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The Locality Property for Compensated Convex Transforms

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We establish the locality property for compensated convex transforms to functions under various continuity and smoothness. More precisely, we generalise the locality property and the density property for compensated convex transforms obtained recently by K. Zhang, A. Orlando and E. Crooks [*Compensated convexity and Hausdorff stable geometric singularity extractions*, Math. Models Methods Appl. Sci. 25/4 (2015) 747–801] for bounded functions to functions under various continuity and smoothness assumptions such as C^{α}, $C^{1,\beta}$. We also establish some localised versions of such locality property.

Keywords: Convex functions, convex envelope, compensated convex transforms, bounded functions, C^{α}-functions, $C^{1,\beta}$-functions, locality property, density property.

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1. Introduction and main results

The aim of this paper is to generalise the locality property and the density property for compensated convex transforms obtained in [10, 11] for bounded and Lipschitz functions to other types of functions under various continuity and smoothness assumptions such as C^{α}, $C^{1,\beta}$. We give quantitative estimates for the radius of locality to show that the value of lower/upper transform for functions under various continuity and smoothness assumptions depend only on the value of function $f(y)+\lambda|y|^2$ in the closed ball $\overline{B}(x_0;\mathbf{R}_\lambda)$ with $\mathbf{R}_\lambda \to 0$ at different rates for different continuity/smoothness assumptions for f as $\lambda \to +\infty$. A consequence of the locality property is the density property. It means when we approximate a function under various continuity and smoothness conditions by using compensated convex transforms, then there are some points x_i inside the ball where the compensated convex transforms are equal to the original function at those points. We have also established some localized versions of such locality property.

Throughout the paper, we denote by $\mathbb{R}^n$ the standard n-dimensional Euclidean space with the standard inner product $x \cdot y$ and norm $|x|$ for $x,\ y \in \mathbb{R}^n$. We denote by $B(x_0;r)$ and $\overline{B}(x_0;r)$ the open and closed balls centered at x_0 with radius $r>0$ respectively. Also $\mathbf{co}[K]$ denotes the convex hull of K [5, 6] which is the smallest convex set containing K. For a given function $f:\mathbb{R}^n \to \mathbb{R}$ bounded below, the convex envelope $\mathbf{Co}[f]$ of f is the largest convex function whose graph lies below f [5, 6]. We denote the class of globally *Hölder* continuous functions by C^{α}, and the class of functions whose gradient is *Hölder continous* by $C^{1,\beta}$ [4] and the upper/lower semicontinuous envelope of f by $\overline{f}(x)$, $\underline{f}(x)$ see [5, 6].

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Let us first recall the notions of quadratic compensated convex transforms for bounded functions in $\mathbb{R}^n$ defined in [1, 7, 9, 10]. For the definition of compensated convex transforms for functions under more general growth conditions, see [7, Definition 1.1].

Let $f : \mathbb{R}^n \to \mathbb{R}$ satisfy the growth condition

$$|f(x)| \leq C_f |x|^2 + C_1, \tag{1}$$

for some constants $C_f > 0,\ C_1 > 0$ and for all $x \in \mathbb{R}^n$.

The *quadratic lower compensated convex transform* for f (lower transform for short) is defined by

$$\mathcal{C}^l_\lambda(f)(x) = \mathbf{Co}[\lambda|\cdot|^2 + f(\cdot)](x) - \lambda|x|^2, \quad x \in \mathbb{R}^n,\ \text{for } \lambda > C_f, \tag{2}$$

where $\mathbf{Co}[\lambda|\cdot|^2 + f(\cdot)](x)$ is the value of the convex envelope of the function $y \mapsto f(y) + \lambda|y|^2$ at $x \in \mathbb{R}^n$ [5, 6].

The *quadratic upper compensated convex transform* for f (upper transform for short) is defined by

$$\mathcal{C}^u_\lambda(f)(x) := -\mathbf{Co}[\lambda|\cdot|^2 - f(\cdot)](x) + \lambda|x|^2, \quad x \in \mathbb{R}^n,\ \text{for } \lambda > C_f. \tag{3}$$

The two *quadratic mixed compensated convex transforms* for f (mixed transforms for short) are defined, respectively, by $\mathcal{C}^u_\lambda(\mathcal{C}^l_\tau(f))$, and $\mathcal{C}^l_\lambda(\mathcal{C}^u_\tau(f))$, $x \in \mathbb{R}^n$, when $\lambda, \tau > C_f$. Evidently our growth condition for f implies $\mathcal{C}^u_\lambda(f)(x) = -\mathcal{C}^l_\lambda(-f)(x)$.

The present work is motivated by [10]. Our main results generalise those obtained in [10]. We establish the locality property for such transforms under various continuity and smoothness conditions, including some of the borderline cases such as the bounded lower semicontinuous functions and the uniformly continuous functions.

We generalise the locality property of compensated convex transforms established for bounded functions in [10, Theorem 3.10(i)] and for Lipschitz functions in [8, Proposition 2.3] to other types of functions under various continuity and smoothness assumptions such as C^α, $C^{1,\beta}$. We give quantitative estimates for the radius of locality $\mathbf{R}_\lambda$ to show that the values of lower and upper transform depends only on the values of the function $f(y) + \lambda|y|^2$ in the closed ball $\overline{B}(x_0; \mathbf{R}_\lambda)$.

A consequence of the locality property is the density property. It means when we approximate a function under various continuity and smoothness conditions using compensated convex transforms, then there are some points x_i inside the ball where the compensated convex transforms equal to the original function at those points. We also establish some localized versions of such locality property for functions defined on a subset of $\mathbb{R}^n$ by using appropriate extensions to $\mathbb{R}^n$. For localized versions the property only holds at one point x_0 in the domain. We study the locality property of the compensated convex transforms for locally Lipschitz function at x_0 by extending the original function f to $\mathbb{R}^n$ by a constant.

Before we state our main results, we need the following technical lemma.

Lemma 1.1. *Let $A = \{x > 0 : x^\mu \le ax + b\}$ for $a, b \in \mathbb{R}^+$ and $\mu > 1$. If there is some $x^* > 0$ such that $(x^*)^\mu > ax^* + b$, then we have $x \le x^*$ for all $x \in A$, that is , x^* is an upper bound of the set A.*

Now we state the main locality result under various global continuity /smoothness assumptions. These results show the quality of approximation of functions by compensated convex transforms. We will use a simple example to show that some of our estimates are sharp.

Theorem 1.2. *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ satisfies the growth condition (1) and is a function under the continuity/smoothness assumptions below. Let $\lambda > 0$. Then for every $x_0 \in \mathbb{R}^n$, the lower transform of f at x_0 is given by*

$$\mathcal{C}^l_\lambda(f)(x_0) = \boldsymbol{Co}_{\overline{B}(x_0;\boldsymbol{R}_\lambda)}[f(\cdot) + \lambda|\cdot - x_0|^2](x_0), \tag{4}$$

where the radius $\boldsymbol{R}_\lambda \ge 0$ depends on the following continuity/smoothness conditions respectively.

(i) [11, Proposition 2.3] *If f is a Lipschitz function with Lipschitz constant $L > 0$, then*

$$\boldsymbol{R}_\lambda := \boldsymbol{R}^L_\lambda = \frac{L(2+\sqrt{2})}{\lambda},$$

(ii) *If f is a C^α-function for $0 < \alpha < 1$ with a constant $L > 0$, then*

$$\boldsymbol{R}_\lambda := \boldsymbol{R}^\alpha_\lambda = 2^{\left(\frac{5-2\alpha}{2-\alpha}\right)} \left(\frac{L}{\lambda}\right)^{\left(\frac{1}{2-\alpha}\right)},$$

(iii) *If f is a $C^{1,\beta}$-function for $0 < \beta < 1$ with $\alpha = 1 + \beta$ and a constant $L > 0$, then*

$$\boldsymbol{R}_\lambda := \boldsymbol{R}^{1,\beta}_\lambda = (M(\alpha))^{1/(2-\alpha)} \left(\frac{L}{\lambda}\right)^{1/(2-\alpha)},$$

where $M(\alpha) > 0$ is a constant depending on $1 < \alpha < 2$.

(iv) *If f is a bounded lower semicontinuous function with its oscillation in $\mathbb{R}^n$ defined by* $\operatorname{osc}(f) = \sup(f) - \inf(f)$, *then*

$$\boldsymbol{R}_\lambda := \boldsymbol{R}^B_\lambda = \frac{(1+\sqrt{2})\sqrt{\operatorname{osc}(f)}}{\sqrt{\lambda}},$$

(v) *If f is a bounded uniformly continuous function in $\mathbb{R}^n$, then vspm1*

$$\boldsymbol{R}_\lambda := \boldsymbol{R}^U_\lambda = \frac{3\sqrt{\tilde{\omega}_f\left(\sqrt{\frac{\operatorname{osc}(f)}{\lambda}}(1+\sqrt{2})\right)} + 2\sqrt{\tilde{\omega}_f\left(\frac{a^*}{\lambda} + \sqrt{\frac{b^*}{\lambda}}\right)}}{\sqrt{\lambda}},$$

where $\tilde{\omega}_f(\cdot)$, a^*, b^* are given by Definition 2.5.

(vi) *If f is a differentiable function, and its gradient is bounded and uniformly continuous, then*

$$\boldsymbol{R}_\lambda := \boldsymbol{R}^D_\lambda = \frac{4\,\omega_{Df}\left(\frac{L}{\lambda}(2+\sqrt{2})\right)}{\lambda},$$

where ω_{Df} is the modulus of continuity of Df (see also Proposition 1.5 *below.)*

(vii) [7, Theorem (2.3,v)] *If f is $C^{1,1}$ with Lipschitz constant L for Df such that*

$$|Df(x) - Df(y)| \le L|x - y|,$$

then $\mathcal{C}^l_\lambda(f)(x) = f(x)$ for all $x \in \mathbb{R}^n$ whenever $\lambda \ge L$, that is, $\boldsymbol{R}_\lambda = 0$ for $\lambda \ge L$.

The estimates in Theorem 1.2 also apply to the upper transforms under the same continuity / smoothness conditions expect in iv lower semicontinuous is replaced by upper semicontinuous. For (vii), this is a direct consequence of Theorem (2.3,v) in [7].

Remark 1.3. The various $\mathbf{R}'_\lambda s$ above are in fact upper bounds for radius of a ball on which (4) holds. However we can show later that asymptotically those bounds are sharp (see Example 1.6).

The following table shows the rate of decay of $\mathbf{R}_\lambda$ to 0 when $\lambda \to \infty$ from the slowest to fastest. We say that $f(x) = O(|x|^\alpha)$ for some $\alpha > 0$ if $|f(x)|/|x|^\alpha \leq M$ for some constant $M > 0$ and $f(x) = o(|x|^\alpha)$ if $\lim_{x\to 0}(|f(x)|/|x|^\alpha) = 0$.

Order	**Continuous/Smoothness of function**	**Rate of decay when $\lambda \to \infty$**
1	Bounded lower semicontinuous function	$\mathbf{R}^B_\lambda = O(\frac{1}{\sqrt{\lambda}})$
2	Bounded uniformly continuous function	$\mathbf{R}^U_\lambda = o(\frac{1}{\sqrt{\lambda}})$
3	C^α-function for $0 < \alpha < 1$	$\mathbf{R}^\alpha_\lambda = O(\frac{1}{\lambda^{1/(2-\alpha)}})$
4	Lipschitz function [11]	$\mathbf{R}^L_\lambda = O(\frac{1}{\lambda})$
5	C^1-function with bounded uniformly continuous gradient	$\mathbf{R}^D_\lambda = o(\frac{1}{\lambda})$
6	$C^{1,\alpha-1}$-function for $1 < \alpha < 2$	$\mathbf{R}^{1,\alpha-1}_\lambda = O(\frac{1}{\lambda^{1/(2-\alpha)}})$
8	$C^{1,1}$-function [7]	$\mathbf{R}_\lambda = 0$ for $\lambda \geq L$

Table 1.1: A general observation from Theorem 1.2 is that the smoother the function, the faster the rate of decay of $\mathbf{R}_\lambda$ to 0 is when $\lambda \to \infty$ based on the power of $\frac{1}{\lambda}$.

For the borderline case (vi) in Theorem 1.2 where Df is bounded and uniformly continuous, we also need the following lemma and proposition.

Lemma 1.4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function, and its gradient be bounded by $L \geq 0$, that is, $|Df(x)| \leq L$ for $x \in \mathbb{R}^n$ and Df is uniformly continuous. Then*

(i) *The function f is Lipschitz continuous with constant $L \geq 0$,*

(ii) $|f(y) - f(x) - Df(x) \cdot (y - x)| \leq \omega_{Df}(|y - x|)\,|y - x|$, *where ω_{Df} is the modulus of continuity of Df.*

Proposition 1.5. *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function, and its gradient is bounded and uniformly continuous in $\mathbb{R}^n$. Suppose ω_{Df} is the modulus of continuity of Df such that for all $x,\ y \in \mathbb{R}^n$,*

$$|Df(x)| \leq L \quad \textit{and} \quad |Df(y) - Df(x)| \leq \omega_{Df}(|y - x|).$$

Then for $\lambda > 0$, we have for all $x \in \mathbb{R}^n$,

$$\mathcal{C}_\lambda^l(f)(x) \le f(x) \le \mathcal{C}_\lambda^l(f)(x) + \frac{\left(\omega_{Df}\left(\frac{L}{\lambda}(2+\sqrt{2})\right)\right)^2}{4\lambda}. \tag{5}$$

The following example shows that our locality properties are asymptotically sharp for the power of $\frac{1}{\lambda}$ as $\lambda \to +\infty$ for the cases (i), (ii), (iii) (see Table 1.1). Sharpness in terms of asymptotic behavior means that they have both big O of the same power $\frac{1}{2-\alpha}$. The best power that we obtain for the decay rate of $\mathbf{R}_\lambda$ to zero when $\lambda \to +\infty$.

Example 1.6. Assume $f(x) = -|x|^\alpha$ for $x \in \mathbb{R}$ and $0 < \alpha < 2$, then f is Lipschitz for $\alpha = 1$, C^α-function for $0 < \alpha < 1$ and is $C^{1,\alpha-1}$-function for $1 < \alpha < 2$. The lower transform can be calculated explicitly as follows: for $0 < \alpha < 2$

$$\mathcal{C}_\lambda^l(f)(x) = \begin{cases} \lambda\left(\frac{\alpha}{2\lambda}\right)^{\frac{2}{2-\alpha}} - \left(\frac{\alpha}{2\lambda}\right)^{\frac{\alpha}{2-\alpha}} - \lambda|x|^2, & \text{for } |x| \le \left(\frac{\alpha}{2\lambda}\right)^{\frac{1}{2-\alpha}}, \\ -|x|^\alpha, & \text{for } |x| \ge \left(\frac{\alpha}{2\lambda}\right)^{\frac{1}{2-\alpha}}. \end{cases}$$

At $x_0 = 0$, $\alpha = 1$, $\mathbf{R}_\lambda = \left(\frac{1}{2\lambda}\right) = O\left(\frac{1}{\lambda}\right)$.

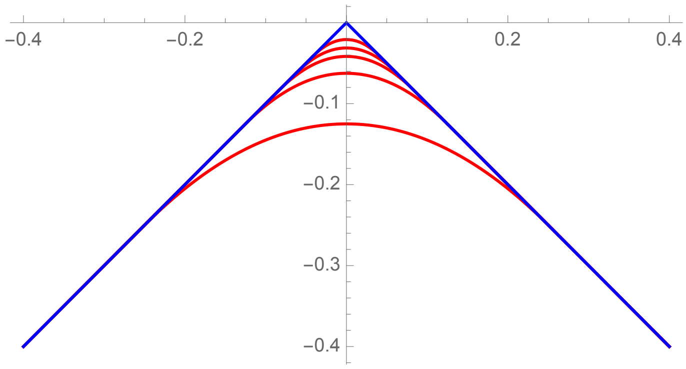

Figure 1.1: This figure shows the bigger λ is the closer the graph of the lower transform to the orignal function.

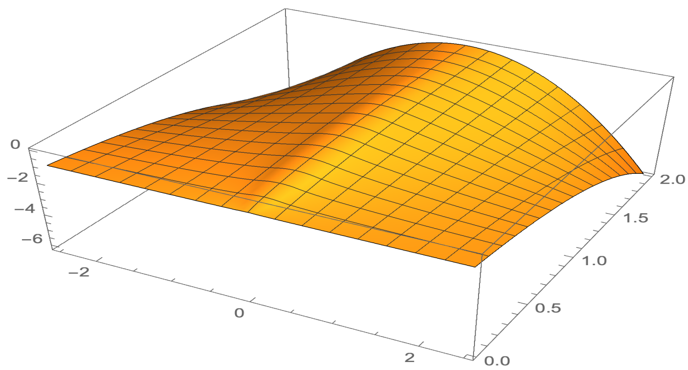

Figure 1.2: This figure shows the evolution of the graph of the lower transform of $f(x,\alpha) = -|x|^\alpha$ in $[-2,2] \times (0,2)$ when α increases.

Next we state the density theorem under various continuity and smoothness conditions assumed in Theorem 1.2. The result follows from Theorem 1.2 and the proofs in [10].

Theorem 1.7. *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a function. Let $\boldsymbol{R}_\lambda$ be one of $\boldsymbol{R}_\lambda^L$, $\boldsymbol{R}_\lambda^\alpha$, $\boldsymbol{R}_\lambda^{1,\beta}$, $\boldsymbol{R}_\lambda^B$, $\boldsymbol{R}_\lambda^U$, $\boldsymbol{R}_\lambda^D$ under the corresponding continuity/smoothness conditions in Theorem* 1.2. *We define the lower touching set of f by*

$$\mathcal{T}_\lambda^l(f) = \{x \in \mathbb{R}^n,\ \mathcal{C}_\lambda^l f(x) = f(x)\}.$$

Then for $\forall\ x_0 \in \mathbb{R}^n$:

$$x_0 \in \boldsymbol{co}(\mathcal{T}_\lambda^l(f) \cap \overline{B}(x_0, \boldsymbol{R}_\lambda)). \tag{6}$$

More precisely, if $x_0 \notin \mathcal{T}_\lambda^l(f)$, then there are $x_i \in \overline{B}(x_0; \boldsymbol{R}_\lambda)$ with $x_i \neq x_0$, and $\lambda_i > 0$ for $i = 1, \dots, k$ with $2 \leq k \leq n+1$ satisfying $\sum_{i=1}^k \lambda_i = 1$ and $\sum_{i=1}^k \lambda_i x_i = x_0$, such that $\mathcal{C}_\lambda^l(f)(x_i) = f(x_i)$.

From Theorem 1.7, we see that $\mathcal{T}_\lambda^l(f)$ is not an empty set and the convex density of $\mathcal{T}_\lambda^l(f)$ can be measured by (6). Similarly, we can introduce the upper touching set $\mathcal{T}_\lambda^u(f)$ of f and establish similar results.

Next we state the localized versions of the locality properties obtained from Theorem 1.2 for the Lipschitz case. We need the following Lemma.

Lemma 1.8. *Suppose $f : \overline{B}(x_0; \delta) \to \mathbb{R}$ is lower semicontinuous and Lipschitz at x_0 in $\overline{B}(x_0; \delta)$ with Lipschitz constant $L \geq 0$ in the sense that $|f(x) - f(x_0)| \leq L|x - x_0|$ for $x \in \overline{B}(x_0; \delta)$. Let $\widehat{f} : \mathbb{R}^n \to \mathbb{R}$ be the extension of f outside $B(x_0; \delta)$, defined by*

$$\widehat{f}(x) = \begin{cases} f(x), & \text{for } x \in B(x_0; \delta) \\ \\ f(x_0) - L\delta, & \text{for } x \notin B(x_0; \delta). \end{cases}$$

Then the extension of f is bounded and lower semicontinuous in $\mathbb{R}^n$.

Theorem 1.9. *Let $f : \overline{B}(x_0; \delta) \to \mathbb{R}$ be defined as in Lemma* 1.8 *and $\widehat{f} : \mathbb{R}^n \to \mathbb{R}$ be the extension of f defined in Lemma* 1.8. *Then there is $\Lambda > 0$ such that when $\lambda > \Lambda$ we have $\boldsymbol{R}_\lambda^L := \frac{L(2+\sqrt{2})}{\lambda} < \delta$, and*

$$\mathcal{C}_\lambda^l(\widehat{f})(x_0) = \boldsymbol{Co}_{\overline{B}(x_0; \boldsymbol{R}_\lambda^L)}[f(\cdot) + \lambda|\cdot - x_0|^2](x_0).$$

The plan of the rest of the paper is as follows. Section 2 will present some preliminary results which are needed for the proofs. We present the proofs of our main results in Section 3.

2. Preliminary results

We first collect some basic results from convex analysis which will be used in the following, and refer to [5, 6] for further details and proof.

Definition 2.1. [2] For a function $f : \mathbb{R}^n \to \mathbb{R}$ bounded below, we define its *convex envelope at $x \in \mathbb{R}^n$*, denoted by $\mathbf{Co}[f]$, by

$$\mathbf{Co}[f](x) = \sup\{g(x),\ g(y) \leq f(y),\ \text{for all } y \in \mathbb{R}^n,\ g \text{ is convex}\}.$$

We often use the following characterization of convex envelope in the proofs.

Proposition 2.2. [10, Proposition 2.1] *Let $f : \mathbb{R}^n \to \mathbb{R}$ be bounded below and $x_0 \in \mathbb{R}^n$. Then*

(i) *The convex envelope $\boldsymbol{Co}[f](\cdot)$ of f at $x_0 \in \mathbb{R}^n$ is given by*

$$\boldsymbol{Co}[f](x_0) = \inf \left\{ \sum_{i=1}^{n+1} \lambda_i f(x_i) : \sum_{i=1}^{n+1} \lambda_i = 1, \sum_{i=1}^{n+1} \lambda_i x_i = x_0, \lambda_i \geq 0, x_i \in \mathbb{R}^n \right\}. \quad (7)$$

In addition, if f is lower semicontinuous and coercive in the sense that we have $f(x)/|x| \to +\infty$ as $|x| \to +\infty$, then the infimum can be attained by some (λ_i^*, x_i^*) for $0 < \lambda_i^* \leq 1$, $x_i^* \in \mathbb{R}^n$, for $i = 1, ..., k$ with $1 \leq i \leq k \leq n+1$, satisfy $\sum_{i=1}^{k} \lambda_i^* = 1$, and $\sum_{i=1}^{k} \lambda_i^* x_i^* = x_0$, such that

$$\boldsymbol{Co}[f](x_0) = \sum_{i=1}^{k} \lambda_i^* f(x_i^*).$$

(ii) *We can replace the general convex functions g in Definition* 2.1 *by affine functions* [7, 10]:

$$\boldsymbol{Co}[f](x_0) = \sup \left\{ \ell(x_0) : \ \ell \text{ affine and } \ell(y) \leq f(y) \text{ for all } y \in \mathbb{R}^n \right\}.$$

In particular if f is lower semicontinuous and coercive, then the sup *is attained by an affine function $\ell^* \in \mathbb{R}^n$.*

Next we introduce the local version of convex envelope at a point $x_0 \in \mathbb{R}^n$ as follows.

Definition 2.3. [10, Definition 2.2] Let $r > 0$ and $x_0 \in \mathbb{R}^n$. Suppose that $f : \overline{B}(x_0; r) \to \mathbb{R}$ is a bounded function in $\overline{B}(x_0; r)$. The *local convex envelope* of f at x_0 in $\overline{B}(x_0; r)$ denoted by $\mathbf{Co}_{\overline{B}(x_0;r)}[f](x_0)$ is defined by

$$\mathbf{Co}_{\overline{B}(x_0;r)}[f](x_0) = \inf \left\{ \sum_{i=1}^{n+1} \lambda_i f(x_i) \ : \ \begin{array}{c} \sum_{i=1}^{n+1} \lambda_i = 1, \ \sum_{i=1}^{n+1} \lambda_i x_i = x_0, \ \lambda_i \geq 0, \\ |x_i - x_0| \leq r, \ x_i \in \mathbb{R}^n, \ i = 1, ..., n+1 \end{array} \right\}. \quad (8)$$

The difference between $\mathbf{Co}[f](x_0)$ and $\mathbf{Co}_{\overline{B}(x_0;r)}[f](x_0)$ is that for the latter the points $\{x_i\}$ are restricted to the ball $\overline{B}(x_0; r)$.

We need the following definitions from [10].

Definition 2.4. [10, Definition 2.11] Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and let $x_0 \in \mathbb{R}^n$.

(i) We say that f is *locally C^α at x_0 for some* $0 < \alpha < 1$ if there are constants $\delta > 0$ and $L > 0$ such that

$$|f(x) - f(x_0)| \leq L|x - x_0|^\alpha \quad \text{whenever } |x - x_0| \leq \delta.$$

(ii) If $\alpha = 1$ in the inequality above, we say that f is *locally Lipschitz at x_0*.

(iii) We say that f is *locally C^α at x_0 for some* $1 < \alpha < 2$ if f is differentiable in $\overline{B}(x_0; \delta)$and there are constants $\delta > 0$ and $L > 0$ such that

$$|Df(x) - Df(x_0)| \leq L|x - x_0|^{\alpha-1} \quad \text{whenever } |x - x_0| \leq \delta.$$

In fact if we let $\beta = \alpha - 1$, our definition gives the local $C^{1,\beta}$ property of f at x_0.

We recall from [3] the definition of modulus of continuity with some of its properties.

Definition 2.5. [3] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded and uniformly continuous function in $\mathbb{R}^n$. Then $\omega_f : [0, \infty) \to [0, \infty)$, defined by

$$\omega_f(t) = \sup\{|f(x) - f(y)| : x,\ y \in \mathbb{R}^n \text{ and } |x - y| \le t\}$$

is called the *modulus of continuity* of f. In particular, there is a concave continuous majorant $\tilde{\omega}_f$ of ω_f of f satisfying that $\tilde{\omega}_f(0) = 0$, $\omega_f(t) \le \tilde{\omega}_f(t)$, and $\tilde{\omega}_f \le a^*t + b^*$ for some constants $a^* > 0,\ b^* \ge 0$.

The modulus of continuity of f has the following properties.

Proposition 2.6. [3] *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded and uniformly continuous function in $\mathbb{R}^n$. Then a modulus of continuity ω_f of f satisfying the following properties:*

(i) $\omega_f(t) \longrightarrow \omega(0) = 0, \quad \text{as } t \longrightarrow 0.$

(ii) *ω_f is non-negative and non-decreasing continuous function on $[0, \infty)$.*

(iii) *ω_f is subadditive $\omega_f(t_1 + t_2) \le \omega_f(t_1) + \omega_f(t_2) \quad \text{for all} \quad t_1, t_2 \ge 0.$*

Next we list some basic properties of compensated convex transforms (see [7, 9, 10] for further details and proofs).

The lower and upper transforms are related to each other when $\lambda > 0$ by the following relation [7]

$$C^l_\lambda(f)(x) = -C^u_\lambda(-f)(x),$$

when both can be defined. Furthermore, the ordering property of compensated convex transform [7] holds for $x \in \mathbb{R}^n$, $0 < \lambda < \tau$,

$$C^l_\lambda(f)(x) \le C^l_\tau(f)(x) \le f(x) \le C^u_\tau(f)(x) \le C^u_\lambda(f)(x), \tag{9}$$

and for $f \le g$ in $\mathbb{R}^n$, we have that [7, 10]

$$C^l_\lambda(f)(x) \le C^l_\lambda(g)(x) \quad \text{and} \quad C^u_\lambda(f)(x) \le C^u_\lambda(g)(x),\ x \in \mathbb{R}^n. \tag{10}$$

In addition, the compensated convex transforms are affine invariant [10], that is,

$$C^l_\lambda(f + \ell)(x) = C^l_\lambda(f)(x) + \ell(x) \quad \text{and} \quad C^u_\lambda(f + \ell)(x) = C^u_\lambda(f)(x) + \ell(x), \tag{11}$$

where ℓ is any affine function and we also have [7, Theorem 2.1(iii)]

$$C^u_\tau(C^u_\lambda(f)) = \begin{cases} C^u_\lambda(f), & \text{if } \tau \ge \lambda, \\ C^u_\tau(f), & \text{if } \tau \le \lambda; \end{cases} \quad \text{and} \quad C^l_\tau(C^l_\lambda(f)) = \begin{cases} C^l_\lambda(f), & \text{if } \tau \ge \lambda, \\ C^l_\tau(f), & \text{if } \tau \le \lambda. \end{cases} \tag{12}$$

We also recall the following properties. If f is a continuous function with quadratic growth [7, Theroem 2.3(i)], then

$$\lim_{\lambda \to \infty} C^l_\lambda(f)(x) = f(x) \quad \text{and} \quad \lim_{\lambda \to \infty} C^u_\lambda(f)(x) = f(x), x \in \mathbb{R}^n.$$

Also, if f and g are both bounded real valued functions in $\mathbb{R}^n$ [10, Proposition 2.9], then for $\lambda > 0$, and $\tau > 0$, we have

$$\mathcal{C}^l_{\lambda+\tau}(f+g)(x) \geq \mathcal{C}^l_\lambda(f)(x) + \mathcal{C}^l_\tau(g)(x) \text{ and } \mathcal{C}^u_{\lambda+\tau}(f+g)(x) \leq \mathcal{C}^l_\lambda(f)(x) + \mathcal{C}^l_\tau(g)(x).$$

The following property of compensated convex transforms in [10, Proposition 3.1] will also be used:

$$\mathcal{C}^u_\lambda(\overline{f})(x) = \mathcal{C}^u_\lambda(f)(x), \tag{13}$$

$$\mathcal{C}^l_\lambda(\underline{f})(x) = \mathcal{C}^l_\lambda(f)(x), \tag{14}$$

where $\overline{f}, \underline{f}$ is the upper and lower semicontinuous envelopes of f [6, 5].

We will use the following simple translation-invariance property in our proofs in order to allow us to consider the point $x_0 = 0$ without loss of generality.

Proposition 2.7. [10, Proposition 2.10] (Translation-invariance property) *For any $f : \mathbb{R}^n \to \mathbb{R}$ bounded below and for any affine function $\ell : \mathbb{R}^n \to \mathbb{R}$ we have $\boldsymbol{Co}[f+\ell] = \boldsymbol{Co}[f] + \ell$. Consequently, both $\mathcal{C}^l_\lambda(f)(x)$ and $\mathcal{C}^u_\lambda(f)(x)$ are translation invariant against the weight function, that is:*

$$\mathcal{C}^l_\lambda(f)(x) = \boldsymbol{Co}[\lambda|\cdot - x_0|^2 + f(\cdot)](x) - \lambda|x - x_0|^2,$$

$$\mathcal{C}^u_\lambda(f)(x) = -\boldsymbol{Co}[\lambda|\cdot - x_0|^2 - f(\cdot)](x) + \lambda|x - x_0|^2,$$

for all $x \in \mathbb{R}^n$ and for every fixed $x_0 \in \mathbb{R}^n$. In particular, at x_0

$$\mathcal{C}^l_\lambda(f)(x_0) = \boldsymbol{Co}[\lambda|\cdot - x_0|^2 + f(\cdot)](x_0), \quad \mathcal{C}^u_\lambda(f)(x_0) = -\boldsymbol{Co}[\lambda|\cdot - x_0|^2 - f(\cdot)](x_0).$$

Now we state some results of compensated convex transforms which are used later in our proofs. The following estimates for lower transform can be found in [10]. Similar results hold for the upper transform.

Theorem 2.8. [10, Theorem 2.12]

(i) *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is lower semicontinuous, f maps bounded sets to bounded sets and satisfies $f(x) \geq -C_f|x|^2 - C_1$. If at some $x_0 \in \mathbb{R}^n$, f is locally C^α at x_0 for $0 < \alpha < 1$ with constant $L \geq 0$, or $C^{1,\beta}$ with $\alpha = 1 + \beta$ and $1 < \alpha < 2$ then for $\lambda > 0$ sufficiently large,*

$$\mathcal{C}^l_\lambda(f)(x_0) \leq f(x_0) \leq \mathcal{C}^l_\lambda(f)(x_0) + L^{2/(2-\alpha)} \left(\frac{\alpha}{2\lambda}\right)^{\alpha/(2-\alpha)} \left(1 - \frac{\alpha}{2}\right). \tag{15}$$

(ii) *Suppose that the $f : \mathbb{R}^n \to \mathbb{R}$ is lower semicontinuous, f maps bounded sets to bounded sets and satisfies $f(x) \geq -C_f|x|^2 - C_1$. If at some $x_0 \in \mathbb{R}^n$, f is differentiable, then for any $\epsilon > 0$, there is a $\Lambda > 0$, such that when $\lambda \geq \Lambda$,*

$$\mathcal{C}^l_\lambda(f)(x_0) \leq f(x_0) \leq \mathcal{C}^l_\lambda(f)(x_0) + \frac{\epsilon^2}{4\lambda}. \tag{16}$$

(iii) *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a globally Lipschitz function with Lipschitz constant $L \geq 0$. Then for $\lambda > 0$ and for $x \in R^n$ we have*

$$\mathcal{C}^l_\lambda(f)(x) \leq f(x) \leq \mathcal{C}^l_\lambda(f)(x) + \frac{L^2}{4\lambda}. \tag{17}$$

The following result gives the estimates for a bounded uniformly continuous function f with the modulus of continuity $\tilde{\omega}_f$.

Theorem 2.9. [10, Theorem 2.15]. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded and uniformly continuous function and $\tilde{\omega}_f$ is the least concave majorant of the modulus of continuity ω_f of f. Assume that $a > 0, b \geq 0$ are such that $0 \leq \tilde{\omega}_f(t) \leq at + b$ for $t \in [0, \infty)$. Then we have for $\lambda > 0$:*

$$f(x) - \tilde{\omega}_f\left(\frac{a}{\lambda} + \sqrt{\frac{b}{\lambda}}\right) \leq \mathcal{C}^l_\lambda(f)(x) \leq f(x) \quad \textit{for } x \in \mathbb{R}^n, \tag{18}$$

$$\mathcal{C}^u_\lambda(f)(x) \leq f(x) + \tilde{\omega}_f\left(\frac{a}{\lambda} + \sqrt{\frac{b}{\lambda}}\right) \ \textit{for } x \in \mathbb{R}^n.$$

3. Proofs of the main results

Proof of Lemma 1.1: Consider $f(x) = |x|^\mu$ for some $\mu > 1, x \in \mathbb{R}$. Since the first derivative is strictly positive for $x \in (0, \infty)$, then the function f is strictly increasing. Since the second derivative is strictly positive for all x except at $x = 0$, then the function is strictly convex. Now we prove by contradiction by assuming that there exists $x_0 > 0$ with ${x_0}^\mu \leq ax_0 + b$ such that we have $x_0 > x^*$. Since f is strictly increasing and $x_0 > x^*$, then ${x_0}^\mu > (x^*)^\mu$. Moreover,

$$x_0^{\mu-1} > (x^*)^{\mu-1} \tag{19}$$

Also, since $x_0 > x^*$, then $\frac{1}{x_0} < \frac{1}{x^*}$. Since x_0 satisfies ${x_0}^\mu \leq ax_0 + b$, this is also equivalent to ${x_0}^{\mu-1} \leq a + \frac{b}{x_0}$. Then by the above we have

$$x_0^{\mu-1} \leq a + \frac{b}{x_0} < a + \frac{b}{x^*} < (x^*)^{\mu-1}.$$

This means $x_0^{\mu-1} < (x^*)^{\mu-1}$ which contradicts to (19). Hence for every $x > 0$ with $x^\mu \leq ax + b$ we have $x \leq x^*$. □

Proof of Theorem 1.2: By the translation invariance property [10, Proposition 2.10] (also see Proposition 2.7), we assume without loss of generality that $x_0 = 0$. Since f is bounded and continuous, the function $y \mapsto f(y) + \lambda|y|^2$ is continuous and coercive. Therefore, by [10, Proposition 2.1] (also see Proposition 2.2), we have

$$\mathcal{C}^l_\lambda(f)(0) = \mathbf{Co}[f(\cdot) + \lambda|\cdot|^2](0) = \sum_{i=1}^{k} \lambda_i[f(x_i) + \lambda|x_i|^2], \tag{20}$$

for some $2 \leq k \leq n+1$, $\lambda_i \geq 0$, $x_i \in \mathbb{R}^n$ for $i = 1, \ldots, k$ with $\sum_{i=1}^k \lambda_i = 1$ and $\sum_{i=1}^k \lambda_i x_i = 0$. We define $g(y) = f(y) + \lambda|y|^2$ for $y \in \mathbb{R}^n$, and $\lambda > 0$. Again from Proposition 2.1 in [10] (also see Proposition 2.2), there is an affine function $\ell(y) = a \cdot y + b$, $y \in \mathbb{R}^n$ with some $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that

$$\ell(y) \leq g(y) \qquad \text{for all } y \in \mathbb{R}^n, \tag{21}$$

$$\ell(x_i) = g(x_i) \qquad \text{for } i = 1, \ldots, k. \tag{22}$$

The condition (21), means that, there is a support plane of the convex envelope of g, and the condition (22), means that, the support plane touch the graph of g at some x_i, for $i = 1, \cdots, k$.

Since $\ell(y)$ is an affine function, then we conclude that by (21) and (22),

$$\ell(x_i) = \mathbf{Co}[g](x_i) \qquad \text{for } i = 1, \ldots, k.$$

Hence at the origin point, and by (22) and (20), we have

$$\mathcal{C}^l_\lambda(f)(0) = \mathbf{Co}[g](0) = \ell(0) = b. \qquad \square$$

Proof of (i): The proof of the Lipschitz case can be found in [11, Prop. 2.3]. $\square$

Proof of (ii): Due to the translation invariance property [10, Proposition 2.10] (also see Proposition 2.7), we only need to consider the case for $x_0 = 0$. By [10, Theorem 2.12 (i)] (see also Theorem 2.8 i), since f is a C^α- function for $0 < \alpha < 1$ with constant $L \geq 0$, we have at $x_0 = 0 \in \mathbb{R}^n$,

$$\mathcal{C}^l_\lambda(f)(0) \leq f(0) \leq \mathcal{C}^l_\lambda(f)(0) + L^{2/(2-\alpha)} \left(\frac{\alpha}{2\lambda}\right)^{\alpha/(2-\alpha)} \left(1 - \frac{\alpha}{2}\right). \tag{23}$$

Furthermore, inequality (23) will take the form

$$b \leq f(0) \leq b + L^{2/(2-\alpha)} \left(\frac{\alpha}{2\lambda}\right)^{\alpha/(2-\alpha)} \left(1 - \frac{\alpha}{2}\right).$$

Then
$$0 \leq f(0) - b \leq L^{2/(2-\alpha)} \left(\frac{\alpha}{2\lambda}\right)^{\alpha/(2-\alpha)} \left(1 - \frac{\alpha}{2}\right). \tag{24}$$

Thus
$$-L^{2/(2-\alpha)} \left(\frac{\alpha}{2\lambda}\right)^{\alpha/(2-\alpha)} \left(1 - \frac{\alpha}{2}\right) \leq b - f(0) \leq 0. \tag{25}$$

Recall that the affine function $\ell(y) = a \cdot y + b$ for $y \in \mathbb{R}^n$ with some $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ satisfies (21) and (22). To derive the bound $\mathbf{R}_\lambda$ we evaluate (21) at $y = \frac{a}{2\lambda}$ to obtain an upper bound of $|a|$ as follows:

$$\frac{a \cdot a}{2\lambda} + b \leq f\left(\frac{a}{2\lambda}\right) + \lambda \left|\frac{a}{2\lambda}\right|^2.$$

So that
$$\begin{aligned}
\frac{|a|^2}{4\lambda} &\leq f\left(\frac{a}{2\lambda}\right) - b \leq f\left(\frac{a}{2\lambda}\right) - f(0) + f(0) - b \\
&\leq \left|f\left(\frac{a}{2\lambda}\right) - f(0)\right| + L^{2/(2-\alpha)} \left(\frac{\alpha}{2\lambda}\right)^{\alpha/(2-\alpha)} \left(1 - \frac{\alpha}{2}\right) \\
&\leq L\left|\frac{a}{2\lambda}\right|^\alpha + L^{2/(2-\alpha)} \left(\frac{\alpha}{2\lambda}\right)^{\alpha/(2-\alpha)} \left(1 - \frac{\alpha}{2}\right).
\end{aligned} \tag{26}$$

Here we have used the fact that f is a Hölder continuous function and (24). Now we aim at showing that

$$|a| \leq 2^{\frac{3-\alpha}{2-\alpha}} L^{1/(2-\alpha)} \lambda^{\frac{1-\alpha}{2-\alpha}}.$$

Let $|a| = \lambda^\beta t$, then we can rewrite (26) as follows:

$$\frac{\lambda^{2\beta} t^2}{4\lambda} \leq \frac{L t^\alpha \lambda^{\alpha\beta}}{\lambda^\alpha \, 2^\alpha} + L^{2/(2-\alpha)} \left(\frac{\alpha}{2\lambda}\right)^{\alpha/(2-\alpha)} \left(1 - \frac{\alpha}{2}\right). \tag{27}$$

Let $2\beta-1=\alpha(\beta-1)$, then we have $\beta=\frac{1-\alpha}{2-\alpha}$. Moreover, we have

$$2\beta-1=2\left(\tfrac{1-\alpha}{2-\alpha}\right)-1=\tfrac{-\alpha}{2-\alpha}.$$

Hence we can rewrite (27) as follows

$$\frac{\lambda^{2\beta-1}t^2}{4}\leq L\left(\frac{t}{2}\right)^{\alpha}\lambda^{\alpha(\beta-1)}+L^{2/(2-\alpha)}\left(\frac{\alpha}{2\lambda}\right)^{\alpha/(2-\alpha)}\left(1-\frac{\alpha}{2}\right). \tag{28}$$

Since $2\beta-1=\alpha(\beta-1)=\frac{-\alpha}{2-\alpha}$, then by dividing (28) by $\lambda^{2\beta-1}$, we have

$$\frac{t^2}{4}\leq L\left(\frac{t}{2}\right)^{\alpha}+L^{2/(2-\alpha)}\left(\frac{\alpha}{2}\right)^{\alpha/(2-\alpha)}\left(1-\frac{\alpha}{2}\right). \tag{29}$$

Assume that t is given by $t=gL^{\delta}$. Then we can rewrite (29) as follows:

$$\frac{g^2L^{2\delta}}{4}\leq L^{1+\alpha\delta}\left(\frac{g}{2}\right)^{\alpha}+L^{2/(2-\alpha)}\left(\frac{\alpha}{2}\right)^{\alpha/(2-\alpha)}\left(1-\frac{\alpha}{2}\right). \tag{30}$$

Similarly, let $2\delta=\alpha\delta+1$, then we have $\delta=\frac{1}{2-\alpha}$. Moreover, we have the equalities $2\delta=\alpha\delta+1=\frac{2}{2-\alpha}$. Then by dividing inequality (30) by $L^{2\delta}$, we have

$$\frac{g^2}{4}\leq\left(\frac{g}{2}\right)^{\alpha}+\left(\frac{\alpha}{2}\right)^{\alpha/(2-\alpha)}\left(1-\frac{\alpha}{2}\right). \tag{31}$$

Consider $C(\alpha)=\left(\frac{\alpha}{2}\right)^{\alpha/(2-\alpha)}\left(1-\frac{\alpha}{2}\right)$. Then we have

$$\frac{g^2}{4}\leq\left(\frac{g}{2}\right)^{\alpha}+C(\alpha). \tag{32}$$

Let $\left(\frac{g}{2}\right)^{\alpha}=y$, then $\frac{g}{2}=y^{1/\alpha}$. Now we can rewrite inequality (32) as follows:

$$y^{2/\alpha}\leq y+C(\alpha)\quad\text{with}\quad y\geq 0.$$

Assume that $y=x\ C(\alpha)$, then we can rewrite the previous inequality as follows:

$$x^{2/\alpha}(C(\alpha))^{2/\alpha}\leq x\ C(\alpha)+C(\alpha).$$

Therefore
$$x^{2/\alpha}\leq(C(\alpha))^{\frac{\alpha-2}{\alpha}}(x+1).$$

Assume that $C^*(\alpha)=(C(\alpha))^{\frac{\alpha-2}{\alpha}}$. Then we have

$$x^{2/\alpha}\leq C^*(\alpha)(x+1). \tag{33}$$

Now by applying lemma 1.1 to the above inequality (33). If $x^*>0$ satisfies

$${x^*}^{2/\alpha}>C^*(\alpha)(x^*+1), \tag{34}$$

then for every $x>0$ which satisfies that inequality (33), we have $x\leq x^*$. Inequality (34) is equivalent to $\frac{{x^*}^{2/\alpha}}{1+x^*}>C^*(\alpha)$, which is equivalent to

$$\frac{{x^*}^{\frac{2-\alpha}{\alpha}}}{1+\frac{1}{x^*}}>C^*(\alpha). \tag{35}$$

Let ${x^*}^{\frac{2-\alpha}{\alpha}}=2C^*(\alpha)$, then $x^*=(2C^*(\alpha))^{\alpha/(2-\alpha)}$. Now we need to check whether inequality (35) holds for x^* or not.

From inequality (35), we have

$$\frac{2C^*(\alpha)}{1+\frac{1}{(2C^*(\alpha))^{\alpha/(2-\alpha)}}} > C^*(\alpha) \iff \frac{2}{1+\frac{1}{(2C^*(\alpha))^{\alpha/(2-\alpha)}}} > 1$$

$$\iff 2 > 1 + \frac{1}{(2C^*(\alpha))^{\alpha/(2-\alpha)}} \iff (2C^*(\alpha))^{\alpha/(2-\alpha)} > 1, \tag{36}$$

and this holds if and only if $2C^*(\alpha) > 1$. Since for $0 < \alpha < 1$ we have

$$\begin{aligned} 2C^*(\alpha) = 2(C(\alpha))^{\frac{\alpha-2}{\alpha}} &= 2\left[\left(\frac{\alpha}{2}\right)^{-1}\left(1-\frac{\alpha}{2}\right)^{\frac{\alpha-2}{\alpha}}\right] \\ &= 2\left[\left(\frac{2}{\alpha}\right)\left(1-\frac{\alpha}{2}\right)^{\frac{\alpha-2}{\alpha}}\right] > 1. \end{aligned}$$

Thus inequality (36) holds, implying that (35) holds for $x^* = (2C^*(\alpha))^{\alpha/(2-\alpha)}$. Thus x^* is an upper bound of x which satisfies inequality (33), that means,

$$x \le (2C^*(\alpha))^{\alpha/(2-\alpha)} \le \frac{2^{\alpha/(2-\alpha)}}{C(\alpha)}.$$

Furthermore, since $y = xC(\alpha)$, then $y \le 2^{\alpha/(2-\alpha)}$. Also since $y = \left(\frac{g}{2}\right)^{\alpha}$, then

$$\frac{g}{2} \le 2^{1/(2-\alpha)}.$$

Hence, $g \le 2^{\frac{3-\alpha}{2-\alpha}}$. By using $t = gL^{\delta}$, we have $\frac{t}{L^{\delta}} \le 2^{\frac{3-\alpha}{2-\alpha}}$ where $\delta = \frac{1}{2-\alpha}$. Therefore $t \le 2^{\frac{3-\alpha}{2-\alpha}} L^{1/(2-\alpha)}$. By using $|a| = \lambda^{\beta} t$, then we have

$$\frac{|a|}{\lambda^{\beta}} \le 2^{\frac{3-\alpha}{2-\alpha}} L^{1/(2-\alpha)},$$

where $\beta = \left(\frac{1-\alpha}{2-\alpha}\right)$. Thus we have

$$|a| \le 2^{\frac{3-\alpha}{2-\alpha}} L^{1/(2-\alpha)} \lambda^{\frac{1-\alpha}{2-\alpha}}. \tag{37}$$

Now we use (22) to obtain an upper bound of $|x_i|$ for $i = 1, \ldots, k$.

$$\begin{aligned} \lambda |x_i|^2 = a \cdot x_i + b - f(x_i) &= a \cdot x_i + b - f(0) + f(0) - f(x_i), \\ &\le |a \cdot x_i| + |f(x_i) - f(0)| \le |a||x_i| + L|x_i|^{\alpha}. \end{aligned}$$

Here we have used the facts that f is a Hölder continuous function and $b - f(0) \le 0$ by (25). By (37), we can deduce that

$$|x_i|^2 \le \frac{|a|}{\lambda}|x_i| + \frac{L}{\lambda}|x_i|^{\alpha} \le \frac{|x_i|}{\lambda}\, 2^{\frac{3-\alpha}{2-\alpha}} L^{1/(2-\alpha)} \lambda^{\frac{1-\alpha}{2-\alpha}} + \frac{L}{\lambda}|x_i|^{\alpha}.$$

If $x_i \neq 0$, by dividing the previous inequality by $|x_i|^{\alpha}$, then we have

$$|x_i|^{2-\alpha} \le \frac{|x_i|^{1-\alpha}}{\lambda}\, 2^{\frac{3-\alpha}{2-\alpha}} L^{1/(2-\alpha)} \lambda^{\frac{1-\alpha}{2-\alpha}} + \frac{L}{\lambda}. \tag{38}$$

Write $|x_i|$ as $|x_i| = t^{1/(1-\alpha)}$, so $t = |x_i|^{1-\alpha}$. Then we can rewrite inequality (38) as

$$t^{\frac{2-\alpha}{1-\alpha}} \leq \frac{t}{\lambda}\ 2^{\frac{3-\alpha}{2-\alpha}} L^{1/(2-\alpha)} \lambda^{\frac{1-\alpha}{2-\alpha}} + \frac{L}{\lambda} \leq t\ 2^{\frac{3-\alpha}{2-\alpha}} L^{1/(2-\alpha)} \lambda^{-1/(2-\alpha)} + \frac{L}{\lambda}. \tag{39}$$

Assume that $t = s\lambda^m$, then we can rewrite inequality (39) as follows:

$$s^{\left(\frac{2-\alpha}{1-\alpha}\right)} \lambda^{m\left(\frac{2-\alpha}{1-\alpha}\right)} \leq\ s\ 2^{\frac{3-\alpha}{2-\alpha}} L^{1/(2-\alpha)} \lambda^{\left(m-\frac{1}{(2-\alpha)}\right)} + \frac{L}{\lambda}. \tag{40}$$

Let $m\left(\frac{2-\alpha}{1-\alpha}\right) = m - \left(\frac{1}{2-\alpha}\right)$, then we have $m = -\left(\frac{1-\alpha}{2-\alpha}\right)$. Moreover,

$$m\left(\frac{2-\alpha}{1-\alpha}\right) = m - \left(\frac{1}{2-\alpha}\right) = -1.$$

Hence by dividing inequality (40) by $\lambda^{m\left(\frac{2-\alpha}{1-\alpha}\right)}$, we have

$$s^{\frac{2-\alpha}{1-\alpha}} \leq s\ \ 2^{\frac{3-\alpha}{2-\alpha}} L^{1/(2-\alpha)} + L. \tag{41}$$

Similarly we set $s = qL^{\gamma}$, then we can rewrite inequality (41) as follows:

$$q^{\frac{2-\alpha}{1-\alpha}} L^{\gamma\left(\frac{2-\alpha}{1-\alpha}\right)} \leq\ q\ 2^{\frac{3-\alpha}{2-\alpha}} L^{\left(\gamma+\frac{1}{2-\alpha}\right)}\ + L. \tag{42}$$

Also, let $\gamma\left(\frac{2-\alpha}{1-\alpha}\right) = \gamma + \frac{1}{2-\alpha}$, then $\gamma = \left(\frac{1-\alpha}{2-\alpha}\right)$. Moreover, $\gamma\left(\frac{2-\alpha}{1-\alpha}\right) = \gamma + \frac{1}{2-\alpha} = 1$. Hence by dividing inequality (42) by $L^{\gamma\left(\frac{2-\alpha}{1-\alpha}\right)}$, we have

$$q^{\frac{2-\alpha}{1-\alpha}} \leq\ 2^{\frac{3-\alpha}{2-\alpha}} q\ + 1. \tag{43}$$

Using Lemma 1.1 again to (43) to find an upper bound of q denoted by q^*. If $q^* > 0$ satisfies

$$q^{*\frac{2-\alpha}{1-\alpha}} >\ 2^{\frac{3-\alpha}{2-\alpha}}\ q^* + 1, \tag{44}$$

then for every $q > 0$ which satisfies inequality (43), we have $q \leq q^*$. From inequality (44) we obtain the following chain of equivalences:

$$\frac{q^{*\frac{2-\alpha}{1-\alpha}}}{2^{\frac{3-\alpha}{2-\alpha}} q^* + 1} > 1 \iff \frac{q^{*\left(\frac{1}{1-\alpha}\right)}}{2^{\frac{3-\alpha}{2-\alpha}} + \frac{1}{q^*}} > 1 \iff q^{*\frac{1}{1-\alpha}} > 2^{\frac{3-\alpha}{2-\alpha}} + \frac{1}{q^*}. \tag{45}$$

Let $q^{*\frac{1}{1-\alpha}} = 2\left(2^{\frac{3-\alpha}{2-\alpha}}\right) = 2^{\frac{5-2\alpha}{2-\alpha}}$, then $q^* = 2^{\frac{(5-2\alpha)(1-\alpha)}{2-\alpha}}$.

Now we need to check whether inequality (45) holds for q^* or not. From (45), we have

$$q^{*\frac{1}{1-\alpha}} = 2^{\frac{5-2\alpha}{2-\alpha}} > 2^{\frac{3-\alpha}{2-\alpha}} + \frac{1}{2^{\left(\frac{(5-2\alpha)(1-\alpha)}{2-\alpha}\right)}}.$$

This holds if and only if

$$2^{\frac{5-2\alpha}{2-\alpha}} - 2^{\frac{3-\alpha}{2-\alpha}} > \frac{1}{2^{\left(\frac{(5-2\alpha)(1-\alpha)}{2-\alpha}\right)}} \iff 2^{\frac{3-\alpha}{2-\alpha}}(2-1) > \frac{1}{2^{\left(\frac{(5-2\alpha)(1-\alpha)}{2-\alpha}\right)}},$$

and this holds if and only if $2^{\frac{3-\alpha}{2-\alpha}}\left(2^{\frac{(5-2\alpha)(1-\alpha)}{2-\alpha}}\right) > 1$. Since for $0 < \alpha < 1$ we have

$$2^{\frac{3-\alpha}{2-\alpha}}\left(2^{\frac{(5-2\alpha)(1-\alpha)}{2-\alpha}}\right) = 2^{-2(\alpha-2)} > 1,$$

inequality (45) then holds for $q^* = 2^{\frac{(5-2\alpha)(1-\alpha)}{2-\alpha}}$. Thus q^* is an upper bound of q which satisfies the inequality (43), that means,

$$q \leq 2^{\frac{(5-2\alpha)(1-\alpha)}{2-\alpha}}.$$

Then we conclude by $q = \frac{s}{L^\gamma}$ that $s \leq 2^{\frac{(5-2\alpha)(1-\alpha)}{2-\alpha}} L^\gamma$, where $\gamma = \frac{1-\alpha}{2-\alpha}$. Since $s = \frac{t}{\lambda^m}$, with $m = -\left(\frac{1-\alpha}{2-\alpha}\right)$, which can be written as

$$t \leq L^{\frac{1-\alpha}{2-\alpha}} \lambda^{\left(-\frac{1-\alpha}{2-\alpha}\right)} 2^{\frac{(5-2\alpha)(1-\alpha)}{2-\alpha}}.$$

From the formula $t = |x_i|^{(1-\alpha)}$, we obtain

$$|x_i|^{(1-\alpha)} \leq 2^{\left(\frac{(5-2\alpha)(1-\alpha)}{2-\alpha}\right)} \left(\frac{L}{\lambda}\right)^{\frac{1-\alpha}{2-\alpha}}.$$

Thus we obtain the estimate of x_i

$$|x_i| \leq 2^{\frac{5-2\alpha}{2-\alpha}} \left(\frac{L}{\lambda}\right)^{\frac{1}{2-\alpha}}.$$

Therefore we can take $\mathbf{R}_\lambda^\alpha = 2^{\frac{5-2\alpha}{2-\alpha}} \left(\frac{L}{\lambda}\right)^{\frac{1}{2-\alpha}}$ for all $i = 1, \cdots, k$. □

Proof of (iii): The proof is very similar to the C^α case but we need to consider that $\ell(y) = a^* \cdot y + Df(0) \cdot y + b$ for $y \in \mathbb{R}^n$ with some $a^* \in \mathbb{R}^n$ and $b \in \mathbb{R}$, then the rest of the the proof will be very similar after this. Due to the translation invariance property [10, Proposition 2.10] (also see Proposition 2.7), we only need to consider the case for $x_0 = 0$. By [10, Theorem 2.12(i)] (see also Theorem 2.8 i), since f is a $C^{1,\beta}$-function with $\alpha = 1 + \beta,\ 1 < \alpha < 2$, we have at $x_0 = 0 \in \mathbb{R}^n$,

$$\mathcal{C}_\lambda^l(f)(0) \leq f(0) \leq \mathcal{C}_\lambda^l(f)(0) + L^{2/(2-\alpha)} \left(\frac{\alpha}{2\lambda}\right)^{\alpha/(2-\alpha)} \left(1 - \frac{\alpha}{2}\right). \tag{46}$$

Furthermore, inequality (46) will take the form,

$$b \leq f(0) \leq b + L^{2/(2-\alpha)} \left(\frac{\alpha}{2\lambda}\right)^{\alpha/(2-\alpha)} \left(1 - \frac{\alpha}{2}\right).$$

Then
$$0 \leq f(0) - b \leq L^{2/(2-\alpha)} \left(\frac{\alpha}{2\lambda}\right)^{\alpha/(2-\alpha)} \left(1 - \frac{\alpha}{2}\right). \tag{47}$$

To derive the bound $\mathbf{R}_\lambda$ we evaluate (21) at $y = \frac{a^*}{2\lambda}$ to derive an upper bound of $|a^*|$ as follows:

$$a^* \cdot \left(\frac{a^*}{2\lambda}\right) + Df(0) \cdot \left(\frac{a^*}{2\lambda}\right) + b \leq f\left(\frac{a^*}{2\lambda}\right) + \lambda \left|\frac{a^*}{2\lambda}\right|^2.$$

Hence, we obtain

$$
\begin{aligned}
\frac{|a^*|^2}{4\lambda} &\le f\left(\frac{a^*}{2\lambda}\right) - Df(0)\cdot\frac{a^*}{2\lambda} - f(0) + f(0) - b \\
&\le \left| f\left(\frac{a^*}{2\lambda}\right) - f(0) - Df(0)\cdot\frac{a^*}{2\lambda}\right| + f(0) - b. \qquad (48)
\end{aligned}
$$

Note that, by the fundamental theorem of calculus, we have

$$
\begin{aligned}
&|f(y) - f(0) - Df(0)\cdot y| = \left|\int_0^1 (Df(ty)\cdot y - Df(0)\cdot y)\mathrm{d}t\right| \\
&\le \int_0^1 |(Df(ty) - Df(0))\cdot y \ \mathrm{d}t| \le \int_0^1 |Df(ty) - Df(0)|\ |y|\ \mathrm{d}t \\
&\le \int_0^1 L|ty|^\beta\ |y|\ \mathrm{d}t = L|y|^{\beta+1}\int_0^1 |t|^\beta \mathrm{d}t = \frac{L|y|^{\beta+1}}{\beta+1} = \frac{L|y|^\alpha}{\alpha}. \qquad (49)
\end{aligned}
$$

Here we have used the fact that $f \in C^{1,\beta}(\mathbb{R}^n)$. Thus, by (47) and (49) we can rewrite (48) as follows:

$$
\frac{|a^*|^2}{4\lambda} \le \frac{L}{\alpha}\left|\frac{a^*}{2\lambda}\right|^\alpha + L^{2/(2-\alpha)}\left(\frac{\alpha}{2\lambda}\right)^{\alpha/(2-\alpha)}\left(1-\frac{\alpha}{2}\right). \qquad (50)
$$

We first show that $|a^*| \le 2\left(L\lambda^{(1-\alpha)}\left(\frac{1+n(\alpha)}{\alpha}\right)\right)^{\frac{1}{2-\alpha}}$.

Set $|a^*|$ as $|a^*| = \lambda^\delta t$. Then we can rewrite (50) as follows:

$$
\frac{\lambda^{2\delta-1}t^2}{4} \le \frac{Lt^\alpha\lambda^{\alpha(\delta-1)}}{\alpha\ 2^\alpha} + \lambda^{-\alpha/(2-\alpha)}L^{2/(2-\alpha)}\left(\frac{\alpha}{2\lambda}\right)^{\alpha/(2-\alpha)}\left(1-\frac{\alpha}{2}\right). \qquad (51)
$$

Let $2\delta - 1 = \alpha(\delta-1)$, then we have $\delta = \frac{1-\alpha}{2-\alpha}$. Moreover, we have

$$
2\delta - 1 = \alpha(\delta - 1) = \frac{-\alpha}{2-\alpha}.
$$

Hence we can rewrite (51) as follows:

$$
\frac{t^2}{4} \le \frac{L}{\alpha}\left(\frac{t}{2}\right)^\alpha + L^{2/(2-\alpha)}\left(\frac{\alpha}{2}\right)^{\alpha/(2-\alpha)}\left(1-\frac{\alpha}{2}\right). \qquad (52)
$$

Let $t = gL^\gamma$, we have

$$
\frac{g^2L^{2\gamma}}{4} \le \frac{L^{\alpha\gamma+1}}{\alpha}\left(\frac{g}{2}\right)^\alpha + L^{2/(2-\alpha)}\left(\frac{\alpha}{2}\right)^{\alpha/(2-\alpha)}\left(1-\frac{\alpha}{2}\right).
$$

Let $2\gamma = \alpha\gamma + 1$, then $\gamma = \frac{1}{2-\alpha}$. Moreover we have $2\gamma = \alpha\gamma + 1 = \frac{2}{2-\alpha}$. Hence by dividing the previous inequality by $L^{2\gamma}$, we obtain

$$
\left(\frac{g}{2}\right)^2 \le \frac{1}{\alpha}\left(\frac{g}{2}\right)^\alpha + \left(\frac{\alpha}{2}\right)^{\alpha/(2-\alpha)}\left(1-\frac{\alpha}{2}\right).
$$

Let $\left(\frac{g}{2}\right)^{\alpha} = y$, then $\frac{g}{2} = y^{1/\alpha}$, and

$$y^{2/\alpha} \le \frac{y}{\alpha} + C(\alpha), \tag{53}$$

where $C(\alpha) = \left(\frac{\alpha}{2}\right)^{\alpha/(2-\alpha)}\left(1 - \frac{\alpha}{2}\right)$. Again by applying Lemma 1.1 to (53) to find an upper bound of y denoted by y^*. If $y^* > 0$ satisfies

$$y^{*2/\alpha} > \frac{y^*}{\alpha} + C(\alpha), \tag{54}$$

then for every $y > 0$ which satisfies inequality (53), we have $y \le y^*$. Inequality (54) is equivalent to

$$\frac{y^{*\frac{2-\alpha}{\alpha}}}{\frac{1}{\alpha} + \frac{C(\alpha)}{y^*}} > 1. \tag{55}$$

Let $y^{*\frac{2-\alpha}{\alpha}} = \frac{n+1}{\alpha}$, then $y^* = \left(\frac{n+1}{\alpha}\right)^{\alpha/(2-\alpha)}$. Also, we can rewrite (55) as follows:

$$\frac{n+1}{1 + \frac{\alpha C(\alpha)}{y^*}} > 1,$$

from which we receive the following chain of equivalences:

$$n + 1 > 1 + \frac{\alpha C(\alpha)}{y^*} \iff n > \frac{\alpha C(\alpha)}{y^*} \iff y^* > \frac{\alpha C(\alpha)}{n}. \tag{56}$$

The last inequality of (56) holds if and only if

$$\left(\frac{n+1}{\alpha}\right)^{\alpha/(2-\alpha)} > \frac{\alpha C(\alpha)}{n} \iff (n+1)^{\alpha/(2-\alpha)} > \frac{\alpha^{2/(2-\alpha)}C(\alpha)}{n+1}$$
$$\iff (n+1)^{2/(2-\alpha)} > \alpha^{2/(2-\alpha)}C(\alpha) \iff n + 1 > \alpha\left(C(\alpha)\right)^{\frac{2-\alpha}{2}}$$
$$\iff n > \alpha\left(C(\alpha)\right)^{\frac{2-\alpha}{2}} - 1.$$

Take $n = n(\alpha) = \alpha\left(C(\alpha)\right)^{\frac{2-\alpha}{2}}$, then (54) holds if

$$y^* = \left(\frac{1 + n(\alpha)}{\alpha}\right)^{\alpha/(2-\alpha)}. \tag{57}$$

Thus y^* is an upper bound of y which satisfies inequality (53). That means

$$y \le \left(\frac{1 + n(\alpha)}{\alpha}\right)^{\alpha/(2-\alpha)}.$$

Then from $y = (\frac{g}{2})^{\alpha}$, we have $$\left(\frac{g}{2}\right)^{\alpha} \le \left(\frac{1 + n(\alpha)}{\alpha}\right)^{\alpha/(2-\alpha)}. \tag{58}$$

This implies that $$\left(\frac{g}{2}\right) \le \left(\frac{1 + n(\alpha)}{\alpha}\right)^{1/(2-\alpha)}.$$

By using $t = gL^\gamma$ where $\gamma = \frac{1}{2-\alpha}$, we have

$$t \le 2L^{1/(2-\alpha)} \left(\frac{1+n(\alpha)}{\alpha}\right)^{1/(2-\alpha)}. \tag{59}$$

By using $|a^*| = \lambda^\delta t$, where $\delta = \frac{1-\alpha}{2-\alpha}$, we conclude

$$\begin{aligned} &|a^*| \le 2\lambda^{\frac{1-\alpha}{2-\alpha}} L^{1/(2-\alpha)} \left(\frac{1+n(\alpha)}{\alpha}\right)^{1/(2-\alpha)} \\ \iff \quad &|a^*| \le 2\left(L\lambda^{(1-\alpha)}\left(\frac{1+n(\alpha)}{\alpha}\right)\right)^{\frac{1}{2-\alpha}}. \end{aligned} \tag{60}$$

Now we use (22) to obtain an upper bound of $|x_i|$ for $i = 1, \dots, k$.

$$\begin{aligned} \lambda|x_i|^2 &= a^* \cdot x_i + Df(0) \cdot x_i + b - f(x_i) \\ &= a^* \cdot x_i + Df(0) \cdot x_i + b - f(0) + f(0) - f(x_i) \\ &\le |a^* \cdot x_i| + |f(x_i) - Df(0) \cdot x_i - f(0)|, \end{aligned}$$

as $b - f(0) \le 0$ by (47) and then from (49), we have

$$\lambda|x_i|^2 \le |a^*|\ |x_i| + \frac{L}{\alpha}|x_i|^\alpha. \tag{61}$$

Then we deduce that
$$|x_i| \le \frac{|a^*|}{\lambda} + \frac{L}{\alpha\lambda}|x_i|^{\alpha-1}. \tag{62}$$

Take $|x_i| = s^{1/(\alpha-1)}$ so that $s = |x_i|^{(\alpha-1)}$, then from (60) we can rewrite inequality (62) as follows:

$$s^{1/(\alpha-1)} \le \frac{2}{\lambda}\left(L\lambda^{(1-\alpha)}\left(\frac{1+n(\alpha)}{\alpha}\right)\right)^{\frac{1}{2-\alpha}} + \frac{L}{\alpha\lambda}s. \tag{63}$$

Using Lemma 1.1 to the above inequality (63) to find an upper bound of s denoted by s^*. If $s^* > 0$ satisfies

$$s^{*1/(\alpha-1)} > \frac{2}{\lambda}\left(L\lambda^{(1-\alpha)}\left(\frac{1+n(\alpha)}{\alpha}\right)\right)^{\frac{1}{2-\alpha}} + \frac{L}{\alpha\lambda}s^*, \tag{64}$$

then for every $s > 0$ which satisfies inequality (63) , we have $s \le s^*$. Inequality (64) holds if and only if

$$s^{*\left(\frac{2-\alpha}{\alpha-1}\right)} > \frac{2}{\lambda s^*}\left(L\lambda^{(1-\alpha)}\left(\frac{1+n(\alpha)}{\alpha}\right)\right)^{\frac{1}{2-\alpha}} + \frac{L}{\alpha\lambda},$$

which again holds if and only if

$$\frac{s^{*\left(\frac{2-\alpha}{\alpha-1}\right)}}{\frac{2}{\lambda s^*}\left(L\lambda^{(1-\alpha)}\left(\frac{1+n(\alpha)}{\alpha}\right)\right)^{\frac{1}{2-\alpha}} + \frac{L}{\alpha\lambda}} > 1.$$

Let $s^{*\left(\frac{2-\alpha}{\alpha-1}\right)} = \dfrac{(m+1)L}{\alpha\lambda}$, then $s^* = \left(\dfrac{(m+1)L}{\alpha\lambda}\right)^{\left(\frac{\alpha-1}{2-\alpha}\right)}$.

We can rewrite the previous inequality equivalently as follows:

$$\frac{(m+1)L}{L+\frac{2\alpha}{s^*}\left(L\lambda^{(1-\alpha)}\left(\frac{1+n(\alpha)}{\alpha}\right)\right)^{\frac{1}{2-\alpha}}}>1 \iff mL>\frac{2\alpha\left(L\lambda^{(1-\alpha)}\left(\frac{1+n(\alpha)}{\alpha}\right)\right)^{\frac{1}{2-\alpha}}}{s^*}$$

$$\iff s^*>\frac{2\alpha\left(L\lambda^{(1-\alpha)}\left(\frac{1+n(\alpha)}{\alpha}\right)\right)^{\frac{1}{2-\alpha}}}{mL}$$

$$\iff \left(\frac{(m+1)L}{\alpha\lambda}\right)^{\frac{\alpha-1}{2-\alpha}}>\frac{2\alpha\left(L\lambda^{(1-\alpha)}\left(\frac{1+n(\alpha)}{\alpha}\right)\right)^{\frac{1}{2-\alpha}}}{mL}$$

$$\iff (m+1)^{\frac{\alpha-1}{2-\alpha}}>\left(\frac{\alpha\lambda}{L}\right)^{\frac{\alpha-1}{2-\alpha}}\frac{2\alpha\left(L\lambda^{(1-\alpha)}\left(\frac{1+n(\alpha)}{\alpha}\right)\right)^{\frac{1}{2-\alpha}}}{(m+1)L}$$

$$\iff (m+1)^{1/(2-\alpha)}>\left(\frac{\alpha}{L}\right)^{1/(2-\alpha)}\lambda^{\frac{\alpha-1}{2-\alpha}}\quad 2\left(L\lambda^{(1-\alpha)}\left(\frac{1+n(\alpha)}{\alpha}\right)\right)^{\frac{1}{2-\alpha}}.$$

Then we obtain $m+1>\left(\frac{\alpha}{L}\right)\lambda^{(\alpha-1)}2^{(2-\alpha)}\left(L\lambda^{(1-\alpha)}\left(\frac{1+n(\alpha)}{\alpha}\right)\right)$, thus

$$m>\left(\frac{\alpha}{L}\right)\lambda^{(\alpha-1)}2^{(2-\alpha)}\left(L\lambda^{(1-\alpha)}\left(\frac{1+n(\alpha)}{\alpha}\right)\right)-1.$$

Take $m=m(\alpha)=2^{(2-\alpha)}\,(1+n(\alpha))$, then (64) holds if

$$s^*=\left(\frac{(1+m(\alpha))L}{\alpha\lambda}\right)^{\frac{\alpha-1}{2-\alpha}}.$$

Thus s^* is an upper bound of s which satisfies inequality (63), that means,

$$s\leq\left(\frac{(1+m(\alpha))L}{\alpha\lambda}\right)^{\frac{\alpha-1}{2-\alpha}}.$$

From $s=|x_i|^{(\alpha-1)}$, we obtain $|x_i|^{(\alpha-1)}\leq\left(\frac{(1+m(\alpha))L}{\alpha\lambda}\right)^{\frac{\alpha-1}{2-\alpha}}$, thus

$$|x_i|\leq\left(\frac{(1+m(\alpha))L}{\alpha\lambda}\right)^{1/(2-\alpha)}.$$

Take $M(\alpha)=\left(\frac{(1+m(\alpha)}{\alpha}\right)$, then we have $|x_i|\leq\left(M(\alpha)\frac{L}{\lambda}\right)^{1/(2-\alpha)}$.

Hence we deduce that for each x_i with $i=1,\cdots,k$, we get $|x_i|\leq\left(M(\alpha)\frac{L}{\lambda}\right)^{1/(2-\alpha)}$.

Therefore $\mathbf{R}_{\lambda}^{1,\beta}=(M(\alpha))^{1/(2-\alpha)}\left(\frac{L}{\lambda}\right)^{1/(2-\alpha)}$. $\square$

Proof of (iv): Without loss of generality we may assume that $x_0 = 0$. Consider $\ell^*(x) = \inf(f)$, then $\ell^*(x) \leq f(x) + \lambda|x|^2$ for all $x \in \mathbb{R}^n$. So that

$$\ell^*(0) \leq \mathbf{Co}[f + \lambda|\cdot|^2](0) = \mathbf{Co}[g](0) = \mathcal{C}^l_\lambda(f)(0) = b.$$

Then $b \geq \inf(f)$. To derive the bound $\mathbf{R}_\lambda$ we evaluate (21) at $y = \frac{a}{2\lambda}$ to derive an upper bound of a as follows:

$$\begin{aligned} \frac{a\cdot a}{2\lambda} + b &\leq f\left(\frac{a}{2\lambda}\right) + \lambda\left|\frac{a}{2\lambda}\right|^2, \\ \frac{|a|^2}{4\lambda} &\leq f\left(\frac{a}{2\lambda}\right) - b \leq f\left(\frac{a}{2\lambda}\right) - \inf(f) \leq \mathrm{osc}(f). \end{aligned}$$

Thus we can deduce that $|a| \leq 2\sqrt{\lambda\ \mathrm{osc}(f)}$.

Now we use (22) to obtain an upper bound of $|x_i|$ for $i = 1, \dots, k$.

$$\begin{aligned} \lambda|x_i|^2 &= a\cdot x_i + b - f(x_i) \leq |a\cdot x_i| + |f(x_i) - b| \\ &\leq |a||x_i| + f(x_i) - \inf(f) \leq |a||x_i| + \mathrm{osc}(f) \leq 2\sqrt{\lambda\ \mathrm{osc}(f)}\ |x_i| + \mathrm{osc}(f). \end{aligned}$$

Thus $\lambda|x_i|^2 - 2\sqrt{\lambda\ \mathrm{osc}(f)}\ |x_i| - \mathrm{osc}(f) \leq 0$.

Hence we can deduce that for each x_i with $i = 1, \dots, k$,

$$|x_i| \leq \sqrt{\frac{\mathrm{osc}(f)}{\lambda}}(1+\sqrt{2}).$$

Thus $\mathbf{R}^B_\lambda = \sqrt{\frac{\mathrm{osc}(f)}{\lambda}}(1+\sqrt{2}) = \frac{(1+\sqrt{2})\sqrt{\mathrm{osc}(f)}}{\sqrt{\lambda}}$. □

Proof of (v): Due to the translation invariance property [10, Proposition 2.10] (also see Proposition 2.7), we only need to consider the case for $x_0 = 0$. By [10, Theorem 2.15] (also see Theorem 2.9), since f is a bounded uniformly continuous function, we have at $x_0 = 0 \in \mathbb{R}^n$,

$$f(0) - \tilde{\omega}_f\left(\frac{a^*}{\lambda} + \sqrt{\frac{b^*}{\lambda}}\right) \leq b \leq f(0), \tag{65}$$

where $a^* > 0$ and $b^* \geq 0$ and $\tilde{\omega}_f$ is the concave continuous majorant of ω_f of f. As above, we have from [10, Theorem (2.15)],

$$0 \leq f(0) - b \leq \tilde{\omega}_f\left(\frac{a^*}{\lambda} + \sqrt{\frac{b^*}{\lambda}}\right). \tag{66}$$

To derive the bound $\mathbf{R}_\lambda$ we evaluate (21) at $y = \frac{a}{2\lambda}$ to derive an upper bound of a as follows:

$$\frac{a\cdot a}{2\lambda} + b \leq f\left(\frac{a}{2\lambda}\right) + \lambda\left|\frac{a}{2\lambda}\right|^2$$

This implies

$$\frac{|a|^2}{4\lambda} \le f\left(\frac{a}{2\lambda}\right) - f(0) + f(0) - b \implies \frac{|a|^2}{4\lambda} \le \left| f\left(\frac{a}{2\lambda}\right) - f(0)\right| + f(0) - b$$
$$\implies \frac{|a|^2}{4\lambda} \le \tilde{\omega}_f\left(\frac{|a|}{2\lambda}\right) + f(0) - b \implies \frac{|a|^2}{4\lambda} \le \tilde{\omega}_f\left(\sqrt{\frac{\operatorname{osc}(f)}{\lambda}}\right) + \tilde{\omega}_f\left(\frac{a^*}{\lambda} + \sqrt{\frac{b^*}{\lambda}}\right),$$

hence we deduce that

$$|a|^2 \le 4\ \lambda\ \left(\tilde{\omega}_f\left(\sqrt{\frac{\operatorname{osc}(f)}{\lambda}}\right) + \tilde{\omega}_f\left(\frac{a^*}{\lambda} + \sqrt{\frac{b^*}{\lambda}}\right)\right).$$

Here we have used the estimate of $|a|$ by $|a| \le 2\sqrt{\lambda \operatorname{osc}(f)}$ when f is bounded and lower semicontinuous, the definition of the modulus of continuity of f, and inequality (66). Thus we have

$$|a| \le\ 2\ \sqrt{\lambda\left(\tilde{\omega}_f\left(\sqrt{\frac{\operatorname{osc}(f)}{\lambda}}\right) + \tilde{\omega}_f\left(\frac{a^*}{\lambda} + \sqrt{\frac{b^*}{\lambda}}\right)\right)}. \tag{67}$$

Now we use (22) to obtain an upper bound of $|x_i|$ for $i = 1, \ldots, k$.

$$\begin{aligned}\lambda|x_i|^2 &= a \cdot x_i + b - f(x_i) = a \cdot x_i + b - f(0) + f(0) - f(x_i)\\ &\le |a \cdot x_i| + |f(x_i) - f(0)| \le |a||x_i| + \tilde{\omega}_f(|x_i|)\\ &\le |a||x_i| + \tilde{\omega}_f\left(\sqrt{\frac{\operatorname{osc}(f)}{\lambda}}\left(1+\sqrt{2}\right)\right).\end{aligned}$$

Here we have used the estimate of $|x_i|$ obtained in iv as f is bounded and lower semicontinuous, the definition of the modulus of continuity of f and that $b - f(0) \le 0$ by (66). Hence

$$\lambda|x_i|^2 - |a||x_i| - \tilde{\omega}_f\left(\sqrt{\frac{\operatorname{osc}(f)}{\lambda}}\left(1+\sqrt{2}\right)\right) \le 0.$$

Thus we can deduce that

$$|x_i| \le \frac{|a| + \sqrt{|a|^2 + 4\lambda\tilde{\omega}_f}}{2\lambda} \;\Rightarrow\; |x_i| \le \frac{2|a| + 2\sqrt{\lambda\tilde{\omega}_f}}{2\lambda} \;\Rightarrow\; |x_i| \le \frac{|a|}{\lambda} + \sqrt{\frac{\tilde{\omega}_f}{\lambda}}. \tag{68}$$

Hence from inequality (67), we obtain

$$|x_i| \le \frac{2\sqrt{\lambda\left(\tilde{\omega}_f\left(\sqrt{\frac{\operatorname{osc}(f)}{\lambda}}\right) + \tilde{\omega}_f\left(\frac{a^*}{\lambda} + \sqrt{\frac{b^*}{\lambda}}\right)\right)}}{\lambda} + \sqrt{\frac{\tilde{\omega}_f\left(\sqrt{\frac{\operatorname{osc}(f)}{\lambda}}\left(1+\sqrt{2}\right)\right)}{\lambda}}$$

$$|x_i| \le \frac{2\sqrt{\tilde{\omega}_f\left(\sqrt{\frac{\operatorname{osc}(f)}{\lambda}}\right) + \tilde{\omega}_f\left(\frac{a^*}{\lambda} + \sqrt{\frac{b^*}{\lambda}}\right)} + \sqrt{\tilde{\omega}_f\left(\sqrt{\frac{\operatorname{osc}(f)}{\lambda}}\left(1+\sqrt{2}\right)\right)}}{\sqrt{\lambda}}$$

$$|x_i| \le \frac{2\sqrt{\tilde{\omega}_f\left(\sqrt{\frac{\mathrm{osc}(f)}{\lambda}}\right)} + \sqrt{\tilde{\omega}_f\left(\sqrt{\frac{\mathrm{osc}(f)}{\lambda}}\,(1+\sqrt{2})\right)} + 2\sqrt{\tilde{\omega}_f\left(\frac{a^*}{\lambda} + \sqrt{\frac{b^*}{\lambda}}\right)}}{\sqrt{\lambda}}$$

$$|x_i| \le \frac{3\sqrt{\tilde{\omega}_f\left(\sqrt{\frac{\mathrm{osc}(f)}{\lambda}}\,(1+\sqrt{2})\right)} + 2\sqrt{\tilde{\omega}_f\left(\frac{a^*}{\lambda} + \sqrt{\frac{b^*}{\lambda}}\right)}}{\sqrt{\lambda}}. \tag{69}$$

Therefore we can take $\mathbf{R}_\lambda^U = \dfrac{3\sqrt{\tilde{\omega}_f\left(\sqrt{\frac{\mathrm{osc}(f)}{\lambda}}\,(1+\sqrt{2})\right)} + 2\sqrt{\tilde{\omega}_f\left(\frac{a^*}{\lambda} + \sqrt{\frac{b^*}{\lambda}}\right)}}{\sqrt{\lambda}}.$ $\square$

Proof of (vi): Consider $\ell(y) = a^* \cdot y + Df(0) \cdot y + b$ for $y \in \mathbb{R}^n$ with some $a^* \in \mathbb{R}^n$ and $b \in \mathbb{R}$. By Proposition 1.5 at the origin 0, we have

$$0 \le f(0) - b \le \frac{\left(\omega_{Df}\left(\frac{L}{\lambda}(2+\sqrt{2})\right)\right)^2}{4\lambda}. \tag{70}$$

To derive the bound $\mathbf{R}_\lambda$ we evaluate (21) at $y = \frac{a^*}{2\lambda}$ and derive an upper bound of $|a^*|$:

$$\frac{a^* \cdot a^*}{2\lambda} + Df(0) \cdot \frac{a^*}{2\lambda} + b \le f\left(\frac{a^*}{2\lambda}\right) + \lambda\left|\frac{a^*}{2\lambda}\right|^2,$$

so that
$$\begin{aligned}\frac{|a^*|^2}{4\lambda} &\le f\left(\frac{a^*}{2\lambda}\right) - Df(0) \cdot \frac{a^*}{2\lambda} - f(0) + f(0) - b \\ &\le \left|f\left(\frac{a^*}{2\lambda}\right) - f(0) - Df(0) \cdot \frac{a^*}{2\lambda}\right| + f(0) - b.\end{aligned} \tag{71}$$

Moreover, from Lemma 1.4(ii), we have

$$\left|f\left(\frac{a^*}{2\lambda}\right) - f(0) - Df(0) \cdot \frac{a^*}{2\lambda}\right| \le \int_0^1 \left|Df\left(\frac{a^*}{2\lambda}t\right) - Df(0)\right| \left|\frac{a^*}{2\lambda}\right| \mathrm{d}t.$$

From the definition of modulus of continuity of Df, we obtain

$$\begin{aligned}\left|f\left(\frac{a^*}{2\lambda}\right) - f(0) - Df(0) \cdot \frac{a^*}{2\lambda}\right| &\le \int_0^1 \omega_{Df}\left(\left|\frac{a^*}{2\lambda}t\right|\right) \mathrm{d}t \ \left|\frac{a^*}{2\lambda}\right| \\ &\le \omega_{Df}\left(\left|\frac{a^*}{2\lambda}\right|\right)\left|\frac{a^*}{2\lambda}\right| \le \left|\frac{a^*}{2\lambda}\right| \omega_{Df}\left(\frac{L}{2\lambda}(1+\sqrt{2})\right),\end{aligned} \tag{72}$$

where the last inequality follows from the fact that f is a Lipschitz function so that we can use the estimate of $|a^*|$ in the Lipschitz case, that is $|a^*| \le L(1+\sqrt{2})$. Now by substituting (70) and (72) into (71), we have

$$\frac{|a^*|^2}{4\lambda} \le \frac{|a^*|}{2\lambda}\,\omega_{Df}\left(\frac{L}{2\lambda}(1+\sqrt{2})\right) + \frac{\left(\omega_{Df}\left(\frac{L}{\lambda}(2+\sqrt{2})\right)\right)^2}{4\lambda}. \tag{73}$$

Let $\Lambda = \omega_{Df}\left(\frac{L}{2\lambda}(1+\sqrt{2})\right)$ and $L_\lambda = \omega_{Df}\left(\frac{L}{\lambda}(2+\sqrt{2})\right)$.

Then we have $|a^*|^2 - 2|a^*|\Lambda - {L_\lambda}^2 \leq 0$, hence $|a^*| \leq 2\Lambda + L_\lambda$.
Now we use (22) to obtain an upper bound of $|x_i|$ for $i = 1, \dots, k$.

$$\begin{aligned} \lambda|x_i|^2 &= a^* \cdot x_i + Df(0) \cdot x_i + b - f(x_i) \\ &= a^* \cdot x_i + Df(0) \cdot x_i + b - f(0) + f(0) - f(x_i). \\ &\leq |a^* \cdot x_i| + |f(x_i) - Df(0) \cdot x_i - f(0)|. \end{aligned} \tag{74}$$

In the above inequality we have used that $b - f(0) \leq 0$ by (70). Therefore, from Lemma 1.4(ii), it follows that

$$|f(x_i) - f(0) - Df(0) \cdot x_i| \leq \omega_{Df}\left(|x_i - 0|\right)|x_i - 0|. \leq \omega_{Df}\left(|x_i|\right)|x_i|.$$

Then by using the estimate of $|x_i|$ from the Lipschitz continuous case, we have

$$|f(x_i) - f(0) - Df(0) \cdot x_i| \leq \omega_{Df}\left(\frac{L}{\lambda}(2 + \sqrt{2}\right)|x_i|$$

and $$|f(x_i) - f(0) - Df(0) \cdot x_i| \leq |x_i| L_\lambda.$$

Now we can rewrite inequality (74) as follows:

$$\lambda|x_i|^2 \leq |a^*|\ |x_i| + |x_i|\ L_\lambda, \leq (2\Lambda + L_\lambda)|x_i| + L_\lambda|x_i|, \leq (2\Lambda + 2L_\lambda)|x_i|.$$

Since $\omega_{\mathcal{D}f}(\cdot)$ is a non decreasing function, we have

$$\omega_{Df}\left(\frac{L}{2\lambda}(1 + \sqrt{2}\right) \leq \omega_{Df}\left(\frac{L}{\lambda}(2 + \sqrt{2}\right). \tag{75}$$

This means $\Lambda \leq L_\lambda$. Thus $|x_i|^2 \leq \dfrac{4L_\lambda|x_i|}{\lambda}$.

Hence we can deduce that for each x_i with $k = 1, \cdots, k$,

$$|x_i| \leq \frac{4\ \omega_{Df}\left(\frac{L}{\lambda}(2 + \sqrt{2}\right)}{\lambda}. \tag{76}$$

Therefore we can choose $\mathbf{R}_\lambda^D = \dfrac{4\ \omega_{Df}\left(\frac{L}{\lambda}(2 + \sqrt{2})\right)}{\lambda}$. □

Proof of (vii): This is a direct consequence of Theorem 2.3(v) in [7]. □

Proof of Lemma 1.4: (i) Consider $g(t) = f(x + t(y - x))$. Since f is a differentiable function, by using the fundamental theorem of calculus,

$$g(1) - g(0) = \int_0^1 g'(t)dt.$$

Since $g'(t) = Df\left(x + t(y - x)\right) \cdot (y - x)$, we have

$$f(y) - f(x) = \int_0^1 Df\left(x + t(y - x)\right) \cdot (y - x)\mathrm{d}t.$$

Since the gradient of f is bounded, there exists $L \geq 0$ such that $|Df| \leq L$. Hence

$$|f(y) - f(x)| \leq \int_0^1 |Df\,(x + (y-x)t)\,|\,|y-x|\mathrm{d}t \leq L|y-x|,$$

proving (i).

(ii) By using the fundamental theorem of calculus, we have

$$\begin{aligned}
&|f(y) - f(x) - Df(x)\cdot(y-x)| \\
&= \left| \int_0^1 (Df(x+(y-x)t)\cdot(y-x) - Df(x)\cdot(y-x))\mathrm{d}t \right| \\
&\leq \int_0^1 |(Df(x+(y-x)t) - Df(x))\cdot(y-x)\mathrm{d}t| \\
&\leq \int_0^1 |Df(x+(y-x)t) - Df(x)|\,|y-x|\mathrm{d}t. \qquad (77)
\end{aligned}$$

Since the derivative of f is bounded and uniformly continuous, then we are able to define the modulus of continuity of Df by $\omega_{Df(t)}$. Hence we obtain

$$|Df(y) - Df(x)| \leq \omega_{Df}(|y-x|).$$

Next we can rewrite inequality (77) as follows:

$$\begin{aligned}
|f(y) - f(x) - Df(x)\cdot(y-x)| &\leq |y-x| \int_0^1 \omega_{Df}(t|y-x|)\mathrm{d}t \\
&\leq \omega_{Df}(|y-x|)|y-x|, \qquad (78)
\end{aligned}$$

which completes the proof. □

Proof of Proposition 1.5: Since f is differentiable and its gradient is bounded and uniformly continuous, by Lemma 1.4(i), f is a Lipschitz function. By [10, Proposition 2.1](see also Proposition 2.2), there exists $\lambda_i \in \mathbb{R}$, and $x_i \in \mathbb{R}^n$ with $2 \leq k \leq n+1$, such that the lower transform is given by

$$\mathcal{C}_\lambda^l(f)(x_0) = \sum_{i=1}^k \lambda_i(f(x_i) + \lambda|x_i - x_0|^2),$$

with $\sum_{i=1}^k \lambda_i = 1$, $\sum_{i=1}^k \lambda_i(x_i - x_0) = 0$, $\lambda_i \geq 0$, $x_i \in \mathbb{R}^n$. Therefore

$$\begin{aligned}
f(x_0) - \mathcal{C}_\lambda^l(f)(x_0) &= \sum_{i=1}^k \lambda_i(f(x_0) - f(x_i) - \lambda|x_i - x_0|^2) \\
&= -\sum_{i=1}^k \lambda_i \left((f(x_i) - f(x_0) - Df(x_0)\cdot(x_i - x_0)) + \lambda|x_i - x_0|^2 \right). \qquad (79)
\end{aligned}$$

In (79) above we have used the fact that $\sum_{i=1}^k \lambda_i Df(x_0)\cdot(x_i - x_0) = 0$.

Furthermore, from Lemma 1.4(ii), we have

$$|f(x_i) - f(x_0) - Df(x_0)\cdot(x_i - x_0)| \leq \omega_{Df}\left(|x_i - x_0|\right)|x_i - x_0|. \qquad (80)$$

By substituting (80) into (79), we have

$$
\begin{aligned}
f(x_0) - \mathcal{C}^l_\lambda(f)(x_0) &\le \sum_{i=1}^{k} \lambda_i(\omega_{Df}\left(|x_i - x_0|\right)|x_i - x_0| - \lambda|x_i - x_0|^2),\\
&\le \sum_{i=1}^{k} \lambda_i(\omega_{Df}\left(\frac{L}{\lambda}(2+\sqrt{2})\right)|x_i - x_0| - \lambda|x_i - x_0|^2).
\end{aligned}
$$

Here we have used the facts that f is a Lipschitz function and we have used the estimate of $|x_i|$ in the Lipschitz case. Now we consider

$$g(t) = \omega_{Df}\left(\frac{L}{\lambda}(2+\sqrt{2})\right)t - \lambda t^2,$$

and
$$g'(t) = \omega_{Df}\left(\frac{L}{\lambda}(2+\sqrt{2})\right) - 2\lambda t.$$

Let $g'(t^*) = 0$, then $t^* = \frac{\omega_{Df}\left(\frac{L}{\lambda}(2+\sqrt{2})\right)}{2\lambda}$. Therefore the function $g(t)$ attains the maximum value at the point t^*, hence

$$f(x_0) - \mathcal{C}^l_\lambda(f)(x_0) \le \sum_{i=1}^{k} \lambda_i g(t^*),$$

and
$$f(x_0) - \mathcal{C}^l_\lambda(f)(x_0) \le \frac{\left(\omega_{Df}\left(\frac{L}{\lambda}(2+\sqrt{2})\right)\right)^2}{4\lambda}.$$

Thus we conclude that the inequality on the right side of (5) holds and the left part of (5) is given by the definition of the lower transform. The proof is complete. □

Proof of Theorem 1.7: This is very similar to the proof of Theorem 3.10(ii) in [10] but for completeness we will give the proof below.
By [10, Proposition 3.1] (14), we have $\mathcal{C}^l_\lambda(f)(x) = \mathcal{C}^l_\lambda(\underline{f})(x)$, and by our assumption of the theorem, we have $\mathcal{C}^l_\lambda(\underline{f})(x_0) < \underline{f}(x_0)$. Furthermore,

$$\mathcal{C}^l_\lambda(\underline{f})(x_0) = \mathbf{Co}[\underline{f} + \lambda|\cdot - x_0|^2](x_0) < \underline{f}(x_0). \tag{81}$$

Now, let $g(y) = \underline{f}(y) + \lambda|y - x_0|^2$, since g is continuous and coercive, then by [10, Proposition 2.1] (also see Proposition 2.2), there exists $x_i \in \mathbb{R}^n$ with $x_i \neq x_0$ because of (81), which belong to $\overline{B}(x_0; \mathbf{R}_\lambda)$, and $\lambda_i \ge 0$ for $i = 1, \dots, k$ satisfying $\sum_{i=1}^k \lambda_i = 1$ and $\sum_{i=1}^k \lambda_i x_i = x_0$. Moreover, there is an affine function $\ell(y) = a \cdot (y - x_0) + b$ for $y \in \mathbb{R}^n$ with some $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that

$$\ell(y) \le g(y) \quad \text{for all } y \in \mathbb{R}^n \text{ and } \quad \ell(x_i) = g(x_i) \qquad \text{for } i = 1, \dots, k. \tag{82}$$

Since $\ell(y)$ is an affine function, we conclude that

$$\ell(x_i) = \mathbf{Co}[g](x_i) \qquad \text{for } i = 1, \dots, k.$$

By accounting for (82), we conclude that

$$\mathbf{Co}[\underline{f} + \lambda| \cdot - x_0|^2](x_i) = \ell(x_i) = g(x_i) = \underline{f}(x_i) + \lambda|x_i - x_0|^2. \tag{83}$$

Since the convex envelope is invariant with respect to affine functions, we have

$$\mathbf{Co}[\underline{f}+\lambda|\cdot - x_i + x_i - x_0|^2](x) = \mathbf{Co}[\underline{f}+\lambda|\cdot - x_i|^2](x) + \lambda|x_i - x_0|^2 + 2\lambda(x - x_i)\cdot(x_i - x_0).$$

Therefore at $x = x_i$ we deduce that

$$\mathbf{Co}[\underline{f} + \lambda| \cdot - x_0|^2](x_i) = \mathbf{Co}[\underline{f} + \lambda| \cdot - x_i|^2](x_i) + \lambda|x_i - x_0|^2. \tag{84}$$

By comparing (83) and (84), we have for $i = 1, \ldots, k$,

$$\mathbf{Co}[\underline{f} + \lambda| \cdot - x_0|^2](x_i) = \mathbf{Co}[\underline{f} + \lambda| \cdot - x_i|^2](x_i) + \lambda|x_i - x_0|^2 = \underline{f}(x_i) + \lambda|x_i - x_0|^2.$$

Hence we obtain

$$\mathcal{C}^l_\lambda(\underline{f})(x_i) = \mathbf{Co}[\underline{f} + \lambda| \cdot - x_0|^2](x_i) = \underline{f}(x_i) \qquad \text{for } i = 1, \ldots, k,$$

which completes the proof. □

Proof of Lemma 1.8: Since f is Lipschitz at x_0, then we see that f is bounded. In fact, for $x \in \overline{B}(x_0; \delta)$, we have

$$|f(x)| - |f(x_0)| \le |f(x) - f(x_0)| \le L|x - x_0| \le L\delta, \tag{85}$$

then f is bounded in $\overline{B}(x_0; \delta)$ by $|f(x_0)| + L\delta$. Also,

$$f(x_0) - L\delta \le f(x) \le f(x_0) + L\delta.$$

Thus $\widehat{f}$ is bounded in $\mathbb{R}^n$. Now we show that $\widehat{f}(x)$ is lower semicontinuous in $\mathbb{R}^n$.

Since f is lower semicontinuous inside $B(x_0, \delta)$, then $\widehat{f}$ is lower semicontinuous in $B(x_0, \delta)$. Moreover, $\widehat{f}$ is constant in the open set $(\overline{B(x_0, \delta)})^c$, then it is clearly continuous in the open set , hence it is lower semicontinuous, that means,

$$\lim_{\substack{y \to x \\ y \in B(x_0,\delta)}} \widehat{f}(y) = f(x_0) - L\delta = \widehat{f}(x).$$

Now we need to show that $\widehat{f}(x)$ is lower semicontinuous on ∂B. Let $x_1 \in \partial B$ and consider two cases:

Case 1: $\liminf\limits_{\substack{x \to x_1 \\ x \in B(x_0,\delta)}} \widehat{f}(x)$.

Since $x \in B(x_0, \delta)$, $f(x) \ge f(x_0) - L\delta$, we have

$$\liminf_{\substack{x \to x_1 \\ x \in B(x_0,\delta)}} \widehat{f}(x) = \liminf_{\substack{x \to x_1 \\ x \in B(x_0,\delta)}} f(x) \ge f(x_0) - L\delta = \widehat{f}(x_1) \tag{86}$$

Case 2: $\liminf\limits_{\substack{x \to x_1 \\ x \in B(x_0,\delta)^c}} \widehat{f}(x)$.

Since $x \in B(x_0, \delta)^c$, $\widehat{f}(x) = f(x_0) - L\delta$, so

$$\liminf_{\substack{x \to x_1 \\ x \in B(x_0,\delta)^c}} \widehat{f}(x) = \lim_{\substack{x \to x_1 \\ x \in B(x_0,\delta)^c}} f(x_0) - L\delta = f(x_0) - L\delta = \widehat{f}(x_1). \tag{87}$$

Thus we conclude that $\widehat{f}(x)$ is lower semicontinuous in $\mathbb{R}^n$. □

Proof of Theorem 1.9: Due to the translation invariance property [10, Proposition 2.10] (also see Proposition 2.7), we only need to consider the case for $x_0 = 0$. Consider the function $g(y) = \widehat{f}(y) + \lambda|y|^2$ and by Proposition 2.1 in [10] (also see Proposition 2.2), there is an affine function $\ell(y) = a \cdot y + b$ for $y \in \mathbb{R}^n$ with some $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$, such that (82) hold.

Our aim is to improve gradually the estimate of $|a|$ in order to reduce the size of $\mathbf{R}_\lambda$. By Lemma 1.8, we have $\widehat{f}(x)$ is bounded and lower semicontinuous in $\mathbb{R}^n$. Then, by following Theorem 1.2iv, we have $|a| \le c\sqrt{\lambda}$ where $c = 2\sqrt{\operatorname{osc}(\widehat{f})}$. Moreover,

$$\frac{|a|}{2\lambda} \le \frac{c}{\sqrt{\lambda}} < \delta, \tag{88}$$

If $\sqrt{\lambda} > \frac{c}{\delta}$, hence $\lambda > \left(\frac{c}{\delta}\right)^2$. Let $\Lambda_1 = \left(\frac{c}{\delta}\right)^2$, so that when $\lambda > \Lambda_1$, we have $\frac{a}{2\lambda} \in B(0,\delta)$. Thus $\widehat{f}\left(\frac{a}{2\lambda}\right) = f\left(\frac{a}{2\lambda}\right)$. From (82), we also have

$$a \cdot y + b \le \widehat{f}\left(\frac{a}{2\lambda}\right) + \lambda|y|^2,$$

and
$$\frac{a \cdot a}{2\lambda} + b \le f\left(\frac{a}{2\lambda}\right) + \frac{|a|^2}{4\lambda}.$$

Hence,
$$\begin{aligned}\frac{|a|^2}{4\lambda} &\le f\left(\frac{a}{2\lambda}\right) - b \le f\left(\frac{a}{2\lambda}\right) - f(0) + f(0) - b\\ &\le \left|f\left(\frac{a}{2\lambda}\right) - f(0)\right| + \frac{L^2}{4\lambda} \le \frac{L|a|}{2\lambda} + \frac{L^2}{4\lambda}.\end{aligned}$$

Here we have used the fact that f is Lipschitz at x_0, hence $|a|^2 - 2L|a| - L^2 \le 0$. Thus we have $|a| \le L(1+\sqrt{2})$.

So, we have improved the estimate of $|a|$. Again, since $\widehat{f}$ is bounded and lower semicontinuous in $\mathbb{R}^n$ then by Theorem 1.2iv, we have

$$|x_i| \le \frac{c^*}{\sqrt{\lambda}} < \delta, \quad \text{where } c^* = (1+\sqrt{2})\sqrt{\operatorname{osc}(\widehat{f})}.$$

If $\sqrt{\lambda} > \frac{c^*}{\delta}$,we conclude $\lambda > \left(\frac{c^*}{\delta}\right)^2$. Let $\Lambda_2 = \left(\frac{c^*}{\delta}\right)^2$, so that when $\lambda > \Lambda_2$, we have $x_i \in B(0,\delta)$. Therefore if we take $\Lambda = \max\{\Lambda_1, \Lambda_2\}$ and $\lambda > \Lambda$, we have $\widehat{f}(x_i) = f(x_i)$. Now we use (82) $a \cdot x_i + b = f(x_i) + \lambda|x_i|^2$ to obtain

$$\begin{aligned}\lambda|x_i|^2 &= a \cdot x_i + b - f(x_i) = a \cdot x_i + b - f(0) + f(0) - f(x_i)\\ &\le |a||x_i| + |f(x_i) - f(0)| \le |a||x_i| + L|x_i|,\end{aligned}$$

as $b - f(0) \le 0$. Thus we can deduce that for each x_i with $i = 1, 2, \cdots, k$,

$$|x_i| \le \frac{L + |a|}{\lambda} \le \frac{L}{\lambda}(2+\sqrt{2}).$$

Therefore $\mathbf{R}_\lambda^L := \frac{L}{\lambda}(2+\sqrt{2})$.

That means $\mathcal{C}_\lambda^l(\widehat{f})(x_0) = \mathbf{Co}_{\overline{B}(x_0;\mathbf{R}_\lambda^L)}[f(\cdot) + \lambda|\cdot - x_0|^2](x_0)$, concluding the proof. $\square$

References

[1] M. Alatawi, K. Zhang: *Hausdorff stability and error estimates for compensated convexity based methods for approximation and interpolation for functions in* $\mathbb{R}^n$, J. Convex Analysis 27/4 (2020) 1277–1302.

[2] B. Dacorogna: *Direct Methods in the Calculus of Variations*, Springer, Berlin (1989).

[3] R. A. DeVore, G. G. Lorentz: *Constructive Approximation*, Springer, Berlin (1993).

[4] D. Gilbarg, N. S. Trudinger: *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin (1983).

[5] J.-B. Hiriart-Urruty, C. Lemaréchal: *Fundamentals of Convex Analysis*, Springer, Berlin (2001).

[6] R. T. Rockafellar: *Convex Analysis*, Princeton University Press, Princeton (1970).

[7] K. Zhang: *Compensated convexity and its applications*, Ann. l'Inst. H. Poincaré (C), Analyse Non Linear 25/4 (2008) 743–771.

[8] K. Zhang, E. Crooks, A. Orlando: *Compensated convexity methods for approximations and interpolations of sampled functions in Euclidean spaces: Theoretical foundations*, SIAM J. Math. Analysis 48/6 (2016) 4126–4154.

[9] K. Zhang, A. Orlando, E. Crooks: *Compensated convexity and Hausdorff stable extraction of intersections for smooth manifolds*, Math. Models Methods Appl. Sci. 25/5 (2015) 839–873.

[10] K. Zhang, A. Orlando, E. Crooks: *Compensated convexity and Hausdorff stable geometric singularity extractions*, Math. Models Methods Appl. Sci. 25/4 (2015) 747–801.

[11] K. Zhang, A. Orlando, E. Crooks: *Compensated convex transforms and geometric singularity extraction from semiconvex functions*, Sci. Sin. Math. 46/5 (2016) 747–768.