Journal of Convex Analysis
Volume 29 (2022), No. 3, 921–928

Approximation of Spherical Bodies of Constant Width and Reduced Bodies

Marek Lassak
University of Science and Technology, Bydgoszcz, Poland
lassak@pbs.edu.pl

Received: June 15, 2021
Accepted: December 27, 2021

We present a spherical version of the theorem of Blaschke that every body of constant width $w < \pi/2$ can be approximated by a body of constant width w whose boundary consists only of pieces of circles of radius w as well as we wish in the sense of the Hausdorff distance. This is a special case of our theorem about approximation of spherical reduced bodies.

Keywords: Sphere, spherical convex body, body of constant width, reduced body, Hausdorff distance, approximation.

2010 Mathematics Subject Classification: 52A55.

1. Introduction

A theorem of Blaschke says that for every convex body of constant width w in the Euclidean plane E^2 and every $\varepsilon > 0$ there exists a convex body of constant width w whose boundary consists only of pieces of circles of radius w such that the Hausdorff distance between the two bodies is at most ε (see [1] and also §65 of [2]). A generalization of this fact for normed planes is given in [6], where also the approximation of reduced bodies is considered. A corollary at the end of this note presents an analog of this theorem for bodies of constant width on the sphere S^2, while Theorem 4.1 from Section 4 of the present paper gives a general version for reduced convex bodies on S^2.

The method of the proof of Theorem 4.1 in Section 4 is similar to the proofs from [1] (by Blaschke) and [6]. In order to facilitate a comparison of the present proof with Theorem 3.3 from [6] for normed planes we use in Section 4 a notation similar to that from [6]. In the present paper we need a number of lemmas and claims for the spherical situation. They are given in Sections 2 and 3.

2. Auxiliary facts from spherical geometry

Let S^2 be the unit sphere of the three-dimensional Euclidean space E^3. By a *great circle* we mean the intersection of S^2 with any two-dimensional subspace of E^3. By a pair of *antipodes* we mean any pair of points obtained as the intersection of S^2 with a one-dimensional subspace of E^3. Observe that if two different points a, b are not antipodes, then there is exactly one great circle containing them. By the *arc of great circle*, or shortly *arc*, ab connecting them we understand the shorter part of

ISSN 0944-6532 / \$ 2.50

the great circle between a and b. By the *distance* $|ab|$, of these points we mean the length of ab.

The set of points of S^2 whose distance from a point $c \in S^2$ equals (is at most) $r \leq \frac{\pi}{2}$ is called the *circle* (respectively, *disk*) *of radius* r *and center* c. Disks of radius $\frac{\pi}{2}$ are called *hemispheres*. If p belongs to a great circle, then the set of points which are at distances at most $\frac{\pi}{2}$ from p is called a *semicircle*. We say that p is the *center* of this semicircle.

A subset of S^2 is called *convex* if it does not contain any pair of antipodes and if together with every two points it contains the arc connecting them. For a set A contained in the interior of a hemisphere we denote by $\mathrm{conv}(A)$ the smallest convex set containing A. By *a convex body* we mean a closed convex set with non-empty interior. Let p be a boundary point of a set A contained in the interior of a hemisphere. We say that a hemisphere K containing A *supports* it if $A \cap \mathrm{bd}(K)$ is non-empty. If $p \in A \cap K$, we say that K supports A at p.

Lemma 2.1. *Let $A \subset S^2$ be a closed set with non-empty interior. If at every boundary point the set A is supported by a hemisphere, then A is convex.*

Proof. The proof is analogous to the proof of Theorem 9 of [3]. Namely, suppose that A is not convex. Then there are points $x_1, x_2 \in A$ and a point $y \in x_1x_2$ such that $y \notin A$. There exists an interior point z of A such that $z \notin x_1x_2$. Hence there is $x_f \in \mathrm{bd}(A) \cap yz$. Every great circle through x_f separates one of the points x_1, x_2, z from at least one other of these points. So there is no supporting hemisphere of A through x_f. A contradiction. □

Recall a few notions and facts from [7]. We say that e is an *extreme point* of C provided the set $C \setminus \{e\}$ is convex. If hemispheres G and H are different and their centers are not antipodes, then $L = G \cap H$ is called a *lune*. The semicircles bounding L and contained respectively in $\mathrm{bd}(G)$ and $\mathrm{bd}(H)$ are denoted by G/H and H/G. The *thickness* $\Delta(L)$ of L is defined as the distance of the centers of G/H and H/G. For every hemisphere K supporting a convex body C we find hemispheres K^* supporting C such that the lunes $K \cap K^*$ are of the minimum thickness. At least one such a K^* exists. By *the width* $\mathrm{width}_K(C)$ *of* C *determined by* K we mean the thickness of the lune $K \cap K^*$. It changes continuously, as the position of K changes. From Part I of Theorem 1 of [7] we know that such K^* is unique if $\mathrm{width}_K(C) < \frac{\pi}{2}$.

When we go on $\mathrm{bd}(C)$, then always counterclockwise. Let us recall, but in a slightly different form than in [10], that if hemispheres X, Y, Z support a convex body C in this order, then we write $\prec XYZ$. Let X and Z be hemispheres supporting C at p and let $\preceq XYZ$ for every hemisphere Y supporting C at p. Then X is said to be the *right supporting* hemisphere at p and Z is said to be the *left supporting* hemisphere at p.

By the *thickness* $\Delta(C)$ of C we mean the minimum of $\mathrm{width}_K(C)$ over all supporting hemispheres K of C.

Assume that a lune L of thickness $\Delta C)$ contains a convex body C. Then by Claim 2 of [7] both the centers of the semicircles bounding L belong to $\mathrm{bd}(C)$. We call such a lune L a *supporting lune of* C. We say that the lune *supports* C *at* the mentioned centers. The arc connecting the centers of the semicircles bounding L is called a *thickness chord of* C. This notion is an analog of the notion of a thickness chord of a convex body in E^d considered by many authors under this name or without any name, as for instance in [2].

If for all hemispheres K supporting C the numbers $\mathrm{width}_K(C)$ are equal, we say that C is *of constant width.* Of course, a convex body $C \subset S^2$ with $\Delta(C) < \frac{\pi}{2}$ is of constant width if and only if for every hemisphere K supporting L the lune $K \cap K^*$ has thickness $\Delta(C)$.

Lemma 2.2. *Consider a non-degenerate spherical triangle $abd \subset S^2$ and a point $e \in ab$ such that de is orthogonal to ab. The distance between d and ab is at most*

$$\arcsin \frac{\tan \frac{1}{2}|ab|}{\tan \frac{1}{2}\angle adb} .$$

Proof. Let e' be the center of ab and d' be such that $|d'e'| = |de|$ with $d'e'$ orthogonal to ab. Clearly, $\angle e'd'a = \frac{1}{2}\angle ad'b$ and $|ae'| = \frac{1}{2}|ab|$. Since $ae'd'$ is a right triangle, $\sin |d'e'| \cdot \tan \frac{1}{2}\angle adb = \tan \frac{1}{2}|ab|$ and our thesis for d' instead of d is true. By $|d'e'| = |de|$ we get the thesis for our d. □

Here is a spherical version of the classic Blaschke selection theorem.

Claim 2.3. *From every sequence of convex bodies on S^d of thickness at most a fixed constant smaller than π we may select a subsequence which tends to a spherical convex body.*

Proof. First we select a subsequence which is contained in a certain spherical disk of a radius below π. Then from this subsequence we select the final subsequence (see [7] p. 558 and [5]). □

3. Reduced bodies on sphere

After [7] we say that a spherical convex body R is *reduced* if $\Delta(Z) < \Delta(R)$ for every convex body $Z \subset R$ different from R. For basic properties of reduced bodies on S^2 see [10]. The class of spherical reduced bodies is larger than the class of bodies of constant width. Earlier many papers considered reduced bodies in the Euclidean and normed d-dimensional spaces; see the survey articles [8] and [9].

Later we tacitly assume that all considered reduced bodies $R \subset S^2$ are of thickness below $\frac{\pi}{2}$.

Theorem 3.1 of [10] presents the boundary structure of a reduced body $R \subset S^2$. Namely, *assume that M_1, M_2 are supporting hemispheres of R with the property* $\mathrm{width}_{M_1}(R) = \Delta(R) = \mathrm{width}_{M_2}(R)$ *such that* $\mathrm{width}_M(R) > \Delta(R)$ *for every M fulfilling $\prec M_1MM_2$. Consider the lunes $L_1 = M_1 \cap M_1^*$ and $L_2 = M_2 \cap M_2^*$. Then the arcs a_1a_2 and b_1b_2 are in* $\mathrm{bd}(R)$, *where a_i is the center of M_i/M_i^* and b_i is the center of M_i^*/M_i for $i = 1, 2$.*

Denote by c the intersection point of the arcs a_1a_2 and b_1b_2. The union of the triangles a_1a_2c and b_1b_2c is called a *butterfly*, while a_1a_2 and b_1b_2 its *arms*.

Consider any maximum piece $\widehat{gh}$ of the boundary of R which does not contain any arc Let K be any hemisphere supporting R at a point of $\widehat{gh}$. But at g we agree only for the right, and at h we agree only for the left. By Part I of Theorem 1 of [7] for every K there exists exactly one hemisphere K^* supporting R such that the lune $K \cap K^*$ has thickness $width_K(R)$. From this theorem we also conclude that K^* touches R at a unique point. In particular, for the right hemisphere supporting R at g denote this unique point by g', and for the left hemisphere supporting R at h by h'.

Claim 3.1. *For any hemisphere K supporting R at a point of $\widehat{gh}$, we have the relation* $\mathrm{width}_K(R) = \Delta(R)$.

Proof. Clearly, $\mathrm{width}_K(R)$ cannot be smaller than $\Delta(R)$, because then we get a contradiction with the definition of the thickness of R.

Suppose that $\mathrm{width}_K(R) > \Delta(R)$. Then by continuity arguments (see Theorem 2 of [7]) we obtain that if K supports R at a point different from g and h, then there are supporting hemispheres K_1 and K_2 with $\prec K_1KK_2$ such that for every supporting hemisphere H fulfilling $\prec K_1HK_2$ we have $\Delta(H) > \Delta(R)$. By the just recalled Theorem 3.1 of [10], the piece $\widehat{gh}$ of the boundary of R contains an arc. A contradiction with the assumption on $\widehat{gh}$ from the paragraph preceding our claim. If K supports R at g or h, then similarly we show that $\widehat{gh}$ contains an arc whose one end-point is g or h, which again gives a contradiction with the choice of $\widehat{gh}$.

By the two preceding paragraphs we get $\mathrm{width}_K(R) = \Delta(R)$. □

From this claim and Lemma 2.2 of [10] we see that all the points at which all our hemispheres K^* touch R form a curve $\widehat{g'h'}$ being a piece of $\mathrm{bd}(R)$. We call it the curve *opposite to the curve* $\widehat{gh}$. Vice-versa, from this lemma we obtain that $\widehat{g'h'}$ determines $\widehat{gh}$.

So we say that $\widehat{gh}$ and $\widehat{g'h'}$ is a *pair of opposite curves of constant width* $\Delta(R)$.

Let us summarize the above consideration as the following claim analogous to Corollary 11 of [4] in the situation of a normed plane.

Claim 3.2. *The boundary of a reduced body $R \subset S^2$ consists of countably many pairs of arms of butterflies and of countably many pairs of opposite pairs of curves of constant width $\Delta(R)$.*

Clearly any particular pair of opposite pieces of curves of constant width $\Delta(R)$ consists of end-points of thickness chords; they are the centers of the semicircles bounding the lunes of thickness $\Delta(R)$ supporting R at the points of these pieces of curves of constant width.

In particular, when R from Claim 3 is a body of constant width, then its boundary is the union of one pair of curves of constant width $\Delta(R)$. We may present $\mathrm{bd}(R)$ on infinitely many ways as such an union. Each time the curves end at the end-points of a thickness chord of R.

4. Approximation of reduced bodies

Theorem 4.1. *Let $R \subset S^2$ be a reduced body of thickness below $\frac{\pi}{2}$. For any $\varepsilon > 0$ there exists a reduced body $R_\varepsilon \subset S^2$ with $\Delta(R_\varepsilon) = \Delta(R)$ whose boundary consists only of arms of butterflies and arcs of circles of radius $\Delta(R)$ such that the Hausdorff distance between R_ε and R is at most ε.*

Proof. We omit the trivial case when the boundary of R consists only of arms of butterflies.

Take any positive $\varepsilon < \frac{1}{2}\pi$. Put $\rho_\varepsilon = 2 \cdot \arctan(\sin \varepsilon)$.

Consider any pair F, G of opposite curves of constant width $\Delta(R)$ in the boundary of R. Exceptionally, when R is a body of constant width, we divide $\mathrm{bd}(R)$ into a pair of curves F and G by an arbitrary thickness chord. Denote the endpoints of F by f' and f'', and the endpoints of G by g' and g'', in both cases according to the positive orientation.

Part A. The aim of this part is to construct the set R_ε.

From Claim 3.2 we know that $\mathrm{bd}(R)$ contains countably many pairs of opposite pieces of curves of constant width $\Delta(R)$. For each such a pair F, G we provide a number of different thickness chords $f_1g_1, \dots, f_ng_n$ of R such that $f_1, \dots, f_n \in F$ (with $f_1 = f'$ and $f_n = f''$), and $g_1, \dots, g_n \in G$ (with $g_1 = g'$ and $g_n = g''$), taking care that $|f_if_{i+1}|$ and $|g_ig_{i+1}|$ be below ε for $i = 1, \dots, n-1$, and that the positively oriented angle between every two successive chords is below $\frac{\pi}{2}$. Clearly, some of points $f_1, \dots, f_n$ (some of $g_1, \dots, g_n$) may coincide.

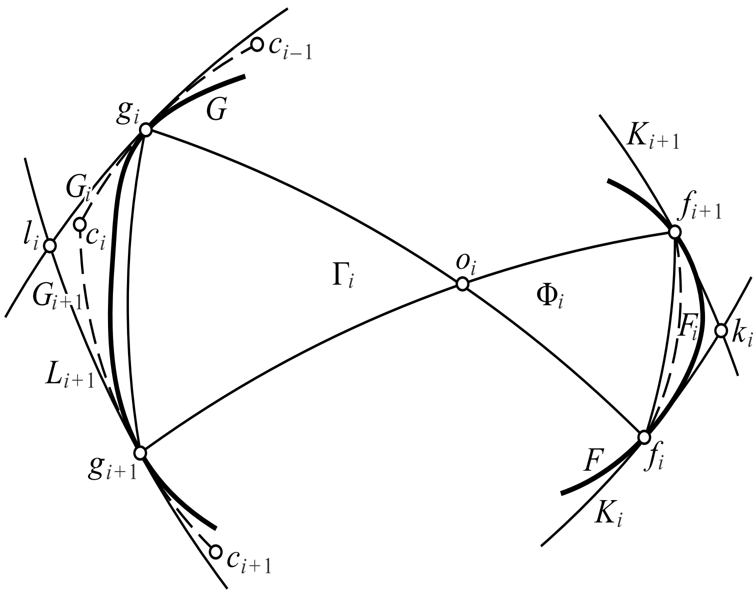

Figure: Illustration to the proof of Theorem 4.1

Denote by o_i the intersection point of f_ig_i and $f_{i+1}g_{i+1}$ for $i = 1, \dots, n-1$ (see Figure). Moreover, denote by Φ_i (respectively, Γ_i) the angle between the rays from o_i through f_i and f_{i+1} (respectively, through g_i and g_{i+1}) for $i = 1, \dots, n-1$. Let c_i be the point of intersection of circles of radius $\Delta(R)$ with centers f_i and f_{i+1} (so c_i is in equal distances from f_i and f_{i+1}). Such c_i exists since, thanks to Claim 3.1, we have $|f_i g_{i+1}| \le \Delta(R)$ and $|f_{i+1} g_i| \le \Delta(R)$. Moreover, by c_0 we mean g', and by c_n we mean g''.

For every $i \in \{1, \dots, n-1\}$ take the arc F_i of the circle F_i° of radius $\Delta(R)$ with center c_i and endpoints f_i and f_{i+1} which is in Φ_i. Moreover, for every $i \in \{1, \dots, n\}$ take the arc G_i of the circle G_i° of radius $\Delta(R)$ with center f_i which begins at c_{i-1} and ends at c_i. Created arcs are marked by broken lines in Figure. Clearly, $G_1 \subset \Gamma_1$, $G_i \subset \Gamma_{i-1} \cup \Gamma_i$ for $i = 2, \dots, n-1$, and $G_n \subset \Gamma_{n-1}$. We have constructed the pair of curves $F^* = F_1 \cup \dots \cup F_{n-1}$ and $G^* = G_1 \cup \dots \cup G_n$.

Denote by U_ε the closure of the union of all arms of the butterflies of R and of all pairs of curves of the form F^* and G^*. We see that U_ε is obtained from $\mathrm{bd}(R)$ by exchanging all pairs of opposite curves F and G by the constructed pairs of curves F^* and G^*.

We define R_ε as the set bounded by U_ε.

Part B. We define a sequence of sets $R^j \subset S^2$ and show that they are convex bodies.

We define $R_0, R_1, \dots$ by induction.

Put $R^0 = R$. Clearly it is a convex body.

Assume that R^{j-1} is a convex body, where $j \ge 1$. We get the boundary of R^j by exchanging a pair of opposite curves F, G, if any remains, from $\mathrm{bd}(R^{j-1})$ into a pair F^*, G^* as in Part A.

At every boundary point p, the set R^j is supported by a hemisphere. If $p \in F^* \cup G^*$ this follows from the construction of F^* and G^*. If $p \in \mathrm{bd}(R^j)$ does not belong to any curves F^* or G^*, then in the part of the supporting hemisphere take this supporting R^{j-1}. From Lemma 2.1 we conclude that R^j is convex. Clearly, R^j is a convex body.

Part C. We show that R_ε is a convex body and $\Delta(R_\varepsilon) = \Delta(R)$.

If after a finite number of steps in Part B we get R_ε, we see that it is a convex body.

In the opposite case, $R_\varepsilon = \lim_{j\to\infty} R^j$. By Claim 2.3 we see that R_ε is also a convex body.

From the construction of R^j we see that for every supporting hemisphere K of R^j at every point of $F^* \cup G^*$ we have $\mathrm{width}_K = \Delta(R)$. The remaining part of $\mathrm{bd}(R^j)$ does not differ from $\mathrm{bd}(R^{j-1})$. So by induction we get $\Delta(R^j) = \Delta(R)$. Thus if after a finite number of steps we obtain R_ε, it has thickness $\Delta(R)$. By Lemma 4 of [7], if $R_\varepsilon = \lim_{j\to\infty} R^j$, then the same is true.

Part D. Let us prove that R_ε is a reduced body.

We should show that for any convex body $Z \subset R_\varepsilon$ different from R_ε the inequality $\Delta(Z) < \Delta(R_\varepsilon)$ holds true. The body Z does not contain an extreme point e of R_ε as it follows from the spherical analog of the Krein-Milman theorem formulated on p. 565 of [7]. Consequently, Z is disjoint with an open disk D centered at e. Now exactly as in Part 4 of [6] we show that $\Delta(Z) < \Delta(R_\varepsilon)$, so that R_ε is a reduced body.

Part E. We show that every F^* and every G^* are in the union of some triangles $f_ik_if_{i+1}$ and $g_il_ig_{i+1}$

Take a pair of curves F^* and G^* constructed in Part A. The bounding semicircle of the first lune supporting R_ε at f_i is denoted by K_i, and that at g_i by L_i (again see Figure).

For $i \in \{1, \dots, n-1\}$, by k_i denote the point of intersection of K_i with K_{i+1} (if they are subsets of a great circle, take k_i as the center of f_if_{i+1}). By l_i denote the point of intersection of L_i with L_{i+1} (if they are subsets of a great circle, take l_i as the center of g_ig_{i+1}).

For every $c \in G_i$ take the lune supporting R_ε such that cf_i is the thickness chord and denote by $T(c)$ the bounding semicircle of this lune through f_i. In particular, $T(g_i) = K_i$. When we move $c \in G_i$ counterclockwise from g_i to c_i, the lune and thus also $T(c)$ "rotate" counterclockwise. So since the distance between c_i and any point of $T(c_i)$ is at least $\Delta(R)$ we see that the distance from c_i to every point of the arc f_ik_i is at least $\Delta(R)$. Analogously, the distance from c_i to any point of $f_{i+1}k_i$ is at least $\Delta(R)$. So since every point of F_i is at the distance $\Delta(R)$ from c_i, we get $F_i \subset f_ik_if_{i+1}$. Thus F^* is contained in the union of triangles $f_ik_if_{i+1}$, where $i = 1, \dots, n$. Similarly, G^* is in the union of triangles $g_il_ig_{i+1}$, where $i = 1, \dots, n$.

Part F. We show that the Hausdorff distance between R and R_ε is at most ε.

Denote by P the closure of the convex hull of all points f_i and g_i and of endpoints of all arms of butterflies of R. Denote by Q the union of P and all triangles $f_ik_if_{i+1}$ and $g_il_ig_{i+1}$.

Part E implies inclusions $P \subset R \subset Q$ and $P \subset R_\varepsilon \subset Q$. So in order to estimate the Hausdorff distance between R and R_ε by ε it is sufficient to estimate the Hausdorff distance between P and Q by ε. Since $P \subset Q$, the Hausdorff distance between them is $\sup_{q \in Q} \inf_{p \in P} |pq|$. Hence it is sufficient to show that all (i.e., for all pairs F, G and all i) distances between k_i and f_if_{i+1}, and also between l_i and g_ig_{i+1}, are at most ε.

Since the sum of angles of any quadrangle $o_if_ik_if_{i+1}$ is over 2π, from the assumption in Part A on the angle between two successive chords below $\frac{\pi}{2}$, we get $\angle f_io_if_{i+1} < \frac{\pi}{2}$. So $\angle f_ik_if_{i+1} > 2\pi - 3 \cdot \frac{\pi}{2} = \frac{\pi}{2}$. Hence $\tan \frac{1}{2} \angle f_ik_if_{i+1} > 1$. By Lemma 2.2 the distance from k_i to f_if_{i+1} is at most

$$\arcsin \frac{\tan \frac{1}{2}|f_if_{i+1}|}{\tan \frac{1}{2}\angle f_ik_if_{i+1}} < \arcsin \tan \frac{1}{2}|f_if_{i+1}|.$$

By $|f_if_{i+1}| \le \rho_\varepsilon$ (see Part A) and $\rho_\varepsilon = 2 \cdot \arctan(\sin \varepsilon)$ we get that it is at most

$$\arcsin \tan \frac{\rho_\varepsilon}{2} = \arcsin \tan \frac{2 \cdot \arctan(\sin \varepsilon)}{2} = \varepsilon.$$

Analogously, the distance between l_i and g_ig_{i+1} is at most ε. We see that the Hausdorff distance between P and Q, and thus between R and R_ε, is at most ε. □

If R is a body of constant width, from Parts A and B of this proof we see that R_ε (so R^1 in this case) is also a body of constant width. So we get the following corollary.

Corollary 4.2. *For every body $W \subset S^2$ of constant width and for arbitrary $\varepsilon > 0$ there exists a body $W_\varepsilon \subset S^2$ of constant width $\Delta(W_\varepsilon) = \Delta(W)$ whose boundary consists only of arcs of circles of radius $\Delta(W)$, such that the Hausdorff distance between W and W_ε is at most ε.*

Applications for Barbier's theorem in Part 4 of [6] do not hold true here for the sphere.

Let us correct misprints in this Part 4 on p. 873: three times 2π should be changed into π.

References

[1] W. Blaschke: *Konvexe Bereiche gegebener konstanter Breite und kleinsten Inhalts*, Math. Ann. 76 (1915) 504–513.

[2] T. Bonnesen, W. Fenchel: *Theorie der konvexen Körper*, Springer, Berlin (1934); English translation: *Theory of Convex Bodies*, BCS Associated, Moscow, Idaho USA (1987).

[3] H. G. Eggleston: *Convexity*, Cambridge University Press, Cambridge (1958).

[4] E. Fabińska, M. Lassak: *Reduced bodies in normed spaces*, Israel J. Math. 161 (2007) 75–88.

[5] N. N. Hai, P. T. Ann: *A generalization of Blaschke's converegence theorem in metric space*, J. Convex Analysis 20 (2013) 1013–1024.

[6] M. Lassak: *Approximation of bodies of constant width and reduced bodies in a normed plane*, J. Convex Analysis 19 (2012) 865–874.

[7] M. Lassak: *Width of spherical convex bodies*, Aeq. Math. 89 (2015) 555–567.

[8] M. Lassak, H. Martini: *Reduced convex bodies in Euclidean space – a survey*, Expo. Math. 29/2 (2011) 204–219.

[9] M. Lassak, H. Martini: *Reduced convex bodies in finite-dimensional normed spaces – a survey*, Results Math. 66/3-4 (2014) 405–426.

[10] M. Lassak, M. Musielak: *Reduced spherical convex bodies*, Bull. Pol. Acad. Sci., Math. 66 (2018) 87–97.