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Periodic Problems for Doubly Nonlinear Evolution Equations

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We are concerned with the time-periodic problem of some doubly nonlinear equations governed by differentials of two convex functionals over uniformly convex Banach spaces. G. Akagi and U. Stefanelli [*Weighted energy-dissipation functionals for doubly nonlinear evolution*, J. Funct. Analysis 260/9 (2011) 2541–2578] considered the Cauchy problem of the same equation via the so-called WED functional approach. The main purpose of this paper is to show the existence of the time-periodic solution under the same growth conditions on functionals and differentials as those imposed in Akagi-Stefanelli [loc.cit]. Because of the difference in nature between the Cauchy problem and the periodic problem, we can not apply the WED functional approach directly, so we here adopt standard compactness methods with suitable approximation procedures.

Keywords: Doubly nonlinear evolution equation, time-periodic problem, subdifferential, p-Laplacian, doubly nonlinear parabolic equation.

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1. Introduction

Let V be a uniformly convex real Banach space and V^* be its uniformly convex dual. In this paper, we are concerned with the following time-periodic problem of a doubly nonlinear evolution equation:

$$\text{(AP)} \qquad \begin{cases} d\psi(u'(t)) + \partial\phi(u(t)) \ni f(t), & t \in (0,T) \quad \text{in } V^*, \\ u(0) = u(T), \end{cases}$$

where $d\psi$ and $\partial\phi$ are the Gâteaux differential and the subdifferential of convex functionals ψ and ϕ which are mappings from V into $(-\infty, +\infty]$. Here and henceforth, u' denotes the time derivative of u and f is a given external force belonging to

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$L^{p'}(0,T;V^*)$, where $p' := p/(p-1)$ and $p \in (1,\infty)$ (precise definitions and assumptions will be given in the next section). A typical example which can be reduced to (AP) is given by the following doubly nonlinear parabolic equation:

$$\alpha(u'(x,t)) - \Delta_m u(x,t) = f(x,t), \tag{1}$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is a non-decreasing function and $\Delta_m u := \nabla \cdot \left(|\nabla u|^{m-2}\nabla u\right)$ (so-called m-Laplacian).

The Cauchy problem for (AP) in Hilbert spaces has been studied by Barbu [13], where the elliptic regularization technique by adding the term $-\varepsilon(d_V\psi(u'))'$ is employed and by Arai [10], Senba [42], Colli-Visintin [21] via the standard relaxation procedure with the additional term $\varepsilon u'$. Stefanelli [45] discussed a variational characterization of the solutions to gradient systems relying on the Brézis-Ekeland principle. Investigation of global solvability in Banach spaces began with Colli [20], where the time discretization and the polygonal chain approximation are exploited. In Akagi-Stefanelli [5], the authors adopted another approach based on the fact that the solution to the Cauchy problem for (AP) with elliptic regularization can be characterized by the minimizer of some functional with the weight $\exp(-t/\varepsilon)$, the so-called Weighted Energy-Dissipation (WED, for short) functional (see also [3][4][8][23][36][37][38] and references therein). By this procedure, they proved the existence of global solutions to the Cauchy problem assuming some growth conditions on ψ and ϕ (see Remark 2.8 below).

We here comment on the following variant type of doubly nonlinear equation, studied more vigorously than (AP):

$$(Au(t))' + Bu(t) \ni f(t), \tag{2}$$

where A and B are maximal monotone operators. The Cauchy problem of (2) is investigated in, e.g., [24][28][29][35]. Akagi-Stefanelli attempt the WED functional approach to (2) in [7]. Moreover, the papers [9][16][27][31][32][40][41][47][48] are devoted to the initial boundary value problem of specific parabolic PDEs such as $\partial_t u - \Delta_m u^p = f$ and $\partial_t u^p - \Delta_m u = f$.

Compared with the Cauchy problem, there are only few results on the existence of time-periodic solutions to doubly nonlinear equations. The periodic problem for (2) is considered in [1][30][33][34] and concrete PDEs of the same type as (2) in [25][26][46][49][50][51]. However, to the best of our knowledge, the investigation into other types of equations different from (2) can be found only in Akagi-Stefanelli [6]. They consider the solvability and the structural stability of the periodic problem of the following abstract equation:

$$A(u'(t)) + \partial\phi(u(t)) \ni f(t), \tag{3}$$

where A is a possibly multi-valued maximal monotone operator.

In [6], they restrict their discussion to the Hilbert space framework and impose the linear growth condition on A. These conditions seem to be more restrictive than those imposed for the Cauchy problem in [5] when $A = d\psi$. The main purpose of this paper is to show that we can discuss the periodic problem for (AP) under almost the same growth conditions as those in [5]. In the next section, we fix several notations, present a few auxiliary tools, and state our main results more precisely.

Section 3 is devoted to our proofs. In Section 4, we show that estimates established in the previous section are immediately applicable to the study for some structural stability of (AP). Finally, we exemplify the applicability of our setting by dealing with the doubly nonlinear parabolic equation (1).

Some parts of our arguments for estimates and convergences rely on [5]. However, our main strategy seems to be indispensable in order to cope with difficulties arising from the difference in nature between the Cauchy problem and the time-periodic problem, for which one would perceive that it is hard to find a suitable variational structure to apply the WED functional approach by [5]. So here we depart from the WED functional setting and adopt the standard compactness method with a suitable approximation procedure (see $(\mathrm{AP})_\varepsilon$ in Theorem 2.6 given later). In establishing a priori estimates for solutions of approximate equations, the coercivity of $\partial\phi$ to be assumed plays an essential role, since $u(0)$ is an unknown value which may possibly depend on approximation parameters for the periodic problem.

In proving our main results, we first deal with the easier case where ϕ dominates ψ in Sections 3.2–3.4 and the other case will be treated in Section 3.5.

2. Main Results

2.1. Preliminary

We first fix some notations and recall basic properties (see [14][15][18]), which will be used later.

Let E be a real Banach space and E^* be its dual. The norms of E and E^* are denoted by $|\cdot|_E$ and $|\cdot|_{E^*}$, respectively, and the duality pairing by $\langle u^*, u\rangle_E$, where $[u, u^*] \in E \times E^*$. We define the duality mapping F_E of E by

$$F_E(u) := \left\{u^* \in E^*;\ \langle u^*, u\rangle_E = |u|_E^2 = |u^*|_{E^*}^2\right\}.$$

By virtue of the Hahn-Banach theorem, $F_E(u)$ is non-empty for every $u \in E$. If E^* is strictly convex, then F_E is a single-valued demi-continuous mapping. We also know the following (see [39]):

Proposition 2.1. *A Banach Space E is uniformly convex if and only if for each $R > 0$ there exists a non-decreasing function $m_R : [0,\infty) \to [0,\infty)$ which satisfies $m_R(0) = 0$, $m_R(\rho) > 0$ for any $\rho > 0$, and*

$$\langle u_1^* - u_2^*, u_1 - u_2\rangle_E \geq m_R(|u_1 - u_2|_E)|u_1 - u_2|_E,$$

for any $u_1, u_2 \in B_R$ and $u_1^* \in F_E(u_1)$, $u_2^* \in F_E(u_2)$, where $B_R := \{u \in E;\ |u|_E \leq R\}$.

Let A be a (possibly) multivalued operator from E into 2^{E^*}(the power set of E^*) and we often identify A with its graph $G(A)$, more precisely, we write $[u, u^*] \in A$ if $u \in D(A) := \{\, v \in E\,;\ Av \neq \varnothing \,\}$ and $u^* \in Au$. An operator A is said to be *monotone* if

$$\langle u_1^* - u_2^*, u_1 - u_2\rangle_E \geq 0 \quad \forall [u_i, u_i^*] \in A \quad (i = 1, 2)$$

and a monotone operator A is said to be *maximal monotone* if there is no monotone operator which contains A properly. If E and E^* are reflexive and strictly convex, maximality of A is equivalent to $R(F_E + A) = E^*$ (see also Lemma A.2 below).

Let $\phi : E \to (-\infty, +\infty]$ be a proper (i.e., $\phi \not\equiv +\infty$) lower semi-continuous (l.s.c., for short) and convex functional. The set $D(\phi) := \{u \in E;\ \phi(u) < +\infty\}$ is called the effective domain of ϕ and the subdifferential operator $\partial\phi$ from E into 2^{E^*} is defined by

$$\partial\phi : u \mapsto \{\eta \in E^*;\ \phi(v) \geq \phi(u) + \langle \eta, v - u \rangle_E\ \ \forall v \in D(\phi)\}. \tag{4}$$

Then $\partial\phi$ becomes a maximal monotone operator from E into 2^{E^*} (see, e.g., Theorem 2.8 of [15]). Here we recall the following fundamental feature of closedness of maximal monotone operators (see Lemma 1.2 in [19]).

Proposition 2.2. *Let E be a real Banach space and $A : E \to 2^{E^*}$ be a maximal monotone operator. Assume that the sequences $\{u_n\}_{n\in\mathbb{N}} \subset E$ and $\{v_n\}_{n\in\mathbb{N}} \subset E^*$ satisfy $u_n \rightharpoonup u$ weakly in E, $v_n \rightharpoonup v$ weakly in E^*, $[u_n, v_n] \in A$, and either*

$$\limsup_{n,m\to\infty} \langle v_n - v_m, u_n - u_m \rangle_E \leq 0$$

or

$$\limsup_{n\to\infty} \langle v_n, u_n \rangle_E \leq \langle v, u \rangle_E .$$

Then $[u, v] \in A$ and $\langle v_n, u_n \rangle_E \to \langle v, u \rangle_E$ as $n \to \infty$.

Let Z be another Banach space which is densely embedded in E and $D(\phi) \subset Z$. Then we can consider the restriction of ϕ onto Z, denoted by ϕ_Z and regard ϕ_Z as a proper l.s.c. convex functional on Z. To distinguish between the two subdifferentials over E and Z, we write

$$\begin{aligned}\partial_E\phi(u) &:= \partial\phi(u) = \{\eta \in E^*;\ \phi(v) \geq \phi(u) + \langle \eta, v - u \rangle_E\ \ \forall v \in D(\phi)\},\\ \partial_Z\phi_Z(u) &:= \{\eta \in Z^*;\ \phi_Z(v) \geq \phi_Z(u) + \langle \eta, v - u \rangle_Z\ \ \forall v \in D(\phi_Z)\}.\end{aligned}$$

In general, we have $\partial_E\phi \subset \partial_Z\phi_Z$. More precisely it is shown that (see Proposition 2.1 in [5])

$$\begin{aligned}D(\partial_E\phi) &= \{\, u \in D(\partial_Z\phi_Z)\, ;\, \partial_Z\phi_Z(u) \cap E^* \neq \varnothing \,\},\\ \partial_E\phi(u) &= \partial_Z\phi_Z(u) \cap E^* \quad \forall u \in D(\partial_E\phi).\end{aligned} \tag{5}$$

We next define another functional derivative. Let ψ be a functional defined over E. Then ψ is said to be *Gâteaux differentiable* at $u \in E$ if there exists $\xi \in E^*$ such that

$$\lim_{h\to 0} \frac{\psi(u + he) - \psi(u)}{h} = \langle \xi, e \rangle_E \quad \forall e \in E.$$

We call ξ the *Gâteaux derivative* of ψ at u and write $\xi = d_E\psi(u)$. When there exists $d_E\psi(u)$ for every $u \in E$, ψ is said to be *Gâteaux differentiable over E*. It is well known that if ψ is proper l.s.c. convex and Gâteaux differentiable at u, then the Gâteaux derivative $d_E\psi(u)$ coincides with the subdifferential $\partial_E\psi(u)$ (i.e., $\partial_E\psi(u) = \{d_E\psi(u)\}$ holds).

For a proper l.s.c. convex functional ϕ on E, we define

$$\phi_\lambda(u) := \inf_{v\in E} \left\{ \frac{|u - v|_E^2}{2\lambda} + \phi(v) \right\} = \frac{|u - J_\lambda u|_E^2}{2\lambda} + \phi(J_\lambda u) \quad \lambda > 0,\ u \in E,$$

where $J_\lambda : E \to E$ stands for the resolvent of $\partial_E\phi$, i.e., $J_\lambda u$ is the unique solution of $F_E(J_\lambda u - u) + \lambda\partial_E\phi(J_\lambda u) \ni 0$.

This functional ϕ_λ, called the *Moreau-Yosida regularization* of ϕ, is convex, continuous, and Gâteaux differentiable over E. We can show that the Gâteaux differential $d_E\phi_\lambda$ coincides with the Yosida approximation $(\partial_E\phi)_\lambda$ of the subdifferential $\partial_E\phi$, which is defined by

$$(\partial_E\phi)_\lambda u := -\lambda^{-1}F_E(J_\lambda u - u).$$

Moreover, we have $D(\phi_\lambda) = E$ and for any $u \in D(\phi)$

$$d_E\phi_\lambda(u) \in \partial_E\phi(J_\lambda u), \quad \phi(J_\lambda u) \leq \phi_\lambda(u) \leq \phi(u), \quad \lim_{\lambda\to+0}\phi_\lambda(u) = \phi(u). \tag{6}$$

We here present some useful tools, the chain rule and a formula concerning the integration by parts.

Proposition 2.3. (see Lemma 4.1 of [20]) *Let E be a strictly convex reflexive Banach space and $\phi : E \to (-\infty, +\infty]$ be a proper lower semi-continuous convex function. Assume that $u \in W^{1,p}(0,T;E)$ with $p \in (1,\infty)$, $\eta \in L^{p'}(0,T;E^*)$, and $\eta(t) \in \partial\phi(u(t))$ a.e. $t \in (0,T)$. Then $t \mapsto \phi(u(t))$ is absolutely continuous on $[0,T]$ and*

$$\frac{d}{dt}\phi(u(t)) = \langle \eta(t), u'(t)\rangle_E \qquad \text{for a.e. } t \in (0,T).$$

Proposition 2.4. (see Proposition 2.3 of [5]) *Let $m, p \in (1,\infty)$ and E, Z be reflexive Banach spaces such that $Z \hookrightarrow E$ and $E^* \hookrightarrow Z^*$ are dense. Moreover, let $u \in L^m(0,T;Z) \cap W^{1,p}(0,T;E)$, $\xi \in L^{p'}(0,T;E^*)$, and its derivative of distribution ξ' belong to $L^{m'}(0,T;Z^*) + L^{p'}(0,T;E^*)$. Provided that $t_1, t_2 \in (0,T)$ are Lebesgue points of the function $t \mapsto \langle \xi(t), u(t)\rangle_E$, then we have*

$$\langle \xi', u\rangle_{L^m(t_1,t_2;Z)\cap L^p(t_1,t_2;E)} = \langle \xi(t_2), u(t_2)\rangle_E - \langle \xi(t_1), u(t_1)\rangle_E - \langle \xi, u'\rangle_{L^p(t_1,t_2;E)}.$$

2.2. Assumptions

In order to formulate our results, we introduce a suitable Banach space V and its subspace X.

(A.0) Both V and its dual V^* are uniformly convex real Banach spaces and X is a subspace of V such that X and its dual X^* are real reflexive Banach spaces. Furthermore the embeddings $X \hookrightarrow V$ and $V^* \hookrightarrow X^*$ hold with densely defined compact canonical injections.

We also assume the following growth conditions:

(A.1) The functional $\phi : V \to (-\infty, +\infty]$ is proper lower semi-continuous convex and $\psi : V \to (-\infty, +\infty)$ is convex and Gâteaux differentiable over V. Assume that there exists a constant $C > 0$ satisfying

$$|u|_V^p \leq C(\psi(u) + 1) \quad \forall u \in V, \tag{7}$$

$$|d_V\psi(u)|_{V^*}^{p'} \leq C(|u|_V^p + 1) \quad \forall u \in V, \tag{8}$$

$$|u|_X^m \leq C(\phi(u) + 1) \quad \forall u \in D(\phi), \tag{9}$$

$$|\eta|_{X^*}^{m'} \leq C(|u|_X^m + 1) \quad \forall\, [u, \eta] \in \partial_X\phi_X, \tag{10}$$

with some exponents $p, m \in (1,\infty)$, where $p' := p/(p-1)$ and $m' := m/(m-1)$ (the Hölder conjugate exponents).

Remark 2.5. (i) From (7) and (9), we assume that $\phi \geq 0$ and $\psi \geq 0$ without loss of generality henceforth (replace ϕ and ψ with $\phi + 1$ and $\psi + 1$, respectively).

(ii) Combining (7) with (8), we can immediately see that

$$|d_V\psi(u)|_{V^*}^{p'} \leq C(\psi(u) + 1) \quad \forall u \in V. \tag{11}$$

Moreover, since $\psi(u) \leq \psi(0) + \langle d_V\psi(u), u\rangle_V$ holds by the definition of the subdifferential, (8) leads to

$$\psi(u) \leq C(|u|_V^p + 1) \qquad \forall u \in V, \tag{12}$$

$$|u|_V^p \leq C(|d_V\psi(u)|_{V^*}^{p'} + 1) \qquad \forall u \in V. \tag{13}$$

From (9) and (10), we can derive analogous inequalities for ϕ:

$$|\eta|_{X^*}^{m'} \leq C(\phi(u) + 1) \qquad \forall\, [u,\eta] \in \partial_X\phi_X, \tag{14}$$

$$\phi(u) \leq C(|u|_X^m + 1) \qquad \forall u \in D(\partial_X\phi_X), \tag{15}$$

$$|u|_X^m \leq C(|\eta|_{X^*}^{m'} + 1) \qquad \forall\, [u,\eta] \in \partial_X\phi_X. \tag{16}$$

2.3. Statements of the main results

The following result is concerned with the solvability of the elliptic regularization problem for (AP).

Theorem 2.6. *Assume* (A.0) *and* (A.1) *with $m > p$. Then for any $\varepsilon > 0$ and $f \in L^{p'}(0,T;V^*)$, the time-periodic problem*

$$(\mathrm{AP})_\varepsilon \begin{cases} -\varepsilon(d_V\psi(u_\varepsilon'(t)))' + d_V\psi(u_\varepsilon'(t)) + \partial_X\phi_X(u_\varepsilon(t)) \\ \qquad\qquad + \varepsilon F_V(u_\varepsilon(t)) + \varepsilon d_V\psi(u_\varepsilon(t)) \ni f(t) \quad t \in (0,T) \quad \text{in } X^*, \\ u_\varepsilon(0) = u_\varepsilon(T), \quad d_V\psi(u_\varepsilon'(0)) = d_V\psi(u_\varepsilon'(T)), \end{cases}$$

possesses at least one solution u_ε satisfying

$$\begin{aligned} &u_\varepsilon \in W^{1,p}(0,T;V) \cap L^m(0,T;X), \\ &d_V\psi(u_\varepsilon), d_V\psi(u_\varepsilon'), F_V(u_\varepsilon) \in L^{p'}(0,T;V^*), \\ &\eta_\varepsilon \in L^{m'}(0,T;X^*), \\ &(d_V\psi(u_\varepsilon'))' \in L^{p'}(0,T;V^*) + L^{m'}(0,T;X^*), \end{aligned} \tag{17}$$

and
$$\int_0^T \langle d_V\psi(u_\varepsilon'(t)), u_\varepsilon'(t)\rangle_V \, dt \leq \int_0^T \langle f(t), u_\varepsilon'(t)\rangle_V \, dt,$$

where η_ε is the section of $\partial_X\phi_X(u_\varepsilon)$ satisfying $(\mathrm{AP})_\varepsilon$*, that means, η_ε satisfies $\eta_\varepsilon(t) \in \partial_X\phi_X(u_\varepsilon(t))$ and*

$$-\varepsilon(d_V\psi(u_\varepsilon'(t)))' + d_V\psi(u_\varepsilon'(t)) + \eta_\varepsilon(t) + \varepsilon F_V(u_\varepsilon(t)) + \varepsilon d_V\psi(u_\varepsilon(t)) = f(t)$$

for a.e. $t \in (0,T)$.

Via this approximation, the solvability of the original problem is assured as follows:

Theorem 2.7. *Assume* (A.0) *and* (A.1). *Then for every* $f \in L^{p'}(0,T;V^*)$, (AP) *possesses at least one solution* u *satisfying*

$$
\begin{aligned}
&u \in W^{1,p}(0,T;V) \cap L^\infty(0,T;X),\\
&\phi(u(t)) \ \ \text{is absolutely continuous on } [0,T],\\
&d_V\psi(u'),\ \eta \in L^{p'}(0,T;V^*),
\end{aligned}
\tag{18}
$$

where η *is the section of* $\partial_V\phi(u)$ *satisfying* (AP), *i.e.,* η *satisfies* $\eta(t) \in \partial_V\phi(u(t))$ *and* $d_V\psi(u'(t)) + \eta(t) = f(t)$ *for a.e.* $t \in (0,T)$.

Remark 2.8. (i) In so far as A is given as a Gâteaux differential, i.e., $A = d_V\psi$ in (3), we see that our result can cover that of Akagi-Stefanelli [6] by letting $V = X$ be Hilbert spaces and $p = 2$.

(ii) In Akagi-Stefanelli [5] (the Cauchy problem), they assume almost the same growth condition as (A.1) and allow that constants C in (9) and (10) depend on $|u|_V$. In order to reveal a difference in nature between the Cauchy problem and the periodic problem, we here check how to derive a priori estimates of solutions to (AP). Testing (AP) by u' and applying Proposition 2.3, we get

$$
\langle d_V\psi(u'(t)), u'(t)\rangle_V + \frac{d}{dt}\phi(u(t)) = \langle f(t), u'(t)\rangle_V .
$$

Integrating over $[0,t]$ and using (7), we have for every $t \in [0,T]$

$$
\int_0^t |u'(s)|_V^p ds + \phi(u(t)) \le C\left(|f|^{p'}_{L^{p'}(0,T;V^*)} + \phi(u(0)) + T\right)
$$

with some suitable constant $C > 0$. Hence for the Cauchy problem, since $u(0)$ is given data, one can derive estimates of $|u'|_{L^p(0,T;V)}$ and $\sup_t \phi(u(t))$ immediately. For the periodic problem, however, since $u(0)$ is unknown, the boundedness of $|u'|_{L^p(0,T;V)}$ can be assured but one can not get any information on $\sup_t \phi(u(t))$ from the procedure above.

3. Proofs of the main results

3.1. Scheme of the proofs

Our proofs consist of the following four steps:

Step 1: We first deal with the case where $m > p$ and show that

$$
(\mathrm{AP})^h_\lambda
\begin{cases}
-\varepsilon(d_V\psi(u_\lambda'(t)))' + \varepsilon d_V\psi(u_\lambda(t)) + \partial_V\phi_\lambda(u_\lambda(t)) + \varepsilon F_V(u_\lambda(t))\\
\qquad\qquad\qquad\qquad = f(t) + h(t) \qquad t \in (0,T) \quad \text{in } V^*,\\
u_\lambda(0) = u_\lambda(T),\\
d_V\psi(u_\lambda'(0)) = d_V\psi(u_\lambda'(T)),
\end{cases}
$$

possesses a unique solution for every $f, h \in L^{p'}(0,T;V^*)$ and $\lambda, \varepsilon > 0$, where ϕ_λ denotes the Moreau-Yosida regularization of ϕ and $\cdot\,'$ stands for the time derivative. By letting $\lambda \to 0$, we next show the following result.

Lemma 3.1. *Let* $1 < p < m$ *and* $f, h \in L^{p'}(0,T;V^*)$. *Then for any* $\varepsilon > 0$,

$$
(\mathrm{AP})^h \begin{cases} -\varepsilon(d_V\psi(u_h'(t)))' + \varepsilon d_V\psi(u_h(t)) + \partial_X\phi_X(u_h(t)) + \varepsilon F_V(u_h(t)) \\ \qquad\qquad\qquad\qquad \ni f(t) + h(t) \quad t \in (0,T) \quad \text{in } X^*, \\ u_h(0) = u_h(T), \quad d_V\psi(u_h'(0)) = d_V\psi(u_h'(T)), \end{cases}
$$

has a unique solution satisfying

$$
\begin{aligned}
&u_h \in W^{1,p}(0,T;V) \cap L^m(0,T;X), \\
&d_V\psi(u_h), d_V\psi(u_h'), F_V(u_h) \in L^{p'}(0,T;V^*), \\
&\eta_h \in L^{m'}(0,T;X^*), \\
&(d_V\psi(u_h'))' \in L^{p'}(0,T;V^*) + L^{m'}(0,T;X^*),
\end{aligned} \tag{19}
$$

where η_h *is the section of* $\partial_X\phi_X(u_h)$ *satisfying* $(\mathrm{AP})^h$, *i.e.,* $\eta_h(t) \in \partial_X\phi_X(u_h(t))$ *and* $-\varepsilon(d_V\psi(u_h'(t)))' + \varepsilon d_V\psi(u_h(t)) + \eta_h(t) + \varepsilon F_V(u_h(t)) = f(t) + h(t)$ *for a.e.* $t \in (0,T)$.

Note that the equality $L^{p'}(0,T;V^*) + L^{m'}(0,T;X^*) = L^{m'}(0,T;X^*)$ holds, since we here assume that $p < m$. However, we still write as (19) for the sake of consistency with the notation in [5].

Step 2: Let u_h be the unique solution of $(\mathrm{AP})^h$. Define an operator β by

$$
\beta : h \mapsto u_h \mapsto -d_V\psi(u_h'), \tag{20}
$$

namely, $\beta(h) := -d_V\psi(u_h')$. We apply the Schauder-Tychonoff fixed point theorem to β in $\Xi := L^{p'}(0,T;V^*)$ endowed with its weak topology. Let $\bar{h}$ be the fixed point of β, then $u_{\bar{h}}$ gives a solution of $(\mathrm{AP})_\varepsilon$ satisfying properties given in Theorem 2.6.

Step 3: By establishing a priori estimates for solutions of $(\mathrm{AP})_\varepsilon$ independent of the parameter ε and discussing the convergence of solutions, we show the solvability of (AP) for $p < m$.

Step 4: We deal with the case where $m \leq p$.

3.2. Step 1: Existence of solutions of the auxiliary equations

Since our argument in this step is based on the procedure in Section 3 of Akagi-Stefanelli [5], one may find some duplications in this subsection. We first prepare the following result on some variational problem associated with $(\mathrm{AP})_\lambda^h$.

Lemma 3.2. *Let* $p < m$, $\bar{p} := \max\{p, 2\}$, $\Gamma := L^{\bar{p}}(0,T;V)$, $\Gamma^* := L^{(\bar{p})'}(0,T;V^*)$, $\langle\cdot,\cdot\rangle_\Gamma := \int_0^T \langle\cdot,\cdot\rangle_V \, dt$, *and let* $h, f \in \Gamma^*$. *Then for any* $\varepsilon, \lambda > 0$ *the functional over* Γ *defined by*

$$
I_{\varepsilon,\lambda}(u) := \begin{cases} \displaystyle\int_0^T \Big(\phi_\lambda(u(t)) + \varepsilon\psi(u'(t)) + \varepsilon\psi(u(t)) + \frac{\varepsilon}{2}|u(t)|_V^2 - \langle f(t) + h(t), u(t)\rangle_V\Big)\, dt \\ \qquad\qquad \text{if } u \in W^{1,p}(0,T;V),\ u(0) = u(T),\ \psi(u(\cdot)), \psi(u'(\cdot)) \in L^1(0,T), \\ +\infty \quad \text{otherwise,} \end{cases}
$$

admits a global minimizer u_λ on Γ. Moreover, $u_\lambda \in W^{1,\bar{p}}(0,T;V)$ is a unique solution to $(\mathrm{AP})^h_\lambda$ satisfying

$$
\begin{aligned}
&u_\lambda(0) = u_\lambda(T), \quad d_V\psi(u'_\lambda(0)) = d_V\psi(u'_\lambda(T)),\\
&d_V\psi(u'_\lambda) \in W^{1,(\bar{p})'}(0,T;V^*),\ d_V\psi(u_\lambda),\ \partial_V\phi_\lambda(u_\lambda),\ F_V(u_\lambda) \in L^{(\bar{p})'}(0,T;V^*).
\end{aligned}
\tag{21}
$$

Proof. It is easy to see that $I_{\varepsilon,\lambda}$ is proper lower semi-continuous and strictly convex on Γ. By assumptions (7) and (9), $I_{\varepsilon,\lambda}$ is coercive and bounded from below. Then the standard argument guarantees the existence of a global minimizer u_λ of $I_{\varepsilon,\lambda}$.

Divide $I_{\varepsilon,\lambda}$ into the following five parts:

$$
I^1_{\varepsilon,\lambda}(u) := \begin{cases} \varepsilon \displaystyle\int_0^T \psi(u'(t))dt & \text{if } u \in D(I^1_{\varepsilon,\lambda}) := \{u \in W^{1,\bar{p}}(0,T;V); u(0) = u(T)\},\\ +\infty & \text{otherwise,} \end{cases}
$$

$$
I^2_{\varepsilon,\lambda}(u) := \int_0^T \phi_\lambda(u(t))dt,
$$

$$
I^3_{\varepsilon,\lambda}(u) := \begin{cases} \varepsilon \displaystyle\int_0^T \psi(u(t))dt & \text{if } \ \psi(u(\cdot)) \in L^1(0,T),\\ +\infty & \text{otherwise,} \end{cases}
$$

$$
I^4_{\varepsilon,\lambda}(u) := \frac{\varepsilon}{2}\int_0^T |u(t)|^2_V dt \quad \text{and} \quad I^5_{\varepsilon,\lambda}(u) := \int_0^T \langle f(t) + h(t), u(t)\rangle_V\, dt.
$$

We first show that $\partial_\Gamma I^1_{\varepsilon,\lambda}$ coincides with the operator $A : \Gamma \to \Gamma^*$ defined by

$$
Au(t) := -\left(\varepsilon d_V\psi(u'(t))\right)'
$$

with the domain

$$
D(A) := \left\{u \in D(I^1_{\varepsilon,\lambda});\ \ d_V\psi(u'(\cdot)) \in W^{1,(\bar{p})'}(0,T;V^*),\ \ d_V\psi(u'(0)) = d_V\psi(u'(T))\right\}.
$$

Since $d_V\psi = \partial_V\psi$, we have for every $u \in D(A)$ and $v \in D(I^1_{\varepsilon,\lambda})$

$$
\begin{aligned}
I^1_{\varepsilon,\lambda}(v) - I^1_{\varepsilon,\lambda}(u) &\geq \varepsilon \int_0^T \langle d_V\psi(u'(t)), v'(t) - u'(t)\rangle_V\, dt\\
&= -\varepsilon \int_0^T \langle (d_V\psi(u'(t)))', v(t) - u(t)\rangle_V\, dt = \langle Au, v-u\rangle_\Gamma\,,
\end{aligned}
$$

which implies $A \subset \partial_\Gamma I^1_{\varepsilon,\lambda}$. To prove the inverse inclusion, we set $\Lambda := W^{1,\bar{p}}(0,T;V)$ and define two functionals $L, K : \Lambda \to [0,+\infty]$ by

$$
L(u) := \varepsilon \int_0^T \psi(u'(t))dt, \quad K(u) := \begin{cases} 0 & \text{if } u(0) = u(T),\\ +\infty & \text{otherwise.} \end{cases}
\tag{22}
$$

Remark that the restriction of $I^1_{\varepsilon,\lambda}$ onto Λ, denoted by I^1_Λ henceforth, coincides with the sum of L and K. Fix $h \in \mathbb{R}$ and $u, e \in \Lambda$ arbitrarily.

By the definition of the subdifferential,

$$h \langle d_V\psi(u'(t)), e'(t)\rangle_V \le \psi(u'(t)+he'(t)) - \psi(u'(t)) \le h \langle d_V\psi(u'(t)+he'(t)), e'(t)\rangle_V$$

holds for a.e. $t \in (0,T)$. Thanks to (8), we can apply Lebesgue's dominated convergence theorem and we obtain

$$\begin{aligned}\frac{L(u+he)-L(u)}{h} &= \varepsilon \int_0^T \frac{\psi(u'(t)+he'(t)) - \psi(u'(t))}{h} dt \\ &\longrightarrow \int_0^T \langle \varepsilon d_V\psi(u'(t)), e'(t)\rangle_V \, dt \ = \langle d_\Lambda L(u), e\rangle_\Lambda \quad \text{as} \quad h \to 0.\end{aligned}$$

Hence the functional L is Gâteaux differentiable over Λ. On the other hand, since K is a proper indicator function of the closed linear subset $\{u \in \Lambda; u(0) = u(T)\}$ of Λ, the subdifferential of K is well defined and $\langle v, e\rangle_\Lambda = 0$ holds for every $[u,v] \in \partial_\Lambda K$ and $e \in \Lambda$ s.t. $e(0) = e(T)$. Since $D(d_\Lambda L) = \Lambda$ and $D(\partial_\Lambda K) = D(K)$, we can derive $\partial_\Lambda I^1_\Lambda = \partial_\Lambda(L+K) = d_\Lambda L + \partial_\Lambda K$ and

$$D(\partial_\Lambda I^1_\Lambda) = D(d_\Lambda L) \cap D(\partial_\Lambda K) = \{u \in \Lambda;\ u(0) = u(T)\}$$

(see, e.g., Theorem 2.10 in [15]). Let $[u,v] \in \partial_\Gamma I^1_{\varepsilon,\lambda}$ and $e \in \Lambda$ s.t. $e(0) = e(T)$. By the general relationship $\partial_\Gamma I^1_{\varepsilon,\lambda} \subset \partial_\Lambda I^1_\Lambda$, we have

$$\langle v, e\rangle_\Lambda = \langle d_\Lambda L(u), e\rangle_\Lambda = \int_0^T \langle \varepsilon d_V\psi(u'(t)), e'(t)\rangle_V \, dt.$$

In addition, since $\langle v, e\rangle_\Lambda = \langle v, e\rangle_\Gamma = \int_0^T \langle v(t), e(t)\rangle_V \, dt$ is verified by $v \in \Gamma^* \hookrightarrow \Lambda^*$ and $e \in \Lambda \hookrightarrow \Gamma$, the equality

$$\int_0^T \langle v(t), e(t)\rangle_V \, dt = \int_0^T \langle \varepsilon d_V\psi(u'(t)), e'(t)\rangle_V \, dt$$

holds for every $e \in \Lambda$ with $e(0) = e(T)$. Hence by testing this by $e \in C_0^\infty((0,T);V)$, which is dense in Γ, we conclude that $v = -(\varepsilon d_V\psi(u'(\cdot)))' \in \Gamma^* = L^{(\bar{p})'}(0,T;V^*)$. Combining this with the integration by parts, we obtain

$$\langle d_V\psi(u'(T)), e(T)\rangle_V - \langle d_V\psi(u'(0)), e(0)\rangle_V = \langle d_V\psi(u'(T)) - d_V\psi(u'(0)), e(0)\rangle_V = 0.$$

Therefore $d_V\psi(u'(T)) = d_V\psi(u'(0))$ in V^* by the arbitrariness of $e(0)$ and then $\partial_\Gamma I^1_{\varepsilon,\lambda} \subset A$ is shown.

We now check the remainders $I^i_{\varepsilon,\lambda}(\cdot)$ $(i = 2,3,4,5)$. Obviously, these are proper lower semi-continuous convex functionals defined on Γ. We first get

$$I^2_{\varepsilon,\lambda}(u) \le \int_0^T \left(\frac{|u(t)-v|_V^2}{2\lambda} + \phi(v) \right) dt \quad \forall u \in \Gamma \text{ and } \forall v \in D(\phi)$$

from the definition of the Moreau-Yosida regularization, and the definition of $\partial\psi$ and (8) yield

$$\begin{aligned} I^3_{\varepsilon,\lambda}(u) &\le \varepsilon T\psi(v) + \varepsilon \int_0^T |d_V\psi(u(t))|_{V^*} |u(t) - v|_V dt \\ &\le \varepsilon T\psi(v) + \varepsilon C \int_0^T (|u(t)|_V^p + |v|_V^p + 1) \, dt \quad \forall u \in \Gamma \text{ and } \forall v \in D(\psi), \end{aligned}$$

where C is some suitable constant independent of ε and λ.

Hence $D(I^3_{\varepsilon,\lambda}) = \Gamma$ and $I^4_{\varepsilon,\lambda}$, $I^5_{\varepsilon,\lambda}$ are well defined on Γ. Moreover, by the standard argument (see Appendice I in [17]), we have

$$\partial_\Gamma I^2_{\varepsilon,\lambda}(u) = \partial_V \phi_\lambda(u(t)), \quad \partial_\Gamma I^3_{\varepsilon,\lambda}(u) = \varepsilon d_V \psi(u(t)),$$
$$\partial_\Gamma I^4_{\varepsilon,\lambda}(u) = \varepsilon F_V(u(t)), \quad \partial_\Gamma I^5_{\varepsilon,\lambda}(u) = -f(t) - h(t),$$

for any $u \in \Gamma$ and a.e. $t \in (0,T)$. Hence we immediately have

$$\begin{aligned}\partial_\Gamma I_{\varepsilon,\lambda} &= \partial_\Gamma(I^1_{\varepsilon,\lambda} + I^2_{\varepsilon,\lambda} + I^3_{\varepsilon,\lambda} + I^4_{\varepsilon,\lambda} + I^5_{\varepsilon,\lambda}) \\ &= \partial_\Gamma I^1_{\varepsilon,\lambda} + \partial_\Gamma I^2_{\varepsilon,\lambda} + \partial_\Gamma I^3_{\varepsilon,\lambda} + \partial_\Gamma I^4_{\varepsilon,\lambda} + \partial_\Gamma I^5_{\varepsilon,\lambda}.\end{aligned}$$

Therefore, since the global minimizer u_λ satisfies $0 = \partial_\Gamma I_{\varepsilon,\lambda}(u_\lambda)$, we conclude that u_λ is the unique solution to $(\mathrm{AP})^h_\lambda$. □

Remark 3.3. If one tries to apply the WED approach to $I^1_{\varepsilon,\lambda}$ by the same argument as that in [5], one gets a solution with the condition $d_V\psi(u'_\lambda(0)) = e^{-T/\varepsilon} d_V \psi(u'_\lambda(T))$. Taking the limit as $\varepsilon, \lambda \to +0$, one is forced to face with the over-determined problem with $d_V\psi(u'(0)) = d_V\psi(u'(T)) = 0$.

We next take the limit $\lambda \to 0$ in $(\mathrm{AP})^h_\lambda$. In this procedure, we note that if we multiply the equation of $(\mathrm{AP})^h_\lambda$ by u'_λ as in [5], we get for any t_1 and t_2,

$$\left[\varepsilon\psi^*(d_V\psi(u'(t))) + \varepsilon\psi(u_\lambda(t)) + \phi_\lambda(u_\lambda(t)) + \frac{\varepsilon}{2}|u_\lambda(t)|^2_V\right]_{t=t_1}^{t=t_2} \leq \int_{t_1}^{t_2} \langle f(t) + h(t), u'_\lambda(t)\rangle_V \, dt,$$

where ψ^* is the Legendre-Fenchel transform of ψ (precise treatment of the first term of L.H.S. will be given in (56) and (57) below). However, this inequality seems to be not useful for the periodic problem, since $u_\lambda(0)$ is still an unknown value. So we here adopt a different manner to prove Lemma 3.1.

Proof of Lemma 3.1. Let $h, f \in L^{p'}(0,T;V^*)$. Since $L^{p'}(0,T;V^*) \subset L^{(\bar p)'}(0,T;V^*)$, we note that Lemma 3.2 assures the existence of the unique solution u_λ of $(\mathrm{AP})^h_\lambda$ satisfying $u_\lambda \in W^{1,\bar p}(0,T;V) \subset W^{1,p}(0,T;V)$. We first multiply the equation of $(\mathrm{AP})^h_\lambda$ by u_λ. Then using the integration by parts and the periodicity of u_λ, we have

$$\begin{aligned}&\varepsilon\int_0^T \langle d_V\psi(u'_\lambda(t)), u'_\lambda(t)\rangle_V \, dt + \varepsilon\int_0^T \langle d_V\psi(u_\lambda(t)), u_\lambda(t)\rangle_V \, dt \\ &+ \int_0^T \langle \partial_V\phi_\lambda(u_\lambda(t)), u_\lambda(t)\rangle_V \, dt + \varepsilon\int_0^T |u_\lambda(t)|^2_V dt = \int_0^T \langle f(t) + h(t), u_\lambda(t)\rangle_V \, dt.\end{aligned} \tag{23}$$

The definition of subdifferential and (7) yield

$$\int_0^T \langle d_V\psi(v(t)), v(t)\rangle_V \, dt \geq \int_0^T (\psi(v(t)) - \psi(0)) \, dt \geq c_1\left(|v|^p_{L^p(0,T;V)} - T\right), \tag{24}$$

and hence by (24) with $v = u'_\lambda$ and $v = u_\lambda$, we get

$$\begin{aligned}
&\varepsilon c_1 \int_0^T (|u'_\lambda(t)|_V^p + |u_\lambda(t)|_V^p)\, dt + \int_0^T \langle \partial_V \phi_\lambda(u_\lambda(t)), u_\lambda(t)\rangle_V \, dt + \varepsilon \int_0^T |u_\lambda(t)|_V^2 dt \\
&\leq \int_0^T \langle f(t) + h(t), u_\lambda(t)\rangle_V \, dt + 2\varepsilon C_1 T. \qquad (25)
\end{aligned}$$

Here c_1 and C_1 denote positive general constants independent of the parameters λ and ε. From this and Young's inequality, we can derive

$$\begin{aligned}
&\varepsilon c_1 \int_0^T (|u'_\lambda(t)|_V^p + |u_\lambda(t)|_V^p)\, dt + \int_0^T \langle \partial_V \phi_\lambda(u_\lambda(t)), u_\lambda(t)\rangle_V \, dt + \varepsilon \int_0^T |u_\lambda(t)|_V^2 dt \\
&\leq C_\varepsilon \int_0^T |f(t) + h(t)|_{V^*}^{p'} dt + 2\varepsilon C_1 T, \qquad (26)
\end{aligned}$$

where C_ε is a general constant depending only on ε. Then we obtain the following estimate independent of λ:

$$\int_0^T \langle \partial_V \phi_\lambda(u_\lambda(t)), u_\lambda(t)\rangle_V \, dt + \varepsilon \int_0^T \left(|u'_\lambda(t)|_V^p + |u_\lambda(t)|_V^p + |u_\lambda(t)|_V^2\right) dt \leq C_\varepsilon. \qquad (27)$$

Hence for any fixed $v \in D(\phi) \subset X$, the definition of subdifferential and the fact that $\phi_\lambda(v) \leq \phi(v)$ yield

$$\begin{aligned}
0 &\leq \int_0^T \phi_\lambda(u_\lambda(t)) dt \leq \int_0^T \langle \partial_V \phi_\lambda(u_\lambda(t)), u_\lambda(t) - v\rangle_V \, dt + T\phi(v) \\
&\leq C_\varepsilon + |v|_X \int_0^T |\partial_V \phi_\lambda(u_\lambda(t))|_{X^*} dt.
\end{aligned}$$

Then since $\partial_V \phi(J_\lambda u_\lambda) \subset \partial_X \phi_X(J_\lambda u_\lambda)$ and $\phi(J_\lambda u_\lambda) \leq \phi_\lambda(u_\lambda) \leq \phi(u_\lambda)$ (recall (6)), we have from (9) and (10)

$$\begin{aligned}
\int_0^T \phi_\lambda(u_\lambda(t)) dt &\leq C_\varepsilon + C|v|_X \int_0^T (\phi(J_\lambda u_\lambda) + 1)^{1/m'} dt \\
&\leq C_\varepsilon + CT^{1/m} |v|_X \left(\int_0^T (\phi_\lambda(u_\lambda) + 1) dt\right)^{1/m'},
\end{aligned}$$

whence follows
$$0 \leq \int_0^T \phi_\lambda(u_\lambda(t)) dt \leq C_\varepsilon. \qquad (28)$$

Hence, again by (6), (9) and (10), we obtain

$$\int_0^T \left(|J_\lambda u_\lambda(t)|_X^m + |\partial_V \phi_\lambda(u_\lambda(t))|_{X^*}^{m'}\right) dt \leq C_\varepsilon,$$

which together with (27) yields

$$\begin{aligned}
&|\partial_V \phi_\lambda(u_\lambda)|_{L^{m'}(0,T;X^*)} + |u_\lambda|_{W^{1,p}(0,T;V)} \\
&\qquad + |u_\lambda|_{L^2(0,T;V)} + |J_\lambda u_\lambda|_{L^m(0,T;X)} \leq C_\varepsilon. \qquad (29)
\end{aligned}$$

Therefore (8) leads to

$$\int_0^T |d_V\psi(u_\lambda(t))|_{V^*}^{p'}dt + \int_0^T |d_V\psi(u'_\lambda(t))|_{V^*}^{p'}dt \le C_\varepsilon \tag{30}$$

and (12) implies
$$\int_0^T \psi(u_\lambda(t))dt + \int_0^T \psi(u'_\lambda(t))dt \le C_\varepsilon. \tag{31}$$

Returning to $(\mathrm{AP})^h_\lambda$, we get

$$\begin{aligned} &|\varepsilon(d_V\psi(u'_\lambda))'|_{L^{p'}(0,T;V^*)+L^{m'}(0,T;X^*)} \\ &\quad \le |\varepsilon d_V\psi(u_\lambda)|_{L^{p'}(0,T;V^*)} + |\partial_V\phi_\lambda(u_\lambda)|_{L^{m'}(0,T;X^*)} \\ &\qquad + |\varepsilon F_V(u_\lambda)|_{L^{p'}(0,T;V^*)} + |f+h|_{L^{p'}(0,T;V^*)} \le C_\varepsilon. \end{aligned} \tag{32}$$

Here we used the fact that $W^{1,p}(0,T;V) \subset L^\infty(0,T;V)$ and $|F_V(u_\lambda)|_{V^*} = |u_\lambda|_V$.

Then by (29), (30) and (32), we can extract a subsequence $\{\lambda_n\}_{n\in\mathbb{N}}$ such that $\lambda_n \to 0$ as $n \to \infty$,

$$\begin{aligned} u_{\lambda_n} &\rightharpoonup \exists u && \text{weakly in } W^{1,p}(0,T;V), \\ J_{\lambda_n}u_{\lambda_n} &\rightharpoonup \exists v && \text{weakly in } L^m(0,T;X), \\ F_V(u_{\lambda_n}) &\rightharpoonup \exists w && \text{weakly in } L^{p'}(0,T;V^*), \end{aligned} \tag{33}$$

and

$$\begin{aligned} \partial_V\phi_{\lambda_n}(u_{\lambda_n}) &\rightharpoonup \exists \eta && \text{weakly in } L^{m'}(0,T;X^*), \\ d_V\psi(u_{\lambda_n}) &\rightharpoonup \exists a && \text{weakly in } L^{p'}(0,T;V^*), \\ d_V\psi(u'_{\lambda_n}) &\rightharpoonup \exists \xi && \text{weakly in } L^{p'}(0,T;V^*), \\ (d_V\psi(u'_{\lambda_n}))' &\rightharpoonup \xi' && \text{weakly in } L^{p'}(0,T;V^*) + L^{m'}(0,T;X^*), \end{aligned} \tag{34}$$

where ξ' stands for the derivative of ξ in the sense of distributions. Taking the limit in $(\mathrm{AP})^h_{\lambda_n}$, we have

$$-\varepsilon\xi' + \varepsilon a + \eta + \varepsilon w = f + h \quad \text{in} \quad L^{p'}(0,T;V^*) + L^{m'}(0,T;X^*).$$

Now we show that u gives a solution of $(\mathrm{AP})^h$. We first note that $\{u_\lambda\}_{\lambda>0}$ forms an equi-continuous family in $C([0,T];V)$, which is assured by $|u'_\lambda|_{L^p(0,T;V)} \le C_\varepsilon$. Moreover, since $u_\lambda \in C([0,T];V)$, there exists $t_0 \in [0,T)$ such that

$$T^{1/p}|u_\lambda(t_0)|_V = T^{1/p}\min_{t\in[0,T]}|u_\lambda(t)|_V \le |u_\lambda|_{L^p(0,T;V)} \le C_\varepsilon.$$

Then for any $t \in [t_0, t_0+T]$, we get

$$|u_\lambda(t)|_V \le |u_\lambda(t_0)|_V + \int_{t_0}^t |u'_\lambda(s)|_V\,ds \le |u_\lambda(t_0)|_V + T^{1/p'}|u'_\lambda|_{L^p(0,T;V)} \le C_\varepsilon,$$

which implies
$$\sup_{t\in[0,T]}|u_\lambda(t)|_V \le C_\varepsilon. \tag{35}$$

Thanks to the general property $|J_\lambda u_\lambda|_V \le C_1(|u_\lambda|_V + 1)$, we also get

$$\sup_{t\in[0,T]} |J_\lambda u_\lambda(t)|_V \le C_\varepsilon. \tag{36}$$

We recall here that $J_\lambda u_\lambda$ is defined by the unique solution of $F_V(J_\lambda u_\lambda - u_\lambda) \in -\lambda\partial_V\phi(J_\lambda u_\lambda)$. Then from the monotonicity of $\partial_V\phi$,

$$\langle F_V(J_\lambda u_\lambda(t+h) - u_\lambda(t+h)) - F_V(J_\lambda u_\lambda(t) - u_\lambda(t)), J_\lambda u_\lambda(t+h) - J_\lambda u_\lambda(t)\rangle_V \le 0$$

holds for any $h \in \mathbb{R}$. On the other hand, (35) and (36) yield

$$\begin{aligned}&\langle F_V(J_\lambda u_\lambda(t+h) - u_\lambda(t+h)) - F_V(J_\lambda u_\lambda(t) - u_\lambda(t)), -u_\lambda(t+h) + u_\lambda(t)\rangle_V \\ &\le C_\varepsilon |u_\lambda(t+h) - u_\lambda(t)|_V.\end{aligned}$$

Then by adding two inequalities above, we obtain by Proposition 2.1

$$\begin{aligned}&m_{C_\varepsilon}(|\,(J_\lambda u_\lambda(t+h) - u_\lambda(t+h)) - (J_\lambda u_\lambda(t) - u_\lambda(t))\,|_V) \\ &\quad\times |\,(J_\lambda u_\lambda(t+h) - u_\lambda(t+h)) - (J_\lambda u_\lambda(t) - u_\lambda(t))\,|_V \;\le C_\varepsilon |u_\lambda(t+h) - u_\lambda(t)|_V,\end{aligned}$$

which implies that $\{J_\lambda u_\lambda - u_\lambda\}_{\lambda>0}$ forms an equi-continuous family in $C([0,T];V)$. Hence so does $\{J_\lambda u_\lambda\}_{\lambda>0}$. Therefore Theorem 3 of [43] assures the relative compactness of $\{J_\lambda u_\lambda\}_{\lambda>0}$ in $C([0,T];V)$, i.e.,

$$J_{\lambda_n} u_{\lambda_n} \to v \quad \text{strongly in } C([0,T];V)$$

(recall that v is the weak limit $\{J_{\lambda_n} u_{\lambda_n}\}_{n\in\mathbb{N}}$ in $L^m(0,T;X)$). Since $J_\lambda u_\lambda(0) = J_\lambda u_\lambda(T)$, the limit v is also time-periodic. Furthermore, by the definition of ϕ_λ and the fact that $\phi \ge 0$, we have

$$\int_0^T |u_\lambda(t) - J_\lambda u_\lambda(t)|_V^2 dt \le 2\lambda \int_0^T \phi_\lambda(u_\lambda(t))dt \le 2\lambda C_\varepsilon,$$

which implies that $\{u_\lambda - J_\lambda u_\lambda\}_{\lambda>0}$ converges to 0 strongly in $L^2(0,T;V)$ and then $u = v$. Using the strong convergence of $\{J_{\lambda_n} u_{\lambda_n}\}_{n\in\mathbb{N}}$ and uniform boundedness (35) and (36), we also have

$$u_{\lambda_n} \to u \quad \text{strongly in } L^r(0,T;V) \;\; \forall r \in [1,\infty) \tag{37}$$

and then by the demiclosedness of F_V and $d_V\psi$, we get

$$w = F_V(u), \quad a = d_V\psi(u). \tag{38}$$

Here we can extract a subsequence of $\{u_{\lambda_n}\}_{n\in\mathbb{N}}$ (we omit relabeling) such that

$$u_{\lambda_n}(t) \to u(t) \quad \text{strongly in } V \quad \text{for a.e. } t \in (0,T). \tag{39}$$

Since (30), (32), and the compact embedding $V^* \hookrightarrow X^*$ hold, Theorem 3 of [43] is also applicable to the sequence $\{d_V\psi(u'_{\lambda_n})\}_{n\in\mathbb{N}}$. Readily,

$$d_V\psi(u'_{\lambda_n}) \to \xi \quad \text{strongly in } C([0,T];X^*). \tag{40}$$

Remark that $\xi(0) = \xi(T)$ also can be deduced from the periodicity of $d_V\psi(u'_{\lambda_n})$.

In order to complete this step, we have to check $\xi = d_V\psi(u')$ and $\eta \in \partial_X\phi_X(u)$. We here define a subset Υ of $(0,T)$ by

$$\Upsilon := \left\{ t \in (0,T) \;\middle|\; \begin{array}{l} t \text{ is a Lebesgue point of } t \mapsto \langle \xi(t), u(t)\rangle_V \text{ and} \\ \{\lambda_n\} \text{ has a subsequence } \{\lambda^t_{n'}\} \text{ tending to 0 as } n' \to \infty \\ \text{s.t. } \left\langle d_V\psi(u'_{\lambda^t_{n'}}(t)), u_{\lambda^t_{n'}}(t)\right\rangle_V \to \langle \xi(t), u(t)\rangle_V \text{ as } n' \to \infty. \end{array} \right\}.$$

Since $\langle \xi(\cdot), u(\cdot)\rangle_V \in L^1(0,T)$, almost every point of $(0,T)$ satisfies the first requirement. Moreover, thanks to Fatou's lemma and (30), $t \mapsto \liminf_{n\to\infty} |d_V\psi(u_{\lambda_n}(t))|^{p'}_{V^*}$ belongs to $L^1(0,T)$, and then takes a finite value at a.e. $t \in (0,T)$. Hence from (39) and (40), the second requirement is also assured by a.e. $t \in (0,T)$. Therefore Υ has full Lebesgue measure in $(0,T)$.

Fix $t_1, t_2 \in \Upsilon$ with $t_1 < t_2$ arbitrarily. Since $\lambda\partial_V\phi_\lambda(u_\lambda) = F_V(u_\lambda - J_\lambda u_\lambda)$ holds by the definition of the Yosida approximation, we have for every $\lambda > 0$

$$\begin{aligned}\int_{t_1}^{t_2} \langle \partial_V\phi_\lambda(u_\lambda(t)), J_\lambda u_\lambda(t)\rangle_X \,dt &= \int_{t_1}^{t_2} \langle \partial_V\phi_\lambda(u_\lambda(t)), J_\lambda u_\lambda(t)\rangle_V \,dt \\ = \int_{t_1}^{t_2} \langle \partial_V\phi_\lambda(u_\lambda(t)), u_\lambda(t)\rangle_V \,dt &- \lambda^{-1}\int_{t_1}^{t_2} |u_\lambda - J_\lambda u_\lambda|^2_V dt \\ \leq \int_{t_1}^{t_2} \langle \partial_V\phi_\lambda(u_\lambda(t)), u_\lambda(t)\rangle_V \,dt.& \end{aligned} \tag{41}$$

Repeating the same calculation as that for (23), i.e., multiplying the equation of $(\mathrm{AP})^h_\lambda$ by u_λ and integrating over (t_1,t_2), we get

$$\begin{aligned}&\varepsilon\int_{t_1}^{t_2} \langle d_V\psi(u'_\lambda(t)), u'_\lambda(t)\rangle_V \,dt - \varepsilon\,\langle d_V\psi(u'_\lambda(t_2)), u_\lambda(t_2)\rangle_V + \varepsilon\,\langle d_V\psi(u'_\lambda(t_1)), u_\lambda(t_1)\rangle_V \\ &\quad + \varepsilon\int_{t_1}^{t_2} \langle d_V\psi(u_\lambda(t)), u_\lambda(t)\rangle_V \,dt + \int_{t_1}^{t_2} \langle \partial_V\phi_\lambda(u_\lambda(t)), u_\lambda(t)\rangle_V \,dt \\ &\qquad + \varepsilon\int_{t_1}^{t_2} \langle F_V(u_\lambda(t)), u_\lambda(t)\rangle_V \,dt = \int_{t_1}^{t_2} \langle f(t) + h(t), u_\lambda(t)\rangle_V \,dt.\end{aligned} \tag{42}$$

Monotonicity of $d_V\psi$ in $L^p(t_1,t_2;V)$ yields

$$\liminf_{n\to\infty} \int_{t_1}^{t_2} \left\langle d_V\psi(u'_{\lambda_n}(t)), u'_{\lambda_n}(t)\right\rangle_V dt \geq \int_{t_1}^{t_2} \langle \xi(t), u'(t)\rangle_V \,dt. \tag{43}$$

Furthermore, in view of (33), (34), (37) and (38), we get

$$\begin{aligned}\lim_{n\to\infty} \int_{t_1}^{t_2} &\langle d_V\psi(u_{\lambda_n}(t)) + F_V(u_{\lambda_n}(t)), u_{\lambda_n}(t)\rangle_V \,dt \\ &= \int_{t_1}^{t_2} \langle d_V\psi(u(t)) + F_V(u(t)), u(t)\rangle_V \,dt.\end{aligned} \tag{44}$$

Thus choosing a suitable subsequence of $\{\lambda_n\}$ in (43), denoted by $\{\lambda_k\}$, and taking the limit in (42) with $\lambda = \lambda_k \to 0$ as $k \to \infty$, we obtain by (41), (42), (43) and (44)

$$\begin{aligned}
&\limsup_{k\to\infty} \int_{t_1}^{t_2} \langle \partial_V \phi_{\lambda_k}(u_{\lambda_k}(t)), J_{\lambda_k} u_{\lambda_k}(t)\rangle_X \, dt \\
&\leq -\varepsilon \int_{t_1}^{t_2} \langle \xi(t), u'(t)\rangle_V \, dt + \varepsilon \langle \xi(t_2), u(t_2)\rangle_V - \varepsilon \langle \xi(t_1), u(t_1)\rangle_V \\
&\qquad - \int_{t_1}^{t_2} \langle \varepsilon d_V \psi(u(t)) + \varepsilon F_V(u(t)) - f(t) - h(t), u(t)\rangle_V \, dt. \qquad (45)
\end{aligned}$$

From the fact that $u = v \in W^{1,p}(0,T;V) \cap L^m(0,T;X)$, $\xi \in L^{p'}(0,T;V^*)$, and $\xi' \in L^{p'}(0,T;V^*) + L^{m'}(0,T;X^*)$, we can apply Proposition 2.4 and derive

$$\begin{aligned}
&\limsup_{k\to\infty} \int_{t_1}^{t_2} \langle \partial_V \phi_{\lambda_k}(u_{\lambda_k}(t)), J_{\lambda_k} u_{\lambda_k}(t)\rangle_X \, dt \\
&\leq \varepsilon \langle \xi'(t), u(t)\rangle_{L^p(t_1,t_2;V)\cap L^m(t_1,t_2;X)} \\
&\qquad - \int_{t_1}^{t_2} \langle \varepsilon d_V \psi(u(t)) + \varepsilon F_V(u(t)) - f(t) - h(t), u(t)\rangle_V \, dt. \qquad (46)
\end{aligned}$$

Substituting $\varepsilon\xi' = \varepsilon d_V \psi(u) + \varepsilon F_V(u) + \eta - f - h$ in (46), we have

$$\limsup_{k\to\infty} \int_{t_1}^{t_2} \langle \partial_V \phi_{\lambda_k}(u_{\lambda_k}(t)), J_{\lambda_k} u_{\lambda_k}(t)\rangle_X \, dt \leq \int_{t_1}^{t_2} \langle \eta(t), u(t)\rangle_X \, dt. \qquad (47)$$

Since $\partial_V \phi_{\lambda_k}(u_{\lambda_k}) \in \partial_V \phi(J_{\lambda_k} u_{\lambda_k}) \subset \partial_X \phi_X(J_{\lambda_k} u_{\lambda_k})$ and $\partial_X \phi_X$ is maximal monotone in $L^m(t_1,t_2;X)$, then (47) implies that (recall Proposition 2.2)

$$\lim_{k\to\infty} \int_{t_1}^{t_2} \langle \partial_V \phi_{\lambda_k}(u_{\lambda_k}(t)), J_{\lambda_n} u_{\lambda_k}(t)\rangle_X \, dt = \int_{t_1}^{t_2} \langle \eta(t), u(t)\rangle_X \, dt, \qquad (48)$$

where $\eta(t) \in \partial_X \phi_X(u(t))$ for a.e. $t \in (t_1,t_2)$. Moreover, since t_1, t_2 are chosen arbitrarily in Υ which has full Lebesgue measure in $(0,T)$, we see that $\eta \in \partial_X \phi_X(u)$ in $L^m(0,T;X)$. Let $t_1, t_2 \in \Upsilon$. Recall that (41) and (42) give us

$$\begin{aligned}
&\varepsilon \int_{t_1}^{t_2} \langle d_V \psi(u'_\lambda(t)), u'_\lambda(t)\rangle_V \, dt \\
&\leq \varepsilon \langle d_V \psi(u'_\lambda(t_2)), u_\lambda(t_2)\rangle_V - \varepsilon \langle d_V \psi(u'_\lambda(t_1)), u_\lambda(t_1)\rangle_V - \int_{t_1}^{t_2} \langle \partial_V \phi_\lambda(u_\lambda(t)), J_\lambda u_\lambda(t)\rangle_X \, dt \\
&\quad - \int_{t_1}^{t_2} \langle \varepsilon d_V \psi(u_\lambda(t)) + \varepsilon F_V(u_\lambda(t)) - f(t) - h(t), u_\lambda(t)\rangle_V \, dt.
\end{aligned}$$

Taking the limit $\lambda = \lambda_k \to 0$ in the inequality above and using (48), the definition of Υ, and Proposition 2.4 (integration by parts), we obtain

$$
\begin{aligned}
&\varepsilon \limsup_{k\to\infty} \int_{t_1}^{t_2} \left\langle d_V\psi(u'_{\lambda_k}(t)), u'_{\lambda_k}(t)\right\rangle_V dt \\
&\le \varepsilon \left\langle \xi(t_2), u(t_2)\right\rangle_V - \varepsilon \left\langle \xi(t_1), u(t_1)\right\rangle_V - \int_{t_1}^{t_2} \left\langle \eta(t), u(t)\right\rangle_X dt \\
&\quad - \int_{t_1}^{t_2} \left\langle \varepsilon d_V\psi(u(t)) + F_V(u(t)) - f(t) - h(t), u(t)\right\rangle_V dt \qquad (49)\\
&\le \varepsilon \left\langle \xi', u\right\rangle_{L^m(t_1,t_2;X)\cap L^p(t_1,t_2;V)} + \varepsilon \int_{t_1}^{t_2} \left\langle \xi, u'\right\rangle_V dt - \int_{t_1}^{t_2} \left\langle \eta(t), u(t)\right\rangle_X dt \\
&\quad - \int_{t_1}^{t_2} \left\langle \varepsilon d_V\psi(u(t)) + \varepsilon F_V(u(t)) - f(t) - h(t), u(t)\right\rangle_V dt.
\end{aligned}
$$

Substituting $\varepsilon\xi' - \eta - \varepsilon d_V\psi(u) - \varepsilon F_V(u) + f + h = 0$ in (49), we have

$$
\limsup_{k\to\infty} \int_{t_1}^{t_2} \left\langle d_V\psi(u'_{\lambda_k}(t)), u'_{\lambda_k}(t)\right\rangle_V dt \le \int_{t_1}^{t_2} \left\langle \xi, u'\right\rangle_V dt. \qquad (50)
$$

Hence we can repeat the same argument as that for $\eta \in \partial_X\phi_X(u)$ with the aid of the maximal monotonicity of $d_V\psi$ and we can show that $\xi = d_V\psi(u')$ in $L^{p'}(0,T;V^*)$. Therefore, we conclude that u is the desired solution to $(\mathrm{AP})^h$ satisfying (19). The uniqueness of the solution can be assured by the monotonicity of operators in $(\mathrm{AP})^h$ and Proposition 2.1. □

3.3. Step 2 (Proof of Theorem 2.6)

Let $\Xi := L^{p'}(0,T;V^*)$ with the weak topology. By Step 1, β defined by the relationship (20) is well defined as an operator acting on Ξ. In order to apply the Schauder-Tychonoff fixed point theorem, we shall check:

#1. Let $K_R := \{h \in L^{p'}(0,T;V^*);\ |h|_{L^{p'}(0,T;V^*)} \le R\}$. Then there exists $R > 0$ such that β maps K_R into itself.

#2. For every sequence $\{h_n\}_{n\in N}$ which weakly converges to h in $L^{p'}(0,T;V^*)$, $\{\beta(h_n)\}_{n\in\mathbb{N}}$ weakly converges to $\beta(h)$ in $L^{p'}(0,T;V^*)$.

Then by virtue of Theorem 1 in Arino-Gautier-Penot [11], β possesses at least one fixed point $h_0 \in L^{p'}(0,T;V^*)$. Obviously, u_{h_0} is a solution to $(\mathrm{AP})_\varepsilon$ satisfying regularities (17) in Theorem 2.6.

In this subsection, c_2 and C_2 stand for general constants independent $\varepsilon > 0$ and h.

Proof of #1. Let η_h be the section of $\partial_X\phi_X(u_h)$ satisfying $(\mathrm{AP})^h$. Repeating the same calculation as that for (25), i.e., multiplying the equation of $(\mathrm{AP})^h$ by u_h and using

$$
\begin{aligned}
&\int_0^T \left\langle \eta_h(t), u_h(t)\right\rangle_X dt = \int_0^T \left\langle \eta_h(t), u_h(t) - v\right\rangle_X dt + \int_0^T \left\langle \eta_h(t), v\right\rangle_X dt \\
&\ge \int_0^T \phi(u_h(t))dt - T\phi(v) - |v|_X \int_0^T |\eta_h(t)|_{X^*} dt \quad \forall v \in D(\phi) \\
&\ge c_2 |u_h|^m_{L^m(0,T;X)} - C_2, \qquad (51)
\end{aligned}
$$

we obtain by Young's inequality

$$
\begin{aligned}
&c_2\varepsilon|u_h|^p_{W^{1,p}(0,T;V)} + c_2|u_h|^m_{L^m(0,T;X)} + \varepsilon|u_h|^2_{L^2(0,T;V)} \\
&\le \int_0^T \langle f(t)+h(t), u_\lambda(t)\rangle_V \, dt + 2(\varepsilon+1)\, C_2 \qquad (52)\\
&\le C_2\left(|f|^{p'}_{L^{p'}(0,T;V^*)} + 1\right) + \delta|h|^{p'}_{L^{p'}(0,T;V^*)} + C_\delta|u_h|^p_{L^p(0,T;V)},
\end{aligned}
$$

where $\delta > 0$ is an arbitrarily fixed constant and C_δ is a general constant determined only by $\delta > 0$. Since we assume that $X \hookrightarrow V$ and $m > p$, we get

$$
C_\delta|u_h|^p_{L^p(0,T;V)} \le C_\delta|u_h|^p_{L^m(0,T;X)} \le \delta|u_h|^m_{L^m(0,T;X)} + C_\delta.
$$

Hence for every $\delta \le c_2$, we have

$$
c_2\varepsilon|u_h'|^p_{L^p(0,T;V)} \le C_\delta\left(1 + |f|^{p'}_{L^{p'}(0,T;V^*)}\right) + \delta|h|^{p'}_{L^{p'}(0,T;V^*)}.
$$

From (8) we can derive

$$
\begin{aligned}
c_2\varepsilon|\beta(h)|^{p'}_{L^{p'}(0,T;V^*)} &= c_2\varepsilon|d_V\psi(u_h')|^{p'}_{L^{p'}(0,T;V^*)} \qquad (53)\\
&\le C\, C_\delta\left(1 + |f|^{p'}_{L^{p'}(0,T;V^*)}\right) + C\,\delta|h|^{p'}_{L^{p'}(0,T;V^*)} + C\, C_\delta\varepsilon\, T,
\end{aligned}
$$

where C is the constant appearing in (8).

Here we fix R and δ such that

$$
C\, C_\delta\left(1 + |f|^{p'}_{L^{p'}(0,T;V^*)}\right) + C\, C_\delta\varepsilon\, T = \frac{c_2\,\varepsilon}{2}R^{p'}, \quad \delta = \min(c_2, \frac{c_2\varepsilon}{2\,C}).
$$

Then by (53), we get

$$
|\beta(h)|^{p'}_{L^{p'}(0,T;V^*)} \le \frac{1}{c_2\varepsilon}\left(\frac{c_2\varepsilon}{2} + C\,\delta\right)R^{p'} \le R^{p'},
$$

which implies $\beta(h) \in K_R$. □

Proof of #2. Let $\{h_n\}_{n\in\mathbb{N}}$ be a sequence such that $h_n \rightharpoonup h$. Let $u_n = u_{h_n}$ be the solution of $(\mathrm{AP})^{h_n}$ and denote by η_n the section of $\partial_X\phi_X(u_n)$ satisfying $(\mathrm{AP})^{h_n}$. Moreover, let C' denote a general constant independent of n, which may depend on ε. Noting that $|h_n|_{L^{p'}(0,T;V^*)} \le C'$, we repeat the same procedure to establish a priori estimates as in the previous step. Indeed, multiplying the equation of $(\mathrm{AP})^{h_n}$ by u_n, we have (see (52))

$$
|u_n|^p_{W^{1,p}(0,T;V)} + |u_n|^2_{L^2(0,T;V)} + \int_0^T \langle \eta_n(t), u_n(t)\rangle_X \, dt \le C',
$$

which yields (see (51))

$$
|F_V(u_n)|_{L^{p'}(0,T;V^*)} + \int_0^T \phi(u_n(t))dt \le C'. \qquad (54)
$$

Then by (8), (9), and (14), we get

$$|d_V\psi(u_n)|_{L^{p'}(0,T;V^*)} + |d_V\psi(u_n')|_{L^{p'}(0,T;V^*)} + |u_n|_{L^m(0,T;X)} + |\eta_n|_{L^{m'}(0,T;X^*)} \le C'.$$

From the boundedness of remainder terms in $(\mathrm{AP})^{h_n}$, we obtain

$$|(d_V\psi(u_n'))'|_{L^{p'}(0,T;V^*)+L^{m'}(0,T;X^*)} \le C'.$$

Arguments for the convergence given in the previous step can be repeated for this step. In fact, applying Theorem 3 of [43] to $\{u_n\}_{n\in\mathbb{N}}$, which is uniformly bounded in $W^{1,p}(0,T;V)\cap L^m(0,T;X)$, we can extract a subsequence denoted by $\{u_l\}_{l\in\mathbb{N}}$ which strongly converges in $C([0,T];V)$. Let u stands for its limit (remark $u(0) = u(T)$ holds). Furthermore, there exists a subsequence of $\{u_l\}_{l\in\mathbb{N}}$ (we still employ the same index) such that

$$\begin{aligned}
F_V(u_l) &\rightharpoonup \exists w && \text{weakly in } L^{p'}(0,T;V^*),\\
\eta_l &\rightharpoonup \exists \eta && \text{weakly in } L^{m'}(0,T;X^*),\\
d_V\psi(u_l) &\rightharpoonup \exists a && \text{weakly in } L^{p'}(0,T;V^*),\\
d_V\psi(u_l') &\rightharpoonup \exists \xi && \text{weakly in } L^{p'}(0,T;V^*),\\
(d_V\psi(u_l'))' &\rightharpoonup \xi' && \text{weakly in } L^{p'}(0,T;V^*) + L^{m'}(0,T;X^*).
\end{aligned}$$

By virtue of the demiclosedness of F_V and $d_V\psi$, we can easily see that $w = F_V(u)$ and $a = d_V\psi(u)$. Therefore limit functions given above fulfill the equation

$$-\varepsilon\xi' + \varepsilon d_V\psi(u) + \eta + \varepsilon F_V(u) = f + h \quad \text{in} \quad L^{p'}(0,T;V^*) + L^{m'}(0,T;X^*).$$

By the same reasoning as that for (40), we can deduce the strong convergence of $\{d_V\psi(u_l')\}_{l\in\mathbb{N}}$ to ξ in $C([0,T];X^*)$ and then $\xi(0) = \xi(T)$.

Define Υ_h, which has full Lebesgue measure in $(0,T)$ by

$$\Upsilon_h := \left\{ t\in(0,T) \;\middle|\; \begin{array}{l} t \text{ is a Lebesgue point of } t \mapsto \langle \xi(t), u(t)\rangle_V \text{ and} \\ \{l\} \text{ has a subsequence } \{l_k^t\} \text{ tending to } \infty \text{ as } k\to\infty \\ \text{s.t. } \left\langle d_V\psi(u_{l_k^t}'(t)), u_{l_k^t}(t)\right\rangle_V \to \langle \xi(t), u(t)\rangle_V \text{ as } k\to\infty. \end{array} \right\}.$$

Let $t_1 < t_2$ belong to Υ. Tracing the argument from (42) to (46), we can show that

$$\begin{aligned}
\limsup_{k\to\infty} \int_{t_1}^{t_2} \langle \eta_{l_k}(t), u_{l_k}(t)\rangle_X \, dt &\le \varepsilon \, \langle \xi'(t), u(t)\rangle_{L^p(t_1,t_2;V)\cap L^m(t_1,t_2;X)} \\
-\int_{t_1}^{t_2} \langle \varepsilon d_V\psi(u(t)) + \varepsilon F_V(u(t)) - f(t) - h(t), u(t)\rangle_V \, dt &= \int_{t_1}^{t_2} \langle \eta(t), u(t)\rangle_X \, dt,
\end{aligned}$$

where $\{l_k\}_{k\in\mathbb{N}}$ is a suitable subsequence of $\{l\}$. Hence from Proposition 2.2, we derive $\eta \in \partial_X\phi_X(u)$ in $L^m(0,T;X)$. Moreover, repeating the same verification as that for (50), we can obtain

$$\limsup_{l\to\infty} \int_{t_1}^{t_2} \langle d_V\psi(u_l'(t)), u_l'(t)\rangle_V \, dt \le \int_{t_1}^{t_2} \langle \xi(t), u'(t)\rangle_V \, dt,$$

which implies $\xi = d_V\psi(u')$ in $L^{p'}(0,T;V^*)$.

Since the uniqueness of the solution to $(\mathrm{AP})^h$ assures that the above argument does not depend on the choice of subsequences, the original sequence $\{d_V\psi(u_n')\}_{n\in\mathbb{N}} = \{\beta(h_n)\}_{n\in\mathbb{N}}$ also converges to $d_V\psi(u') = \beta(h)$ weakly in $L^{p'}(0,T;V^*)$. $\square$

Remark 3.4. In order to derive the uniqueness of the solution of $(\mathrm{AP})_\varepsilon$ by the standard argument, we need to handle the following terms concerning the difference of two solutions u_1 and u_2:

$$\langle d_V\psi(u_1') - d_V\psi(u_2'), u_1 - u_2\rangle_V\,, \qquad \langle \partial\phi(u_1) - \partial\phi(u_2), u_1' - u_2'\rangle_V\,.$$

Even for the Cauchy problem, assumptions (7)-(10) are not enough to control these terms. The characterization of the Cauchy problem by the Euler-Lagrange equation of the WED functional copes with this difficulty arising in the doubly nonlinear evolution equations and enables us to assure the uniqueness of the solution to the approximate problem when either ϕ or ψ is strictly convex (see Theorem 4.2 of Akagi-Stefanelli [5]). If one leaves the variational approach, however, the uniqueness can not be assured only by such a natural condition especially in the case of the time-periodic problem, where the technical assumptions might be more aggravated than those in the Cauchy problem in general.

In order to complete the proof of Theorem 2.6, we verify the following:

Lemma 3.5. *There exists a solution u_ε to* $(\mathrm{AP})_\varepsilon$ *which satisfies*

$$\int_0^T \langle d_V\psi(u_\varepsilon'(t)), u_\varepsilon'(t)\rangle_V\,dt \le \int_0^T \langle f(t), u_\varepsilon'(t)\rangle_V\,dt. \tag{55}$$

Proof. We return to Step 1. Let u_λ be the unique solution to $(\mathrm{AP})^h_\lambda$. Multiplying $(\mathrm{AP})^h_\lambda$ by u_λ' and integrating over $(0,T)$ with respect to t, we have

$$-\varepsilon\int_0^T \langle (d_V\psi(u_\lambda'(t)))', u_\lambda'(t)\rangle_V\,dt - \int_0^T \langle h(t), u_\lambda'(t)\rangle_V\,dt = \int_0^T \langle f(t), u_\lambda'(t)\rangle_V\,dt.$$

Here we used Proposition 2.3 (chain rule) and the periodicity of u_λ.

By formal calculation, we get

$$\begin{aligned}
&-\int_0^T \langle (d_V\psi(u_\lambda'(t)))', u_\lambda'(t)\rangle_V\,dt\\
&= -\langle d_V\psi(u_\lambda'(0)), u_\lambda'(T) - u_\lambda'(0)\rangle_V + \int_0^T \langle d_V\psi(u_\lambda'(t)), u_\lambda''(t)\rangle_V\,dt\\
&\ge \psi(u_\lambda'(0)) - \psi(u_\lambda'(T)) + \int_0^T \frac{d}{dt}\psi(u_\lambda'(t))dt = 0.
\end{aligned}$$

Although we can not assure the existence of $u_\lambda''(t)$ in a proper way, we can rigorously justify the following:

$$-\int_0^T \langle (d_V\psi(u_\lambda'(t)))', u_\lambda'(t)\rangle_V\,dt \ge 0. \tag{56}$$

Indeed, by the same procedure as that in the proof for Lemma A.1 in [5], one can see that

$$\int_{t_1}^{t_2} \langle (d_V\psi(u_\lambda'(t)))', u_\lambda'(t)\rangle_V \, dt \le \psi^*(d_V\psi(u_\lambda'(t_2))) - \psi^*(d_V\psi(u_\lambda'(t_1))) \tag{57}$$

holds for every t_1, t_2 belonging to

$$\Theta_\lambda := \left\{ t \in (0,T) \;\middle|\; \begin{array}{l} t \text{ is a Lebesgue point of} \\ t \mapsto \langle (d_V\psi(u_\lambda'(t)))', u_\lambda'(t)\rangle_V \\ \text{and } u_\lambda \text{ is differentiable at } t. \end{array} \right\},$$

ψ^* denoting the Legendre-Fenchel transform of ψ. Since $d_V\psi(u_\lambda'(t)) \in W^{1,p'}(0,T;V^*)$ and $u_\lambda \in W^{1,p}(0,T;V)$ (recall (21)), Θ_λ has full Lebesgue measure in $(0,T)$. Moreover, from the fundamental facts that ψ^* is a proper l.s.c. convex functional on V^* and $[v,u] \in \partial_{V^*}\psi^*$ is equivalent to $v = \partial_V\psi(u) = d_V\psi(u)$, we can derive the absolute continuity of $\psi^*(d_V\psi(u_\lambda'(\cdot)))$ on $[0,T]$. Then for $t_1 \to 0$ and $t_2 \to T$, we obtain (56).

Hence u_λ satisfies
$$-\int_0^T \langle h(t), u_\lambda'(t)\rangle_V \, dt \le \int_0^T \langle f(t), u_\lambda'(t)\rangle_V \, dt. \tag{58}$$

Since we already showed that $u_\lambda' \rightharpoonup u_h'$ in $L^p(0,T;V)$ as $\lambda \to +0$, where u_h stands for the unique solution to $(\mathrm{AP})^h$, (58) leads to

$$-\int_0^T \langle h(t), u_h'(t)\rangle_V \, dt \le \int_0^T \langle f(t), u_h'(t)\rangle_V \, dt \tag{59}$$

for every $h \in L^{p'}(0,T;V^*)$. Now let $h_0 \in L^{p'}(0,T;V^*)$ be a fixed point of β, i.e., $h_0 = \beta(h_0) := -d_V\psi(u_{h_0}')$. Then substituting h_0 for h in (59), we can assure that u_{h_0} is one of solutions to $(\mathrm{AP})_\varepsilon$ satisfying (55). $\square$

3.4. Step 3 (Proof of Theorem 2.7 : Case $p < m$)

Henceforth, let u_ε be the solution of $(\mathrm{AP})_\varepsilon$ given in Theorem 2.6 and C_3 stand for a general constant independent of the parameter $\varepsilon \in (0,1)$. We first note that the definition of the subdifferential and (7) yield

$$\langle d_V\psi(u_\varepsilon'(t)), u_\varepsilon'(t)\rangle_V \ge \psi(u_\varepsilon'(t)) - \psi(0) \ge \frac{1}{C}\,|u_\varepsilon'(t)|_V^p - 1 - \psi(0).$$

Hence by (55), we get

$$\int_0^T |u_\varepsilon'(t)|_V^p dt \le C_3\Big(\int_0^T \langle f(t), u_\varepsilon'(t)\rangle_V \, dt + 1\Big) \le \frac{1}{2}\int_0^T |u_\varepsilon'(t)|_V^p dt + C_3\big(|f|_{L^{p'}(0,T;V^*)}^{p'} + 1\big),$$

whence follows the a priori bound for $|u_\varepsilon'|_{L^p(0,T;V)}$. Then by (8) and (12), we obtain

$$\int_0^T |u_\varepsilon'(t)|_V^p dt + \int_0^T |d_V\psi(u_\varepsilon'(t))|_{V^*}^{p'} dt + \int_0^T \psi(u_\varepsilon'(t)) dt \le C_3. \tag{60}$$

Next multiplying $(\mathrm{AP})_\varepsilon$ by u_ε and repeating the same argument as that for (52), we have

$$\begin{aligned}
&\varepsilon|u_\varepsilon|^p_{W^{1,p}(0,T;V)} + |u_\varepsilon|^m_{L^m(0,T;X)} + \varepsilon|u_\varepsilon|^2_{L^2(0,T;V)} \\
&\le C_3\left(|f|^{p'}_{L^{p'}(0,T;V^*)} + |d_V\psi(u'_\varepsilon(t))|^{p'}_{L^{p'}(0,T;V^*)} + |u_\varepsilon|^p_{L^p(0,T;V)} + 1\right) \\
&\le C_3 + \frac{1}{2}|u_\varepsilon|^m_{L^m(0,T;X)},
\end{aligned}$$

that is,
$$\varepsilon|u_\varepsilon|^p_{W^{1,p}(0,T;V)} + |u_\varepsilon|^m_{L^m(0,T;X)} + \varepsilon|u_\varepsilon|^2_{L^2(0,T;V)} \le C_3. \tag{61}$$

Hence the canonical embedding $L^m(0,T;X) \hookrightarrow L^p(0,T;V)$ and (60) yield

$$|u_\varepsilon|^p_{W^{1,p}(0,T;V)} \le C_3. \tag{62}$$

Moreover, from (8) and (10), we can derive

$$\int_0^T |\eta_\varepsilon(t)|^{m'}_{X^*}dt + \int_0^T |d_V\psi(u_\varepsilon(t))|^{p'}_{V^*}dt \le C_3, \tag{63}$$

where η_ε is the section of $\partial_X\phi_X(u_\varepsilon(t))$ satisfying $(\mathrm{AP})_\varepsilon$. Hence by the equation of $(\mathrm{AP})_\varepsilon$, we also have

$$|\varepsilon(d_V\psi(u'_\varepsilon))'|_{L^{p'}(0,T;V^*)+L^{m'}(0,T;X^*)} \le C_3. \tag{64}$$

By using (60)–(64), we discuss the convergence of u_ε. To begin with, (61) and (62) enable us to apply Theorem 3 of [43] and extract a subsequence $\{u_{\varepsilon_n}\}_{n\in\mathbb{N}}$ which converges strongly in $C([0,T];V)$. Its limit, denoted by u, clearly satisfies $u(0) = u(T)$. We are going to show that u is the desired periodic solution of (AP). By (62) and (63), we get

$$|\varepsilon d_V\psi(u_\varepsilon)|_{L^{p'}(0,T;V^*)} + |\varepsilon F_V(u_\varepsilon)|_{L^{p'}(0,T;V^*)} \le \varepsilon C_3,$$

which implies that $\{\varepsilon_n d_V\psi(u_{\varepsilon_n})\}_{n\in\mathbb{N}}$ and $\{\varepsilon_n F_V(u_{\varepsilon_n})\}_{n\in\mathbb{N}}$ converge to zero strongly in $L^{p'}(0,T;V^*)$. Furthermore, there exists a subsequence, still denoted by $\{u_{\varepsilon_n}\}_{n\in\mathbb{N}}$, such that

$$\begin{aligned}
u_{\varepsilon_n} &\rightharpoonup u && \text{weakly in } L^m(0,T;X),\\
\eta_{\varepsilon_n} &\rightharpoonup \exists\eta && \text{weakly in } L^{m'}(0,T;X^*),\\
d_V\psi(u'_{\varepsilon_n}) &\rightharpoonup \exists\xi && \text{weakly in } L^{p'}(0,T;V^*),\\
\varepsilon(d_V\psi(u'_{\varepsilon_n}))' &\rightharpoonup \exists\zeta && \text{weakly in } L^{p'}(0,T;V^*) + L^{m'}(0,T;X^*).
\end{aligned}$$

For every $v \in W^{1,m}(0,T;X)$ with $v(0) = v(T)$, we get by (60)

$$\left\langle \varepsilon_n(d_V\psi(u'_{\varepsilon_n}))', v\right\rangle_{L^p(0,T;V)\cap L^m(0,T;X)} = -\varepsilon_n\int_0^T \left\langle d_V\psi(u'_{\varepsilon_n}(t)), v'(t)\right\rangle_V dt \to 0$$

as $n \to \infty$ (use Proposition 2.4). Then $\zeta = 0$ holds by the density. Therefore, we obtain $\xi + \eta = f$. Immediately, η belongs to $L^{p'}(0,T;V^*)$, since the remainders ξ and f are both members of $L^{p'}(0,T;V^*)$.

Multiplying $(\mathrm{AP})_{\varepsilon_n}$ by u_{ε_n}, we have

$$\begin{aligned}
&\int_0^T \langle \eta_{\varepsilon_n}(t), u_{\varepsilon_n}(t)\rangle_X \, dt \\
&= -\varepsilon_n \int_0^T \left\langle d_V\psi(u'_{\varepsilon_n}(t)), u'_{\varepsilon_n}(t)\right\rangle_V dt - \varepsilon_n \int_0^T \langle d_V\psi(u_{\varepsilon_n}(t)), u_{\varepsilon_n}(t)\rangle_V \, dt \\
&\quad - \varepsilon_n |u_{\varepsilon_n}|^2_{L^2(0,T;V)} - \int_0^T \left\langle d_V\psi(u'_{\varepsilon_n}(t)), u_{\varepsilon_n}(t)\right\rangle_V dt + \int_0^T \langle f(t), u_{\varepsilon_n}(t)\rangle_V \, dt.
\end{aligned}$$

Taking the limit as $n \to \infty$, we obtain

$$\begin{aligned}
\limsup_{n\to\infty} \int_0^T \langle \eta_{\varepsilon_n}(t), u_{\varepsilon_n}(t)\rangle_X \, dt &= -\int_0^T \langle \xi(t), u(t)\rangle_V \, dt + \int_0^T \langle f(t), u(t)\rangle_V \, dt \\
&= \int_0^T \langle \eta(t), u(t)\rangle_V \, dt = \int_0^T \langle \eta(t), u(t)\rangle_X \, dt,
\end{aligned}$$

which implies $\eta \in \partial_X \phi_X(u)$ thanks to Proposition 2.2. Hence by virtue of (5) together with the fact that $\eta \in L^{p'}(0,T;V^*)$, we can conclude

$$\eta \in \partial_V \phi(u). \tag{65}$$

Finally, letting $\varepsilon_n \to 0$ in (55) of Lemma 3.5, we can see that

$$\begin{aligned}
\limsup_{n\to\infty} \int_0^T \left\langle d_V\psi(u'_{\varepsilon_n}(t)), u'_{\varepsilon_n}(t)\right\rangle_V dt &\le \int_0^T \langle f(t), u'(t)\rangle_V \, dt \\
&= \int_0^T \langle f(t) - \eta(t), u'(t)\rangle_V \, dt = \int_0^T \langle \xi(t), u'(t)\rangle_V \, dt.
\end{aligned}$$

Here we used Proposition 2.3 and (65). Then the maximal monotonicity of $d_V\psi$ leads to $\xi = d_V\psi(u')$ in $L^{p'}(0,T;V^*)$, hence it follows that u is a solution to (AP).

Furthermore Proposition 2.3 together with (62) and (65) assures that $\phi(u(t))$ is absolutely continuous on $[0,T]$ and hence (9) assures that $u \in L^\infty(0,T;X)$. □

3.5. Step 4 (Proof of Theorem 2.7 : Case $m \le p$)

In this subsection, we consider the excluded case, i.e., the case where $m \le p$. Put

$$\Phi := \phi + \frac{\mu}{1+\alpha}\, \phi^{1+\alpha},$$

where $\mu \in (0,1)$ and α is some fixed exponent with $\alpha > p/m - 1$. It is easy to see that Φ is a proper l.s.c. convex functional over V. Since ϕ satisfies (9),

$$\Phi(u) \ge c_\mu |u|_X^m (1 + |u|_X^{m\alpha}) - C_\mu \ge c_\mu |u|_X^{m(\alpha+1)} - C_\mu$$

holds for every $u \in D(\Phi) = D(\phi)$ with some constants $c_\mu, C_\mu > 0$ (which may depend on the parameter μ). Moreover, since we have

$$\partial_V \Phi = \partial_V \phi + \mu \phi^\alpha \partial_V \phi, \quad \partial_X \Phi_X = \partial_X \phi_X + \mu \phi_X^\alpha \partial_X \phi_X \tag{66}$$

(see Appendix) and ϕ satisfies (10) and (15), then for any $[u,\eta]\in\partial_X\Phi_X$ there exists some $C>0$ independent of μ such that

$$|\eta|_{X^*}^{m'\left(\frac{\alpha+1}{m'\alpha+1}\right)}\le C\left[|u|_X^m\left(1+|u|_X^{\alpha mm'}\right)\right]^{\frac{\alpha+1}{m'\alpha+1}}\le C(|u|_X^{m(\alpha+1)}+1). \tag{67}$$

Hence Φ fulfills (9) and (10) with $\bar m:=m(\alpha+1)>p$ (note that the Hölder conjugate of $\bar m$ coincides with $\bar m'=m'(\alpha+1)/(m'\alpha+1)$). Then we can carry out the same argument given above with ϕ replaced by Φ. That is to say, for every $\mu\in(0,1)$ and $f\in L^{p'}(0,T;V^*)$, the time-periodic problem

$$(\mathrm{AP})_\mu^\star\qquad\begin{cases} d_V\psi(u_\mu')+(1+\mu\phi^\alpha(u_\mu))\,\partial_V\phi(u_\mu)\ni f, & t\in(0,T)\quad\text{in } V^*,\\ u_\mu(0)=u_\mu(T),\end{cases}$$

possesses at least one solution satisfying

$$\begin{aligned}&u_\mu\in W^{1,p}(0,T;V)\cap L^{\bar m}(0,T;X),\\ &d_V\psi(u_\mu'),\ (1+\mu\phi^\alpha(u_\mu))\eta_\mu\in L^{p'}(0,T;V^*),\end{aligned}\tag{68}$$

where η_μ is the section of $\partial_V\varphi(u_\mu)$ satisfying $(\mathrm{AP})_\mu^\star$.

We shall establish a priori estimates of u_μ independent of μ by repeating the same calculation in Step 3. Let $c_4,C_4>0$ denote general constants independent of $\mu\in(0,1)$. Multiplying $(\mathrm{AP})_\mu^\star$ by u_μ', using the chain rule

$$\Phi'(u(t))=\left\langle(1+\mu\varphi^\alpha(u_\mu(t)))\eta_\mu,u_\mu'(t)\right\rangle_V,$$

and integrating over $[0,T]$, we have by (7), (8) and (12) (see (24))

$$\int_0^T|u_\mu'(t)|_V^p dt+\int_0^T|d_V\psi(u_\mu'(t))|_{V^*}^{p'}dt+\int_0^T\psi(u_\mu'(t))dt\le C_4. \tag{69}$$

By the equation of $(\mathrm{AP})_\mu^\star$, we obtain

$$|\eta_\mu|_{L^{p'}(0,T;V^*)}+|\mu\phi^\alpha(u_\mu)\eta_\mu|_{L^{p'}(0,T;V^*)}\le|d_V\psi(u_\mu')|_{L^{p'}(0,T;V^*)}+|f|_{L^{p'}(0,T;V^*)}\le C_4. \tag{70}$$

From (15), (16) and the canonical embedding $|\eta_\mu|_{V^*}\ge c_4|\eta_\mu|_{X^*}$, we can derive

$$\int_0^T\phi^{p'/m'}(u_\mu(t))dt\le C_4. \tag{71}$$

By Proposition 2.3, $\phi(u_\mu(\cdot))$ is absolutely continuous on $[0,T]$ and hence there is $t_0=t_0^\mu\in[0,T]$ at which $\phi(u_\mu(\cdot))$ attains its minimum. Immediately, (71) implies $\phi(u_\mu(t_0))\le C_4$. Hence testing $(\mathrm{AP})_\mu^\star$ by u_μ' again and integrating over $[t_0,t]$ with $t\in[t_0,t_0+T]$, we obtain

$$\sup_{0\le t\le T}\phi(u_\mu(t))\le\sup_{0\le t\le T}\Phi(u_\mu(t))\le C_4. \tag{72}$$

Combining (72) with (9) and (10), we can show that

$$c_4\sup_{0\le t\le T}|u_\mu(t)|_V\le\sup_{0\le t\le T}|u_\mu(t)|_X+\sup_{0\le t\le T}|\eta_\mu(t)|_{X^*}\le C_4. \tag{73}$$

Note that (69), (70) and (73) enable us to follows the same convergence argument as that developed in Step 3. Indeed, (70) and (72) yield

$$|\mu\phi^\alpha(u_\mu)\eta_\mu|_{L^{p'}(0,T;V^*)} \leq \mu C_4 \to 0 \quad \text{as } \mu \to 0. \tag{74}$$

Applying Theorem 3 of [43] and standard argument, we can extract a subsequence of $\{u_\mu\}_{\mu\in\mathbb{N}}$ (we skip relabeling) such that

$$\begin{aligned} u_\mu &\to u && \text{strongly in } C([0,T];V), \\ & && \text{weakly in } W^{1,p}(0,T;V), \\ & && *\text{-weakly in } L^\infty(0,T;X), \\ d_V\psi(u'_\mu) &\rightharpoonup \xi && \text{weakly in } L^{p'}(0,T;V^*), \\ \eta_\mu &\rightharpoonup \eta && \text{weakly in } L^{p'}(0,T;V^*), \end{aligned}$$

and $\xi + \eta = f$ in $L^{p'}(0,T;V^*)$. Using (74), we have

$$\begin{aligned} \limsup_{\mu\to 0} \int_0^T \langle \eta_\mu(t), u_\mu(t)\rangle_V \, dt &= -\int_0^T \langle \xi(t), u(t)\rangle_V \, dt + \int_0^T \langle f(t), u(t)\rangle_V \, dt \\ &= \int_0^T \langle \eta(t), u(t)\rangle_V \, dt, \end{aligned}$$

which together with Proposition 2.2 implies $\eta \in \partial_V\phi(u)$. Furthermore, since

$$\begin{aligned} \int_0^T \left\langle d_V\psi(u'_\mu(t)), u'_\mu(t)\right\rangle_V dt &= \Phi(u_\mu(0)) - \Phi(u_\mu(T)) + \int_0^T \left\langle f(t), u'_\mu(t)\right\rangle_V dt \\ &\longrightarrow \phi(u(0)) - \phi(u(T)) + \int_0^T \langle f(t), u'(t)\rangle_V \, dt \quad \text{as } \mu \to 0, \end{aligned}$$

we obtain

$$\begin{aligned} \limsup_{\mu\to 0} \int_0^T \left\langle d_V\psi(u'_\mu(t)), u'_\mu(t)\right\rangle_V dt &= -\int_0^T \langle \eta(t), u'(t)\rangle_V \, dt + \int_0^T \langle f(t), u'(t)\rangle_V \, dt \\ &= \int_0^T \langle \xi(t), u'(t)\rangle_V \, dt, \end{aligned}$$

which together with Proposition 2.2 leads to $\xi = d_V\psi(u')$. Thus it is shown that u gives a solution of (AP) satisfying (18) . □

4. Structural stability

In this section, we show that the method developed in the previous section is applicable to the study of the structural stability for solutions to (AP). More precisely, we here consider some perturbation to the functionals ϕ and ψ in the following sense (see Definition 3.17 and Proposition 3.19 in [12]).

Definition 4.1. Let Z be a reflexive Banach space and let φ and φ_n ($n \in \mathbb{N}$) be proper lower semi-continuous convex functions from Z to $(-\infty, +\infty]$. Then it is said that φ_n *converges to* φ *in the sense of Mosco*, if the following two conditions (a) and (b) hold.

(a) (Liminf condition) If $u_n \rightharpoonup u$ weakly in Z, then $\liminf_{n\to\infty} \varphi_n(u_n) \geq \varphi(u)$ holds.

(b) (Existence of recovery sequence) For every $u \in D(\varphi)$, there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ such that $u_n \to u$ strongly in Z and $\varphi_n(u_n) \to \varphi(u)$.

We here present the following fact (see Theorem 3.66 and Proposition 3.59 in [12]), which gives a generalization of Proposition 2.2.

Proposition 4.2. *Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence of proper l.s.c. convex functions on a reflexive Banach space Z which converges to a proper l.s.c. convex function $\varphi : Z \to (-\infty, +\infty]$ on Z in the sense of Mosco. Assume that $[u_n, v_n] \in \partial_Z \varphi_n$ satisfies $u_n \rightharpoonup u$ weakly in Z, $v_n \rightharpoonup v$ weakly in Z^*, and*

$$\limsup_{n\to\infty} \langle v_n, u_n \rangle_Z \leq \langle v, u \rangle_Z .$$

Then $[u, v] \in \partial_Z \varphi$ and $\langle v_n, u_n \rangle_Z \to \langle v, u \rangle_Z$ as $n \to \infty$.

To state our result, we introduce the following growth conditions on ϕ_n and ψ_n, which is similar to (A.1):

(A.2) $\{\phi_n\}_{n\in\mathbb{N}}$ and $\{\psi_n\}_{n\in\mathbb{N}}$ are sequences of proper lower semi-continuous convex functionals and Gâteaux differentiable convex functionals over V, respectively, such that ϕ_n and ψ_n converge to ϕ and ψ in the sense of Mosco on V, respectively. Furthermore there exists some constant $C > 0$ independent of the index n satisfying

$$|u|_V^p \leq C(\psi_n(u) + 1) \quad \forall u \in V, \tag{75}$$

$$|d_V \psi_n(u)|_{V^*}^{p'} \leq C(|u|_V^p + 1) \quad \forall u \in V, \tag{76}$$

$$|u|_X^m \leq C(\phi_n(u) + 1) \quad \forall u \in D(\phi_n), \tag{77}$$

$$|\eta|_{X^*}^{m'} \leq C(|u|_X^m + 1) \quad \forall\, [u, \eta] \in \partial_X (\phi_n)_X, \tag{78}$$

where $p, m \in (1, \infty)$ and $(\phi_n)_X$ stands for the restriction of ϕ_n onto X.

Then our result can be stated as follows.

Theorem 4.3. *Assume* (A.0), (A.1) *and* (A.2) *and let u_n be solutions of*

$$(\mathrm{AP})_n \qquad \begin{cases} d\psi_n(u_n'(t)) + \partial\phi_n(u_n(t)) \ni f_n(t), & t \in (0, T) \quad \text{in } V^*, \\ u_n(0) = u_n(T), \end{cases}$$

where $\{f_n\}_{n\in\mathbb{N}}$ and f satisfy either (i) *or* (ii) *below:*

(i) *$f_n \to f$ strongly in $L^{p'}(0, T; V^*)$,*

(ii) *$f_n \rightharpoonup f$ weakly in $W^{1,p'}(0, T; V^*)$ and $f(0) = f(T)$, $f_n(0) = f_n(T)$ hold for any n.*

Then there exists a subsequence $\{u_j\}_{j\in\mathbb{N}} := \{u_{n_j}\}_{j\in\mathbb{N}}$ and its limit u such that

$$u_j \to u \qquad \text{strongly in } C([0, T]; V), \ \text{weakly in } W^{1,p}(0, T; V),$$
$$\qquad\qquad * \text{-weakly in } L^\infty(0, T; X),$$

$$d_V\psi_j(u_j) \rightharpoonup d_V\psi(u) \qquad \textit{weakly in } L^{p'}(0,T;V^*),$$
$$\eta_j \rightharpoonup \eta \qquad \textit{weakly in } L^{p'}(0,T;V^*),$$

and u is a solution to (AP)*. Here η and $\eta_j = \eta_{n_j}$ denote the sections of $\partial\phi(u)$ and $\partial\phi_{n_j}(u_{n_j})$ satisfying the equation of* (AP) *and* $(\mathrm{AP})_{n_j}$*, respectively.*

Proof. It is clear that assumptions (75) and (76) yield (11)–(13) with ψ replaced by ψ_n and C independent of n, and (14) with ϕ replaced by ϕ_n is a direct consequence of (77) and (78). Moreover, (15)–(16) also hold with ϕ replaced by ϕ_n by virtue of (b) of Definition 4.1. Indeed, let $v \in D(\phi)$, then there exists a recovery sequence $\{v_n\}_{n\in\mathbb{N}}$ which satisfies $v_n \to v$ in V and $\phi_n(v_n) \to \phi(v)$. Remark that (77) implies uniform boundedness of $|v_n|_X$. By the definition of subdifferential,

$$\begin{aligned}\phi_n(u) &\le \phi_n(v_n) + \langle \eta, u - v_n\rangle_X \le \phi_n(v_n) + |\eta|_{X^*}(|u|_X + |v_n|_X)\\ &\le C + |\eta|_{X^*}(|u|_X + C),\end{aligned}$$

where $[u,\eta] \in \partial_X(\phi_n)_X$ and C is independent of n. Hence ϕ_n fulfills (15) by (78) and (16) by (77).

Then repeating exactly the same manipulations as those for (69)–(73) in Step 4 of the previous section, we can establish the following estimates for the solutions u_n of $(\mathrm{AP})_n$:

$$|u_n'|_{L^p(0,T;V)} + |d_V\psi_n(u_n')|_{L^p(0,T;V)} + \int_0^T \psi_n(u_n'(t))dt \le C_5,$$
$$\sup_{0\le t\le T}|u_n(t)|_X + \sup_{0\le t\le T}|\eta_n(t)|_{X^*} + \sup_{0\le t\le T}\phi_n(u_n(t)) \le C_5,$$

where $C_5 > 0$ is a general constant independent of n. Therefore we can reprise the same discussion of convergence as that in Step 4 of the previous section and obtain

$$\limsup_{n\to\infty}\int_0^T \langle \eta_n, u_n(t)\rangle_V\, dt \le \int_0^T \langle \eta(t), u(t)\rangle_V\, dt,$$
$$\limsup_{n\to\infty}\int_0^T \langle d_V\psi_n(u_n'(t)), u_n'(t)\rangle_V \le \int_0^T \langle \xi(t), u'(t)\rangle_V\, dt, \tag{79}$$

where η and ξ are limits of (a suitable subsequence of) $\{\eta_n\}_{n\in\mathbb{N}}$ and $\{d_V\psi_n(u_n'(t))\}_{n\in\mathbb{N}}$, respectively. Here assumption (i) on f_n is used to assure

$$\int_0^T \langle f_n(t), u_n'(t)\rangle_V\, dt \to \int_0^T \langle f(t), u'(t)\rangle_V\, dt$$

as $n \to \infty$ and obtain (79). Assumption (ii) also leads to

$$\begin{aligned}\int_0^T \langle f_n(t), u_n'(t)\rangle_V\, dt &= \langle f_n(T), u_n(T)\rangle_V - \langle f_n(0), u_n(0)\rangle_V - \int_0^T \langle f_n'(t), u_n(t)\rangle_V\, dt\\ &\longrightarrow -\int_0^T \langle f'(t), u(t)\rangle_V\, dt = \int_0^T \langle f(t), u'(t)\rangle_V\, dt.\end{aligned}$$

Therefore, it follows our result with the aid of Proposition 4.2. □

5. Application

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. As in Akagi-Stefanelli [5], we consider the following doubly nonlinear parabolic equation:

$$\text{(DNP)} \qquad \begin{cases} \alpha(u'(x,t)) - \Delta_m^a u(x,t) = f(x,t) & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \end{cases}$$

where

$$\Delta_m^a u(x) := \nabla \cdot \left(a(x)|\nabla u(x)|^{m-2}\nabla u(x)\right), \quad x \in \Omega,\ m \in (1,\infty), \quad a : \Omega \to \mathbb{R}.$$

In [5], it is assumed that a, α, and the exponent m satisfy the followings:

(a.1) There exist some constants $a_1, a_2 > 0$ such that $a_1 \le a(x) \le a_2$ for a.e. $x \in \Omega$.

(a.2) $\alpha : \mathbb{R} \to \mathbb{R}$ is a single-valued maximal monotone function with $D(\alpha) = \mathbb{R}$. Moreover, there exist $p \in (1,\infty)$ and constants $c, C > 0$ such that

$$c\,|s|^p - \frac{1}{c} \le A(s), \quad |\alpha(s)|^{p'} \le C(|s|^p + 1) \quad \forall s \in \mathbb{R},$$

where $A(s) := \int_0^s \alpha(\sigma)d\sigma$ (primitive function of α).

(a.3) $p < m^* := \dfrac{Nm}{(N-m)_+}$.

To reduce (DNP) to (AP), set $V := L^p(\Omega)$ and $X := W_0^{1,m}(\Omega)$ and define functionals ψ and ϕ on V by

$$\begin{aligned} \psi(u) &:= \int_\Omega A(u(x))dx, \\ \phi(u) &:= \begin{cases} \dfrac{1}{m}\displaystyle\int_\Omega a(x)|\nabla u(x)|^m dx & \text{if } u \in W_0^{1,m}(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \end{aligned} \tag{80}$$

Note that V, X, and $V^* = L^{p'}(\Omega)$ are uniformly convex and the embedding $X \hookrightarrow V$ is compact by (a.3). From the growth condition assumed in (a.2), we have $D(\psi) = L^p(\Omega)$. It is easy to obtain the convexity and differentiability of ψ on V and see that its derivative coincides with $d_V\psi(u) = \alpha(u)$. Since a is assumed to be a non-degenerate coefficient on Ω, we can show that ϕ_X is differentiable on X and $\partial_X\phi_X(u) = -\Delta_m^a u$ by the standard variational argument (immediately $\partial_V\phi(u) = -\Delta_m^a u$ by $\partial_V\phi \subset \partial_X\phi_X$). Moreover, assumptions (a.1) and (a.2) lead to growth conditions (7)–(10).

Therefore, (A.0) and (A.1) are verified for (DNP), so Theorem 2.7 assures that the following result holds.

Theorem 5.1. *Assume* (a.1)–(a.3). *Then for every $f \in L^{p'}(0,T;L^{p'}(\Omega))$,* (DNP) *possesses at least one time-periodic solution satisfying*

$$\begin{aligned} &u \in W^{1,p}(0,T;L^p(\Omega)) \cap C([0,T];W_0^{1,m}(\Omega)), \\ &\alpha(u'), \Delta_m^a u \in L^{p'}(0,T;L^{p'}(\Omega)). \end{aligned}$$

Proof. The assertions above follow from the direct application of Theorem 2.7 except $u \in C([0,T]; W_0^{1,m}(\Omega))$. To verify this, we first note that (a.1) assures that $L_a^m(\Omega)$ with norm $|w|_{L_a^m} := (\int_\Omega a\, |w|^m\, dx)^{1/m}$ is uniformly convex and that the function $m\,\phi(u(t)) = |\nabla u(t)|_{L_a^m}^m$ is (absolutely) continuous on $[0,T]$. Therefore we immediately derive $\nabla u \in C([0,T]; (L_a^m(\Omega))^d)$, which together with (a.1) assures that $u \in C([0,T]; W_0^{1,m}(\Omega))$. $\square$

We next consider the structural stability of (DNP). For this purpose, we introduce the following conditions:

(a.4) $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of functions $a_n : \Omega \to \mathbb{R}$ such that $a_n(x) \to a(x)$ as $n \to \infty$ for a.e. $x \in \Omega$. In addition, (a.1) with a replaced by a_n holds for all n.

(a.5) $\{\alpha_n\}_{n\in\mathbb{N}}$ is a sequence of functions $\alpha_n : \mathbb{R} \to \mathbb{R}$ such that the sequence of primitive functions $A_n(s) := \int_0^s \alpha_n(\sigma)d\sigma$ converges to $A(s) := \int_0^s \alpha(\sigma)d\sigma$ in the following sense:
   (i) If $s_n \to s$ in $\mathbb{R}$, then $\liminf_{n\to\infty} A_n(s_n) \geq A(s)$.
   (ii) For every $s \in \mathbb{R}$, there exists a sequence $\{s_n\}_{n\in\mathbb{R}}$ such that $A_n(s_n) \to A(s)$ as $n \to \infty$.

   In addition, (a.2) holds with functions α and A replaced by α_n and A_n, respectively by the same $c, C > 0$ (independent of n).

Define ψ_n and ϕ_n by (80) with $A(s)$ and $a(x)$ replaced by $A_n(s)$ and $a_n(x)$, respectively. Then according to [2] and standard facts in [12], (a.4) and (a.5) assure the Mosco convergence of $\{\psi_n\}_{n\in\mathbb{N}}$ and $\{\phi_n\}_{n\in\mathbb{N}}$ to ψ and ϕ on V, respectively. Hence the following theorem follows from Theorem 4.3.

Theorem 5.2. *Assume* (a.1)–(a.5) *and* $\{f_n\}_{n\in\mathbb{N}}$ *and* f *satisfy either* (i) *or* (ii) *below:*
(i) $f_n \to f$ *strongly in* $L^{p'}(0,T; L^{p'}(\Omega))$,
(ii) $f_n \rightharpoonup f$ *weakly in* $W^{1,p'}(0,T; L^{p'}(\Omega))$ *and* $f(0) = f(T)$, $f_n(0) = f_n(T)$ *for any* n.
Let u_n *be a solution of*

$$
(\mathrm{DNP})_n \begin{cases} \alpha_n(u_n'(x,t)) - \Delta_m^{a_n} u_n(x,t) = f_n(x,t) & (x,t) \in \Omega \times (0,T), \\ u_n(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u_n(x,0) = u_n(x,T), & x \in \Omega, \end{cases}
$$

whose existence is assured by Theorem 5.1*. Then the following holds:*
There exists a subsequence $\{u_j\}_{j\in\mathbb{N}} := \{u_{n_j}\}_{j\in\mathbb{N}}$ *and its limit* u *such that*

$$
\begin{aligned}
u_j &\to u && \text{strongly in } C([0,T]; L^p(\Omega)), \\
& && \text{weakly in } W^{1,p}(0,T; L^p(\Omega)), \\
& && \ast\text{-weakly in } L^\infty(0,T; W_0^{1,m}(\Omega)), \\
\alpha(u_j') &\rightharpoonup \alpha(u') && \text{weakly in } L^{p'}(0,T; L^{p'}(\Omega)), \\
\Delta_m^{a_j} u_j &\rightharpoonup \Delta_m^a u && \text{weakly in } L^{p'}(0,T; L^{p'}(\Omega)),
\end{aligned}
$$

and u *is a solution to* (DNP).

A. Appendix

In order to complete our argument in Section 3.5 (Step 4), we here consider a chain rule of subdifferential. Let φ be a l.s.c. non-negative convex function and $\theta : [0,\infty) \to [0,\infty)$ be a continuously differentiable non-decreasing convex function. Then one may expect that the subdifferential of $\Phi := \theta \circ \varphi$ coincides with $\partial\Phi(u) = \theta'(\varphi(u))\partial\varphi(u)$ on the analogy of the standard calculus. Actually, this fact is already proved by Corollary 3.5 of [22] within a fairly general framework.

However, for the sake of completeness and independent interest, we here give another proof of this result in our Banach space setting. Since $\partial\Phi \supset \theta'(\varphi(\cdot))\partial\varphi$ is clear, the derivation of this chain rule is reduced to showing the maximal monotonicity of $u \mapsto \theta'(\varphi(u))\partial\varphi(u)$. The most often used criterion for the maximality of the monotone operator $A : E \to 2^{E^*}$ is given by $R(A+F_E) = E^*$. However this criterion is valid only when E and E^* are reflexive and strictly convex Banach spaces (see Theorem 2.2 of [15] and Remark 10.8 of [44]) and we here work in E, which is assumed to be only reflexive.

So we here aim to verify this chain rule by using only the reflexivity of E under the restriction to the specific case where $\theta(s) := s + \frac{\mu}{1+\alpha}s^{1+\alpha}$ $(\alpha,\mu > 0)$.

Lemma A.1. *Let E be a real reflexive Banach space and $\varphi : E \to [0,\infty]$ be a proper lower semi-continuous convex functional. Define*

$$\Phi(u) := \varphi(u) + \frac{\mu}{1+\alpha}\varphi^{1+\alpha}(u) \qquad u \in E$$

for some $\alpha,\mu > 0$. Then the followings hold:

$$D(\Phi) = D(\varphi), \;\; D(\partial_E\Phi) = D(\partial_E\varphi), \;\; \partial_E\Phi(u) = (1+\mu\varphi^\alpha(u))\partial_E\varphi(u) \quad \forall u \in D(\partial_E\varphi).$$

Proof. First, $D(\Phi) = D(\varphi)$ is trivial since $\varphi \geq 0$. Moreover, we can easily see that $(1+\mu\varphi^\alpha)\partial_E\varphi \subset \partial_E\Phi$. Indeed, for every $[u_1, v_1] \in \partial_E\varphi$ and $u_2 \in D(\Phi) = D(\varphi)$, we have

$$\begin{aligned}
&\langle (1+\mu\varphi^\alpha(u_1))v_1, u_1 - u_2\rangle_E \geq (1+\mu\varphi^\alpha(u_1))(\varphi(u_1) - \varphi(u_2)) \\
&\geq \varphi(u_1) - \varphi(u_2) + \mu\varphi^{\alpha+1}(u_1) - \mu\left(\frac{\alpha}{\alpha+1}\varphi^{\alpha\cdot\frac{\alpha+1}{\alpha}}(u_1) + \frac{1}{\alpha+1}\varphi^{\alpha+1}(u_2)\right) \\
&\geq \Phi(u_1) - \Phi(u_2).
\end{aligned}$$

Hence we only have to check the maximality of $A := (1+\mu\varphi^\alpha)\partial_E\varphi$.

To this end, we rely on the following criterion (see Theorem 10.6 of [44]).

Lemma A.2. *Let E be a reflexive Banach space and $A : E \to 2^{E^*}$ be a monotone operator. Then A is maximal monotone if and only if*

$$\begin{aligned}
&G(A) + G(-F_E) = E \times E^*, \quad \textit{i.e.}, \\
&\forall (w,w^*) \in E \times E^*, \; \exists (u,u^*) \in G(A) \;\; \textit{such that} \;\; w^* \in u^* + F_E(u-w).
\end{aligned} \tag{81}$$

According to this lemma, it suffices to show that for any $(w,w^*) \in E \times E^*$ there exists $u \in D(A)$ such that

$$F_E(u-w) + (1+\mu\varphi^\alpha(u))\partial_E\varphi(u) \ni w^*. \tag{82}$$

To see this, for any $(w,w^*)\in E\times E^*$ and $\lambda\ge 0$, we consider the following auxiliary equation:

$$F_E(u_\lambda - w) + (1+\lambda)\partial_E\varphi(u_\lambda) \ni w^*. \tag{83}$$

Since $(1+\lambda)\partial_E\varphi$ is maximal monotone in $E\times E^*$, Lemma A.2 assures that (83) admits at least one solution. In our setting, however, the uniqueness of the solution is not ensured.

Define the set of all solutions of (83) by $\mathcal{X}_\lambda \subset E$ and a set-valued mapping $\gamma : [0,\infty)\to 2^{[0,\infty)}$ by

$$\gamma(\lambda) := \{\, \mu\,\varphi^\alpha(u_\lambda)\,;\, u_\lambda\in\mathcal{X}_\lambda\,\}.$$

We are going to show that γ has a fixed point λ_0. It is easy to see that $u=u_{\lambda_0}$ gives a solution of (82), which completes the proof.

In order to show the existence of the fixed point of γ, we rely on Kakutani's fixed-point theorem. To do this, it suffices to verify the following facts:

#1 There exists $L>0$ such that $\gamma([0,L])\subset[0,L]$.

#2 The graph of γ is closed, i.e., if $\lambda_n\to\lambda$ and $y_n\to y$ with $y_n\in\gamma(\lambda_n)$, then $y\in\gamma(\lambda)$ holds true.

#3 $\gamma(\lambda)$ is convex and non-empty for each $\lambda\in[0,\infty)$.

We first establish a priori estimates for u_λ and $\varphi(u_\lambda)$. Let g_λ and h_λ be the sections of $\partial_E\varphi(u_\lambda)$ and $F_E(u_\lambda-w)$ satisfying (83), namely,

$$g_\lambda := \frac{w^*-h_\lambda}{1+\lambda}\in\partial_E\varphi(u_\lambda). \tag{84}$$

By the definition of subdifferential,

$$\varphi(v)-\varphi(u_\lambda)\ge \langle g_\lambda, v-u_\lambda\rangle_E = \left\langle \frac{w^*-h_\lambda}{1+\lambda}, v-u_\lambda\right\rangle_E \tag{85}$$

holds for any fixed $v\in D(\varphi)$. Hence u_λ satisfies

$$2(1+\lambda)\varphi(v)+|w-v|_E^2+2\,|w^*|_{E^*}(|v|_E+|u_\lambda|_E)\ge 2(1+\lambda)\varphi(u_\lambda)+|u_\lambda-w|_E^2, \tag{86}$$

which implies the uniform boundedness of $|u_\lambda|_E$ and $\varphi(u_\lambda)$ with respect to λ.

Let $u_\nu\in\mathcal{X}_\nu$, i.e., $u_\nu\in D(\partial_E\varphi)$ be a solution of (83) with λ replaced by ν. Substituting u_ν with v in (85), we get

$$(1+\lambda)\varphi(u_\nu)-(1+\lambda)\varphi(u_\lambda)\ge\langle w^*-h_\lambda, u_\nu-u_\lambda\rangle_E\,. \tag{87}$$

Reversing the roles of u_λ and u_ν, we have

$$(1+\nu)\varphi(u_\lambda)-(1+\nu)\varphi(u_\nu)\ge\langle w^*-h_\nu, u_\lambda-u_\nu\rangle_E\,. \tag{88}$$

By adding (87) and (88), we obtain

$$(\nu-\lambda)\varphi(u_\lambda)-(\nu-\lambda)\varphi(u_\nu)\ge\langle h_\lambda-h_\nu, u_\lambda-u_\nu\rangle_E\,. \tag{89}$$

The monotonicity of F_E and (89) yield

$$\varphi(u_0)\ge\varphi(u_\lambda)\ge\varphi(u_\nu)\qquad\text{if }\ \nu\ge\lambda\ge 0, \tag{90}$$

where u_0, u_λ, and u_ν are arbitrary elements of $\mathcal{X}_0$, $\mathcal{X}_\lambda$, and $\mathcal{X}_\nu$, respectively. Then (89) also gives us

$$\langle h_\lambda - h_\nu, u_\lambda - u_\nu \rangle_E \le \varphi(u_0)|\lambda - \nu| \quad \text{for any } \lambda, \nu \ge 0. \tag{91}$$

Multiplying the difference of two equations (83) with $\lambda = \lambda$ and $\lambda = \nu$ by $u_\lambda - u_\nu$ and using the monotonicity of F_E, we have

$$\langle g_\lambda - g_\nu, u_\lambda - u_\nu \rangle_E + \langle \lambda g_\lambda - \nu g_\nu, u_\lambda - u_\nu \rangle_E \le 0 \quad \text{for any } \lambda, \nu \ge 0.$$

Here, if $\lambda \ge \nu$,

$$\begin{aligned} -\langle \lambda g_\lambda - \nu g_\nu, u_\lambda - u_\nu \rangle_E &= -(\lambda - \nu)\langle g_\lambda, u_\lambda - u_\nu \rangle_E - \nu \langle g_\lambda - g_\nu, u_\lambda - u_\nu \rangle_E \\ &\le (\lambda - \nu)(\varphi(u_\nu) - \varphi(u_\lambda)), \end{aligned}$$

and if $\nu \ge \lambda$,

$$\begin{aligned} -\langle \lambda g_\lambda - \nu g_\nu, u_\lambda - u_\nu \rangle_E &= -(\lambda - \nu)\langle g_\nu, u_\lambda - u_\nu \rangle_E - \lambda \langle g_\lambda - g_\nu, u_\lambda - u_\nu \rangle_E \\ &\le (\nu - \lambda)(\varphi(u_\lambda) - \varphi(u_\nu)). \end{aligned}$$

Hence
$$\langle g_\lambda - g_\nu, u_\lambda - u_\nu \rangle_E \le \varphi(u_0)|\lambda - \nu| \quad \text{for any } \lambda, \nu \ge 0. \tag{92}$$

Now from (86), we can assure #1 with

$$L := \mu \left(\varphi(v) + \frac{1}{2}|w - v|_E^2 + |w^*|_{E^*}(|v|_E + |w|_E + |w^*|_{E^*}) \right)^\alpha,$$

where v is an arbitrary element in $D(\varphi)$.

In order to see #2, let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative numbers which converges to $\lambda \ge 0$ and let $y_n \to y$ with $y_n = \mu\varphi(u_{\lambda_n})^\alpha \in \gamma(\lambda_n)$. By (84) and (86), we find that $|u_{\lambda_n}|_E, |h_{\lambda_n}|_{E^*}, |g_{\lambda_n}|_{E^*}$ are all uniformly bounded. Hence there exists a subsequence of $\{\lambda_n\}$, denoted again by the same symbol, such that

$$\begin{aligned} &u_{\lambda_n} \rightharpoonup u_\lambda && \text{weakly in } E, \\ &h_{\lambda_n} \rightharpoonup h_\lambda && \text{weakly in } E^*, \\ &g_{\lambda_n} \rightharpoonup g_\lambda = \frac{w^* - h_\lambda}{1+\lambda} && \text{weakly in } E^*. \end{aligned}$$

Thanks to Proposition 2.2, (91) and (92) with $\lambda = \lambda_n$ and $\nu = \lambda_m$ imply that

$$\begin{aligned} &h_\lambda \in F_E(u_\lambda), \quad g_\lambda = \frac{w^* - h_\lambda}{1+\lambda} \in \partial_E\varphi(u_\lambda) \\ &\langle h_{\lambda_n}, u_{\lambda_n} \rangle_E \quad \to \quad \langle h_\lambda, u_\lambda \rangle_E \quad \text{as } n \to \infty. \end{aligned} \tag{93}$$

Hence $u_\lambda \in \mathcal{X}_\lambda$, i.e., u_λ is a solution of (83). Furthermore from (87) and (88) with $\lambda = \lambda,\ \nu = \lambda_n$ together with (93), we can derive

$$\begin{aligned} |\varphi(u_{\lambda_n}) - \varphi(u_\lambda)| &\le |\langle h_\lambda - w^*, u_{\lambda_n} - u_\lambda \rangle_E| + |\langle h_{\lambda_n} - w^*, u_\lambda - u_{\lambda_n} \rangle_E| \\ &\longrightarrow 0 \quad \text{as} \quad n \to \infty, \end{aligned}$$

which implies that $y = \mu\varphi(u_\lambda)^\alpha \in \gamma(\lambda)$. Thus #2 is verified.

To show #3, we discuss the following lemma.

Lemma A.3. *For each $\lambda \in [0,\infty)$, $\mathcal{X}_\lambda$ forms a non-empty closed convex subset of E. Moreover for any $u_\lambda, \bar{u}_\lambda \in \mathcal{X}_\lambda$, it holds that*

$$\varphi(\tau\, u_\lambda + (1-\tau)\,\bar{u}_\lambda) = \tau\,\varphi(u_\lambda) + (1-\tau)\,\varphi(\bar{u}_\lambda) \quad \forall \tau \in (0,1). \tag{94}$$

Proof of Lemma A.3. We first note that $\mathcal{X}_\lambda$ is not empty, since (83) admits at least one solution. We put

$$\varphi_1(u) := \frac{1}{2}|u-w|_E^2 + (1+\lambda)\varphi(u) \quad u \in D(\varphi_1) := D(\varphi).$$

Obviously, φ_1 is a lower semi-continuous convex function and

$$\partial_E\varphi_1(u) = F_E(u-w) + (1+\lambda)\partial_E\varphi(u) \quad u \in D(\partial_E\varphi_1) = D(\partial_E\varphi).$$

Then $u_\lambda, \bar{u}_\lambda \in \mathcal{X}_\lambda$ is equivalent to $[u_\lambda, w^*], [\bar{u}_\lambda, w^*] \in \partial_E\varphi_1$, that is,

$$\varphi_1(v) - \varphi_1(u_\lambda) \geq \langle w^*, v - u_\lambda\rangle_E\,, \quad \varphi_1(v) - \varphi_1(\bar{u}_\lambda) \geq \langle w^*, v - \bar{u}_\lambda\rangle_E$$

for any $v \in D(\varphi_1)$. These inequalities and the convexity of φ_1 yield

$$\begin{aligned}\varphi_1(v) - \varphi_1(\tau\, u_\lambda + (1-\tau)\,\bar{u}_\lambda) &\geq \varphi_1(v) - \tau\varphi_1(u_\lambda) - (1-\tau)\varphi_1(\bar{u}_\lambda)\\ &\geq \langle w^*, v - (\tau\, u_\lambda + (1-\tau)\,\bar{u}_\lambda)\rangle_E\,,\end{aligned}$$

which means that $\tau\, u_\lambda + (1-\tau)\,\bar{u}_\lambda \in \mathcal{X}_\lambda$ for every $u_\lambda, \bar{u}_\lambda \in \mathcal{X}_\lambda$, i.e., $\mathcal{X}_\lambda$ is convex.

Let $u_\lambda^n \in \mathcal{X}_\lambda$ converge to u_λ strongly in E, then letting $n \to \infty$ in

$$\varphi_1(v) - \varphi_1(u_\lambda^n) \geq \langle w^*, v - u_\lambda^n\rangle_E \quad \forall v \in D(\varphi_1),$$

we have

$$\varphi_1(v) - \varphi_1(u_\lambda) \geq \langle w^*, v - u_\lambda\rangle_E \quad \forall v \in D(\varphi_1),$$

whence follows $u_\lambda \in \mathcal{X}_\lambda$, i.e., $\mathcal{X}_\lambda$ is closed in E.

Let $u_\lambda,\ \bar{u}_\lambda \in \mathcal{X}_\lambda$ and let $g_\lambda,\ \bar{g}_\lambda$ be the sections of $\partial_E\varphi(u_\lambda),\ \partial_E\varphi(\bar{u}_\lambda)$ satisfying (83). Then (92) with $\nu = \lambda,\ u_\nu = \bar{u}_\lambda,\ g_\nu = \bar{g}_\lambda$ implies

$$\langle g_\lambda - \bar{g}_\lambda, u_\lambda - \bar{u}_\lambda\rangle_E = 0, \text{ i.e., } \langle \bar{g}_\lambda, u_\lambda - \bar{u}_\lambda\rangle_E = \langle g_\lambda, u_\lambda - \bar{u}_\lambda\rangle_E\,. \tag{95}$$

Here by the convexity of φ and the definition of subdifferential, we get

$$\begin{aligned}\tau\,(\varphi(u_\lambda) - \varphi(\bar{u}_\lambda)) &= \tau\varphi(u_\lambda) + (1-\tau)\varphi(\bar{u}_\lambda) - \varphi(\bar{u}_\lambda)\\ &\geq \varphi(\tau u_\lambda + (1-\tau)\bar{u}_\lambda) - \varphi(\bar{u}_\lambda)\\ &\geq \langle \bar{g}_\lambda, \tau u_\lambda + (1-\tau)\bar{u}_\lambda - \bar{u}_\lambda\rangle_E = \tau\,\langle \bar{g}_\lambda, u_\lambda - \bar{u}_\lambda\rangle_E\,.\end{aligned}$$

Hence by virtue of (95), we obtain

$$\tau\,(\varphi(u_\lambda) - \varphi(\bar{u}_\lambda)) \geq \tau\,\langle g_\lambda, u_\lambda - \bar{u}_\lambda\rangle_E \geq \tau\,(\varphi(u_\lambda) - \varphi(\bar{u}_\lambda)),$$

whence follows (94). □

We here claim that $\Gamma_\lambda := \{\varphi(u_\lambda)\ ;\ u_\lambda \in \mathcal{X}_\lambda\}$ is bounded, closed and convex in $\mathbb{R}$. The boundedness is obvious from (90) and the closedness can be derived from the same arguments for the verification of #2 above with $\lambda_n \equiv \lambda$. For each $y_1, y_2 \in \Gamma_\lambda$, there exist $u_1, u_2 \in \mathcal{X}_\lambda$ such that $y_i = \varphi(u_i)$ $(i = 1, 2)$. Then from (94), we have

$$\tau y_1 + (1-\tau) y_2 = \tau\varphi(u_1) + (1-\tau)\varphi(u_2) = \varphi(\tau u_1 + (1-\tau)u_2).$$

Hence $\tau y_1 + (1-\tau)y_2 \in \Gamma_\lambda$ follows by Lemma A.3. Thus Γ_λ is convex and there exist $-\infty < a_\lambda \le b_\lambda < +\infty$ such that

$$\Gamma_\lambda = [a_\lambda, b_\lambda].$$

Immediately, we can conclude $\gamma(\lambda) = [\,\mu\, a_\lambda^\alpha,\ \mu\, b_\lambda^\alpha\,]$, whence follows #3. $\square$

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