+\mathrm{R}_{W_2} x}{2} = \mathrm{P}_{W_2} x = \mathrm{P}_{W_1\cap W_2} x$.

Suppose that $x \notin W_1$. Then $x \neq \mathrm{R}_{W_1} x$. Using Lemma 5.6(iv), we know that

$$\|x - \mathrm{P}_{W_1} x\| \geq \|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2} x\| \Leftrightarrow \mathrm{R}_{W_1} x \in W_2. \tag{28}$$

If $\|x - \mathrm{P}_{W_1} x\| \geq \|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2} x\|$, then $\mathrm{R}_{W_1} x \in W_2$ and $\mathrm{R}_{W_2}\mathrm{R}_{W_1} x = \mathrm{R}_{W_1} x$. So $\mathcal{S} = \{\mathrm{Id}, \mathrm{R}_{W_1}\}$, which, by Fact 3.2(ii), implies that

$$CC_{\mathcal{S}}x = CC(\{x, \mathrm{R}_{W_1} x\}) = \frac{x + \mathrm{R}_{W_1} x}{2} = \mathrm{P}_{W_1} x = \mathrm{P}_{H_1} x \notin \mathrm{P}_{W_1\cap W_2} x,$$

where the last inequality is because $\frac{\eta_1}{\|u_1\|} > \frac{\eta_2}{\|u_2\|}$ yields $H_1 \cap W_2 = \varnothing$.

Suppose $\|x - \mathrm{P}_{W_1} x\| < \|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2} x\|$. Then, by (28), $\mathrm{R}_{W_1} x \notin W_2$. Hence, apply Lemma 2.5 with $x = \mathrm{R}_{W_1} x$ and $W = W_2$ to obtain $\mathrm{R}_{W_2}\mathrm{R}_{W_1} x = \mathrm{R}_{H_2}\mathrm{R}_{W_1} x \in \operatorname{int} W_2$. Combine this with $x \in W_1^c \subseteq W_2^c$ to obtain $x \neq \mathrm{R}_{W_2}\mathrm{R}_{W_1} x$. Recall that $x \neq \mathrm{R}_{W_1} x$ and $\mathrm{R}_{W_1} x \neq \mathrm{R}_{W_2}\mathrm{R}_{W_1} x$. So, $\operatorname{card}\{x, \mathrm{R}_{W_1} x, \mathrm{R}_{W_2}\mathrm{R}_{W_1} x\} = 3$. Moreover, by Lemma 5.6(viii), $x, \mathrm{R}_{W_1} x, \mathrm{R}_{W_2}\mathrm{R}_{W_1} x$ are affinely dependent. Therefore, by Fact 3.2(iii)(b), $CC_{\mathcal{S}}x = \varnothing$.

Case 3: If $x \in W_1$, then $\mathcal{S}(x) = \{x, \mathrm{R}_{W_2} x\}$ and so by Fact 3.2(ii) and Lemma 5.6(vi), $CC_{\mathcal{S}}x = CC(\{x, \mathrm{R}_{W_2} x\}) = (x + \mathrm{R}_{W_2} x)/2 = \mathrm{P}_{W_2} x = \mathrm{P}_{W_1 \cap W_2} x$.

Assume that $x \notin W_1$. Then, by Lemma 5.6(v),

$$\|x - \mathrm{P}_{W_1} x\| > \|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2} x\| \Leftrightarrow \mathrm{R}_{W_1} x \notin W_2. \tag{29}$$

If $\|x - \mathrm{P}_{W_1} x\| \leq \|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2} x\|$, then by (29), $\mathrm{R}_{W_1} x \in W_2$ and $\mathcal{S}(x) = \{x, \mathrm{R}_{W_1} x\}$. Moreover, by Fact 3.2(ii) and by Lemma 5.6(vi) with swapping W_1 and W_2, we have $CC_{\mathcal{S}}x = CC(\{x, \mathrm{R}_{W_1} x\}) = (x + \mathrm{R}_{W_1} x)/2 = \mathrm{P}_{W_1} x = \mathrm{P}_{W_1 \cap W_2} x$. Suppose that $\|x - \mathrm{P}_{W_1} x\| > \|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2} x\|$. If $\frac{\eta_1}{\|u_1\|} = -\frac{\eta_2}{\|u_2\|}$, then by Lemma 5.6(iii), $H_1 = H_2$. Note that $\mathrm{P}_{W_1} x = \mathrm{P}_{H_1} x \in H_1 \cap H_2 \subseteq W_1 \cap W_2$. Then apply Fact 2.1 with $A = W_1 \subseteq W_2$ and $B = W_1$ to obtain that $\mathrm{P}_{W_1} x = \mathrm{P}_{W_1 \cap W_2} x$. Moreover, by Fact 2.4 and (29), $\mathrm{R}_{W_1} x = \mathrm{R}_{H_1} x \notin W_2$ and $\mathrm{R}_{W_2} \mathrm{R}_{W_1} x = \mathrm{R}_{H_2} \mathrm{R}_{H_1} x = \mathrm{R}_{H_1} \mathrm{R}_{H_1} x = x$.

Hence, $\mathcal{S}(x) = \{x, \mathrm{R}_{W_1} x\}$ and $CC_{\mathcal{S}}x = CC(\{x, \mathrm{R}_{W_1} x\}) = \mathrm{P}_{W_1} x = \mathrm{P}_{W_1 \cap W_2} x$.

Suppose $\frac{\eta_1}{\|u_1\|} \neq -\frac{\eta_2}{\|u_2\|}$. Then by Lemma 5.6(iii) we obtain $H_1 \cap H_2 = \varnothing$ and $\mathrm{P}_{H_1} x - \mathrm{P}_{H_2} x \neq 0$. Using Lemma 5.6(vii)(c), we know that $x \neq \mathrm{R}_{W_2} \mathrm{R}_{W_1} x$ and that $\|\mathrm{R}_{W_2} \mathrm{R}_{W_1} x - \mathrm{R}_{W_1} x\| < \|x - \mathrm{R}_{W_1} x\|$. Combine this with $x \notin W_1$ and (29) to see that $\operatorname{card}\{x, \mathrm{R}_{W_1} x, \mathrm{R}_{W_2} \mathrm{R}_{W_1} x\} = 3$. Moreover, by Lemma 5.6(viii), $x, \mathrm{R}_{W_1} x, \mathrm{R}_{W_2} \mathrm{R}_{W_1} x$ are affinely dependent. Hence, via Fact 3.2(iii)(b), we obtain that $CC_{\mathcal{S}}x = \varnothing$.

Altogether, the proof is complete. □

According to Proposition 5.9 below, if u_1 and u_2 are linearly dependent, then we don't need any projection-based algorithms to find $\mathrm{P}_{H_1 \cap W_2} x$.

Proposition 5.9. *Assume that $u_1 \neq 0$ and $u_2 \neq 0$, that u_1 and u_2 are linearly dependent, and that $H_1 \cap W_2 \neq \varnothing$. Then $H_1 \subseteq W_2$. Consequently, $(\forall x \in \mathcal{H})$ $\mathrm{P}_{H_1} x = \mathrm{P}_{H_1 \cap W_2} x$.*

Proof. This is clear from Lemma 5.6(i) and the definitions of H_1 and W_2 in (20). □

u_1 and u_2 are linearly independent

Although the following Proposition 5.10 implies that $CC_{\mathcal{S}}$ with $\mathcal{S} := \{\mathrm{Id}, \mathrm{R}_{W_1}, \mathrm{R}_{W_2}\}$ is proper, it is easy to find subsets W_1 and W_2 in $\mathbb{R}^2$ such that there exists some $x \in (W_1 \cap W_2^c) \cup (W_1^c \cap W_2)$ satisfying $(\forall k \in \mathbb{N})$ $CC_{\mathcal{S}}^k x \notin W_1 \cap W_2$.

Proposition 5.10. *Assume that u_1 and u_2 are linearly independent. Define $\mathcal{S} := \{\mathrm{Id}, \mathrm{R}_{W_1}, \mathrm{R}_{W_2}\}$. Let $x \in \mathcal{H}$. Then exactly one of the following cases occurs:*

Case 1: *$x \in W_1 \cup W_2$. Then $(\forall x \in W_1 \cap W_2)$ $CC_{\mathcal{S}}x = \mathrm{P}_{W_1 \cap W_2} x$;*
$(\forall x \in W_1 \cap W_2^c)$ $CC_{\mathcal{S}}x = \mathrm{P}_{H_2} x$; and $(\forall x \in W_1^c \cap W_2)$ $CC_{\mathcal{S}}x = \mathrm{P}_{H_1} x$.

Case 2: *$x \in \mathcal{H} \smallsetminus (W_1 \cup W_2)$. Then $CC_{\mathcal{S}}x = \mathrm{P}_{H_1 \cap H_2} x$.*

Proof. Case 1: Suppose that $x \in W_1 \cup W_2$. If $x \in W_1 \cap W_2$, then we clearly have $CC_{\mathcal{S}}x = x = \mathrm{P}_{W_1 \cap W_2} x$. If $x \in W_1 \cap W_2^c$, then $x = \mathrm{R}_{W_1} x$, $\mathrm{P}_{W_2} x = \mathrm{P}_{H_2} x$, $\mathrm{R}_{W_2} x = \mathrm{R}_{H_2} x$, and $\mathcal{S}(x) = \{x, \mathrm{R}_{W_1} x, \mathrm{R}_{W_2} x\} = \{x, \mathrm{R}_{W_2} x\}$. Hence we have

$$CC_{\mathcal{S}}x = CC(\{x, \mathrm{R}_{W_2} x\}) = (x + \mathrm{R}_{W_2} x)/2 = \mathrm{P}_{W_2} x = \mathrm{P}_{H_2} x.$$

Similarly, if $x \in W_1^c \cap W_2$, then $CC_{\mathcal{S}}x = \mathrm{P}_{H_1} x$.

Case 2: Suppose that $x \in \mathcal{H} \smallsetminus (W_1 \cup W_2)$. Then we have

$$\mathcal{S}(x) = \{x, \mathrm{R}_{W_1} x, \mathrm{R}_{W_2} x\} = \{x, \mathrm{R}_{H_1} x, \mathrm{R}_{H_2} x\}.$$

Because $H_1 \subseteq W_1$ and $H_2 \subseteq W_2$, we know that $x \notin H_1 \cup H_2$. Note that, in view of [16, Lemma 2.3], if u_1 and u_2 are linearly independent, then $H_1 \cap H_2 \neq \varnothing$. Hence, by Theorem 5.2(i) we obtain that $CC_{\mathcal{S}}x = CC(\{x, \mathrm{R}_{H_1} x, \mathrm{R}_{H_2} x\}) = \mathrm{P}_{H_1 \cap H_2} x$. $\square$

The following results are necessary for the proof of Proposition 5.12. We leave the long and mechanical proof in the Appendix.

Lemma 5.11. *Assume that u_1 and u_2 are linearly independent. Let $x \in \mathcal{H}$.*

(i) *Assume that $x \notin W_1$.*
Then $\mathrm{R}_{W_1} x \in W_2 \Leftrightarrow (\langle x, u_2\rangle - \eta_2)\|u_1\|^2 - 2(\langle x, u_1\rangle - \eta_1)\langle u_1, u_2\rangle \leq 0$.

(ii) *Assume that $x \in W_1^c \cap W_2$ and that $\mathrm{R}_{W_1} x \in W_2$. Then $\mathrm{P}_{W_1} x \in W_1 \cap W_2$.*

(iii) *Assume that $\langle u_1, u_2\rangle \geq 0$ and that $x \in W_1 \cap W_2^c$. Then $\mathrm{P}_{W_2} x \in W_1 \cap W_2$ and $\mathrm{R}_{W_2} x \in W_1 \cap \operatorname{int} W_2$.*

(iv) *Assume that $\langle u_1, u_2\rangle \leq 0$ and that $x \in W_1^c \cap W_2^c$. Then $\mathrm{R}_{W_1} x \notin W_2$.*

(v) *Assume that $\langle u_1, u_2\rangle > 0$ and that $\mathrm{P}_{H_1} x \notin W_2$.*
Then $\mathrm{P}_{W_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_2} \mathrm{P}_{H_1} x \in W_1 \cap W_2$.

(vi) *Assume that $\langle u_1, u_2\rangle < 0$, that $x \in H_2$ and that $x \notin W_1$. Then $\mathrm{R}_{H_1} x \notin W_2$.*

The following result elaborates that the CRM induced by $\mathcal{S} := \{\mathrm{Id}, \mathrm{R}_{W_1}, \mathrm{R}_{W_2} \mathrm{R}_{W_1}\}$ needs at most two steps to find a point in $W_1 \cap W_2$.

Proposition 5.12. *Assume that u_1 and u_2 are linearly independent. Then define $\mathcal{S} := \{\mathrm{Id}, \mathrm{R}_{W_1}, \mathrm{R}_{W_2} \mathrm{R}_{W_1}\}$. Let $x \in \mathcal{H}$. Then exactly one of the following cases occurs:*

(i) *$x \in W_1 \cap W_2$. Then $CC_{\mathcal{S}}x = x = \mathrm{P}_{W_1 \cap W_2} x$.*

(ii) *$x \in W_1 \cap W_2^c$. Then $CC_{\mathcal{S}}x = \mathrm{P}_{W_2} x = \mathrm{P}_{H_2} x$ and there exists $k \in \{1, 2\}$ such that $CC_{\mathcal{S}}^k x \in W_1 \cap W_2$.*

(iii) *$x \in W_1^c \cap W_2$ and $(\langle x, u_2\rangle - \eta_2)\|u_1\|^2 - 2(\langle x, u_1\rangle - \eta_1)\langle u_1, u_2\rangle \leq 0$.*
Then $CC_{\mathcal{S}}x = \mathrm{P}_{W_1} x = \mathrm{P}_{H_1} x = \mathrm{P}_{W_1 \cap W_2} x$.

(iv) *$x \in W_1^c \cap W_2$ and $(\langle x, u_2\rangle - \eta_2)\|u_1\|^2 - 2(\langle x, u_1\rangle - \eta_1)\langle u_1, u_2\rangle > 0$ (this case won't occur when $\langle u_1, u_2\rangle \geq 0$). Then $CC_{\mathcal{S}}x = \mathrm{P}_{H_1 \cap H_2} x$.*

(v) *$x \in W_1^c \cap W_2^c$ and $(\langle x, u_2\rangle - \eta_2)\|u_1\|^2 - 2(\langle x, u_1\rangle - \eta_1)\langle u_1, u_2\rangle \leq 0$ (this case won't occur when $\langle u_1, u_2\rangle \leq 0$). Then $CC_{\mathcal{S}}x = \mathrm{P}_{W_1} x = \mathrm{P}_{H_1} x$ and there exists $k \in \{1, 2\}$ such that $CC_{\mathcal{S}}^k x \in W_1 \cap W_2$.*

(vi) *$x \in W_1^c \cap W_2^c$ and $(\langle x, u_2\rangle - \eta_2)\|u_1\|^2 - 2(\langle x, u_1\rangle - \eta_1)\langle u_1, u_2\rangle > 0$.*
Then $CC_{\mathcal{S}}x = \mathrm{P}_{H_1 \cap H_2} x$.

Proof. (i): Assume that $x \in W_1 \cap W_2$. Then $\mathcal{S}(x) = \{x\}$. Hence we clearly have $CC_{\mathcal{S}}x = x = \mathrm{P}_{W_1 \cap W_2} x$.

(ii): Assume that $x \in W_1 \cap W_2^c$. Then $x = \mathrm{R}_{W_1} x$ and

$$\mathcal{S}(x) = \{x, \mathrm{R}_{W_1} x, \mathrm{R}_{W_2} \mathrm{R}_{W_1} x\} = \{x, \mathrm{R}_{W_2} x\}.$$

Hence, by Fact 3.2(ii) and Fact 2.4,

$$CC_{\mathcal{S}}x = CC(\{x, \mathrm{R}_{W_2} x\}) = \tfrac{x + \mathrm{R}_{W_2} x}{2} = \mathrm{P}_{W_2} x = \mathrm{P}_{H_2} x.$$

Assume that $\langle u_1, u_2\rangle \geq 0$. Then by Lemma 5.11(iii), $\mathrm{P}_{W_2} x \in W_1 \cap W_2$. Hence, by Fact 2.1, $CC_{\mathcal{S}}x = \mathrm{P}_{W_2} x = \mathrm{P}_{W_1 \cap W_2} x$.

Suppose $\langle u_1, u_2\rangle < 0$. We shall prove that $CC_{\mathcal{S}}^2 x \in W_1 \cap W_2$. If $CC_{\mathcal{S}}x = \mathrm{P}_{H_2} x \in W_1$, then we are done.

Assume $\mathrm{P}_{H_2} x \notin W_1$. Then apply Lemma 5.11(vi) with $x = \mathrm{P}_{H_2} x$ to obtain that $\mathrm{R}_{H_1} \mathrm{P}_{H_2} x \notin W_2$. Now $\mathrm{P}_{H_2} x \notin W_1$ and $\mathrm{R}_{H_1} \mathrm{P}_{H_2} x \notin W_2$ imply that $\mathrm{P}_{H_2} x \notin H_1$ and $\mathrm{R}_{H_1} \mathrm{P}_{H_2} x \notin H_2$, and that

$CC(\{\mathrm{P}_{H_2} x, \mathrm{R}_{W_1} \mathrm{P}_{H_2} x, \mathrm{R}_{W_2} \mathrm{R}_{W_1} \mathrm{P}_{H_2} x\}) = CC(\{\mathrm{P}_{H_2} x, \mathrm{R}_{H_1} \mathrm{P}_{H_2} x, \mathrm{R}_{H_2} \mathrm{R}_{H_1} \mathrm{P}_{H_2} x\})$. Using these results and applying Corollary 5.1(ii) with $x = \mathrm{P}_{H_2} x$, we obtain that $CC(\{\mathrm{P}_{H_2} x, \mathrm{R}_{H_1} \mathrm{P}_{H_2} x, \mathrm{R}_{H_2} \mathrm{R}_{H_1} \mathrm{P}_{H_2} x\}) = \mathrm{P}_{H_1 \cap H_2} \mathrm{P}_{H_2} x$. Following [8, Lemma 2.3], we achieve $\mathrm{P}_{H_1 \cap H_2} \mathrm{P}_{H_2} x = \mathrm{P}_{H_1 \cap H_2} x$. Altogether,

$$CC_{\mathcal{S}}^2 x = CC_{\mathcal{S}}(CC_{\mathcal{S}} x) = CC_{\mathcal{S}}(\mathrm{P}_{H_2} x) = \mathrm{P}_{H_1 \cap H_2} x \in W_1 \cap W_2.$$

(iii): Assume that $x \in W_1^c \cap W_2$ and $(\langle x, u_2 \rangle - \eta_2)\|u_1\|^2 - 2(\langle x, u_1 \rangle - \eta_1)\langle u_1, u_2 \rangle \leq 0$. Note that Lemma 5.11(i) yields $\mathrm{R}_{W_1} x \in W_2$. Then $\mathrm{R}_{W_2} \mathrm{R}_{W_1} x = \mathrm{R}_{W_1} x$, and

$$\mathcal{S}(x) = \{x, \mathrm{R}_{W_1} x, \mathrm{R}_{W_2} \mathrm{R}_{W_1} x\} = \{x, \mathrm{R}_{W_1} x\}.$$

Hence, by Fact 3.2(ii) and Fact 2.4, we reach

$$CC_{\mathcal{S}} x = CC(\{x, \mathrm{R}_{W_1} x\}) = \tfrac{x + \mathrm{R}_{W_1} x}{2} = \mathrm{P}_{W_1} x = \mathrm{P}_{H_1} x.$$

On the other hand, $x \in W_1^c \cap W_2$ and $\mathrm{R}_{W_1} x \in W_2$. Moreover, via Lemma 5.11(ii), $\mathrm{P}_{W_1} x \in W_1 \cap W_2$. Combine these results with Fact 2.1 we finally deduce that $CC_{\mathcal{S}} x = \mathrm{P}_{W_1} x = \mathrm{P}_{W_1 \cap W_2} x$.

(iv): Assume that $x \in W_1^c \cap W_2$ and $(\langle x, u_2 \rangle - \eta_2)\|u_1\|^2 - 2(\langle x, u_1 \rangle - \eta_1)\langle u_1, u_2 \rangle > 0$. By Lemma 5.11(i), we have that $\mathrm{R}_{W_1} x \notin W_2$. Apply Lemma 5.11(iii) with swapping W_1 and W_2 to know that this case won't happen when $\langle u_1, u_2 \rangle \geq 0$. Moreover, because $x \in W_1^c \cap W_2$ and $\mathrm{R}_{W_1} x \notin W_2$, by Fact 2.4,

$$\mathcal{S}(x) = \{x, \mathrm{R}_{W_1} x, \mathrm{R}_{W_2} \mathrm{R}_{W_1} x\} = \{x, \mathrm{R}_{H_1} x, \mathrm{R}_{H_2} \mathrm{R}_{H_1} x\}.$$

Hence, by Corollary 5.1(ii), $CC_{\mathcal{S}} x = \mathrm{P}_{H_1 \cap H_2} x$.

(v): Assume that $x \in W_1^c \cap W_2^c$ and $(\langle x, u_2 \rangle - \eta_2)\|u_1\|^2 - 2(\langle x, u_1 \rangle - \eta_1)\langle u_1, u_2 \rangle \leq 0$. In view of Lemma 5.11(i), $\mathrm{R}_{W_1} x \in W_2$. By Lemma 5.11(iv), this case won't happen when $\langle u_1, u_2 \rangle \leq 0$. Hence, we have $\langle u_1, u_2 \rangle > 0$.

Because $x \in W_1^c \cap W_2^c$ and $\mathrm{R}_{W_1} x \in W_2$, we have, by Fact 2.4, that $\mathrm{R}_{W_1} x = \mathrm{R}_{H_1} x$, $\mathrm{R}_{W_2} \mathrm{R}_{W_1} x = \mathrm{R}_{W_1} x$, and, by Fact 3.2(ii),

$$CC_{\mathcal{S}} x = CC(\{x, \mathrm{R}_{W_1} x\}) = \tfrac{x + \mathrm{R}_{W_1} x}{2} = \mathrm{P}_{W_1} x = \mathrm{P}_{H_1} x.$$

If $\mathrm{P}_{H_1} x \in W_2$, then we are done. Assume that $\mathrm{P}_{H_1} x \notin W_2$. Then,

$$\begin{aligned} CC_{\mathcal{S}}^2 x &= CC_{\mathcal{S}}(CC_{\mathcal{S}} x) = CC_{\mathcal{S}}(\mathrm{P}_{H_1} x) = CC(\{\mathrm{P}_{H_1} x, \mathrm{R}_{W_1} \mathrm{P}_{H_1} x, \mathrm{R}_{W_2} \mathrm{R}_{W_1} \mathrm{P}_{H_1} x\}) \\ &= CC(\{\mathrm{P}_{H_1} x, \mathrm{R}_{H_2} \mathrm{P}_{H_1} x\}) = \tfrac{\mathrm{P}_{H_1} x + \mathrm{R}_{H_2} \mathrm{P}_{H_1} x}{2} = \mathrm{P}_{H_2} \mathrm{P}_{H_1} x. \end{aligned}$$

Hence, by Lemma 5.11(v), $CC_{\mathcal{S}}^2 x = \mathrm{P}_{H_2} \mathrm{P}_{H_1} x \in W_1 \cap W_2$. Altogether, (v) holds.

(vi): Assume that $x \in W_1^c \cap W_2^c$ and $(\langle x, u_2 \rangle - \eta_2)\|u_1\|^2 - 2(\langle x, u_1 \rangle - \eta_1)\langle u_1, u_2 \rangle > 0$. By Lemma 5.11(i), $\mathrm{R}_{W_1} x \notin W_2$. Hence, via Corollary 5.1(ii),

$$CC_{\mathcal{S}} x = CC(\{\mathrm{P}_{H_1} x, \mathrm{R}_{H_1} x, \mathrm{R}_{H_2} \mathrm{R}_{H_1} x\}) = \mathrm{P}_{H_1 \cap H_2} x.$$

In conclusion, the proof is complete. $\square$

In the following result, we reduce the CRM associated with one hyperplane and one halfspace to the CRM associated with two hyperplanes which was considered in the first subsection of Section 5.

Lemma 5.13. *Assume that u_1 and u_2 are linearly independent. Set $\mathcal{S}_1 := \{\mathrm{Id}, \mathrm{R}_{H_1}, \mathrm{R}_{W_2}\}$, $\mathcal{S}_2 := \{\mathrm{Id}, \mathrm{R}_{H_1}, \mathrm{R}_{W_2}\mathrm{R}_{H_1}\}$ and $\mathcal{S}_3 := \{\mathrm{Id}, \mathrm{R}_{W_2}, \mathrm{R}_{H_1}\mathrm{R}_{W_2}\}$. Let $x \in \mathcal{H}$. Then the following statements hold:*

(i) *If $x \in W_2$, then $CC_{\mathcal{S}_1}x = \mathrm{P}_{H_1}x$; otherwise, $CC_{\mathcal{S}_1}x = CC(\{x, \mathrm{R}_{H_1}x, \mathrm{R}_{H_2}x\})$.*

(ii) *If $\mathrm{R}_{H_1}x \in W_2$, then $CC_{\mathcal{S}_2}x = \mathrm{P}_{H_1}x$; otherwise, $CC_{\mathcal{S}_2}x = CC(\{x, \mathrm{R}_{H_1}x, \mathrm{R}_{H_2}\mathrm{R}_{H_1}x\})$.*

(iii) *If $x \in W_2$, then $CC_{\mathcal{S}_3}x = \mathrm{P}_{H_1}x$; otherwise, $CC_{\mathcal{S}_3}x = CC(\{x, \mathrm{R}_{H_2}x, \mathrm{R}_{H_1}\mathrm{R}_{H_2}x\})$.*

Proof. Note that $x \in W_2 \Leftrightarrow x = \mathrm{R}_{W_2}x$ and that $\mathrm{R}_{H_1}x \in W_2 \Leftrightarrow \mathrm{R}_{H_1}x = \mathrm{R}_{W_2}\mathrm{R}_{H_1}x$. Therefore, the required results follow easily from Fact 3.2(ii) and Definition 3.6. □

To end this section, below we summarize results on the finite convergence of CRMs that we obtained in this section.

Theorem 5.14. *For every $i \in \{1,2\}$, u_i is in $\mathcal{H}$, η_i is in $\mathbb{R}$, and set*

$$W_i := \{x \in \mathcal{H} \;:\; \langle x, u_i\rangle \leq \eta_i\} \quad \text{and} \quad H_i := \{x \in \mathcal{H} \;:\; \langle x, u_i\rangle = \eta_i\}.$$

(i) *Assume that $H_1 \cap H_2 \neq \varnothing$. Then the CRM induced by $\mathcal{S} := \{\mathrm{Id}, \mathrm{R}_{H_1}, \mathrm{R}_{H_2}\mathrm{R}_{H_1}\}$ needs at most 3 steps to find the best approximation point $\mathrm{P}_{H_1\cap H_2}x$ (see Theorem 5.3).*

(ii) *Assume that u_1 and u_2 are linearly dependent and that $W_1 \cap W_2 \neq \varnothing$. Then the CRM induced by $\mathcal{S} := \{\mathrm{Id}, \mathrm{R}_{W_1}, \mathrm{R}_{W_2}\}$ with initial point $x \in W_1 \cup W_2$ finds the best approximation point $\mathrm{P}_{W_1\cap W_2}x$ in one step (see Lemma 5.5 and Proposition 5.7).*

(iii) *Assume that u_1 and u_2 are linearly dependent and that $W_1 \cap W_2 \neq \varnothing$. Then the CRM induced by $\mathcal{S} := \{\mathrm{Id}, \mathrm{R}_{W_1}, \mathrm{R}_{W_2}\mathrm{R}_{W_1}\}$ with initial point $x \in W_1$ finds the best approximation point $\mathrm{P}_{W_1\cap W_2}x$ in one step (see Lemma 5.5 and Proposition 5.8).*

(iv) *Assume that u_1 and u_2 are linearly independent. Then the CRM induced by $\mathcal{S} := \{\mathrm{Id}, \mathrm{R}_{W_1}, \mathrm{R}_{W_2}\mathrm{R}_{W_1}\}$ with initial point $x \in \mathcal{H}$ finds a feasibility point in $W_1 \cap W_2$ in at most two steps (see Proposition 5.12).*

(v) *Assume that $H_1 \cap W_2 \neq \varnothing$. If $u_1 = u_2 = 0$, then $(\forall x \in \mathcal{H})$ $CC_{\mathcal{S}}x = \mathrm{P}_{H_1\cap W_2}x$ with $\mathcal{S}$ being $\{\mathrm{Id}, \mathrm{R}_{H_1}, \mathrm{R}_{W_2}\}$, $\{\mathrm{Id}, \mathrm{R}_{H_1}, \mathrm{R}_{W_2}\mathrm{R}_{H_1}\}$, or $\{\mathrm{Id}, \mathrm{R}_{W_2}, \mathrm{R}_{H_1}\mathrm{R}_{W_2}\}$; if u_1 and u_2 are nonzero and linearly dependent, then $(\forall x \in \mathcal{H})$ $\mathrm{P}_{H_1}x = \mathrm{P}_{H_1\cap W_2}x$; otherwise, the CRM induced by $\mathcal{S} := \{\mathrm{Id}, \mathrm{R}_{H_1}, \mathrm{R}_{H_2}\mathrm{R}_{H_1}\}$ finds the feasibility point $\mathrm{P}_{H_1\cap H_2}x \in H_1 \cap W_2$ in 3 steps (see Theorem 5.3, Proposition 5.9 and Lemma 5.5).*

Appendix

Below, we provide proofs of the Lemmas 5.6 and 5.11.

Proof of Lemma 5.6

Proof. (i): Because u_1 and u_2 are nonzero and linearly dependent, due to [16, Fact 2.1], either $u_2 = \frac{\|u_2\|}{\|u_1\|}u_1$ or $u_2 = -\frac{\|u_2\|}{\|u_1\|}u_1$, which implies required results in (i).

(ii): These results follow easily from (i) and (20).

(iii): Due to (20) and (i), we have clearly

$$\tfrac{\eta_1}{\|u_1\|} = -\tfrac{\eta_2}{\|u_2\|} \Leftrightarrow H_1 = H_2 \quad\text{and}\quad \tfrac{\eta_1}{\|u_1\|} \neq -\tfrac{\eta_2}{\|u_2\|} \Leftrightarrow H_1 \cap H_2 = \varnothing.$$

In view of (i), $\langle u_1, u_2\rangle < 0$ implies that $u_2 = -\frac{\|u_2\|}{\|u_1\|}u_1$. This and (20) yield

$$W_2 = \{x \in \mathcal{H} \ : \ \langle x, u_2\rangle \leq \eta_2\} = \left\{x \in \mathcal{H} \ : \ \langle x, u_1\rangle \geq -\frac{\|u_1\|}{\|u_2\|}\eta_2\right\}. \tag{30}$$

Employing (20) again, we yield $W_1 = \{x \in \mathcal{H} \ : \ \langle x, u_1\rangle \leq \eta_1\}$. Hence,

$$W_1 \cap W_2 \neq \varnothing \Leftrightarrow -\frac{\|u_1\|}{\|u_2\|}\eta_2 \leq \eta_1 \Leftrightarrow \eta_1\|u_2\| + \eta_2\|u_1\| \geq 0. \tag{31}$$

Assume that $W_1 \cap W_2 \neq \varnothing$. Notice that

$$x \notin W_2 \overset{(30)}{\Leftrightarrow} \langle x, u_1\rangle < -\frac{\|u_1\|}{\|u_2\|}\eta_2 \overset{(31)}{\Rightarrow} \langle x, u_1\rangle < \eta_1 \Leftrightarrow x \in \operatorname{int} W_1$$

and that $\quad x \notin W_1 \Leftrightarrow \langle x, u_1\rangle > \eta_1 \overset{(31)}{\Rightarrow} \langle x, u_1\rangle > -\dfrac{\|u_1\|}{\|u_2\|}\eta_2 \overset{(30)}{\Leftrightarrow} x \in \operatorname{int} W_2.$

Hence, $\mathcal{H} = \operatorname{int} W_1 \cup W_2 = W_1 \cup \operatorname{int} W_2$. Because $u_2 = -\frac{\|u_2\|}{\|u_1\|}u_1$ we have

$$(\forall x \in H_1) \ \ \langle x, u_2\rangle - \eta_2 = -\frac{\|u_2\|}{\|u_1\|}\langle x, u_1\rangle - \eta_2 \overset{(20)}{=} -\frac{\|u_2\|}{\|u_1\|}\eta_1 - \eta_2 \overset{(31)}{\leq} 0,$$

which implies that $H_1 \subseteq W_2$. The proof of $H_2 \subseteq W_1$ is similar.

(iv): In consideration of (i), $u_2 = \frac{\|u_2\|}{\|u_1\|}u_1$. Combine this with Fact 2.3, the assumption $\frac{\eta_1}{\|u_1\|} \geq \frac{\eta_2}{\|u_2\|}$, and some easy algebra, to observe that $(\forall x \in \mathcal{H})$

$$\|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2} x\| = \left\|x + \frac{\eta_1 - \langle x, u_1\rangle}{\|u_1\|^2}u_1 - \left(x + \frac{\eta_2 - \langle x, u_2\rangle}{\|u_2\|^2}u_2\right)\right\| = \frac{\eta_1}{\|u_1\|} - \frac{\eta_2}{\|u_2\|}. \tag{32}$$

Let $x \in \mathcal{H} \smallsetminus W_1$. Then via (20) and Fact 2.4, $\langle x, u_1\rangle > \eta_1$ and

$$\mathrm{P}_{W_1} x = \mathrm{P}_{H_1} x = x + \tfrac{\eta_1 - \langle x, u_1\rangle}{\|u_1\|^2}u_1.$$

Combining these results with (32), we derive that

$$\|x - \mathrm{P}_{W_1} x\| \geq \|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2} x\| \Leftrightarrow \frac{\langle x, u_1\rangle - \eta_1}{\|u_1\|} \geq \frac{\eta_1}{\|u_1\|} - \frac{\eta_2}{\|u_2\|}. \tag{33}$$

On the other hand,

$$\begin{aligned}
\mathrm{R}_{W_1} x \in W_2 &\Leftrightarrow \langle \mathrm{R}_{W_1} x, u_2\rangle - \eta_2 \leq 0 \quad \text{(by (20))}\\
&\Leftrightarrow \left\langle 2\left(x + \frac{\eta_1 - \langle x, u_1\rangle}{\|u_1\|^2}u_1\right) - x, u_2\right\rangle - \eta_2 \leq 0 \quad \text{(by Fact 2.4)}\\
&\Leftrightarrow \left\langle x + 2\frac{\eta_1 - \langle x, u_1\rangle}{\|u_1\|^2}u_1, \frac{\|u_2\|}{\|u_1\|}u_1\right\rangle - \eta_2 \leq 0 \quad \left(\text{by } u_2 = \frac{\|u_2\|}{\|u_1\|}u_1\right)\\
&\Leftrightarrow \|u_2\|\left(\langle x, u_1\rangle + 2(\eta_1 - \langle x, u_1\rangle)\right) - \|u_1\|\eta_2 \leq 0\\
&\Leftrightarrow \frac{\langle x, u_1\rangle - \eta_1}{\|u_1\|} \geq \frac{\eta_1}{\|u_1\|} - \frac{\eta_2}{\|u_2\|},
\end{aligned}$$

which, combined with (33) leads to ($x \in \mathcal{H} \smallsetminus W_1$) $\|x - \mathrm{P}_{W_1} x\| \geq \|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2} x\|$ if and only if $\mathrm{R}_{W_1} x \in W_2$.

(v): The proof is similar to that of (iv) and is omitted.

(vi): Because we have $x \in W_1 \cap W_2^c$, invoking Fact 2.4 and (iii), we know that $\mathrm{P}_{W_2} x = \mathrm{P}_{H_2} x \in H_2 \subseteq W_1$. Hence, $\mathrm{P}_{W_2} x = \mathrm{P}_{H_2} x \in W_1 \cap H_2 \subseteq W_1 \cap W_2$. Apply Fact 2.1 with $A = W_1 \cap H_2$ and $B = H_2$ to obtain that $\mathrm{P}_{H_2} x = \mathrm{P}_{W_1 \cap H_2} x$. Moreover, employ Fact 2.1 with $A = W_1 \cap W_2$ and $B = W_2$ to deduce that $\mathrm{P}_{W_2} x = \mathrm{P}_{W_1 \cap W_2} x$. So, (vi) holds.

(vii): By (i), $\langle u_1, u_2 \rangle < 0$ implies $u_2 = -\frac{\|u_2\|}{\|u_1\|} u_1$. Then via Fact 2.3 and (20),

$$\|x - \mathrm{P}_{H_1} x\| = \left\| x - \left(x + \frac{\eta_1 - \langle x, u_1 \rangle}{\|u_1\|^2} u_1 \right) \right\| = \left| \frac{\eta_1 - \langle x, u_1 \rangle}{\|u_1\|} \right|; \tag{34a}$$

$$x \in W_2 \Leftrightarrow \langle x, u_2 \rangle \leq \eta_2 \Leftrightarrow \left\langle x, -\frac{\|u_2\| u_1}{\|u_1\|} \right\rangle \leq \eta_2 \Leftrightarrow \left\langle x, -\frac{u_1}{\|u_1\|} \right\rangle \leq \frac{\eta_2}{\|u_2\|}. \tag{34b}$$

(vii)(a): Assume to the contrary that $x \in W_1$. Then $\langle x, u_1 \rangle \leq \eta_1$. Hence,

$$\|x - \mathrm{P}_{H_1} x\| \overset{(34a)}{=} \left| \tfrac{\eta_1 - \langle x, u_1 \rangle}{\|u_1\|} \right| = \tfrac{\eta_1 - \langle x, u_1 \rangle}{\|u_1\|} \overset{(34b)}{\leq} \tfrac{\eta_1}{\|u_1\|} + \tfrac{\eta_2}{\|u_2\|} \overset{(v)}{=} \|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2} x\|,$$

which contradicts the assumption, $\|x - \mathrm{P}_{H_1} x\| > \|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2} x\|$.

(vii)(b): Due to (vii)(a) and (20), $\langle x, u_1 \rangle > \eta_1$. So, (34a) and (v) lead to

$$\begin{aligned} \|x - \mathrm{P}_{H_1} x\| > \|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2} x\| &\Leftrightarrow \frac{\langle x, u_1 \rangle - \eta_1}{\|u_1\|} > \frac{\eta_1}{\|u_1\|} + \frac{\eta_2}{\|u_2\|} \\ &\Leftrightarrow -\frac{\|u_2\|}{\|u_1\|} (2\eta_1 - \langle x, u_1 \rangle) > \eta_2. \end{aligned} \tag{35}$$

On the other hand, adopting $u_2 = -\frac{\|u_2\|}{\|u_1\|} u_1$ and $\mathrm{R}_{W_1} x = x + 2\frac{\eta_1 - \langle x, u_1 \rangle}{\|u_1\|^2} u_1$, we observe

$$\begin{aligned} \langle \mathrm{R}_{W_1} x, u_2 \rangle = \left\langle \mathrm{R}_{W_1} x, -\frac{\|u_2\| u_1}{\|u_1\|} \right\rangle &= -\frac{\|u_2\|}{\|u_1\|} \left\langle x + 2\frac{\eta_1 - \langle x, u_1 \rangle}{\|u_1\|^2} u_1, u_1 \right\rangle \\ &= -\frac{\|u_2\|}{\|u_1\|} (2\eta_1 - \langle x, u_1 \rangle) \overset{(35)}{>} \eta_2, \end{aligned}$$

which implies that $\mathrm{R}_{W_1} x \notin W_2$.

(vii)(c): Notice that

$$\begin{aligned} &\|\mathrm{R}_{W_2} \mathrm{R}_{W_1} x - \mathrm{R}_{W_1} x\| = \|\mathrm{R}_{H_2} \mathrm{R}_{H_1} x - \mathrm{R}_{H_1} x\| \quad \text{(by (vii)(a), (vii)(b), and Fact 2.4)} \\ &= 2\|2\,\mathrm{P}_{H_2} \mathrm{P}_{H_1} x - \mathrm{P}_{H_2} x - 2\,\mathrm{P}_{H_1} x + x\| \\ &= 2\left| \frac{\eta_2 - \langle x, u_2 \rangle}{\|u_2\|} + 2\frac{\eta_1 - \langle x, u_1 \rangle}{\|u_1\|} \right| \text{(Fact 2.3, [16, Lemma 2.11(i)], and } u_2 = -\tfrac{\|u_2\|}{\|u_1\|} u_1\text{)} \\ &= 2\left| \frac{\eta_2}{\|u_2\|} + 2\frac{\eta_1}{\|u_1\|} - \left\langle x, \frac{u_1}{\|u_1\|} \right\rangle \right| \quad \text{(by } u_2 = -\tfrac{\|u_2\|}{\|u_1\|} u_1\text{)} \end{aligned}$$

$$= 2\left|\frac{\eta_2}{\|u_2\|} + \frac{\eta_1}{\|u_1\|} + \frac{\eta_1 - \langle x, u_1\rangle}{\|u_1\|}\right| = 2\frac{\langle x, u_1\rangle - \eta_1}{\|u_1\|} - 2\Big(\frac{\eta_1}{\|u_1\|} + \frac{\eta_2}{\|u_2\|}\Big) \quad \text{(by (35))}$$
$$= 2\|x - \mathrm{P}_{H_1} x\| - 2\|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2} x\| \quad \text{(by (vii)(a), (34a) and (v))}$$
$$\leq 2\|x - \mathrm{P}_{H_1} x\| = \|x - \mathrm{R}_{W_1} x\|.$$

Therefore, (vii)(c) holds.

(viii): Let $x \in \mathbb{N}$ and $(\forall i \in \{1,2\})$ $y_i \in H_i$. Bearing Fact 2.2 in mind, we have

$$\begin{aligned} \mathrm{R}_{H_i} x - x &= 2(\mathrm{P}_{y_i + \mathrm{par}\, H_i}(x) - x) = -2\,\mathrm{P}_{(\mathrm{par}\, H_i)^\perp}(x - y_i) \in (\mathrm{par}\, H_i)^\perp \\ &= (\ker u_i)^\perp = \mathrm{span}\,\{u_i\}; \end{aligned} \tag{36a}$$
$$\begin{aligned} \mathrm{R}_{H_2}\mathrm{R}_{H_1} x - \mathrm{R}_{H_1} x &= 2(\mathrm{P}_{y_2 + \mathrm{par}\, H_2}(\mathrm{R}_{H_1} x) - \mathrm{R}_{H_1} x) \\ &= -2\,\mathrm{P}_{(\mathrm{par}\, H_2)^\perp}(\mathrm{R}_{H_1} x - y_2) \in \mathrm{span}\,\{u_2\}. \end{aligned} \tag{36b}$$

Recall that u_1 and u_2 are linear dependent. Hence, $x, \mathrm{R}_{H_1} x, \mathrm{R}_{H_2} x$ are affinely dependent, and so do $x, \mathrm{R}_{H_1} x, \mathrm{R}_{H_2}\mathrm{R}_{H_1} x$.

In addition, if $x \in W_1 \cup W_2$, then $\mathrm{R}_{W_1} x = x$ or $\mathrm{R}_{W_2} x = x$, which clearly forces that $x, \mathrm{R}_{W_1} x, \mathrm{R}_{W_2} x$ are affinely dependent. If $x \notin W_1 \cup W_2$, then, by Fact 2.4, $(\forall i \in \{1,2\})$ $\mathrm{R}_{W_i} x = \mathrm{R}_{H_i} x$. Hence, by (36a), $x, \mathrm{R}_{W_1} x, \mathrm{R}_{W_2} x$ are affinely dependent.

Similarly, in view of Fact 2.4 and (36b), $x, \mathrm{R}_{W_1} x, \mathrm{R}_{W_2}\mathrm{R}_{W_1} x$ are affinely dependent. □

Proof of Lemma 5.11

Proof. (i): Because $x \notin W_1$, by Fact 2.4, we derive that

$$\mathrm{R}_{W_1} x = 2\,\mathrm{P}_{W_1} x - x = x + 2\tfrac{\eta_1 - \langle x, u_1\rangle}{\|u_1\|^2} u_1.$$

Then
$$\begin{aligned} \mathrm{R}_{W_1} x \in W_2 &\Leftrightarrow \Big\langle x + 2\frac{\eta_1 - \langle x, u_1\rangle}{\|u_1\|^2} u_1, u_2 \Big\rangle \leq \eta_2 \\ &\Leftrightarrow (\langle x, u_2\rangle - \eta_2)\|u_1\|^2 - 2(\langle x, u_1\rangle - \eta_1)\langle u_1, u_2\rangle \leq 0. \end{aligned}$$

(ii): As a consequence of (20) and $x \in W_2$, $\langle x, u_2\rangle - \eta_2 \leq 0$. Moreover, using $\mathrm{R}_{W_1} x \in W_2$ and (20), we see that

$$\begin{aligned} \langle \mathrm{R}_{W_1} x, u_2\rangle - \eta_2 \leq 0 &\Leftrightarrow \langle 2\,\mathrm{P}_{W_1} x - x, u_2\rangle - \eta_2 \leq 0 \\ &\Rightarrow \langle \mathrm{P}_{W_1} x, u_2\rangle - \eta_2 \leq \tfrac{1}{2}(\langle x, u_2\rangle - \eta_2) \leq 0, \end{aligned}$$

which implies that $\mathrm{P}_{W_1} x \in W_2$. Clearly, $\mathrm{P}_{W_1} x \in W_1$. Hence, $\mathrm{P}_{W_1} x \in W_1 \cap W_2$.

(iii): Because $x \in W_1 \cap W_2^c$, by (20), we know that $\langle x, u_1\rangle \leq \eta_1$ and $\langle x, u_2\rangle > \eta_2$. Combine these inequalities with $\langle u_1, u_2\rangle \geq 0$ to see that

$$\langle x, u_1\rangle - \eta_1 + \frac{\eta_2 - \langle x, u_2\rangle}{\|u_2\|^2}\langle u_1, u_2\rangle \leq 0$$

and
$$\langle x, u_1\rangle - \eta_1 + 2\frac{\eta_2 - \langle x, u_2\rangle}{\|u_2\|^2}\langle u_1, u_2\rangle \leq 0. \tag{37}$$

According to Fact 2.4,

$$\begin{aligned}\langle \mathrm{P}_{W_2} x, u_1\rangle - \eta_1 &= \left\langle x + \frac{\eta_2 - \langle x, u_2\rangle}{\|u_2\|^2} u_2, u_1 \right\rangle - \eta_1 \\ &= \langle x, u_1\rangle - \eta_1 + \frac{\eta_2 - \langle x, u_2\rangle}{\|u_2\|^2}\langle u_1, u_2\rangle \overset{(37)}{\leq} 0\end{aligned}$$

and
$$\begin{aligned}\langle \mathrm{R}_{W_2} x, u_1\rangle - \eta_1 &= \left\langle x + 2\frac{\eta_2 - \langle x, u_2\rangle}{\|u_2\|^2} u_2, u_1 \right\rangle - \eta_1 \\ &= \langle x, u_1\rangle - \eta_1 + 2\frac{\eta_2 - \langle x, u_2\rangle}{\|u_2\|^2}\langle u_1, u_2\rangle \overset{(37)}{\leq} 0,\end{aligned}$$

which imply that $\mathrm{P}_{W_2} x = \mathrm{P}_{H_2} x \in W_1$ and $\mathrm{R}_{W_2} x \in W_1$, respectively. In addition, because $x \notin W_2$, by Lemma 2.5, $\mathrm{R}_{W_2} x \in \operatorname{int} W_2$.

(iv): Note that, by (20), $x \in W_1^c \cap W_2^c$ means that $\langle x, u_1\rangle - \eta_1 > 0$ and $\langle x, u_2\rangle - \eta_2 > 0$. Combine these two inequalities with $\langle u_1, u_2\rangle \leq 0$ to see that

$$\langle x, u_2\rangle - \eta_2 + 2\frac{\eta_1 - \langle x, u_1\rangle}{\|u_1\|^2}\langle u_1, u_2\rangle > 0. \tag{38}$$

Because $x \notin W_1$, by Fact 2.4, $\mathrm{R}_{W_1} x = 2\,\mathrm{P}_{H_1} x - x = x + 2\frac{\eta_1 - \langle x, u_1\rangle}{\|u_1\|^2} u_1$. Now

$$\begin{aligned}\langle \mathrm{R}_{W_1} x, u_2\rangle - \eta_2 &= \left\langle x + 2\frac{\eta_1 - \langle x, u_1\rangle}{\|u_1\|^2} u_1, u_2 \right\rangle - \eta_2 \\ &= \langle x, u_2\rangle - \eta_2 + 2\frac{\eta_1 - \langle x, u_1\rangle}{\|u_1\|^2}\langle u_1, u_2\rangle \overset{(38)}{>} 0,\end{aligned}$$

which entails that $\mathrm{R}_{W_1} x \notin W_2$.

(v): On account of the assumption $\mathrm{P}_{H_1} x \notin W_2$ and Fact 2.3 we have

$$\langle \mathrm{P}_{H_1} x, u_2\rangle > \eta_2 \Leftrightarrow (\eta_2 - \langle x, u_2\rangle)\|u_1\|^2 - (\eta_1 - \langle x, u_1\rangle)\langle u_1, u_2\rangle < 0. \tag{39}$$

According to the formula of $\mathrm{P}_{H_2}\mathrm{P}_{H_1} x$ in [16, Lemma 2.11(i)] we obtain

$$\begin{aligned}&\langle \mathrm{P}_{H_2}\mathrm{P}_{H_1} x, u_1\rangle - \eta_1 \\ &= \langle x, u_1\rangle - \eta_1 + \frac{\eta_1 - \langle x, u_1\rangle}{\|u_1\|^2}\langle u_1, u_1\rangle \\ &\qquad + \frac{1}{\|u_1\|\|u_2\|}\Big((\eta_2 - \langle x, u_2\rangle)\|u_1\|^2 - (\eta_1 - \langle x, u_1\rangle)\langle u_1, u_2\rangle\Big)\langle u_1, u_2\rangle \\ &= \frac{1}{\|u_1\|\|u_2\|}\Big((\eta_2 - \langle x, u_2\rangle)\|u_1\|^2 - (\eta_1 - \langle x, u_1\rangle)\langle u_1, u_2\rangle\Big)\langle u_1, u_2\rangle < 0, \\ &\qquad\qquad (\text{by (39) and } \langle u_1, u_2\rangle > 0)\end{aligned}$$

which, by (20) and Fact 2.4, guarantees that $\mathrm{P}_{W_2}\mathrm{P}_{H_1} x = \mathrm{P}_{H_2}\mathrm{P}_{H_1} x \in W_1$. Clearly, $\mathrm{P}_{H_2}\mathrm{P}_{H_1} x \in H_2 \subseteq W_2$. Hence, $\mathrm{P}_{H_2}\mathrm{P}_{H_1} x \in W_1 \cap W_2$.

(vi): Note that $x \notin W_1$ and $\langle u_1, u_2\rangle < 0$ ensure $\frac{\eta_1 - \langle x, u_1\rangle}{\|u_1\|^2}\langle u_1, u_2\rangle > 0$. On the other hand, using (20), Fact 2.3, and $x \in H_2$, we obtain that

$$\begin{aligned}\mathrm{R}_{H_1} x \notin W_2 &\Leftrightarrow \langle \mathrm{R}_{H_1} x, u_2\rangle > \eta_2 \Leftrightarrow \left\langle x + 2\frac{\eta_1 - \langle x, u_1\rangle}{\|u_1\|^2} u_1, u_2 \right\rangle > \eta_2 \\ &\Leftrightarrow 2\frac{\eta_1 - \langle x, u_1\rangle}{\|u_1\|^2}\langle u_1, u_2\rangle > 0.\end{aligned}$$

Hence, (vi) is true. □

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