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New Proofs for some Results on Spherically Convex Sets

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In Guo and Peng's article [*Spherically convex sets and spherically convex functions*, J. Convex Analysis 28 (2021) 103–122], one defines the notions of spherical convex sets and functions on "general curved surfaces" in $\mathbb{R}^n$ ($n \geq 2$), one studies several properties of these classes of sets and functions, and one establishes analogues of Radon, Helly, Carathéodory and Minkowski theorems for spherical convex sets, as well as some properties of spherical convex functions which are analogous to those of usual convex functions. In obtaining such results, the authors use an analytic approach based on their definitions. Our aim in this note is to provide simpler proofs for the results on spherical convex sets; our proofs are based on some characterizations/representations of spherical convex sets by usual convex sets in $\mathbb{R}^n$. Moreover, we provide a new proof of the recent result of Han and Nashimura [arXiv:2002.06558] on the separation of spherical convex sets established. Our proof is based on a result stated in locally convex spaces.

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1. Preliminaries

As in Guo and Peng's article [5] we consider $\mathbb{R}^n$ with $n \geq 2$ endowed with its usual inner product $\langle \cdot, \cdot \rangle$ and the corresponding Euclidean norm $\|\cdot\|$, as well as a continuous function $\Phi : \mathbb{R}^n \to \mathbb{R}_+ := [0, \infty[$, such that $\Phi(tx) = t\Phi(x)$ for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$, and $\Phi(x) = 0$ if and only if $x = o$, where o denotes the zero element of $\mathbb{R}^n$. The set $\mathbb{S} := \mathbb{S}_\Phi := \{x \in \mathbb{R}^n \mid \Phi(x) = 1\}$ is called Φ-sphere, or simply sphere when there is no danger of confusion. We also consider the mapping $\rho := \rho_\Phi : \mathbb{R}^n \to \{o\} \cup \mathbb{S}$ defined by $\rho(x) := [\Phi(x)]^{-1}x$ if $x \neq o$ and $\rho(o) := o$; clearly, $\rho(tx) = \rho(x)$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{P} :=]0, \infty[$. On $\mathbb{S}$ we consider the trace of the (usual) topology of $\mathbb{R}^n$. It follows that $\mathbb{S}$ is compact. Clearly, when $\Phi := \|\cdot\|$, $\mathbb{S}_\Phi = \mathbb{S}^{n-1}$ is the $n-1$-dimensional (unit) sphere.

Having $x_1, ..., x_k \in \mathbb{S}$ and $\lambda_1, ..., \lambda_k \in \mathbb{R}_+$ ($k \geq 1$) such that $\sum_{i=1}^k \lambda_i = 1$,

$$(\mathrm{s})\sum_{i=1}^k \lambda_i x_i := (\mathrm{s}_\Phi)\sum_{i=1}^k \lambda_i x_i := \rho\big(\sum_{i=1}^k \lambda_i x_i\big)$$

is called a Φ-*spherical convex* (*s-convex* for short) *combination* of $x_1, ..., x_k$; when $x, y \in \mathbb{S}$ and $\lambda \in [0, 1]$, $\rho(\lambda x + (1-\lambda)y)$ is written $\lambda x +_s (1-\lambda)y$. The nonempty subset $S \subseteq \mathbb{S}$ is called *s-convex* if $\lambda x +_s (1-\lambda)y \in S$ for all $x, y \in S$ and $\lambda \in [0, 1]$; of course, we may take $\lambda \in]0, 1[$ in the preceding definition.

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We denote by $\mathcal{C}_{\mathrm{s}}$ the class of the nonempty subsets of $\mathbb{S}$ which are s-convex.

For $\emptyset \neq A \subseteq \mathbb{R}^n$ we denote by $\operatorname{lin} A$, $\operatorname{aff} A$, $\operatorname{conv} A$, $\overline{\operatorname{conv}} A$, $\operatorname{cone} A$, $\overline{\operatorname{cone}} A$, $\operatorname{cl} A$, $\operatorname{bd} A$ the *linear hull*, *affine hull*, the *convex hull*, the *closed convex hull*, the *convex conic hull*, the *closed convex conic hull*, the *closure* and the *boundary* of A, respectively; recall that a cone $K \subseteq \mathbb{R}^n$ is *pointed* if $K \cap (-K) = \{o\}$. Moreover, $\Gamma A := \{\gamma a \mid \gamma \in \Gamma,\ a \in A\}$ and $A + B := \{a + b \mid a \in A,\ b \in B\}$ for $\emptyset \neq \Gamma \subseteq \mathbb{R}$ and $\emptyset \neq A, B \subseteq \mathbb{R}^n$. Clearly, $\operatorname{cone} A = \mathbb{R}_+ \cdot \operatorname{conv} A = \operatorname{conv}(\mathbb{R}_+ A)$. Furthermore, having a nonempty set E, the function $f : E \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$, $\sigma \in \{<, \leq, =, \geq, >\}$ and $\alpha \in \mathbb{R}$, one sets $[f \,\sigma\, \alpha] := \{x \in E \mid f(x) \,\sigma\, \alpha\}$.

Proposition 1.1. *Let $\emptyset \neq S \subseteq \mathbb{S}$. The following assertions are equivalent:*

(i) *S is s-convex;*

(ii) *$\forall x_1, x_2 \in \mathbb{P}S : x_1 + x_2 \in \mathbb{P}S$;*

(iii) *$\mathbb{P}S$ is convex;*

(iv) *$\mathbb{R}_+ S$ is a pointed, convex cone;*

(v) *there exists a pointed convex cone $K \subseteq \mathbb{R}^n$ such that $S = K \cap \mathbb{S}$;*

(vi) *there exists a convex set $C \subseteq \mathbb{R}^n$ such that $S = \rho(C)$;*

(vii) *there exists a bounded convex set $C \subseteq \mathbb{R}^n$ such that $S = \rho(C)$;*

(viii) *$S = \rho\,(\operatorname{conv} S)$.*

Assume, moreover, that Φ is convex. Then (i) $\Leftrightarrow$ (ix), *where*

(ix) *$]0, 1]S$ is convex.*

Proof. The implications (ix) $\Rightarrow$ (vii) $\Rightarrow$ (vi), (viii) $\Rightarrow$ (vii) and (iv) $\Rightarrow$ (v) are obvious.

(i) $\Rightarrow$ (ii): Take $x_1, x_2 \in \mathbb{P}S$; then there exist $u_i \in S$ and $t_i \in \mathbb{P}$ such that $x_i = t_i u_i$ for $i = 1, 2$. Then $x_1 + x_2 = (t_1 + t_2)u$, where $u := \frac{t_1}{t_1+t_2} u_1 + \frac{t_2}{t_1+t_2} u_2$. By the s-convexity of S, $\rho(u) \in S$, and so $u = \Phi(u) \cdot \rho(u) \in \mathbb{P}S$.

(ii) $\Rightarrow$ (iii): It is obvious that $tx \in \mathbb{P}S$ if $t \in \mathbb{P}$ and $x \in \mathbb{P}S$. Take $x_1, x_2 \in \mathbb{P}S$ and $\lambda \in \,]0, 1[$. Then λx_1, $(1 - \lambda)x_2 \in \mathbb{P}S$ and so $\lambda x_1 + (1 - \lambda)x_2 \in \mathbb{P}S$. Hence $\mathbb{P}S$ is convex.

(iii) $\Rightarrow$ (iv): The convexity of $\mathbb{R}_+ S$ is (almost) obvious. Let $\pm x \in (\mathbb{R}_+ S) \setminus \{o\}$; then $\pm x \in \mathbb{P}S$, whence the contradiction $o = \frac{1}{2}x + \frac{1}{2}(-x) \in \mathbb{P}S$. Therefore, $K := \mathbb{R}_+ S$ is a pointed convex cone and, obviously, $S = K \cap \mathbb{S}$.

(v) $\Rightarrow$ (i): Take $x_1, x_2 \in S$ $(\subseteq K \setminus \{o\})$ and $\lambda \in \,]0, 1[$. Then $u := \lambda x_1 + (1 - \lambda)x_2 \in K$. We claim that $u \neq o$; in the contrary case $K \setminus \{o\} \ni x := \lambda x_1 = (\lambda - 1)x_2 = -x \in K$, whence the contradiction $x = o$. It follows that $\Phi(u) > 0$, and so $\rho(u) = \frac{u}{\Phi(u)} \in K \cap \mathbb{S} = S$. Hence S is s-convex.

(iii) $\Rightarrow$ (viii): Because $\mathbb{P}S$ is convex and $S \subseteq \mathbb{P}S$, we have $(S \subseteq)$ $\operatorname{conv} S \subseteq \mathbb{P}S$, and so $S = \rho(S) \subseteq \rho(\operatorname{conv} S) \subseteq \rho(\mathbb{P}S) = \rho(S)$, whence $S = \rho(\operatorname{conv} S)$.

(vi) $\Rightarrow$ (iii): We have $\mathbb{P}S = \mathbb{P} \cdot \rho(C) = \mathbb{P}C$, and so $\mathbb{P}S$ is convex because C is so.

(i) $\Rightarrow$ (ix): Assume that Φ is convex. Take $x_1, x_2 \in \,]0, 1]S$; then there exist $u_i \in S$ and $t_i \in \,]0, 1]$ such that $x_i = t_i u_i$ for $i = 1, 2$. Consider $\lambda \in \,]0, 1[$. Then we have $x := \lambda x_1 + (1 - \lambda)x_2 = \gamma y$, where $\gamma := \lambda t_1 + (1 - \lambda)t_2 \in \,]0, 1]$ and $y := \mu u_1 + (1 - \mu)u_2$

with $\mu := \gamma^{-1}\lambda t_1 \in]0,1[$. Because S is s-convex, we have that $\rho(y) \in S$, while because Φ is convex one has $(0 <)\ \Phi(y) \leq 1$, and so $\gamma\Phi(y) \in]0,1]$. It follows that $x = \gamma\Phi(y)\rho(y) \in]0,1]S$. □

The equivalence of assertions (i)–(iv) from Proposition 1.1 is established in [5, Prop. 2.7]. However, the equivalence (i) $\Leftrightarrow$ (iv) in the case $\Phi := \|\cdot\|$ seems to be known for a long time (see e.g. [3, p. 158] where one says: "Due to the close and obvious relationship between spherical convexity in S^n and convex cones in R^{n+1}, many results can be interpreted in both the spherical and the Euclidean setting"); this equivalence is mentioned without proof in [1, p. 11] and with proof in [4, Prop. 2].

Remark 1.2. Clearly, the convex sets C from (vi) and (vii) do not contain the origin o of $\mathbb{R}^n$, and so $o \notin \operatorname{conv} S$ if $S \in \mathcal{C}_{\mathrm{s}}$. Moreover, the equivalences (i) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii) describe completely which are the s-convex subsets of $\mathbb{S}$; more precisely,

$$\begin{aligned}\mathcal{C}_{\mathrm{s}} &= \left\{K \cap \mathbb{S} \mid \mathbb{R}^n \supseteq K = \operatorname{cone} K \neq \{o\},\ K \cap (-K) = \{o\}\right\} \\ &= \{\rho(C) \mid \mathbb{R}^n \setminus \{o\} \supseteq C = \operatorname{conv} C \neq \emptyset\} \\ &= \{\rho(C) \mid \mathbb{R}^n \setminus \{o\} \supseteq C = \operatorname{conv} C \neq \emptyset,\ C \text{ is bounded}\}. \end{aligned} \tag{1}$$

From Proposition 1.1 one obtains rapidly several characterizations of the closed s-convex subsets of $\mathbb{S}$; just observe that $\mathbb{R}_+B$ is closed whenever B is a bounded and closed subset of $\mathbb{R}^n$ such that $o \notin B$, the closed subsets of $\mathbb{S}$ are compact and $\rho(E)$ is compact for every compact subset E of $\mathbb{R}^n \setminus \{o\}$ (because $\rho|_{\mathbb{R}^n \setminus \{o\}}$ is continuous).

Corollary 1.3. *Let $\emptyset \neq S \subseteq \mathbb{S}$. The following assertions are equivalent:*
(i) *S is closed and s-convex;*
(ii) *$\mathbb{R}_+S$ is a pointed, closed and convex cone;*
(iii) *there exists a pointed, closed and convex cone $K \subseteq \mathbb{R}^n$ such that $S = K \cap \mathbb{S}$;*
(iv) *$S = \rho\left(\overline{\operatorname{conv}} S\right)$;*
(v) *there exists a compact and convex set $C \subseteq \mathbb{R}^n$ such that $S = \rho(C)$.*

We denote by $\mathcal{C}_{\mathrm{sc}}$ the class of the closed and s-convex subsets of $\mathbb{S}$; hence

$$\begin{aligned}\mathcal{C}_{\mathrm{sc}} &= \left\{K \cap \mathbb{S} \mid \mathbb{R}^n \supseteq K = \overline{\operatorname{cone}} K \neq \{o\},\ K \cap (-K) = \{o\}\right\} \\ &= \{\rho(C) \mid \mathbb{R}^n \setminus \{o\} \supseteq C = \overline{\operatorname{conv}} C \neq \emptyset,\ C \text{ is bounded}\}.\end{aligned}$$

2. Proofs of some results from [5]

In the sequel we provide proofs of the main results on spherical convex sets established in [5]; they are based on Proposition 1.1, as well as on other results stated in the sequel.

Theorem 2.1. ([5, Th. 3.1]) *Let $C \subseteq \mathbb{S}$ be an s-convex set. Then for $(x_i)_{i=1}^k \subseteq C$ and $(\lambda_i)_{i=1}^k \subseteq [0,1]$ with $\sum_{i=1}^k \lambda_i = 1$, one has $(\mathrm{s})\sum_{i=1}^k \lambda_i x_i \in C$. In particular, $o \notin \operatorname{conv} C$.*

Proof. Take $(x_i)_{i=1}^k \subseteq C$ and $(\lambda_i)_{i=1}^k \subseteq \mathbb{R}_+$ with $\sum_{i=1}^k \lambda_i = 1$. Then we have $x := \sum_{i=1}^k \lambda_i x_i \in \operatorname{conv} C$. Using the implication (i) $\Rightarrow$ (viii) of Proposition 1.1 we obtain that $(\mathrm{s})\sum_{i=1}^k \lambda_i x_i = \rho(x) \in \rho(\operatorname{conv} C) = C$. From Remark 1.2 one has $o \notin \operatorname{conv} C$. □

Theorem 2.2. ([5, Th. 3.2]) *Let $C \subseteq \mathbb{S}$ be an s-convex set. Then:*

(i) *There is $u_0 \in \mathbb{S}^{n-1}$ such that $C \subseteq \overline{H}^+_{u_0} \cap \mathbb{S}$, where $\overline{H}^+_{u_0} := [\langle \cdot, u_0\rangle \geq 0]$, that is, C is contained in a closed Φ-hemisphere.*

(ii) *If further C is closed, there are $u_0 \in \mathbb{S}^{n-1}$ and $\alpha > 0$ such that $C \subseteq \overline{H}^+_{u_0,\alpha} \cap \mathbb{S}$, where $\overline{H}^+_{u_0,\alpha} := [\langle \cdot, u_0\rangle \geq \alpha]$ $(\subseteq [\langle \cdot, u_0\rangle > 0])$. In particular, C is contained in an open Φ-hemisphere.*

Proof. (i) By Remark 1.2 we have that $o \notin \operatorname{conv} C$, and so there exists $u_0 \in \mathbb{S}^{n-1}$ such that $\langle x, u_0\rangle \geq 0$ for every $x \in \operatorname{conv} C$; hence $C \subseteq \operatorname{conv} C \subseteq \overline{H}^+_{u_0}$, and so $C \subseteq \overline{H}^+_{u_0} \cap \mathbb{S}$.

(ii) Because C is a closed subset of the compact set $\mathbb{S}$, $\operatorname{conv} C$ is compact. Because $o \notin \operatorname{conv} C$, there exist $u_0 \in \mathbb{S}^{n-1}$ and $\alpha \in \mathbb{R}$ such that $\langle x, u_0\rangle \geq \alpha > 0$ for every $x \in \operatorname{conv} C$. As above we get $C \subseteq \overline{H}^+_{u_0,\alpha} \cap \mathbb{S}$. □

The next result is surely known. In the case $\dim X < \infty$, assertions (i) and (ii) follow by Corollaries 17.1.1 and 17.1.2 from [11], respectively.

Lemma 2.3. *Consider X a real nontrivial linear space and $\emptyset \neq A \subseteq X$. Then:*
(i) $\operatorname{conv} A = \left\{ \sum_{k=1}^m \alpha_k x_k \mid m \geq 1,\ (\alpha_k)_{k=1}^m \subseteq \mathbb{P},\ (x_k)_{k=1}^m \subseteq A \text{ a.i.},\ \sum_{k=1}^m \alpha_k = 1\right\}$;
(ii) $\mathbb{P} \cdot \operatorname{conv} A = \left\{ \sum_{k=1}^m \alpha_k x_k \mid m \geq 1,\ (\alpha_k)_{k=1}^m \subseteq \mathbb{P},\ (x_k)_{k=1}^m \subseteq A \text{ l.i.}\right\}$.

In the text above (and also later on), "a.i." and "l.i." are abbreviations for "affinely independent" and "linearly independent", respectively. Of course, if we have $\dim X = n \in \mathbb{N}^*$, one has $m \leq n+1$ in Lemma 2.3(i) and $m \leq n$ in Lemma 2.3(ii).

Proposition 2.4. *Let $\emptyset \neq C \subseteq \mathbb{R}^n \setminus \{o\}$. Consider the following assertions:*
(i) *there exists $u_0 \in \mathbb{R}^n$ such that $\langle x, u_0\rangle > 0$ for all $x \in C$;*
(ii) *for every a.i. family $(x_i)_{i=1}^k \subseteq C$ and $(\alpha_i)_{i=1}^k \subseteq \mathbb{P}$ with $k \geq 1$ one has $\sum_{i=1}^k \alpha_i x_i \neq o$;*
(iii) *for every a.i. family $(x_i)_{i=1}^k \subseteq C$ with $k \geq 1$ one has $\mathbb{P} \cdot \operatorname{conv}\{x_i \mid i \in \overline{1,k}\} \neq \operatorname{lin}\{x_i \mid i \in \overline{1,k}\}$.*

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii)*; moreover, if C is compact, then* (ii) $\Rightarrow$ (i).

Proof. (i) $\Rightarrow$ (ii) Take an a.i. family $(x_i)_{i=1}^k \subseteq C$ and $(\alpha_i)_{i=1}^k \subseteq \mathbb{P}$ with $k \geq 1$. Then $\left\langle \sum_{i=1}^k \alpha_i x_i, u_0\right\rangle = \sum_{i=1}^k \alpha_i \langle x_i, u_0\rangle > 0$, and so $\sum_{i=1}^k \alpha_i x_i \neq o$.

(ii) $\Rightarrow$ (i): Assume that C is compact. Using Lemma 2.3 (i), one obtains that $o \notin \operatorname{conv} C$. Because C is compact, $\operatorname{conv} C$ is closed; using a separation theorem, there exists $u_0 \in \mathbb{R}^n$ such that $\langle x, u_0\rangle > 0$ for all $x \in \operatorname{conv} C$, that is (i) holds.

(ii) $\Leftrightarrow$ (iii): In fact, for a nontrivial real linear space X and a nonempty convex set $A \subseteq X$ the following equivalences hold:

$$0 \in \operatorname{icr} A \iff \mathbb{R}_+ A \text{ is a linear space} \iff \mathbb{R}_+ A = \mathbb{R}_+ A - \mathbb{R}_+ A \iff \mathbb{R}_+ A = \operatorname{lin} A$$
$$\iff \mathbb{P}A \text{ is a linear space} \iff \mathbb{P}A = \mathbb{P}A - \mathbb{P}A \iff \mathbb{P}A = \operatorname{lin} A,$$

where $\operatorname{icr} E := \{x \in X \mid \forall x' \in \operatorname{aff} E,\ \exists \delta > 0,\ \forall t \in [0, \delta] : (1-t)x + tx' \in E\}$ is the relative algebraic interior (or intrinsic core) of $\emptyset \neq E \subseteq X$. Observing that for $\emptyset \neq E \subseteq X$ one has $\operatorname{lin} E = \operatorname{lin}(\operatorname{conv} E)$, from the above equivalences applied for $A := \operatorname{conv}\{x_i \mid i \in \overline{1,k}\}$, one has that (iii) is equivalent to $0 \notin \operatorname{icr}(\operatorname{conv}\{x_i \mid i \in \overline{1,k}\})$ for every a.i. family $(x_i)_{i=1}^k \subseteq C$. Hence (ii) $\Leftrightarrow$ (iii) because for $(x_i)_{i\in\overline{1,k}} \subseteq X$ an a.i. family one has

$$\operatorname{icr}(\operatorname{conv}\{x_i \mid i \in \overline{1,k}\} = \Big\{ \sum\nolimits_{i=1}^k \alpha_i x_i \mid (\alpha_i)_{i=1}^k \subseteq \mathbb{P},\ \sum\nolimits_{i=1}^k \alpha_i = 1 \Big\}. \qquad \square$$

Observe that [5, Th. 3.3] asserts that $\rceil$(i) $\Leftrightarrow$ $\rceil$(ii) $\Leftrightarrow$ $\rceil$(iii) for $C \subseteq \mathbb{S}^{n-1}$ a compact set, where (i)–(iii) are the assertions from Proposition 2.4.

As in [5], one says that $\emptyset \neq S \subseteq \mathbb{S}$ is *hull-addible* if $o \notin \operatorname{conv} S$; moreover, for $S \subseteq \mathbb{S}$ hull-addible one sets $\operatorname{sco} S := \rho(\operatorname{conv} S)$.

Theorem 2.5. ([5, Th. 3.8]) *Let $S \subseteq \mathbb{S}$ be a hull-addible set. Then*

(i) $\operatorname{sco} S$ *is s-convex.*

(ii) $\operatorname{sco} S = \bigcap\{C \in \mathcal{C}_s \mid S \subseteq C\}$, *i.e.* $\operatorname{sco} S$ *is the smallest s-convex set containing S.*

(iii) $\operatorname{sco} S = \mathbb{S} \cap \operatorname{cone} S$.

Proof. (i) Taking into account (1), one has $\operatorname{sco} S \in \mathcal{C}_{\mathrm{s}}$ because $\operatorname{sco} S := \rho(\operatorname{conv} S)$.

(ii) The inclusion $\supseteq$ follows from (i); the inclusion $\subseteq$ is true because for $C \in \mathcal{C}_{\mathrm{s}}$ with $S \subseteq C$ ($\subseteq \mathbb{S}$) we have that $\operatorname{conv} S \subseteq \operatorname{conv} C$, and so

$$\operatorname{sco} S := \rho(\operatorname{conv} S) \subseteq \rho(\operatorname{conv} C) = C,$$

the last equality being given by the implication (i) $\Rightarrow$ (viii) of Proposition 1.1.

(iii) Having in view that

$$\mathbb{P} \cdot \operatorname{sco} S = \mathbb{P} \cdot \rho(\operatorname{conv} S) = \mathbb{P} \cdot \operatorname{conv} S = (\mathbb{R}_+ \cdot \operatorname{conv} S) \setminus \{o\} = (\operatorname{cone} S) \setminus \{o\},$$

we obtain that $\mathbb{S} \cap \operatorname{cone} S = \mathbb{S} \cap (\mathbb{P} \cdot \operatorname{sco} S) = \operatorname{sco} S$. $\square$

Proposition 2.6. ([5, Rem. 3.11]) *Let $S \subseteq \mathbb{S}$ be a nonempty set. Then there is a closed s-convex set containing S if and only if there exist $u \in \mathbb{S}^{n-1}$ and $\alpha > 0$ such that $S \subseteq \mathbb{S} \cap \overline{H}_{u,\alpha}^+$.*

Proof. Assume $S \subseteq \overline{H}_{u,\alpha}^+$ for some $u \in \mathbb{S}^{n-1}$ and $\alpha > 0$. Then $C := \overline{\operatorname{conv}} S \subseteq \overline{H}_{u,\alpha}^+$ is a compact convex set with $o \notin C \supseteq S$. Using the implication (v) $\Rightarrow$ (i) of Corollary 1.3 we have that $S' := \rho(C) \in \mathcal{C}_{\mathrm{sc}}$ and $S \subseteq S'$.

Assume now that $S \subseteq S'$ with $S' \in \mathcal{C}_{\mathrm{sc}}$. By Theorem 2.2(ii), ($S \subseteq$) $S' \subseteq \mathbb{S} \cap \overline{H}_{u,\alpha}^+$ for some $u \in \mathbb{S}^{n-1}$ and $\alpha > 0$. $\square$

Notice that Proposition 2.6 is established in [5, Prop. 3.10] for Φ convex, and stated without proof in [5, Rem. 3.11].

Theorem 2.7. ([5, Th. 3.12]) *Let $\emptyset \neq S \subseteq \mathbb{S} \cap \overline{H}_{u,\alpha}^+$ for some $u \in \mathbb{S}^{n-1}$ and $\alpha > 0$. Then*

$$\overline{\operatorname{sco}} S := \operatorname{cl}(\operatorname{sco} S) = \bigcap\{C \in \mathcal{C}_{sc} \mid S \subseteq C\}. \tag{2}$$

Consequently, $\overline{\operatorname{sco}} S$ is the smallest closed and s-convex subset of $\mathbb{S}$ containing S.

Proof. By Proposition 2.6, there exists $C_0 \in \mathcal{C}_{\rm sc}$ such that $S \subseteq C_0$; it follows that $\operatorname{cl}(\operatorname{sco} S) \subseteq C_0$. Then $(\emptyset \neq)\ S \subseteq \Sigma := \bigcap\{C \in \mathcal{C}_{\rm sc} \mid S \subseteq C\} \in \mathcal{C}_{\rm sc}$, and so we have $\operatorname{cl}(\operatorname{sco} S) \subseteq \Sigma$. On the other hand, it is known (and easy to prove) that $\operatorname{cl}(\mathbb{R}_+ B) = \mathbb{R}_+ \cdot \operatorname{cl} B$ whenever $B \subseteq \mathbb{R}^n$ is bounded and $0 \notin \operatorname{cl} B$. (In fact, this assertion is true for $\mathbb{R}^n$ replaced by an arbitrary Hausdorff topological vector space.) Since $\operatorname{cl}(\operatorname{sco} S) \subseteq C_0$, one has

$$K := \mathbb{R}_+ \cdot \operatorname{cl}(\operatorname{sco} S) = \operatorname{cl}(\mathbb{R}_+ \cdot \operatorname{sco} S) \subseteq \mathbb{R}_+ C_0.$$

Consequently, K is pointed (since $\mathbb{R}_+ C_0$ is so), convex (since $\mathbb{R}_+ \cdot \operatorname{sco} S$ is so) and closed, whence $(S \subseteq)\ \operatorname{cl}(\operatorname{sco} S) \in \mathcal{C}_{\rm sc}$. Therefore, $\Sigma \subseteq \operatorname{cl}(\operatorname{sco} S)$, and so (2) holds. □

Notice that Theorem 2.7 is established in [5, Th. 3.12] for Φ convex.

The following variant of Radon's theorem holds in $\mathbb{R}^n$.

Lemma 2.8. *Let $A := \{u_i \mid i \in I\} \subseteq \mathbb{R}^n$ be such that we have* $\operatorname{card} I \geq n+1$ *and $o \notin \operatorname{conv} A$. Then there exist two nonempty sets $I_1, I_2 \subseteq I$ such that $I_1 \cap I_2 = \emptyset$, $I_1 \cup I_2 = I$ and* $\operatorname{conv} A_1 \cap (]0,1] \operatorname{conv} A_2) \neq \emptyset$*, where $A_k := \{u_i \mid i \in I_k\}$ for $k \in \{1,2\}$.*

Proof. Take $i' \notin I$, and set $u_{i'} := o$, $I' := I \cup \{i'\}$ and $A' := A \cup \{u_{i'}\}$; then we have $\operatorname{card} I' \geq n+2$. By Radon's theorem (see [12, Th. 1.1.5]), there exist $\emptyset \neq I_1', I_2' \subseteq I'$ such that $I_1' \cap I_2' = \emptyset$, $I_1' \cup I_2' = I'$ and $\operatorname{conv} A_1' \cap \operatorname{conv} A_2' \neq \emptyset$, where $A_k' := \{u_i \mid i \in I_k'\}$ for $k \in \{1,2\}$. Clearly $I_k' \neq \{i'\}$ for $k \in \{1,2\}$; otherwise we would have $o \in \operatorname{conv} A$. Let $i' \in I_2'$ and set $I_1 := I_1'$, $I_2 := I_2' \setminus \{i'\}$. Then $\operatorname{conv} A_1' = \operatorname{conv} A_1$ and $\operatorname{conv} A_2' = \operatorname{conv}(A_2 \cup \{o\}) = [0,1] \operatorname{conv} A_2$, whence the conclusion follows because $o \notin \operatorname{conv} A \supseteq \operatorname{conv} A_1 \cup \operatorname{conv} A_2$. □

Theorem 2.9. ([5, Th. 4.1], Radon-type Theorem) *Let $S \subseteq \mathbb{S}$ be such that we have* $\operatorname{card} S \geq n+1$ *and $o \notin \operatorname{conv} S$. Then there exist two nonempty sets $S_1, S_2 \subseteq S$ such that $S_1 \cap S_2 = \emptyset$, $S_1 \cup S_2 = S$ and* $\operatorname{sco} S_1 \cap \operatorname{sco} S_2 \neq \emptyset$.

Proof. Using the preceding lemma there exist two nonempty sets $S_1, S_2 \subseteq S$ such that $S_1 \cap S_2 = \emptyset$, $S_1 \cup S_2 = S$ and $\operatorname{conv} S_1 \cap (]0,1] \operatorname{conv} S_2) \neq \emptyset$. Because $\operatorname{sco} S_1 = \rho(\operatorname{conv} S_1)$ and $\operatorname{sco} S_2 = \rho(\operatorname{conv} S_2) = \rho(]0,1] \operatorname{conv} S_2)$, the conclusion follows. □

The following result is surely known (see [13, p. 713] for a hint). We provide a simple proof for readers convenience.

Lemma 2.10. *Let $(u_i)_{i\in I} \subseteq \mathbb{R}^n$ be an a.i. family, where $I := \overline{1, n+1}$. Assume that $o \in \operatorname{core}(\operatorname{conv}\{u_i \mid i \in I\})$, or, equivalently, there exists $(\overline{\lambda}_l)_{l\in I} \subseteq \mathbb{P}$ such that $\sum_{l\in I} \overline{\lambda}_l = 1$ and $\sum_{l\in I} \overline{\lambda}_l u_l = o$. For each $i \in I$ set*

$$Q_i := \left\{\textstyle\sum_{l\in I} \lambda_l u_l \mid (\lambda_l)_{l\in I} \subseteq \mathbb{R}_+,\ \lambda_i = 0\right\}. \tag{3}$$

Then the following assertions hold:

(a) *Q_i is a pointed convex cone for each $i \in I$;*

(b) *for every $\emptyset \neq J \subseteq I$ one has*

$$\textstyle\bigcap_{j\in J} Q_j = \left\{\sum_{l\in I} \lambda_l u_l \mid (\lambda_l)_{l\in I} \subseteq \mathbb{R}_+,\ \lambda_j = 0\ \forall j \in J\right\}; \tag{4}$$

(c) *$\bigcap_{i\in I\setminus\{j\}} Q_i = \mathbb{R}_+ u_j$ for every $j \in I$ and $\bigcap_{i\in I} Q_i = \{0\}$;*

(d) *$\bigcup_{i\in I} Q_i = \mathbb{R}^n$.*

Proof. Let us observe that having $x \in X := \mathbb{R}^n$ such that

$$x = \sum\nolimits_{l\in I} \lambda_l u_l = \sum\nolimits_{l\in I} \lambda'_l u_l$$

with $\lambda_l, \lambda'_l \geq 0$ for $l \in I$ and $\lambda_i = \lambda'_j = 0$ for some $i, j \in I$, we have that $\lambda_l = \lambda'_l$ for all $l \in I$. Indeed, setting $\gamma := \sum_{l\in I}(\lambda_l - \lambda'_l)$, clearly $\sum_{l\in I}(\lambda_l - \lambda'_l - \gamma\overline{\lambda}_l) = 0$ and $\sum_{l\in I}(\lambda_l - \lambda'_l - \gamma\overline{\lambda}_l)u_l = o$, and so $\lambda_l - \lambda'_l = \gamma\overline{\lambda}_l$ for every $l \in I$. Taking $l \in \{i, j\}$, we get $(0 \geq) -\lambda'_i = \gamma\overline{\lambda}_i$ and $(0 \leq) \lambda_j = \gamma\overline{\lambda}_j$, and so $\gamma = 0$ because $\overline{\lambda}_l > 0$ for all $l \in I$. Hence $\lambda_l = \lambda'_l$ for all $l \in I$.

Assertion (a) is obvious, while (b) follows immediately from the uniqueness of the representation of $x \in X$ as $\sum_{l\in I} \lambda_l u_l$ with $\lambda_l \geq 0$ for $l \in I$ such that $\lambda_i = 0$ for some $i \in I$.

(c) Taking $J := I \setminus \{j\}$ and $J := I$ in (4) one gets the first and the second formula, respectively.

(d) Take $x \in X$. Then there exist $(\lambda_l)_{l\in I} \subseteq \mathbb{R}$ such that $x = \sum_{l\in I} \lambda_l u_l$. Take $\alpha := \min\{\lambda_l/\overline{\lambda}_l \mid l \in I\}$. Then $\mu_l := \lambda_l - \alpha\overline{\lambda}_l \geq 0$ for $l \in I$ and $\mu_j = 0$ for some $j \in I$, and so $x = \sum_{l\in I} \mu_l u_l \in Q_j$. Therefore, $\bigcup_{i\in I} Q_i = X = \mathbb{R}^n$. □

Proposition 2.11. *Let $C_1, C_2, ..., C_m \subseteq \mathbb{R}^n$, $m \geq n+1$, be nonempty convex sets such that $o \notin C_i = \,]0,1]C_i$ for all $i \in \overline{1,m}$. Moreover, assume that we have $\bigcup_{j\in J}(\mathbb{P}C_j) \neq \mathbb{R}^n \setminus \{o\}$ for any $J \subseteq \overline{1,m}$ with* card $J = n+1$. *If $\bigcap_{j\in J} C_j \neq \emptyset$ for any $J \subseteq \overline{1,m}$ with* card $J = n$, *then $\bigcap_{j\in\overline{1,m}} C_j \neq \emptyset$ and $\bigcup_{j\in\overline{1,m}}(\mathbb{P}C_j) \neq \mathbb{R}^n \setminus \{o\}$.*

Proof. Our aim is to prove that $\bigcap_{j\in J} C_j \neq \emptyset$ for any $J \subseteq \overline{1,m}$ with card $J = n+1$.

Consider $n+1$ sets $A_1, A_2, ..., A_{n+1}$ among those m given sets. Using the trick from the proof of Helly's theorem in [12, p. 4], take $u_i \in A_1 \cap ... \cap \breve{A}_i \cap ... \cap A_{n+1}$, where $\breve{A}_i$ indicates that A_i is the (only) missing set; hence $u_i \in A_j$ for all $i, j \in I := \overline{1, n+1}$ with $i \neq j$, and so

$$\operatorname{conv}\left\{u_i \mid i \in I \setminus \{j\}\right\} \subseteq A_j \quad \forall j \in I. \tag{5}$$

We claim that $o \notin \operatorname{conv}\{u_i \mid i \in I\}$. Assume that our claim is true. By Lemma 2.8, after renumbering if necessary, there exist $k \in \overline{1,n}$ and

$$x \in \operatorname{conv}\{u_1, ..., u_k\} \cap \big(]0,1] \operatorname{conv}\{u_{k+1}, ..., u_{n+1}\}\big).$$

As seen above, $u_i \in A_j$ for all $i, j \in I$ with $i \neq j$, and so $u_1, ..., u_k \in \bigcap_{i=k+1}^{n+1} A_i$ and $u_{k+1}, ..., u_{n+1} \in \bigcap_{i=1}^{k} A_i$. It follows that $x \in \bigcap_{i=k+1}^{n+1} A_i$ and $\alpha^{-1}x \in \bigcap_{i=1}^{k} A_i$ for some $\alpha \in \,]0,1]$, that is $x \in \bigcap_{i=1}^{k} \alpha A_i \subseteq \bigcap_{i=1}^{k} A_i$. Hence $x \in \bigcap_{i\in I} A_i$, and so $\bigcap_{i\in I} A_i \neq \emptyset$.

Assume now that our claim is not true, that is $o \in \operatorname{conv}\{u_i \mid i \in I\}$; then there exists $(\lambda_i)_{i\in I} \subseteq \mathbb{R}_+$ such that

$$\sum\nolimits_{i\in I}\lambda_i = 1, \quad \sum\nolimits_{i\in I}\lambda_i u_i = o. \tag{6}$$

(a) Suppose that $\lambda_j = 0$ for some $j \in I$. Then we get the contradiction $o = \sum_{i\in I\setminus\{j\}} \lambda_i u_i \in A_j$ by (5). Hence $\lambda_i > 0$ for all $i \in I$.

(b) Suppose now that the family $(u_i)_{i\in I}$ is not a.i.; then there exists $(\mu_i)_{i\in I} \subseteq \mathbb{R}$ such that

$$\sum_{i\in I}\mu_i = 0, \quad \sum_{i\in I} |\mu_i| \neq 0, \quad \sum_{i\in I}\mu_i u_i = o. \tag{7}$$

From (7) we have that $I_+ := \{i\in I \mid \mu_i > 0\} \neq \emptyset$. Then $\alpha := \min\{\lambda_i/\mu_i \mid i\in I_+\} > 0$, and so $\nu_i := \lambda_i - \alpha\mu_i \geq 0$ for $i \in I$ and there exists $j \in I$ such that $\nu_j = 0$. Using (6) and (7) we obtain that $\sum_{i\in I} \nu_i = 1$ and $\sum_{i\in I} \nu_i u_i = o$, that is (6) holds with λ_i replaced by ν_i. Using (a), this is a contradiction because $\nu_j = 0$. Hence the family $(u_i)_{i\in I}$ is affinely independent.

(c) Because $(u_i)_{i\in I}$ is a.i. and $o \in \operatorname{conv}\{u_i \mid i \in I\}$ we have that $\mathbb{R}^n = \bigcup_{i\in I} Q_i$ by Lemma 2.10, where Q_i is defined in (3). Clearly, $Q_j = \mathbb{R}_+ \cdot \operatorname{conv}\{u_i \mid i \in I \setminus \{j\}\}$ for $j \in I$. Taking into account (5), we obtain that $\mathbb{R}^n = \bigcup_{j\in I} Q_j \subseteq \bigcup_{j\in I} \mathbb{R}_+ A_j$, and so $\mathbb{R}^n \setminus \{o\} = \bigcup_{j\in I} \mathbb{P}A_j$, contradicting our hypothesis.

Therefore, our claim is true, and so $\bigcap_{j\in J} C_j \neq \emptyset$ for any $J \subseteq \overline{1,m}$ with $\operatorname{card} J = n+1$. Because all the sets C_j are nonempty, by (the classic) Helly's theorem (see e.g. [12, Th. 1.1.6]) one obtains that there exists $\overline{x} \in \bigcap_{j\in\overline{1,m}} C_j$. Clearly, $\overline{x} \neq o$, and so $-\overline{x} \notin \mathbb{P}C_j$ for all $j \in \overline{1,m}$, whence $\bigcup_{j\in\overline{1,m}}(\mathbb{P}C_j) \neq \mathbb{R}^n \setminus \{o\}$. Otherwise, for some $j' \in \overline{1,m}$ and $\lambda \in \mathbb{P}$ one has $-\lambda\overline{x} \in C_{j'}$, and so we get the contradiction $o = \frac{\lambda}{1+\lambda}\overline{x} + \frac{1}{1+\lambda}(-\lambda\overline{x}) \in C_{j'}$. The proof is complete. □

The following result (less $\bigcup_{i\in\overline{1,m}} K_i \neq \mathbb{R}^n$) is stated explicitly in [13, Cor. 3], where one refers to [10, Satz II] and [7, Cor. p. 456]; in the proof of [13, Cor. 3] one uses [13, Th. 2].

Corollary 2.12. *Let $K_1, K_2, ..., K_m \subseteq \mathbb{R}^n$, $m \geq n + 1$, be a family of nontrivial pointed convex cones. If $\bigcap_{j\in J} K_j \neq \{o\}$ for any $J \subseteq \overline{1,m}$ with $\operatorname{card} J = n$ and $\bigcup_{j\in J} K_j \neq \mathbb{R}^n$ for any $J \subseteq \overline{1,m}$ with $\operatorname{card} J = n + 1$, then $\bigcap_{i\in\overline{1,m}} K_i \neq \{o\}$ and $\bigcup_{i\in\overline{1,m}} K_i \neq \mathbb{R}^n$.*

Proof. Set $C_i := K_i \setminus \{0\}$ for $i \in \overline{1,m}$ and apply Proposition 2.11. □

Lemma 2.10 shows that the condition "$\bigcup_{j\in J} K_j \neq \mathbb{R}^n$ for any $J \subseteq \overline{1,m}$ with $\operatorname{card} J = n+1$" is essential for the validness of the conclusion of Corollary 2.12; similar remarks are valid for Proposition 2.11 and the next result.

Theorem 2.13. ([5, Th. 4.3], Helly-type Theorem) *Let $C_1, C_2, ..., C_m \subseteq \mathbb{S}$ for $m \geq n+1$ be a family of nonempty s-convex sets. If $\bigcap_{j\in J} C_j \neq \emptyset$ for any $J \subseteq \overline{1,m}$ with $\operatorname{card} J = n$ and $\bigcup_{j\in J} C_j \neq \mathbb{S}$ for any $J \subseteq \overline{1,m}$ with $\operatorname{card} J = n + 1$, then $\bigcap_{i\in I} C_i \neq \emptyset$.*

Proof. Set $P_i := \mathbb{P}C_i$ for $i \in \overline{1,m}$. Because $\mathbb{P} =]0, 1] \cdot \mathbb{P}$, using Proposition 1.1 we have that $o \notin P_i = \operatorname{conv} P_i =]0, 1]P_i \neq \emptyset$ for every $i \in \overline{1,m}$. By hypothesis, for any $J \subseteq \overline{1,m}$ with $\operatorname{card} J = n$ one has $\emptyset \neq \bigcap_{j\in J} C_j \subseteq \bigcap_{j\in J} P_j$, and so $\bigcap_j P_j \neq \emptyset$. Assume that $\bigcup_{j\in J} P_j = \mathbb{R}^n \setminus \{o\}$ for some $J \subseteq \overline{1,m}$ with $\operatorname{card} J = n + 1$; then $\mathbb{S} = \rho\big(\bigcup_{j\in J} P_j\big) = \bigcup_{j\in J} \rho(P_j) = \bigcup_{j\in J} C_j$, a contradiction.

Hence the hypothesis of Proposition 2.11 holds; applying this result we have that there exists $x \in \bigcap_{i=1}^m P_i$, and so $\rho(x) \in \bigcap_{i=1}^m \rho(P_i) = \bigcap_{i=1}^m C_i$. $\square$

Having in view the equivalence (i) $\Leftrightarrow$ (ix) from Proposition 1.1, in the preceding proof we could take $]0,1]C_i$ instead of $\mathbb{P}C_i$ for $i \in \overline{1,m}$ when Φ is, moreover, convex (hence sublinear).

Theorem 2.14. ([5, Th. 4.8], Carathéodory-type Theorem) *Let $S \subseteq \mathbb{S}$ be a nonempty hull-addible set. Then for each $x \in \operatorname{sco} S$, there exist $(x_i)_{i=1}^n \subseteq S$ and $(\lambda_i)_{i=1}^n \subseteq \mathbb{P}$ with $\sum_{i=1}^n \lambda_i = 1$ such that $x = (\mathrm{s})\sum_{i=1}^n \lambda_i x_i$.*

Proof. Take $x \in \operatorname{sco} S$. Then

$$x \in \mathbb{R}_+ \operatorname{sco} S = \mathbb{R}_+ \rho(\operatorname{conv} S) = \mathbb{R}_+ \operatorname{conv} S = \operatorname{conv}(\mathbb{R}_+ S).$$

Because $x \neq o$, using Lemma 2.3 (ii), there exist $(x_i)_{i=1}^n \subseteq S$ and $(\lambda_i)_{i=1}^n \subseteq \mathbb{R}_+$ such that $x = \sum_{i=1}^n \lambda_i x_i$; then $\lambda := \sum_{i=1}^n \lambda_i > 0$. Setting $\lambda_i' := \lambda^{-1}\lambda_i$ (≥ 0), we have that $\sum_{i=1}^n \lambda_i' = 1$ and so $\lambda^{-1}x =: x' = \sum_{i=1}^n \lambda_i' x_i$. Hence $x = \rho(x') = (\mathrm{s})\sum_{i=1}^n \lambda_i' x_i$. $\square$

Consider $C \subseteq \mathbb{S}$ an s-convex set. One says that $x \in C$ is an *extreme point* of $C \subseteq \mathbb{S}$ if $[x_1, x_2 \in C \wedge \lambda \in\,]0,1[\, \wedge\, x = \lambda x_1 +_s (1-\lambda)x_2] \Rightarrow x_1 = x_2$; the set of s-extreme points of C is denoted by $\operatorname{sext} C$.

Proposition 2.15. ([5, Prop. 5.3 and Cor. 5.5]) *Let $C \subseteq \mathbb{S}$ be s-convex and $x \in C$. Then*

(i) *$x \in \operatorname{sext} C$ if and only if $\mathbb{R}_+ x$ is an extreme ray of $\mathbb{R}_+ C$.*

(ii) *Assume that $x := (\mathrm{s})\sum_{i=1}^k \lambda_i x_i$ with $(x_i)_{i=1}^k \subseteq C$, $(\lambda_i)_{i=1}^k \subseteq \mathbb{P}$ and $\sum_{i=1}^k \lambda_i = 1$; if $x \in \operatorname{sext} C$ then $x_i = x$ for $i \in \overline{1,k}$.*

Proof. Notice that having the pointed convex cone $P \subseteq \mathbb{R}^n$, $\mathbb{R}_+ x$ with $x \in P \setminus \{0\}$ is an extreme ray of P if and only if for all $y, z \in P \setminus \{0\}$ one has $x \in \mathbb{P}y + \mathbb{P}z \Rightarrow \mathbb{P}x = \mathbb{P}y = \mathbb{P}z$. By induction one obtains that $\mathbb{R}_+ x$ with $x \in P \setminus \{0\}$ is an extreme ray of P if and only if for all $x_1, ..., x_k \in P \setminus \{0\}$ $(k \geq 2)$ one has $x \in \mathbb{P}x_1 + ... + \mathbb{P}x_k \Rightarrow \mathbb{P}x_i = \mathbb{P}x$ for $i \in \overline{1,k}$.

Using Proposition 1.1 we have that $K := \mathbb{R}_+ C$ is a pointed convex cone.

(i) Assume that $x \in \operatorname{sext} C$ $(\subseteq K \setminus \{0\})$ and take $y, z \in K \setminus \{0\}$ such that $x \in \mathbb{P}y + \mathbb{P}z$; setting $y' := \rho(y)$, $z' := \rho(z)$, we have that $y', z' \in C$ and $x \in \mathbb{P}y' + \mathbb{P}z'$. Hence $x = \alpha y' + \beta z'$ with $\alpha, \beta \in \mathbb{P}$, whence $\frac{1}{\alpha+\beta}x = \frac{\alpha}{\alpha+\beta}y' + \frac{\beta}{\alpha+\beta}z'$. It follows that $x = \rho\big(\frac{1}{\alpha+\beta}x\big) = \frac{\alpha}{\alpha+\beta}y' +_s \frac{\beta}{\alpha+\beta}z'$. Because $\frac{\alpha}{\alpha+\beta} \in\,]0,1[$ and $x \in \operatorname{sext} C$, we obtain that $y' = z'$, whence $\mathbb{P}y' = \mathbb{P}z'$. Hence $\mathbb{P}y = \mathbb{P}z$, which proves that $\mathbb{R}_+ x$ is an extreme ray of K.

Conversely, assume that $\mathbb{R}_+ x$ is an extreme ray of K and take $y, z \in C$, $\lambda \in\,]0,1[$ such that $x = \lambda y +_s (1-\lambda)z$. It follows that there exists $\gamma \in \mathbb{P}$ such that $\gamma x = \lambda y + (1-\lambda)z$, whence $x \in \mathbb{P}y + \mathbb{P}z$; because $\mathbb{R}_+ x$ is an extreme ray of $\mathbb{R}_+ C$ we have that $\mathbb{R}_+ y = \mathbb{R}_+ z$, and so $\{y\} = \mathbb{S} \cap \mathbb{R}_+ y = \mathbb{S} \cap \mathbb{R}_+ z = \{z\}$. Therefore, $x \in \operatorname{sext} C$.

(ii) Clearly, $\gamma x = \sum_{i=1}^k \lambda_i x_i$ for some $\gamma > 0$, and so $x \in \mathbb{P}x_1 + ... + \mathbb{P}x_k$. Because $\mathbb{R}_+ x$ is an extreme ray of $\mathbb{R}_+ C$, we have that $\mathbb{P}x = \mathbb{P}x_i$ for $i \in \overline{1,k}$, whence $x = x_i$ for $i \in \overline{1,k}$. $\square$

Note that assertion (i) of Proposition 2.15 is [5, Prop. 5.3], while assertion (ii) is established in [5, Cor. 5.5] under the additional assumption that C is closed.

Theorem 2.16. ([5, Th. 5.6 and Cor. 5.4]) *Let $C \subseteq \mathbb{S}$ be closed and s-convex. Then $C = \operatorname{sco}(\operatorname{sext} C)$; in particular, $\operatorname{sext} C \neq \emptyset$.*

Proof. Because $\mathbb{R}_+C$ is a pointed closed convex cone, using [11, Cor. 18.5.2] and Proposition 2.15 (i) we have that $\mathbb{R}_+C = \operatorname{cone}(\operatorname{sext} C)$; in particular $\operatorname{sext} C \neq \emptyset$.

Of course we have $\operatorname{sco}(\operatorname{sext} C) \subseteq C$. Take $x \in C$ $[\subseteq (\mathbb{R}_+C) \setminus \{o\}]$; hence there exist $(x_i)_{i=1}^k \subseteq \operatorname{sext} C$ and $(\lambda_i)_{i=1}^k \subseteq \mathbb{P}$ such that $x = \sum_{i=1}^k \lambda_i x_i$. Then

$$x = \rho(x) = \rho\big(\textstyle\sum_{i=1}^k \lambda_i x_i\big) = \rho\big(\sum_{i=1}^k \lambda_i' x_i\big) = \rho\big(\sum_{i=1}^k \lambda_i' x_i\big) = (\mathrm{s})\sum_{i=1}^k \lambda_i' x_i,$$

where $\lambda_i' := \lambda_i / \sum_{i=1}^k \lambda_i'$ for $i \in \overline{1,k}$. Therefore, $C = \operatorname{sco}(\operatorname{sext} C)$. □

3. A separation theorem on spheres

Recall that the support function of the nonempty subset A of the nontrivial real Hausdorff locally convex space X is the function

$$\sigma_A : X^* \to \overline{\mathbb{R}}, \quad \sigma_A(x^*) := \sup\{\langle x, x^*\rangle \mid x \in A\} \quad (x^* \in X^*),$$

where X^* is the topological dual of X, X^* is endowed with its weak-star topology w^* and $\langle x, x^*\rangle := x^*(x)$ for $x \in X$ and $x^* \in X^*$. It is clear that σ_A is convex, positively homogeneous (hence sublinear), w^*-lsc and $\sigma_A = \sigma_{\overline{\operatorname{conv}} A}$; in particular, $[\sigma_A \leq \alpha]$ is w^*-closed for every $\alpha \in \mathbb{R}$.

Proposition 3.1. *Let X be a Hausdorff locally convex space, $A \subseteq X$ a nonempty set and $\alpha \in \mathbb{R}$. Set $D_\alpha := \{x^* \in X^* \mid \langle x, x^*\rangle < \alpha \ \forall x \in A\}$.*

(i) *If $\operatorname{conv} A$ is open, then $[\sigma_A \leq \alpha] \setminus \{o\} \subseteq D_\alpha \subseteq [\sigma_A \leq \alpha]$, where o is the zero element of X^*; consequently, $D_\alpha \cup \{o\}$ is w^*-closed.*

(ii) *If $\operatorname{conv} A$ is w-compact, then D_α is σ-open, where σ is the Mackey topology on X^* (cf. [8, p. 260]).*

Proof. (i) Consider $x^* \in [\sigma_A \leq \alpha] \setminus \{o\}$. Assume that $x^* \notin D_\alpha$; then there exists $x_0 \in A$ such that $\langle x_0, x^*\rangle = \alpha$. It follows that $\langle x - x_0, x^*\rangle = \langle x, x^*\rangle - \alpha \leq 0$ for every $x \in A$, and so $\langle x', x^*\rangle \leq 0$ for every $x' \in \operatorname{conv}(A - x_0) = (\operatorname{conv} A) - x_0 =: V$. Since $\operatorname{conv} A$ is open and $x_0 \in A$, V is a neighborhood of the origin of X, whence V is absorbing; consequently $\langle x', x^*\rangle = 0$ for every $x' \in X$, whence the contradiction $x^* = o$. Hence, $[\sigma_A \leq \alpha] \setminus \{o\} \subseteq D_\alpha$ holds. The second inclusion is obvious.

Consequently, if $(0 =)\ \sigma_A(o) \leq \alpha$, then $D_\alpha \cup \{o\} = [\sigma_A \leq \alpha]$ is w^*-closed because σ_A is w^*-lsc. If $\alpha < 0$, then $D_\alpha = [\sigma_A \leq \alpha]$, and so $D_\alpha \cup \{o\}$ is w^*-closed because w^* is Hausdorff.

(ii) Take $x_0^* \in D_\alpha$. Because $\operatorname{conv} A$ is w-compact, there exists $x_0 \in \operatorname{conv} A$ such that $\langle x_0, x_0^*\rangle = \max_{x \in \operatorname{conv} A} \langle x, x_0^*\rangle$ $(=: \beta)$. It follows easily that $\beta = \max_{x \in A} \langle x, x_0^*\rangle$, and so we (may) take $x_0 \in A$, whence $\beta < \alpha$ because $x_0^* \in D_\alpha$. Then

$$V := \{x^* \in X^* \mid |\langle x, x^*\rangle| \leq \gamma \ \forall x \in \operatorname{conv} A\} = \gamma([-1, 1] \operatorname{conv} A)^\circ$$

is a σ-neighborhood of o, where $\gamma := (\alpha - \beta)/2 > 0$.

Then, for $x^* \in V$ and $x \in A$ one has

$$\langle x, x_0^* + x^* \rangle = \langle x, x_0^* \rangle + \langle x, x^* \rangle \leq \langle x_0, x_0^* \rangle + \langle x, x^* \rangle \leq \langle x_0, x_0^* \rangle + \gamma = \tfrac{1}{2}(\alpha + \langle x_0, x_0^* \rangle) < \alpha;$$

consequently, $x_0^* + V \subseteq D_\alpha$. Therefore, D_α is σ-open. □

In the next result we apply Proposition 3.1 for $X := \mathbb{R}^n$ $(n \geq 2)$ endowed, as in the previous sections, with the usual inner product $\langle \cdot, \cdot \rangle$ and its Euclidean norm $\|\cdot\|$, X^* being identified with $\mathbb{R}^n$; of course, in this case, the topologies w, w^* and σ coincide with the usual topology of $\mathbb{R}^n$.

Theorem 3.2. *Let $B_1, B_2 \subseteq \mathbb{S}^{n-1}$ $(n \geq 2)$ be nonempty sets such that $P_1 := \mathbb{P}B_1$ and $P_2 := \mathbb{P}B_2$ are convex (equivalently, B_1 and B_2 are spherically convex), where $\mathbb{P} :=]0, \infty[$. Consider the set $E := E_1 \cap E_2$, where*

$$E_1 := \{u \in \mathbb{S}^{n-1} \mid \langle b_1, u \rangle > 0 \ \forall b_1 \in B_1\}, \quad E_2 := \{u \in \mathbb{S}^{n-1} \mid \langle b_2, u \rangle < 0 \ \forall b_2 \in B_2\}.$$

Then the following assertions hold:

(i) *If $E \neq \emptyset$, then $B_1 \cap B_2 = \emptyset$.*
(ii) *If $B_1 \cap B_2 = \emptyset$ and P_1, P_2 are open, then E is nonempty and $\mathbb{R}_+E$ is a pointed, closed and convex cone; consequently, E is a closed spherical convex set.*
(iii) *If $B_1 \cap B_2 = \emptyset$ and B_1, B_2 are closed, then E is nonempty and $\mathbb{P}E$ is convex and open; consequently, E is an open spherical convex set.*

Proof. We clearly have

$$\begin{aligned} \mathbb{P}E_1 &= \{u \in \mathbb{R}^n \mid \langle x_1, u \rangle > 0 \ \forall x_1 \in P_1\} = \{u \in \mathbb{R}^n \mid \langle x_1, u \rangle > 0 \ \forall x_1 \in B_1\}, \\ \mathbb{P}E_2 &= \{u \in \mathbb{R}^n \mid \langle x_2, u \rangle < 0 \ \forall x_2 \in P_2\} = \{u \in \mathbb{R}^n \mid \langle x_2, u \rangle < 0 \ \forall x_2 \in B_2\}, \end{aligned} \tag{8}$$

and so $\mathbb{P}E_1$ and $\mathbb{P}E_2$ are both convex sets. Because $B_1, B_2, E_1, E_2 \subset \mathbb{S}^{n-1}$, one has $P_1 \cap P_2 = \mathbb{P}(B_1 \cap B_2)$ and $\mathbb{P}E = \mathbb{P}E_1 \cap \mathbb{P}E_2$; hence $\mathbb{P}E$ is also convex.

(i) The assertion is obviously true.

(ii) Because $B_1 \cap B_2 = \emptyset$, one has $P_1 \cap P_2 = \emptyset$. As P_1 and P_2 are nonempty disjoint open convex sets, by a well-known separation theorem, there exists $u \in \mathbb{S}^{n-1}$ and $\gamma \in \mathbb{R}$ such that $\langle x_1, u \rangle \geq \gamma \geq \langle x_2, u \rangle$ for all $(x_1, x_2) \in P_1 \times P_2$. It follows that $\gamma = 0$ because $P_1 = \mathbb{P}P_1$ and $P_2 = \mathbb{P}P_2$; moreover, because P_1 and P_2 are open, one obtains that $\langle x_1, u \rangle > 0 > \langle x_2, u \rangle$ for $(x_1, x_2) \in P_1 \times P_2$. As $B_1 \subseteq P_1$ and $B_2 \subseteq P_2$, one has $u \in E_1 \cap E_2$, and so $E \neq \emptyset$. Applying Proposition 3.1(i) for P_2 $(= \operatorname{conv} P_2)$ and $\alpha = 0$, and taking into account the first expression of $\mathbb{P}E_2$ in (8), one obtains that $\mathbb{R}_+E_2$ $(= (\mathbb{P}E_2) \cup \{o\})$ is closed; the closedness of $\mathbb{R}_+E_1$ is obtained similarly, and so $\mathbb{R}_+E$ is closed.

(iii) Set $C_i := \operatorname{conv} B_i$ $(\subseteq P_i)$ for $i \in \{1, 2\}$. Then, for each $i \in \{1, 2\}$, C_i is compact (because B_i is a closed subset of $\mathbb{S}^{n-1}$) and $P_i = \mathbb{P}C_i$; hence $C_1 \cap C_2 = \emptyset$ and $o \notin C_1 \cup C_2$.

Using the second expression of $\mathbb{P}E_2$ in (8) and Proposition 3.1(ii), the set $\mathbb{P}E_2$ is open; the openness of $\mathbb{P}E_1$ is obtained similarly, and so $\mathbb{P}E$ is open.

It remains to show that E is nonempty. We claim $[\mathbb{P}(C_1 + rU)] \cap [\mathbb{P}(C_2 + rU)] = \emptyset$ for some $r \in \mathbb{P}$, where U is the unit closed ball of $\mathbb{R}^n$. In the contrary case, having the sequence $(r_m)_{m\geq 1} \subset \mathbb{P}$ with $r_m \to 0$, for every $m \in \mathbb{N}$ and every $i \in \{1,2\}$ there exist $t_m^i \in \mathbb{P}$, $v_m^i \in C_i$ and $x_m^i \in U$ such that $t_m^1(v_m^1 + r_m x_m^1) = t_m^2(v_m^2 + r_m x_m^2)$; clearly, we may take $t_m^1 = 1$ for $m \in \mathbb{N}$. Passing to subsequences, we may assume that $t_m^2 \to t \in [0, \infty]$, $v_m^i \to v_i \in \operatorname{cl} C_i = C_i$ for $i \in \{1,2\}$. If $t = 0$ we get the contradiction $C_1 \ni v_1 = o$; if $t = \infty$ we get the contradiction $C_2 \ni v_2 = o$. Hence $t \in \mathbb{P}$, and so $v_1 = tv_2 \in \mathbb{P}C_1 \cap \mathbb{P}C_2 = \emptyset$. Therefore, our claim is true, that is, there exists $r \in \mathbb{P}$ such that $[\mathbb{P}(C_1 + rU)] \cap [\mathbb{P}(C_2 + rU)] = \emptyset$. Because the sets $\mathbb{P}(C_i + rU)$ with $i \in \{1,2\}$ are convex sets, there exist $u \in \mathbb{S}^{n-1}$ and $\alpha \in \mathbb{R}$ such that

$$\langle t_1(v_1 + rx_1), u\rangle \geq \alpha \geq \langle t_2(v_2 + rx_2), u\rangle \quad \forall t_1, t_2 \in \mathbb{P},\ x_1, x_2 \in U,\ v_1 \in C_1,\ v_2 \in C_2.$$

It follows that $\alpha = 0$ and $\langle v_1, u\rangle - r \geq 0 \geq \langle v_2, u\rangle + r$, whence $u \in E_1 \cap E_2 = E$. $\square$

Of course, taking B_1 and B_2 subsets of $\mathbb{S}_\Phi$ instead of $\mathbb{S}^{n-1}$, the conclusions of Theorem 3.2 remain true.

Theorem 3.2 is established by Han and Nashimura in [6, Th. 1]. Note that from Theorem 3.2(iii) one obtains the strict separation theorem of disjoint closed strongly convex sets on a sphere by a hyperplane through the centre from [2, Th. 1] (attributed to an unpublished paper by BC Rennie), while from Theorem 3.2(ii) one obtains the separation theorem of convex bodies with disjoint interiors stated in [9, Lem. 2].

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