

Unique Minimizers and the Representation of Convex Envelopes in Locally Convex Vector Spaces

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It is well known that a strictly convex minimand admits at most one minimizer. We prove a partial converse: Let X be a locally convex Hausdorff space and $f: X \rightarrow (-\infty, \infty]$ a function with compact sublevel sets and exhibiting some mildly superlinear growth. Then each tilted minimization problem

$$\min_{x \in X} f(x) - \langle x', x \rangle_X \quad (1)$$

admits at most one minimizer as x' ranges over $\text{dom}(\partial f^*)$ if and only if the biconjugate f^{**} is essentially strictly convex and agrees with f at all points where f^{**} is subdifferentiable. We prove this via a representation formula for f^{**} that might be of independent interest.

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1. Introduction

The minimizer of a strictly convex function f is unique since for any minimizers $x_0 \neq x_1$ holds

$$f(\lambda x_1 + (1 - \lambda)x_0) < \lambda f(x_1) + (1 - \lambda)f(x_0) = \inf f \quad \forall \lambda \in (0, 1).$$

Using subdifferential calculus, a slightly refined uniqueness criterion may be derived requiring merely that f be essentially strictly convex, i.e. proper, convex and strictly convex on each line segment contained in $\text{dom}(\partial f)$. The simplicity of these considerations tempts to conjecture that more elaborate general uniqueness criteria for minimizers might exist. As far as the inhomogeneous problem (1) is concerned, this turns out to be wrong in the following precise sense: In order to have a decent existence theory for (1), it seems reasonable to require that linear perturbations of $f: X \rightarrow (-\infty, \infty]$ have compact sublevel sets. Under this condition, we shall prove that the tilted minimization problem (1) admits at most one minimizer for each $x' \in \text{dom}(\partial f^*)$ if and only if f agrees with its biconjugate f^{**} on $\text{dom}(f^{**})$, which is then essentially strictly convex. This implies $f = f^{**}$ if X is a Banach space. Therefore essential strict convexity is sufficient and necessary for uniqueness of minimizers in (1). An interesting consequence is that a possible failure of uniqueness in the pertaining inhomogeneous inclusion

$$x' \in \partial f(x) \quad (2)$$

cannot be restored by employing global minimality in (1) as a selection criterion.

The essential auxiliary tool in our proof will be a representation formula for the biconjugate f^{**} that we prove beforehand. As is well known, there already exist formulas relating $f^{**}(x)$ for $x \in X$ with f via

$$f^{**}(x) = \liminf_{y \rightarrow x} \inf \left\{ \sum_{k=1}^N \lambda_k f(y_k) \mid N \in \mathbb{N}, \sum_{k=1}^N \lambda_k = 1, \lambda_k \geq 0, \sum_{k=1}^N \lambda_k y_k = y \right\}.$$

Our new contribution consists of identifying general sufficient conditions under which the limit may be omitted and the infimum attains. This question has already been investigated for $X = \mathbb{R}^d$, where Carathéodory's theorem bounds the number of points that contribute meaningfully to a convex combination. Obviously, this no longer works if $\dim X = \infty$. We solve this problem by permitting probability measures as continuous convex combinations. The representation result thus obtained will allow simple rigorous proofs of several intuitive relationships between f and f^{**} , from which our main result will eventually follow. We consider it likely that the representation formula has applications beyond the present setting and therefore might be of independent interest.

We do not know a result resembling our main theorem except [5, Thm. 1], where a related result is proved in the particular case of a reflexive Banach space in its weak topology. Our method of proof differs strongly. After our main result will have been proved, we shall obtain [5, Thm. 1] as a corollary.

Remark on notation: Throughout, T is a topological space with Borel σ -algebra $\mathcal{B}(T)$ and $\text{Pr}(T)$ are the Borel probability measures on T . For $t \in T$, let δ_t be the Dirac-measure supported at t . Let V be a (topological) vector space. We denote by V' its topological dual space. If $M \subset V$, then $\text{co } M$ and $\overline{\text{co}} M$ are the convex hull and closed convex hull of M . Similarly, for a function $f: V \rightarrow [-\infty, \infty]$, $\text{co } f$ and $\overline{\text{co}} f$ are the (closed) convex envelope of f , i.e. the largest (lower semi-continuous) convex function below f . For subsets $A \subset B$ of a fixed superset B , we write $A^c = B \setminus A$. The symbol χ_A denotes the function with value 1 on A and 0 elsewhere, I_A is the function with value 0 on A and ∞ on A^c . We denote by ∂f the Fenchel-Moreau subdifferential of convex analysis. Moreover

$$\text{dom}(f) = \{v \in V \mid f(v) \in \mathbb{R}\} \text{ and } \text{dom}(\partial f) = \{v \in V \mid \partial f(v) \neq \emptyset\}.$$

We say that f is essentially strictly convex iff f is proper, convex everywhere and strictly convex on each convex subset of $\text{dom}(\partial f)$. This is the notion of essential strict convexity introduced in [7]. We caution the reader that we never mean essential strict convexity in the sense of [1] unless explicitly stated.

2. Biconjugate representation

In this section, we shall obtain the announced representation formula for the biconjugate. Its proof will require a lower semi-continuity result for integral functionals with respect to the weak convergence of measures, to be applied when T is a non-metrizable locally convex Hausdorff space. As this result is usually only proved for when T is a metric space (via Lipschitz regularization of the integrand), we first briefly point out why this holds for every topological space. We refer the reader to [2, Ch. 4] for basic definitions.

Proposition 2.1. *Let $\psi: T \rightarrow (-\infty, \infty]$ be lower semi-continuous and bounded below. Consider the integral functional*

$$I_\psi: \text{Pr}(T) \rightarrow (-\infty, \infty] : \mu \mapsto \int \psi d\mu.$$

If a net $\mu_\alpha \in \text{Pr}(T)$ converges weakly to a $\mu \in \text{Pr}(T)$ that is τ -additive (e.g. is Radon), then

$$\liminf_{\alpha} I_\psi(\mu_\alpha) \geq I_\psi(\mu).$$

Proof. We will obtain our claim by expressing I_ψ as a supremum of functionals fulfilling the same lower semi-continuity property. For $m \in \mathbb{N}$, consider the approximant

$$\psi_m(x) = \min\{\psi(x), m\},$$

which is bounded and lower semi-continuous hence induces an integral functional for which the desired property holds by [2, Cor. 4.3.4]. Clearly, $\psi_m \uparrow \psi$ as $m \uparrow \infty$ so that monotone convergence implies

$$\sup_{m \in \mathbb{N}} \int \psi_m d\mu = \int \psi d\mu \quad \text{for every } \mu \in \text{Pr}(T). \quad \square$$

We are now prepared to prove

Lemma 2.2. *Let X be a locally convex Hausdorff space and $f: X \rightarrow (-\infty, \infty]$ a function with (closed) compact sublevel sets. We have*

$$f^{**}(x) = \liminf_{y \rightarrow x} \inf \left\{ \int f d\mu \mid \mu \in \text{Pr}(X), \int \omega d\mu(\omega) = y \right\} \quad \forall x \in X. \quad (3)$$

The expectation in (3) is to be understood as a Pettis integral, cf. [8, Def. 3.26]. If $\text{dom}(f^) = X'$ or if $f^{**}(x) = \inf f$ and $\text{dom}(f^*)$ contains a balanced absorbent subset, then (3) simplifies to*

$$f^{**}(x) = \min \left\{ \int f d\mu \mid \mu \in \text{Pr}(X), \int \omega d\mu(\omega) = x \right\}. \quad (4)$$

*At least one minimizer in (4) is then a Radon measure. A μ_x minimizing in (4) is said to originate f^{**} at x .*

Remark: The assumption $f^{**}(x) = \inf f$ is less restrictive than it might seem since for $x' \in X'$ holds $(f + x')^{**} = (f^* - x')^* = f^{**} + x'$.

Proof. Clearly, we may assume that f is proper. We start by checking that the right-hand side in (3) and hence also in (4) is not less than $f^{**}(x)$. Let $\mu_y \in \text{Pr}(X)$ with expectation y . Since f has a non-empty compact sublevel set by properness, it assumes a finite global minimum value. Therefore we may assume $f \geq 0$ so that $f^{**} = \overline{\text{co}}f$ is a convex, lower semi-continuous, proper function and the Jensen inequality yields

$$f^{**}(y) = f^{**} \left(\int \omega d\mu_y(\omega) \right) \leq \int f^{**} d\mu_y \leq \int f d\mu_y. \quad (5)$$

Taking the infimum over such μ_y and sending $y \rightarrow x$ yields

$$f^{**}(x) \leq \liminf_{y \rightarrow x} \inf \left\{ \int f d\mu \mid \mu \in \text{Pr}(X), \int \omega d\mu(\omega) = y \right\}.$$

For proving the converse inequality we may assume

$$f^{**}(x) < \infty. \quad (6)$$

By [4, Thm. 4.84 and Thm. 4.92(iii)] holds

$$\begin{aligned} f^{**}(x) &= \liminf_{y \rightarrow x} \text{co } f(y) \\ &= \liminf_{y \rightarrow x} \inf \left\{ \sum_{k=1}^N \lambda_k f(y_k) \mid N \in \mathbb{N}, \sum_{k=1}^N \lambda_k = 1, \lambda_k \geq 0, \sum_{k=1}^N \lambda_k y_k = y \right\} \\ &\geq \liminf_{y \rightarrow x} \inf \left\{ \int f d\mu \mid \mu \in \text{Pr}(X), \int \omega d\mu(\omega) = y \right\} \end{aligned}$$

so that (3) has been proved. In the last step we used that

$$\sum_{k=1}^N \lambda_k f(y_k) = \int f(\omega) d \sum_{k=1}^N \lambda_k \delta_{y_k}(\omega).$$

Regarding the remaining claims, we need to deduce more precise information from the representation

$$f^{**}(x) = \liminf_{y \rightarrow x} \inf \left\{ \sum_{k=1}^N \lambda_k f(y_k) \mid N \in \mathbb{N}, \sum_{k=1}^N \lambda_k = 1, \lambda_k \geq 0, \sum_{k=1}^N \lambda_k y_k = y \right\}. \quad (7)$$

Let $\mathcal{N}(x)$ be the neighbourhood filter of x . Remember that in a general topological space, the lower limit is defined via

$$\liminf_{y \rightarrow x} \text{co } f(y) = \sup_{U \in \mathcal{N}(x)} \inf_U \text{co } f.$$

Hence, we find a sequence $U_n \in \mathcal{N}(x)$ such that $a_n = \inf_{U_n} \text{co } f$ has $f^{**}(x) = \sup_n a_n = \lim_n a_n$. Let $(V_i)_{i \in I} \subset \mathcal{N}(x)$ be a base of neighbourhoods directed by inclusion. For any $\alpha = (n, i) \in \mathbb{N} \times I =: A$, we may by (7) find a convex combination μ_α of Dirac measures

$$\mu_\alpha = \sum_{k=1}^{N(\alpha)} \lambda_k^\alpha \delta_{y_k^\alpha}; \quad y^\alpha = \sum_{k=1}^{N(\alpha)} \lambda_k^\alpha y_k^\alpha \in U_n \cap V_i$$

$$\text{with} \quad \frac{1}{n} + f^{**}(x) \geq \int f d\mu_\alpha \geq a_n. \quad (8)$$

This defines a net μ_α of discrete probability measures with

- (i) $\lim_{\alpha \in A} y^\alpha = x$;
- (ii) $f^{**}(x) = \lim_{\alpha \in A} \int f d\mu_\alpha$.

If μ_α has a weakly convergent subnet $\mu_\beta \rightharpoonup \mu_x$ whose limit has expectation x , then we can conclude that μ_x originates f^{**} at x and hence (4) holds as follows:

From Proposition 2.1 and (5) we then have

$$f^{**}(x) = \lim_{\beta \in J} \int f d\mu_{\beta} \geq \int f d\mu_x \geq f^{**}(x).$$

This uses that f has closed sublevel sets and hence is lower semi-continuous. Therefore it remains to show that (a) μ_{α} admits a weakly convergent subnet (b) whose limit has expectation x . Compactness of the sublevel sets $S_r = \{y \in X \mid f(y) \leq r\}$ together with $f \geq 0$ and (6) yields the uniform tightness estimate

$$r \sup_{\alpha} \mu_{\alpha}(S_r^c) \leq \sup_{\alpha} \int f d\mu_{\alpha} \leq f^{**}(x) < \infty \quad \forall r > 0. \quad (9)$$

By (9) we may invoke [2, Thm. 4.5.3] to deduce existence of a convergent subnet $\mu_{\beta} \rightharpoonup \mu_x$ weakly in $\text{Pr}(Y)$ with μ_x a Radon measure, hence τ -additive. Regarding the expectation of μ_x , consider first the case $f^{**}(x) = \inf f$. In this case we have from Proposition 2.1 and (ii) that

$$\inf f = f^{**}(x) = \lim_{\beta \in J} \int f d\mu_{\beta} \geq \int f d\mu_x \geq \inf f.$$

Consequently

$$\lim_{\beta \in J} \int f d\mu_{\beta} = \int f d\mu_x. \quad (10)$$

Let $U' \subset \text{dom}(f^*)$ be a balanced absorbent subset and $u' \in U'$. By Proposition 2.1 and the lower bound $f - u' \geq -f^*(u')$ holds

$$\liminf_{\beta \in J} \int f - u' d\mu_{\beta} \geq \int f - u' d\mu_x. \quad (11)$$

As Proposition 2.1 implies $\int f d\mu_x \leq \liminf_{\beta \in J} \int f d\mu_{\beta}$, the terms $\int f d\mu_x$ and $\liminf_{\beta \in J} \int f d\mu_{\beta}$ are finite by (8). Moreover, $\lim_{\beta \in J} \int u' d\mu_{\beta}$ exists by (i). Therefore we may equivalently rearrange (11) to obtain

$$\liminf_{\beta \in J} \int f d(\mu_{\beta} - \mu_x) \geq \lim_{\beta \in J} \int u' d(\mu_{\beta} - \mu_x).$$

In particular $0 \geq \lim_{\beta \in J} \int u' d(\mu_{\beta} - \mu_x)$ by (10) so that absorbency of U' implies $u'(x) = \lim_{\beta \in J} \int u' d\mu_{\beta} = \int u' d\mu_x$ for all $u' \in X'$, i.e. μ_x has expectation x in the sense of Pettis' integral. Finally, consider the case $\text{dom}(f^*) = X'$. Arguing as before we obtain

$$\liminf_{\beta \in J} \int f d(\mu_{\beta} - \mu_x) \geq \limsup_{\beta \in J} \int x' d(\mu_{\beta} - \mu_x) \quad \forall x' \in X'. \quad (12)$$

The upper bound in (12) being finite, this is impossible unless

$$\lim_{\beta \in J} \int x' d\mu_{\beta} = \int x' d\mu_x \quad \text{for all } x' \in X'. \quad \square$$

A very intuitive consequence of Lemma 2.2 is

Corollary 2.3. *For f as in Lemma 2.2 with $\text{dom}(f^*)$ containing a balanced absorbent subset holds*

$$\overline{\text{co}} \underset{x \in X}{\text{Argmin}} f(x) = \underset{x \in X}{\text{Argmin}} f^{**}(x). \quad (13)$$

Proof. $\subseteq$: As f^{**} is convex and lower semi-continuous, the assertion follows from $\inf f = \inf f^{**}$.

$\supseteq$: For $x \in \underset{x \in X}{\text{Argmin}} f^{**}(x)$ exists $\mu_x \in \text{Pr}(X)$ originating f^{**} at x by Lemma 2.2.

We have
$$\int f d\mu_x = f^{**}(x) = \inf f^{**} = \inf f$$

so that μ_x is concentrated on $\underset{x \in X}{\text{Argmin}} f(x)$ and therefore

$$x = \int \omega d\mu_x(\omega) \in \overline{\text{co}} \underset{x \in X}{\text{Argmin}} f(x). \quad \square$$

The next lemma is not a corollary to Lemma 2.2, but nevertheless adds to the utility of Lemma 2.2 by elucidating its consequences.

Lemma 2.4. *Let $f: X \rightarrow (-\infty, \infty]$ be Borel measurable and let $\mu_x \in \text{Pr}(X)$ originate f^{**} at x . For any sequence ℓ_n of affine continuous functions with $\ell_n \leq f^{**}$ and $\lim_{n \rightarrow \infty} \ell_n(x) = f^{**}(x)$, the measure μ_x is concentrated on the set*

$$A = \left\{ a \in X \mid f(a) = f^{**}(a) \text{ and } \lim_n \ell_n(a) = f^{**}(a) \right\}$$

and f^{**} is affine on $\text{co } A$. Moreover

$$f^{**}(a) = f^{**}(x) + \langle x', a - x \rangle \quad \forall a \in \overline{\text{co}} A$$

if ℓ_n is a constant sequence $\ell(x) = f^{**}(x) + \langle x', x - x \rangle$ with $x' \in \partial f^{**}(x)$.

Proof. For any choice of ℓ_n we have $\mu_x(A) = 1$ since

$$\int f^{**} d\mu_x \leq \int f d\mu_x = f^{**}(x) = \lim_n \ell_n(x) = \lim_n \int \ell_n d\mu_x \leq \int f^{**} d\mu_x.$$

Affinity follows by convexity of f^{**} and since for $a_0, a_1 \in A$ and $\lambda \in (0, 1)$ holds

$$\lambda f^{**}(a_1) + (1 - \lambda) f^{**}(a_0) = \lim_n \ell_n(\lambda a_1 + (1 - \lambda) a_0) \leq f^{**}(\lambda a_1 + (1 - \lambda) a_0).$$

The last claim follows by taking ℓ_n as the constant sequence

$$\ell: a \rightarrow f^{**}(x) + \langle x', a - x \rangle$$

for $x' \in \partial f^{**}(x)$ and using that $\{f^{**} = \ell\} = \{f^{**} \leq \ell\}$ is closed by lower semi-continuity. $\square$

3. The main theorem

The stage has been set for

Theorem 3.1. *Let X be a locally convex Hausdorff space and $f: X \rightarrow (-\infty, \infty]$ a function such that, for each $x' \in \text{dom}(\partial f^*)$, the tilted function $f - x'$ has compact sublevel sets and $\text{dom}(f^*) - x'$ contains a balanced absorbent subset. The following are equivalent:*

- (i) *For all $x' \in X'$ exists at most one $\bar{x} \in \text{Argmin}_{x \in X} f(x) - \langle x', x \rangle_X$.*
- (ii) *f^{**} is essentially strictly convex and agrees with f on $\text{dom}(\partial f^{**})$.*

Remark: If X besides its locally convex topology σ carries a Banach space topology τ such that $X'_\sigma = X'_\tau$, then (ii) implies that f and f^{**} agree globally. This follows from the Brøndsted-Rockafellar theorem [3, Thm. 2]: If $X'_\sigma = X'_\tau$, then the σ -subgradients and τ -subgradients of f^{**} coincide. By [3, Thm. 2] one may reconstruct a convex, lower semi-continuous, proper function f on a Banach space as the lower semi-continuous envelope of the function

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in \text{dom}(\partial f); \\ \infty & \text{else;} \end{cases}$$

so that then $f \geq f^{**}$, $f^{**}(x) = f(x)$ for all $x \in \text{dom}(\partial f^{**})$ and lower semi-continuity of f together imply $f = f^{**}$.

Proof. (i) $\implies$ (ii): A function is essentially strictly convex iff it is convex everywhere and is not affine on any line segment where it is subdifferentiable. We argue by contradiction. Let

$$[x, y] \subset \text{dom}(\partial f^{**}) \text{ with } z = \frac{x+y}{2}$$

and suppose f^{**} were affine on $[x, y]$ and pick $z' \in \partial f^{**}(\frac{x+y}{2})$. Definition of the subdifferential and affinity yield

$$f^{**}(y) \geq f^{**}(z) + \langle z', y - z \rangle = \frac{1}{2} [f^{**}(x) + f^{**}(y) + \langle z', y - x \rangle].$$

Consequently $f^{**}(y) \geq f^{**}(x) + \langle z', y - x \rangle$

so that

$$\begin{aligned} f^{**}(a) &\geq f^{**}(z) + \langle z', a - z \rangle \geq \frac{1}{2} [f^{**}(x) + f^{**}(y)] + \langle z', a - z \rangle \\ &\geq f^{**}(x) + \langle z', a - x \rangle \quad \forall a \in X. \end{aligned}$$

In total $z' \in \partial f^{**}(x)$. As $f^{**} - z'$ has at most one minimizer by (i) and Corollary 2.3, we find $[x, y] = \{z\}$ whence essential strict convexity follows.

Second, we prove that f and f^{**} agree on $\text{dom}(\partial f^{**})$. Let $x \in \text{dom}(\partial f^{**})$. By the Fenchel-Young identity, one has $x' \in \text{dom}(\partial f^*)$ for all $x' \in \partial f^{**}(x)$. Therefore our assumption implies that $\text{dom}(f^*) - x'$ contains a balanced absorbent set.

Applying Lemma 2.2 to the function $f - x'$, we find $\mu_x \in \text{Pr}(X)$ originating f^{**} at x . By Lemma 2.4 there exists A such that $\mu_x(A) = 1$ and

$$f^{**}(y) = f^{**}(x) + \langle x', y - x \rangle \quad \forall y \in \overline{\text{co}}A.$$

Therefore

$$x' \in \partial f^{**}(y) \quad \forall y \in \overline{\text{co}}A.$$

As $f^{**} - x'$ has at most one minimizer by Corollary 2.3, essential strict convexity of f^{**} implies that $\overline{\text{co}}A = \{x\}$ whence $f^{**}(x) = f(x)$ for all $x \in \text{dom}(\partial f^{**})$ follows.

(ii) $\implies$ (i): Let $\bar{x}_0, \bar{x}_1 \in \text{Argmin}_{x \in X} f - x'$. From $(f - x')^{**} = f^{**} - x'$ follows $\bar{x}_0, \bar{x}_1 \in \text{Argmin}_{x \in X} f^{**} - x'$ by Corollary 2.3. Hence $\bar{x}_0 = \bar{x}_1$ by essential strict convexity of f^{**} . $\square$

We conclude our investigations by keeping the promise of demonstrating how [5, Thm. 1] follows from Theorem 3.1. We start with a

Proposition 3.2. *Let X be a reflexive Banach space, $J: X \rightarrow (-\infty, \infty]$ a weakly lower semi-continuous function and $MJ: X' \rightrightarrows X: x' \rightarrow \text{Argmin}_{x \in X} J(x) - \langle x', x \rangle$. If J is essentially strictly convex, then*

$$\begin{cases} \text{dom } MJ = \text{dom}(\partial J^*); \\ MJ \text{ is single-valued on its domain.} \end{cases} \quad (14)$$

If $\text{dom}(\partial J^*) = \text{int dom}(\partial J^*) \neq \emptyset$, the converse is true as well.

Proof. $\implies$: If J is essentially strictly convex, then MJ is single valued as explained in the introduction. Convexity of J implies $x \in MJ(x') \iff x' \in \partial J(x) \iff x \in \partial J^*(x')$ so that $MJ = \partial J^*$.

$\impliedby$: Let $\overline{B}_\varepsilon(x') \subset \text{dom}(\partial J^*)$. Then, as in the proof of [5, Prop. 1], we see that since J^* must be continuous at x' , there exist $r, \alpha > 0$ such that

$$J^* \leq I_{\overline{B}_r(x')} + \alpha.$$

Taking convex conjugates of this inequality, we get

$$J \geq J^{**} \geq x'_0 + r \|\cdot\|_X - \alpha.$$

Consequently, the function $x \rightarrow G(x) = J(x) - \langle x', x \rangle$ is coercive, i.e. its sublevel sets are (weakly) compact. As $G^*(y') = J^*(y' - x')$ we may apply Theorem 3.1 to conclude that $J = \overline{\text{co}}J$ is essentially strictly convex. $\square$

In [5], the function J is said to be adequate if, in addition to (14), the set $\text{dom}(\partial J^*)$ is non-empty and open. Moreover, J is essentially strictly convex in the sense of [1, 5] if in addition to J being essentially strictly convex in the sense of [7] the function MJ is locally bounded on its domain. We can now obtain [5, Thm. 1] as a particular case of Theorem 3.1:

Theorem 3.3. *Under the assumptions of Proposition 3.2, J is adequate in the sense of [1, 5] iff J is essentially strictly convex in the sense of [1, 5].*

Proof. $\implies$: Proposition 3.2 implies that J is essentially strictly convex. To prove that J also is essentially strictly convex in the sense of [1, 5], it suffices to observe that $MJ = \partial J^*$ by the Fenchel-Young identity. Now the statement follows as the maximal monotone operator ∂J^* is locally bounded on its open domain.

$\impliedby$: J being essentially strictly convex, Proposition 3.2 yields (14). Moreover, we have $MJ = \partial J^*$. Since $\text{dom}(J^*)$ is convex, the closure of $\text{dom}(\partial J^*)$ is convex by the Brøndsted-Rockafellar theorem [3, Thm. 2] so that by [6, Remarks on Ch. 2] each point where ∂J^* is locally bounded belongs to $\text{int dom}(\partial J^*)$. Since we assume ∂J^* to be locally bounded, it follows that $\text{dom}(\partial J^*)$ is open. Since J is proper, so is J^* and hence $\text{dom}(\partial J^*)$ is non-empty. $\square$

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