Journal of Convex Analysis
Volume 29 (2022), No. 4, 1023–1035

# Which Convexity Properties are Preserved under Linear Mappings?

**Christer Oscar Kiselman***
*Department of Information Technology, Uppsala University, Sweden*
*kiselman@it.uu.se*

Received: November 11, 2021
Accepted: January 31, 2022

We study marginal functions of functions defined on arbitrary subsets of $\mathbb{R}^n$, especially the convexity properties which such functions can possess.

*Keywords:* Convex sets, convex functions, discrete convexity, marginal functions, linear mappings, affine mappings.

*2010 Mathematics Subject Classification:* 26A51, 26B25, 52A40, 52C99.

## 1. Introduction

The shadow of a convex body in three-space is convex. More generally, a convex set is mapped onto a convex set by an affine mapping from one vector space into another. It is convenient to express this simple fact by saying that the marginal function of a convex function is convex, a most useful fact in optimization theory in real vector spaces.

Coming to discrete convexity, these properties become highly nontrivial.

In this paper we study to which degree convexity properties are preserved for functions which are defined on arbitrary subsets of $\mathbb{R}^n$ and are convex extensible or close to being convex extensible, a study which by necessity leads to considerations of discrete convexity.

The special case when the points in a given subset $A$ of $\mathbb{R}^n$ all have integer coordinates have been studied by Kiselman and Samieinia [6, 7]. In the present note it is no longer assumed that the points in a set $A$ have integer coordinates, nor that $A$ is a discrete set. Its image under an affine mapping need not be discrete even if $A$ is.

### Notation

We shall denote by $\mathscr{P}(X)$ the family of all subsets of a given set $X$. Given a mapping $f\colon X \to Y$ from a set $X$ into another set $Y$, we define two mappings on a higher level:

$$f^*\colon \mathscr{P}(Y) \to \mathscr{P}(X), \text{ defined by } f^*(B) = \{x \in X;\ f(x) \in B\}, \quad B \subset Y, \quad \text{and}$$

$$f_*\colon \mathscr{P}(X) \to \mathscr{P}(Y), \text{ defined by } f_*(A) = \{f(x);\ x \in A\}, \quad A \subset X.$$

We have $f_*(f^*(B)) \subset B$ and $f^*(f_*(A)) \supset A$.

* URL: www.cb.uu.se/~kiselman; ORCiD: 0000-0002-0262-8913.

ISSN 0944-6532 / \$ 2.50 

If $f\colon X \to Y$ and $B \subset X$, we define the *restriction of* $f$ *to* $B$, denoted by $f|_B$, as the mapping defined in $B$ and taking the same values as $f$ there. We say that $f$ is an *extension of* $f|_B$.

To any mapping $f\colon X \to Y$ we associate its *graph*, defined as

$$\operatorname{graph}(f) = \{(x, y) \in X \times Y;\ f(x) = y\}.$$

To a function $f\colon X \to [-\infty, +\infty] = \mathbb{R} \cup \{-\infty, +\infty\}$ we define its *epigraph*, *finite epigraph*, *strict epigraph*, and *finite strict epigraph*:

$$\begin{aligned}
\operatorname{epi}(f) &= \{(x, t) \in X \times [-\infty, +\infty];\ t \geqslant f(x)\},\\
\operatorname{epi}^{\text{finite}}(f) &= \{(x, t) \in X \times \mathbb{R};\ t \geqslant f(x)\},\\
\operatorname{epi}_{\text{strict}}(f) &= \{(x, t) \in X \times [-\infty, +\infty];\ t > f(x)\},\\
\operatorname{epi}^{\text{finite}}_{\text{strict}}(f) &= \{(x, t) \in X \times \mathbb{R};\ t > f(x)\}.
\end{aligned}$$

We shall write $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ for, respectively, the semiring of natural numbers, the ring of integers, and the field of real numbers.

## 2. Convex sets and functions

To any set $A \subset \mathbb{R}^n$ we define its *convex hull*, denoted by $\operatorname{cvxh}(A)$, as the smallest convex set containing $A$.

A function $F\colon \mathbb{R}^m \to [-\infty, +\infty]$ is convex if and only if $\operatorname{epi}^{\text{finite}}(F)$ is a convex subset of $\mathbb{R}^m \times \mathbb{R}$; also if and only if $\operatorname{epi}^{\text{finite}}_{\text{strict}}(F)$ is convex.

Given a subset $B$ of $\mathbb{R}^m$ and a function $f\colon B \to [-\infty, +\infty]$, we define its *convex envelope*, denoted by $\operatorname{cvxe}(f)$, to be the largest convex function $g\colon \mathbb{R}^n \to [-\infty, +\infty]$ such that $g|_B \leqslant f$. We note the relations

$$\operatorname{epi}^{\text{finite}}_{\text{strict}}(\operatorname{cvxe}(F)) = \operatorname{cvxh}(\operatorname{epi}^{\text{finite}}_{\text{strict}}(F)) \subset \operatorname{cvxh}(\operatorname{epi}^{\text{finite}}(F)) \subset \operatorname{epi}^{\text{finite}}(\operatorname{cvxe}(F)) \qquad (1)$$

for any function $F\colon \mathbb{R}^n \to [-\infty, +\infty]$. The last two relations can be strict inclusions; see Chapter 9 in [5].

A function $f\colon B \to [-\infty, +\infty]$, where $B$ is a subset of $\mathbb{R}^n$, is called *convex extensible* if there is a convex function $F\colon \mathbb{R}^n \to [-\infty, +\infty]$ which is an extension of $f$. There may exist several convex extensions. The convex envelope is the largest convex extension if there is one.

**Definition 2.1.** Given any subset $S$ of $\mathbb{R}^n$ we shall say that a subset $A$ of $S$ is $S$*-convex* if there is a convex set $C \subset \mathbb{R}^n$ such that $C \cap S = A$. □

For more on this concept of convexity, see Section 10.3 in Kiselman [5].

**Proposition 2.2.** *Given a subset $S$ of $\mathbb{R}^n$ and a subset $A$ of $S$, the following three properties are equivalent.*

($\alpha'$) *$A$ is $S$-convex;*

($\beta'$) $\operatorname{cvxh}(A) \cap S = A$*;*

($\gamma'$) $\operatorname{cvxh}(A) \cap S \subset A$*.*

**Proof.** That $(\gamma') \Leftrightarrow (\beta') \Rightarrow (\alpha')$ is easy to see.

Assume now that $(\alpha')$ holds. Then $\operatorname{cvxh}(A) = \operatorname{cvxh}(C \cap S) \subset C$, so that

$$\operatorname{cvxh}(A) \cap S \subset C \cap S = A,$$

i.e., $(\gamma')$ holds. □

**Proposition 2.3.** *Given a set $B \subset \mathbb{R}^m$, we define $S = B \times \mathbb{R}$. For any function $f\colon B \to [-\infty, +\infty]$ the following three properties are equivalaent.*

$(\delta')$ *$f$ is convex extensible;*

$(\varepsilon')$ $\operatorname{epi}^{\text{finite}}(f)$ *is $S$-convex;*

$(\zeta')$ $\operatorname{epi}^{\text{finite}}_{\text{strict}}(f)$ *is $S$-convex.*

**Proof.** If $f$ is convex extensible, then $\operatorname{epi}^{\text{finite}}(f) = \operatorname{epi}^{\text{finite}}(\operatorname{cvxe}(f)) \cap S$, showing that $\operatorname{epi}^{\text{finite}}(f)$ is $S$-convex.

Conversely, assume that $\operatorname{epi}^{\text{finite}}(f) = C \cap S$ for some convex set $C$ in $\mathbb{R}^m \times \mathbb{R}$ and take any point $(x, t) \in S$. If $t > (\operatorname{cvxe}(f))(x)$, thus if $(x, t)$ belongs to $\operatorname{epi}^{\text{finite}}_{\text{strict}}(\operatorname{cvxe}(f))$, then $(x, t)$ belongs to $\operatorname{cvxh}(\operatorname{epi}(f)) = \operatorname{cvxh}(C \cap S) \cap S = C \cap S = \operatorname{epi}^{\text{finite}}(f)$ according to (1). So $t \geqslant f(x)$. Taking the infimum of all $t > (\operatorname{cvxe}(f))(x)$, we see that $f(x) \leqslant (\operatorname{cvxe}(f)(x)$, thus that $f$ is convex extensible.

For $\operatorname{epi}^{\text{finite}}_{\text{strict}}(f|_B)$ in $(\zeta')$ the reasoning is the same. □

## 3. Convolution

Given two functions $f, g\colon \mathbb{R}^n \to \mathbb{R}$ we define their *convolution product* $h = f * g$ by

$$h(x) = (f * g)(x) = \sum_{y \in \mathbb{R}^n} f(x - y) g(y), \qquad x \in \mathbb{R}^n,$$

provided that the sum can be given a meaning. In the cases we consider, one of the factors is non-zero in a finite set only, so convergence is not an issue.

The *Kronecker delta*, written $\delta_p$, is defined by $\delta_p(p) = 1$ and $\delta_p(x) = 0$ for $x \neq p$. For $p = 0$ it is a neutral element under convolution: $f * \delta_0 = f$ for all $f\colon \mathbb{R}^n \to \mathbb{R}$.

## 4. Jensen's inequality

For an $(N+1)$-tuple $P = (p^{(0)}, \dots, p^{(N)}) \in (\mathbb{R}^n)^{N+1}$ of given elements of $\mathbb{R}^n$, we define functions

$$J_P = \sum_{j=0}^{N} s_j \delta_{-p^{(j)}}\colon \mathbb{R}^n \to [0, 1],$$

where the $s_j$ are nonnegative numbers with sum equal to 1 and where $\sum s_j p^{(j)} = 0$. We get an operator by taking the convolution $f \mapsto J_P * f$,

$$(J_P * f)(x) = \sum s_j f(x + p^{(j)}), \qquad x \in \mathbb{R}^n, \quad f\colon \mathbb{R}^n \to [-\infty, +\infty].$$

Since infinitive values are allowed, we should agree to define

$$0 \cdot (\pm\infty) = 0 \text{ and } s_j \cdot (+\infty) = +\infty,\ s_j \cdot (-\infty) = -\infty \text{ when } s_j > 0.$$

The set of all such functions $J_P$ will be denoted by $\mathscr{J}_N(\mathbb{R}^n)$. (The function $J_P$ depends on the $s_j$ but this need not be indicated – when the points $p^{(j)}$ are linearly independent, which is often the case, the coefficients $s_j$ are determined by the points $p^{(j)}$.) We can take $N = 0 \in \mathbb{N}$ and $P = 0 \in \mathbb{R}^n$, which yields the identity operator $f \mapsto J_0 * f = \delta_0 * f = f$.

It is well known that a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is convex if and only if $J_P * f \geqslant f$ for all pairs $P = (p^{(0)}, p^{(1)}) \in (\mathbb{R}^n)^2$. The inequality $J_P * f \geqslant f$ is known as *Jensen's inequality*[1] [3]. More generally, $J_P * f \geqslant f$ for $P = (p^{(0)}, \dots, p^{(N)})$ holds for any $N \geqslant 1$.

The convex envelope of a function $f\colon \mathbb{R}^n \to [-\infty, +\infty]$ is given by the formula

$$(\operatorname{cvxe}(f))(x) = \inf_P \, (J_P * f)(x + p^{(j)}), \qquad x \in \mathbb{R}^n,$$

where the infimum is taken over all $N$ and all $P = (p^{(0)}, \dots, p^{(N)}) \in (\mathbb{R}^n)^{N+1}$. By Carathéodory's theorem[2], it suffices to take $N = n$ here; see, e.g., Rockafellar [8], and Hiriart-Urruty & Lemaréchal [3].

## 5. Marginal functions

We shall study functions defined on an arbitrary subset $A$ of $\mathbb{R}^n$ and their marginal functions.

Given a subset $A$ of $\mathbb{R}^n$, we define a projection $\pi\colon \mathbb{R}^n \to \mathbb{R}^m$, $1 \leqslant m \leqslant n-1$, by

$$\pi(x_1, \dots, x_n) = (x_1, \dots, x_m), \qquad (x_1, \dots, x_n) \in \mathbb{R}^n, \text{ and}$$

$$\pi_*(A) = \{(x_1, \dots, x_m);\ (x_1, \dots, x_n) \in A \text{ for some point } (x_{m+1}, \dots, x_n) \in \mathbb{R}^{n-m}\}$$

the image of $A$ under $\pi_*$. We shall write the projection as

$$\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m} \ni (x, y) \mapsto x \in \mathbb{R}^m.$$

**Definition 5.1.** Let $f\colon A \to [-\infty, +\infty]$ be any function defined on a nonempty subset $A$ of $\mathbb{R}^n$, and let $\pi$ be the projection $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m} \ni (x, y) \mapsto x \in \mathbb{R}^m$. The *marginal function* $f_{\mathrm{marg}}$ of $f$ is defined by

$$f_{\mathrm{marg}}(x) = \inf_{y \in \mathbb{R}^{n-m}} \left(f(x, y);\ (x, y) \in A\right), \qquad x \in \pi_*(A),$$

thus a function of only $m$ variables, but not always defined in all of the target space $\mathbb{R}^m$. □

The marginal function is also known as the *optimal value function* or the *primal function*; see for example Singer [9].

For the definition and first properties of marginal functions, see for example Rockafellar [8] and Hiriart-Urruty & Lemaréchal [3]. Fiacco & Kyparisis [2] and Andramonov & Ellero [1] have studied generalized convexity properties of marginal functions. However, the works cited above consider functions of real variables or defined on Banach spaces, not on discrete sets.

[1] Named for Johan Jensen (1859–1925).
[2] Named for Constantin Carathéodory (1873–1950).

**Proposition 5.2.** *If $A$ is a convex subset of $\mathbb{R}^n$ and $f\colon A \to [-\infty, +\infty]$ is convex, then its marginal function, defined on the convex set $\pi_*(A)$, is convex as well.*

This result is well known and easy to prove.

When $A$ is not convex, we want to find conditions under which $f_{\mathrm{marg}}$ is convex extensible. To this end we shall use second-order difference operators, establishing a kind of duality between such operators and families of functions; cf. Kiselman [4].

**Proposition 5.3.** *Let $A$ be an arbitrary nonempty subset of $\mathbb{R}^n$ and $f\colon A \to [-\infty, +\infty]$ any function defined on $A$. Then the two functions $(\mathrm{cvxe}(f))_{\mathrm{marg}}$ and $\mathrm{cvxe}(f_{\mathrm{marg}})$, both defined on $\mathbb{R}^m$, are equal.*

**Proof.** It is enough to prove this result for functions which are bounded from below. Indeed, we can define $f_r = f \vee r$ for a real number $r$. Then $f_r \searrow f$ as $r \searrow -\infty$. If the result is true for $f_r \geqslant r$, then it follows for its limit $f$; we have $\mathrm{cvxe}(f_r) \searrow \mathrm{cvxe}(f)$ and $(f_r)_{\mathrm{marg}} \searrow f_{\mathrm{marg}}$. The value $+\infty$ is allowed – it is easier to calculate if not both $-\infty$ and $+\infty$ are present.

Thus we shall now assume that $f$ is bounded from below. Let us write $F$ for $\mathrm{cvxe}(f)$ for brevity. This function is convex – so is its marginal function $F_{\mathrm{marg}}$.

We have $F|_A \leqslant f$, which implies that $(F|_A)_{\mathrm{marg}} \leqslant f_{\mathrm{marg}}$. In consequence we directly have $F_{\mathrm{marg}} = \mathrm{cvxe}(F_{\mathrm{marg}}) \leqslant \mathrm{cvxe}(f_{\mathrm{marg}})$ in all of $\mathbb{R}^m$.

To prove that, conversely, $F_{\mathrm{marg}} \geqslant \mathrm{cvxe}(f_{\mathrm{marg}})$, we let $p$ be an arbitrary point in $\mathbb{R}^m$ and $\varepsilon$ any positive number.

There exists a point $q \in \mathbb{R}^{n-m}$ such that

$$F_{\mathrm{marg}}(p) + \varepsilon > F(p, q). \tag{2}$$

Then, by the definition of the convex envelope, there are points $(p^{(j)}, q^{(j)}) \in \mathbb{R}^n$ with

$$F(p, q) + \varepsilon > \sum s_j f(p^{(j)}, q^{(j)}) \tag{3}$$

for some $s_j \geqslant 0$ with sum equal to 1 and with $\sum s_j q^{(j)} = q$.
Combining (2) and (3) we obtain

$$F_{\mathrm{marg}}(p) + 2\varepsilon > F(p, q) + \varepsilon > \sum s_j f(p^{(j)}, q^{(j)}).$$

Here the right-hand side majorizes

$$\sum s_j f_{\mathrm{marg}}(p^{(j)}) \geqslant \mathrm{cvxe}(f_{\mathrm{marg}})(p).$$

Taking the infimum over all $\varepsilon > 0$, we are done. □

## 6. The partial Fenchel transformation

In the study of marginal functions it will be of interest to put to work a partial Fenchel transformation. First a reminder about the original Fenchel transformation:

**Definition 6.1.** Given any function $f\colon \mathbb{R}^n \to [-\infty, +\infty]$ we define its *Fenchel transform*

$$\tilde{f}(\xi) = \sup_{x \in \mathbb{R}^n} (\xi \cdot x - f(x)), \qquad \xi \in \mathbb{R}^n,$$

actually defined on the space dual to the first space. □

The second transform $\tilde{\tilde{f}}$ is a minorant of $f$. We have equality in the inequality $\tilde{\tilde{f}} \leqslant f$ if and only if $f$ is convex, lower semicontinuous and takes the value minus infinity only if it is equal to $-\infty$ everywhere.

We have $\tilde{\tilde{f}} \leqslant \operatorname{cvxe}(f)$, possibly with strict inequality at some points, since $\operatorname{cvxe}(f)$ need not be lower semicontinuous.

Here we shall need to transform in some of the variables only:

**Definition 6.2.** Given any function $f\colon A \to [-\infty, +\infty]$, where $A$ is a nonempty subset of $\mathbb{R}^n$, and a number $m$, $1 \leqslant m \leqslant n-1$, we shall write the variables as $(x, y) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$. We define its *partial Fenchel transform*

$$f^\diamond(x, \eta) = \sup_{y \in \mathbb{R}^{n-m}} (\eta \cdot y - f(x, y);\ (x, y) \in A), \qquad (x, \eta) \in \pi_*(A) \times \mathbb{R}^{n-m}. \tag{4}$$

Strictly speaking, the variable $\eta$ should belong to the space dual to that of $y$. □

We note the special case

$$f_{\mathrm{marg}}(x) = \inf_{y \in \mathbb{R}^{n-m}} f(x, y) = -f^\diamond(x, 0), \qquad x \in \mathbb{R}^m. \tag{5}$$

The second transform $f^{\diamond\diamond}$ is convex and lower continuous in the second variable, and its restriction to $A$ is a minorant of $f$.

We have $f^{\diamond\diamond}|_A \leqslant f$, implying that $(f^{\diamond\diamond})_{\mathrm{marg}} \leqslant f_{\mathrm{marg}}$, but in fact the two marginal functions are equal, which is an important fact for our study. This is a consequence of formula (5). In fact, if $f_{\mathrm{marg}}(x) \geqslant a$, then $-f^\diamond(x, 0) \geqslant a$ in view of (5), and so

$$f^{\diamond\diamond}(x, y) = \sup_{\eta \in \mathbb{R}^{n-m}} (\eta \cdot y - f^\diamond(x, \eta)) \geqslant -f^\diamond(x, 0) \geqslant a \text{ for all } y \in \mathbb{R}^{n-m},$$

showing that $(f^{\diamond\diamond})_{\mathrm{marg}}(x) \geqslant a$.

## 7. Conditions for convex extensibility

**Proposition 7.1.** *Given any nonempty subset $A$ of $\mathbb{R}^n$, a function $f\colon A \to \mathbb{R}$ is convex extensible if and only if for all $J_P \in \mathscr{J}_n(\mathbb{R}^n)$ we have $(J_P * f)(x) \geqslant f(x)$ at all the points $x$ and $x + p^{(j)}$ for which the expression is defined; i.e., such when $x$ and the $x + p^{(j)}$ belong to $A$.*

**Proof.** If $F$ is convex and defined on the whole space, then $J_P * F \geqslant F$ for all $J_P \in \mathscr{J}_n(\mathbb{R}^n)$. Then this is true for its restriction $f$ when all points $x$ and $x + p^{(j)}$ that occur belong to $A$.

Conversely, if we can find an $(n+1)$-tuple $P$ and a point $x$ such that $(J_P * f)(x)$ and $f(x)$ are defined and satisfy $(J_P * f)(x) < f(x)$, then the convex envelope of $f$ is not an extension of $f$. □

## 8. Functions of two real variables

Let us first, for training, look at functions defined on an arbitrary subset of $\mathbb{R}^2$ and their marginal functions.

It seems not to be easy to study convexity properties of the marginal function of a single function. This can be illustrated by the fact that, for any function $\varphi\colon \mathbb{Z} \to \mathbb{Z}$, the function of $f(x) = |x_2 - \varphi(x_1)|$, $x = (x_1 x_2) \in \mathbb{Z}^2$, has marginal function equal to zero and still can be very irregular. If we instead consider $f_{\pm 1}(x) = f(x) \pm x_2$, the marginal function is equal to $(f_{\pm 1})_{\mathrm{marg}} = \pm\varphi$. We see that the marginal function of $f_{\pm 1}$ is convex extensible if and only if $\varphi$ is the restriction of an affine function. For this reason, we shall study convexity properties for a family of related functions depending on a real parameter.

We denote by $A_{c_1}$ the set of all points $x = (x_1, x_2)$ in $A$ with $x_1 = c_1$.

**Theorem 8.1.** *Let $f\colon A \to \mathbb{R}$ be any function defined on a nonempty subset $A$ of $\mathbb{R}^2$ and define for any real $\beta$: $\quad f_\beta(x_1, x_2) = f(x_1, x_2) + \beta x_2, \qquad (x_1, x_2) \in A.$*

*Then the marginal function $f_{\beta\,\mathrm{marg}}$ of $f_\beta$ is convex extensible for all $\beta$ if and only if*

(A) *for all $a, b, c \in A$ with $a_1 < c_1 < b_1$ there is a number $p_2$ such that the point $p = (p_1, p_2) = (c_1, p_2) = (1-s)a + sb$ belongs to the closure of the convex hull of $A_{c_1}$; and*

(B) *For all points $a, b \in A$ with $a_1 < b_1$ and for any point $p \in \mathbb{R}^2$ in the convex hull of $\{a, b\}$ and with $a_1 < p_1 < b_1$, we have $f^{\diamond\diamond}(p) \leqslant (1-s)f(a) + sf(b)$, where $s$ is determined by the equation $p_1 = (1-s)a_1 + sb_1$.*

We can express condition (B) by saying that $\inf_Q(J_Q * f) \leqslant J_P * f$, where $P = (a, b) \in A^2 \subset (\mathbb{R}^2)^2$ and $Q = (c, d) \in A^2 \subset (\mathbb{R}^2)^2$ satisfying $c_1 = d_1 = p_1$ and $c_2 \leqslant p_2 \leqslant d_2$, so that $p$ belongs to both segments $[a, b]$ and $[c, d]$. Here $J_P$ gives rise to a non-vertical operator (since $a_1 \neq b_1$) and $J_Q$ to a vertical operator (since $c_1 = d_1$). The non-vertical shall win over the vertical.

**Proof.** Since $f^{\diamond\diamond}$ has the same marginal function as $f$, we may, and shall, assume that $f$ itself is defined for all points $x = (x_1, x_2)$ with $x_1 \in \pi_*(A)$, and convex and lower semicontinuous in the second variable. Importantly, this implies that the restriction of $f$ to the closed convex hull of $A_{c_1}$ is real valued and continuous.

**1.** Suppose now that (A) and (B) are satisfied. We note that $-\infty \leqslant f_{\mathrm{marg}}(x) < +\infty$ for all $x \in \pi_*(A)$. As in the proof of Proposition 5.3, we can prove the wanted result assuming that $-\infty < f_{\mathrm{marg}} < +\infty$ in $\pi_*(A)$, and obtain the general result by working with $f_r = f \vee r \geqslant r > -\infty$ and then letting $r \searrow -\infty$.

Given any $a_1, b_1, c_1$ in $\pi_*(A)$ with $a_1 < c_1 < b_1$ we shall prove that

$$f_{\mathrm{marg}}(c_1) \leqslant (1-s)f_{\mathrm{marg}}(a_1) + sf_{\mathrm{marg}}(b_1),$$

where $s$ is determined by the equation $c_1 = (1-s)a_1 + sb_1$. To any positive number $\varepsilon$ we can find $a_2$ and $b_2$ such that $a = (a_1, a_2)$ and $b = (b_1, b_2)$ belong to $A$ and such that we have $f(a) \leqslant f_{\mathrm{marg}}(a_1) + \varepsilon$ and $f(b) \leqslant f_{\mathrm{marg}}(b_1) + \varepsilon$.

If $p$ belongs to the convex hull of $A_{c_1}$, there are points $c, d \in A$ such that $c_2 \leqslant d_2$ and $c_2 \leqslant (1-s)f(a) + sf(b) \leqslant d_2$. Given $\varepsilon > 0$ we can determine a number $t$ by the equation $(1-t)c_2 + td_2 = (1-s)a_2 + sb_2$ so that $(1-t)f(a) + tf(b) < f(p) + \varepsilon$.

(If $c_2 = d_2$, the choice of $t$ is immaterial.) Then by assumption (B), we directly have $f(p) \leqslant (1-t)f(c) + tf(d) \leqslant (1-s)f(a) + sf(b) + \varepsilon$, so that

$$\begin{aligned} f_{\mathrm{marg}}(c_1) &\leqslant f(p) \leqslant (1-t)f(c) + tf(d) \\ &\leqslant (1-s)f(a) + sf(b) + \varepsilon \leqslant (1-s)f_{\mathrm{marg}}(a) + sf_{\mathrm{marg}}(b) + 2\varepsilon. \end{aligned}$$

Since $\varepsilon$ is arbitrary, the convex extensibility of $f_{\mathrm{marg}}$ follows. The argument applies to $f_\beta$ as well.

If $p$ belongs to the closure of the convex hull of $A_{c_1}$ but not to $A_{c_1}$, then there are numbers $c_2$ arbitrarily close to $p_2$ and with $c = (c_1, c_2)$ belonging to $A$. We obtain with $a$ and $b$ as before

$$\begin{aligned} f_{\mathrm{marg}}(c_1) &\leqslant f(c) = f(p) + (f(c) - f(p)) \leqslant (1-s)f(a) + sf(b) + (f(c) - f(p)) \\ &\leqslant (1-s)f_{\mathrm{marg}}(a) + sf_{\mathrm{marg}}(b) + (f(c) - f(p)) + 2\varepsilon. \end{aligned}$$

Now $c$ and $p$ belong to the closure of the convex hull of $A_{c_1}$, where $f$ is continuous, so, by choosing $c$ close enough to $p$, the difference $f(c) - f(p)$ can be made arbitrarily small. Since $\varepsilon$ is arbitrary, convex extensibility of $f_{\mathrm{marg}}$ and $f_{\beta\,\mathrm{marg}}$ follows.

**2.** Conversely, if condition (A) is not satisfied, say that there are no points $(c_1, c_2)$ in $A$ with $c_2 \leqslant (1-s)a_2 + sb_2$ but a point $(d_1, d_2)$ in $A$ with $d_2 = (1-s)a_2 + sb_2 + \theta$ and $\theta > 0$, then we have $f_{\beta\,\mathrm{marg}}(a_1) \leqslant f_\beta(a)$ and $f_{\beta\,\mathrm{marg}}(b_1) \leqslant f_\beta(a)$, and obtain

$$\begin{aligned} &(\mathrm{cvxe}(f_{\beta\,\mathrm{marg}}))(c_1) - f_{\beta\,\mathrm{marg}}(c_1) \\ &\qquad \leqslant (1-s)f_{\beta\,\mathrm{marg}}(a) + sf_{\beta\,\mathrm{marg}}(b) - f_{\beta\,\mathrm{marg}}(d_1) \\ &\qquad \leqslant (1-s)f(a) + sf(b) - f(d) + (f(d) - f_{\beta\,\mathrm{marg}}(d_1)) - \beta\theta. \end{aligned}$$

The difference $f(d) - f_{\beta\,\mathrm{marg}}(d_1)$ is nonnegative and tends to zero as $\beta$ tends to $+\infty$; in particular it is bounded for large $\beta$. It follows that $(\mathrm{cvxe}(f_{\beta\,\mathrm{marg}}))(c_1) - f_{\beta\,\mathrm{marg}}(c_1)$ is negative for $\beta$ positive and large, proving that $f_{\beta\,\mathrm{marg}}$ is not convex extensible.

**3.** Assume now that condition (A) is satisfied but that condition (B) is not, so that, for some choice of $P \in (\mathbb{R}^2)^3$ and $p \in \mathbb{R}^2$ we have $(J_P * f)(p) < f(p)$. Since $f$ is convex in the second variable, we can find a subdifferential of $f|_{\{p_1\}\times\mathbb{R}}$ at the point

$$p = (c_1, (1-t)c_2 + td_2) = ((1-s)a_1 + sb_1, (1-t)c_2 + td_2),$$

meaning that we can choose $\beta$ such that $f(c_1, p_2) + \beta(x_2 - p_2) \leqslant f(c_1, x_2)$ for all $x_2 \in \mathbb{R}$ such that $(c_1, x_2) \in A$; in fact for all points $(c_1, x_2)$. It follows that the marginal function $f_{\beta\,\mathrm{marg}}$ of $f_\beta$ evaluated at $c_1$ is equal to $f_\beta(p)$. (Note that $J_P * f_\beta - f_\beta = J_P * f - f$.) We have

$$f_{\beta\,\mathrm{marg}}(a_1) \leqslant f_\beta(a), \quad f_{\beta\,\mathrm{marg}}(c_1) = f_\beta(p), \text{ and } f_{\beta\,\mathrm{marg}}(b_1) \leqslant f_\beta(b),$$

so that

$$\begin{aligned} (1-s)f_{\beta\,\mathrm{marg}}(a_1) + sf_{\beta\,\mathrm{marg}}(b_1) &\leqslant (1-s)f_\beta(a) + sf_\beta(b) \\ &= (J_P * f)(p) < f_\beta(p) = f_{\beta\,\mathrm{marg}}(c_1). \end{aligned}$$

Here the strict inequality holds by our assumption on $P$ and $p$. This shows that $f_{\beta\,\mathrm{marg}}$ is not convex extensible. □

## 9. Functions with integer values

### 9.1. Applying the floor function

We are interested in studying functions with integer values, obtained by rounding off a real number to the largest integer not exceeding the given number. This is done by the *floor function* $\mathbb{R} \ni t \mapsto \lfloor t \rfloor \in \mathbb{Z}$, defined by $\lfloor t \rfloor \leqslant t < \lfloor t \rfloor + 1$, $t \in \mathbb{R}$. It is also called the *integer part function*.

For integer-valued functions it is not reasonable to require that the marginal functions be convex extensible as is evident from simple examples like the following.

**Example 9.1.** Let $F\colon \mathbb{Z}^2 \to \mathbb{R}$ be defined by $F(x) = F(x_1, x_2) = \alpha x_1$ for a real constant $\alpha$, and let $f = \lfloor F \rfloor \colon \mathbb{Z}^2 \to \mathbb{Z}$ be its largest minorant with integer values. Then the marginal function $f_{\mathrm{marg}}$ of $f$ is convex extensible only when $\alpha$ is an integer.

Here $f \leqslant F < f + 1$, implying that $\operatorname{cvxe}(f) \leqslant F \leqslant \operatorname{cvxe}(f) + 1$. The second inequality is strict if $\alpha$ is rational, and an equality otherwise.

If $\alpha = p/q$ is rational with $p, q$ being positive and relatively prime integers, then $\operatorname{cvxe}(f) = F - 1 + 1/q$. Indeed, we can find an integer $x$ such that

$$f(x) = \lfloor \alpha x \rfloor = \alpha x - 1 + \frac{1}{q} = F(x) - 1 + \frac{1}{q}.$$

It follows that $\operatorname{cvxe}(f) \geqslant F - 1 + 1/q$. Since $f(x) = F(x)$ at all points $x = jq$, we see that the inequality $\operatorname{cvxe}(f)|_A \geqslant f - 1 + 1/q$ cannot be improved.

When $\alpha$ is irrational, we obtain $\operatorname{cvxe}(f) = F - 1$. This can be seen by comparing $F(x) = \alpha x$ with $G(x) = \beta x$, $\beta$ rational and close to $\alpha$, thus with a big denominator when the difference is small. □

**Example 9.2.** Let $F\colon \mathbb{R}^n \to \mathbb{R}$ be any real-valued convex function which is bounded from below and define $f = \lfloor F \rfloor \colon \mathbb{R}^n \to \mathbb{Z}$ as its largest minorant with integer values. We use the projection $\mathbb{R}^n \ni (x_1, \dots, x_n) \mapsto x_1 \in \mathbb{R}$ and observe that the marginal function $F_{\mathrm{marg}}$ of $F$ is convex and real-valued on $\mathbb{R}$. The inequality $f \leqslant F < f + 1$ implies that $f_{\mathrm{marg}} \leqslant F_{\mathrm{marg}} \leqslant f_{\mathrm{marg}} + 1$. Here the second inquality is actually strict; we need to consider three cases:

1. When, given $p_1$, the infimum of $f$ is attained at some point $p = (p_1, p_2)$, then $f_{\mathrm{marg}}(p_1) = \lfloor F_{\mathrm{marg}}(p_1) \rfloor < f_{\mathrm{marg}}(p) + 1$;
2. when the infimum is an integer and is not attained for a given $p_1$, then $f_{\mathrm{marg}}(p_1) = F_{\mathrm{marg}}(p_1) < f_{\mathrm{marg}}(p_1) + 1$;
3. when the infimum is a non-integer and is not attained for a given point $p_1$, then $f_{\mathrm{marg}}(p_1) = \lfloor F_{\mathrm{marg}}(p_1) \rfloor < f_{\mathrm{marg}}(p_1) + 1$. □

### 9.2. Almost convex extensible functions

The two examples in Subsection 9.1 motivate a definition:

**Definition 9.3.** Given any function $w\colon \mathbb{R}^n \to \mathbb{R}$ and a subset $A$ of $\mathbb{R}^n$, we shall say that a function $f\colon A \to \mathbb{R}$ is *$w$-almost convex extensible* if $f \leqslant (\operatorname{cvxe}(f) + w)|_A$. □

Being 0-almost convex extensible is the same thing as being convex extensible. And we have seen in Example 9.1 that a function can be $(1-1/q)$-almost convex extensible for $q = 2, 3, \dots$, and also 1-almost convex extensible.

Obviously it is only of interest to take $w$ nonnegative.

Most often we shall take $w$ as a constant. We can take $w = 1$, motivated by the floor function. It follows that a 1-almost convex extensible function has distance $\frac{1}{2}$ for the $L^\infty$-norm to the convex function $\operatorname{cvxe}(f)|_A + \frac{1}{2}$. It will however be convenient to use the asymmetric inequality $\operatorname{cvxe}(f)|_A \leqslant f \leqslant \operatorname{cvxe}(f)|_A + 1$.

**Remark 9.4.** If $f$ is not $w$-almost convex extensible, then we also have

$$f(q) > (J_P * f)(q) + w(q) \text{ for some point } q \in \mathbb{R}^n \text{ and some } J_P \in \mathscr{J}_n(\mathbb{R}^n), \tag{6}$$

a fact which is quite useful. To prove that a function is not $w$-almost convex extensible it enough to find $J_P \in \mathscr{J}_2(\mathbb{R}^n)$, thus with $P$ consisting of only two elements, such that $f(q) > (J_P * f)(q) + w(q)$ for some point $q$. □

**Proposition 9.5.** *Let $F\colon \mathbb{R}^n \to \mathbb{R}$ be a convex function on $\mathbb{R}^n$ which is bounded from below, and define $f\colon A \to \mathbb{Z}$ as the largest minorant of the restriction of $F$ to the subset $A$ of $\mathbb{R}^n$, $f = \lfloor F|_A \rfloor$. We shall also need $G = \operatorname{cvxe}(F|_A)$. We note that $G \geqslant F$ with equality in $A$. Then we have $F|_A - 1 < f \leqslant F|_A$, equivalently:*
*$G|_A - 1 < f \leqslant G|_A$, and $(G|_A)_{\mathrm{marg}} - 1 < f_{\mathrm{marg}} \leqslant (G|_A)_{\mathrm{marg}}$.*

**Proof.** The function $F$ plays a role here only via its restriction to $A$; its role is taken over by $G$. The convexity of $F$ implies that $G \geqslant F$ and that the two are equal in $A$.

The result follows from the inqualities $G|_A - 1 < f \leqslant G|_A$ and the fact that $(\operatorname{cvxe}(f))_{\mathrm{marg}} = \operatorname{cvxe}(f_{\mathrm{marg}})$; see Proposition 5.3.

The strict inequality $(G|_A)_{\mathrm{marg}} - 1 < f_{\mathrm{marg}}$ follows as in Example 9.2. □

### 9.3. Functions of one variable with integer values

**Proposition 9.6.** *A function $f$ defined on a subset $A$ of $\mathbb{R}$ with integer values is 1-almost convex extensible if and only if $(1-s)f(a) + sf(b) - f(c) + 1 \geqslant 0$ for all $a, b, c \in A$ satisfying $a < c < b$ with $c = (1-s)a + sb$.*

**Proof.** If $f \leqslant \operatorname{cvxe}(f) + 1$, then we easily get

$$(1-s)f(a)+sf(b)-f(c)+1 \geqslant (1-s)(\operatorname{cvxe}(f))(a)+s(\operatorname{cvxe}(f))(b)-(\operatorname{cvxe}(f))(c)+1,$$

which is nonnegative.

Conversely, if $(1-s)f(a)+sf(b)-f(c)+1 \geqslant 0$, then for all $a, b, c \in A$ with $a < c < b$

we get $$f(c) \leqslant 1 + (1-s)f(a) + sf(b), \text{ where } c = (1-s)a + sb. \tag{7}$$

Taking the infimum over all possible $a$ and $b$ in $A$, the right-hand side of (7) will be equal to $1 + (\operatorname{cvxe}(f))(c)$. □

## 10. Projections from $\mathbb{R}^n$ to $\mathbb{R}^m$

Here we shall write the points in $\mathbb{R}^n$ as $(x', x'') \in \mathbb{R}^m \times \mathbb{R}^{n-m}$, so that we have $x' = (x_1, \ldots, x_m)$ and $x'' = (x_{m+1}, \ldots, x_n)$. We shall write $A_{c'}$ for the set of all points $a = (a', a'')$ in $A$ with $a' = c'$.

It will be convenient to treat functions that are convex extensible and functions that are almost convex extensible simultaneously. For this reason we shall consider $f + w$ in the sequel with $w$ as any given function. Most often we have $w = 0$ (for convex extensible functions) and $w = 1$ (for 1-almost extensible functions).

**Theorem 10.1.** *Let $f\colon A \to \mathbb{R}$ be any function defined on a nonempty subset $A$ of $\mathbb{R}^n$ and define for any $\beta \in \mathbb{R}^{n-m}$*

$$f_\beta(x) = f_\beta(x', x'') = f(x', x'') + \beta \cdot x'', \qquad (x', x'') \in A \subset \mathbb{R}^m \times \mathbb{R}^{n-m}.$$

*Let $w$ be any nonnegative constant. Then the marginal function $f_{\beta\,\mathrm{marg}}$ of $f_\beta$ is almost $w$-convex extensible for all $\beta$ if and only if*

(A) *the convex hull of any finite subset $B$ of $A$ intersects the closure of the convex hull of $A_{c'}$ for any $c' \in \pi_*(A)$ such that $c'$ belongs to the convex hull of $\pi_*(B)$; and*

(B) *for every $P = (p^{(0)}, \ldots, p^{(m)}) \in A^{m+1} \subset (\mathbb{R}^n)^{m+1}$ and $p = (p', p'') \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ we have $f^{\diamond\diamond}(p) \leqslant (J_P * f)(p) + w$.*

If all the $p^{(j)\prime}$ are different, we may call the right-hand side in (B) non-vertical. (They may not all be different; this is harmless.) The left-hand side is vertical. So, again, we have a non-vertical operator defined by $P$ and a vertical operator defined by the second partial Fenchel transform of $f$. The non-vertical operator shall win over the vertical.

**Proof.** We shall use also here the partial Fenchel transform of $f$. The partial transform of $f_\beta$ is $(f_\beta)^\diamond(x', \eta) = f^\diamond(x', \eta + \beta)$. After having replaced $f$ by its second transform $f^{\diamond\diamond}$ we assume that $f$ itself is convex and lower semicontinuous in the last $n - m$ variables, which implies that its restriction to the closed convex hull of $A_{c'}$ has finite values and is continuous.

**1.** Suppose now that (A) and (B) are satisfied.

Let us write $P'$ for the $(m + 1)$-tuple of the first $m$ coordinates of the points $p^{(j)}$ in any $(m+1)$-tuple. We shall prove that $f_{\mathrm{marg}}$ satisfies $f_{\mathrm{marg}}(c') \leqslant (J_{P'} * f_{\mathrm{marg}})(c') + w$ for any given point $c' \in \pi_*(A)$. Given any positive number $\varepsilon$, we can complete the $p^{(j)\prime}$ with points $p^{(j)\prime\prime}$ so that at the completed points $p^{(j)} = (p^{(j)\prime}, p^{(j)\prime\prime})$ in $\mathbb{R}^n$ we have the values $f(p^{(j)}) \leqslant f_{\mathrm{marg}}(p^{(j)\prime}) + \varepsilon$.

Assume first that $c''$ belongs to the convex hull of $A_{c'}$. Then there are points $q^{(k)} \in A$, $k = 0, \ldots, n - m$, such that $c'' = \sum t_k q^{(k)\prime\prime}$ for some $t_k \geqslant 0$ with sum equal to 1. By assumption (B), $\sum t_k f(q^{(k)}) \leqslant \sum_j s_j f(p^{(j)}) + w$, so that

$$\begin{aligned} f_{\mathrm{marg}}(c') &\leqslant \min_k f(q^{(k)}) \leqslant \sum_k t_k f(q^{(k)}) \leqslant \sum_j s_j f(p^{(j)}) + w \\ &\leqslant \sum_j s_j f_{\mathrm{marg}}(p^{(j)\prime}) + w + \varepsilon. \end{aligned}$$

Since $\varepsilon$ is arbitrary, $w$-almost convex extensibility of $f_{\mathrm{marg}}$ follows. The argument applies to $f_\beta$ as well.

Next, if we only assume that $c''$ belongs to the closure of the convex hull of $A_{c'}$, we can still find points $a^{(k)\prime} = (a^{(k)}_{m+1}, \ldots, a^{(k)}_n) \in A \subset \mathbb{R}^{n-m}$, $k = 0, \ldots, n-m$, with

$$q = \sum_{k=0}^{n-m} t_k a^{(k)} \text{ for some } t_k \text{ with } t_k \geqslant 0 \text{ and } \sum t_k = 1$$

is as close to $p$ as we want. Since $f$ is now assumed to be continuous in the closure of the convex hull of $A_{c'}$, we can finish just as in Section 8, where $n$ was equal to 2.

**2.** Conversely, if condition (A) is not satisfied, then $p''$ does not belong to the closed convex hull of $A_{c'}$, and we can choose, in view of the Hahn-Banach theorem, a vector $\beta \in \mathbb{R}^{n-m}$ so that so that $\beta \cdot p \geqslant \sum t_j \beta \cdot a^{(j)} + \theta$ with some $\theta > 0$ for all choices of scalars $t_j$ and all choices of the $a^{(j)}$ with $a^{(j)\prime\prime} \in A_{c'}$.

We keep the vector $\beta$ and shall study $f_{\lambda\beta}$ and its marginal function $f_{\lambda\beta\,\mathrm{marg}}$ for $\lambda$ large positive.

We have $f_{\lambda\beta\,\mathrm{marg}}(a') \leqslant f_{\lambda\beta}(a)$ and $f_{\lambda\beta\,\mathrm{marg}}(b') \leqslant f_{\lambda\beta}(a)$, and obtain

$$\begin{aligned}
&(\mathrm{cvxe}(f_{\lambda\beta\,\mathrm{marg}}))(c') - f_{\lambda\beta\,\mathrm{marg}}(c') \\
&\qquad \leqslant (1-s) f_{\lambda\beta\,\mathrm{marg}}(a) + s f_{\lambda\beta\,\mathrm{marg}}(b) - f_{\lambda\beta\,\mathrm{marg}}(d') \\
&\qquad \leqslant (1-s) f(a) + s f(b) - f(d) + (f(d) - f_{\lambda\beta\,\mathrm{marg}}(d')) - \lambda\beta \cdot \theta.
\end{aligned}$$

The difference $f(d) - f_{\lambda\beta\,\mathrm{marg}}(d')$ is nonnegative and tends to zero as $\lambda$ tends to $+\infty$; in particular it is bounded for large $\lambda$. It follows that $(\mathrm{cvxe}(f_{\lambda\beta\,\mathrm{marg}}))(c') - f_{\lambda\beta\,\mathrm{marg}}(c')$ is negative for $\beta$ positive and large, proving that $f_{\lambda\beta\,\mathrm{marg}}$ is not $w$-almost convex extensible – this is even true for any finite constant $w$.

**3.** Assume now that (A) but not (B) is satisfied. In view of Remark 9.4, we have

$$(J_P * f)(q) = \sum s_j f(q + p^{(j)}) < f(q) - w \text{ for some choice of } q \text{ and } P.$$

Since $f$ is convex in the last $n-m$ variables, we can find a subdifferential of $f|_{\{q'\}\times\mathbb{R}^{n-m}}$ at the point

$$q = (q', q'') = (c', q'') = \textstyle\sum \ldots((1-s)a' + sb', \sum t_j a^{(j)\prime},$$

meaning that we can choose $\beta$ such that

$$f(c', q'') + \beta \cdot (q'' - p'') \leqslant f(c', x'')$$

for all $x'' = (x_{m+1}, \ldots, x_n) \in \mathbb{R}^{n-m}$. It follows that the marginal function $f_{\beta\,\mathrm{marg}}$ of $f_\beta$ evaluated at $c' = q'$ is equal to $f_\beta(q)$. So we have

$$f_{\beta\,\mathrm{marg}}(q' + p^{(j)\prime}) \leqslant f_\beta(q + p^{(j)}), \text{ and } f_{\beta\,\mathrm{marg}}(c') = f_\beta(q),$$

yielding

$$\begin{aligned}
\sum s_j f_{\beta\,\mathrm{marg}}(q' + p^{(j)\prime}) - f_{\beta\,\mathrm{marg}}(q') &\leqslant \sum s_j f_\beta(q + p^{(j)}) - f_\beta(q) \\
&= \sum s_j f(q + p^{(j)}) - f(q) = (J_P * f)(q) - f(q) < -w.
\end{aligned}$$

This shows that $f_{\beta\,\mathrm{marg}}$ is not $w$-almost convex extensible. $\square$

## 11. Conclusion

We have generalized results on convexity properties of marginal functions of functions defined on $\mathbb{Z}^n$ to functions defined on arbitrary subsets of $\mathbb{R}^n$. The notion of convex extensibilty plays an important role for real-valued functions but has to be weakened when we consider functions with integer values.

## Acknowledgment

Göran Björck has given me valuable advice and suggestions concerning earlier versions of this text, which have led to important improvements and simplifications.

## References

[1] M. Andramonov, A. Ellero: *Generalised convexity properties of marginal functions*, in: *Progress in Optimization*, A. Eberhard et al. (eds.), Applied Optimization vol. 30, Kluwer, Dordrecht (1999) 59–78.

[2] A. V. Fiacco, J. Kyparisis: *Convexity and concavity properties of the optimal value function in parametric nonlinear programming*, J. Optim. Theory Appl. 48/1 (1986) 95–126.

[3] J.-B. Hiriart-Urruty, C. Lemaréchal: *Fundamentals of Convex Analysis*, Springer, Berlin (2001).

[4] C. O. Kiselman: *Duality of convolution operators. A tool for shape analysis*, Mathematics and Visualization, Springer, Berlin (2021), to appear.

[5] C. O. Kiselman: *Elements of Digital Geometry, Mathematical Morphology, and Discrete Optimization*, World Scientific, Singapore (2022).

[6] C. O. Kiselman, S. Samieinia: *Convexity of marginal functions in the discrete case*, in: *Digital Geometry, Combinatorics, and Discrete Optimization*, S. Samieinia, PhD Thesis, Stockholm University (2010).

[7] C. O. Kiselman, S. Samieinia: *Convexity of marginal functions in the discrete case*, in: *Analysis Meets Geometry. The Mikael Passare Memorial Volume*, M. Anderson et al. (eds.), Birkhäuser, Cham (2017) 287–309.

[8] R. T. Rockafellar: *Convex Analysis*, Princeton University Press, Princeton (1970, 1979).

[9] I. Singer: *Abstract Convex Analysis*, Wiley, New York (1997).