Journal of Convex Analysis
Volume 29 (2022), No. 4, 1065–1082

On the Convex Combination of Volterra Operators

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Received: August 14, 2021
Accepted: February 27, 2022

We consider a convex combination of certain Volterra operators defined on the two-dimensional simplex depending on the parameter $\theta \in [0,1]$ and study their trajectory behaviours. We divide the set of the parameter θ into two subsets such that on the first (resp. second) of them the corresponding Volterra operator is a regular (resp. non-ergodic) transformation.

Keywords: Quadratic stochastic operator, Volterra operator, cubic stochastic operator.

2010 Mathematics Subject Classification: Primary 37N25; secondary 92D10.

1. Introduction

Let $E = \{1, \dots, m\}$ be a finite set and the set of all probability distributions on E

$$S^{m-1} = \Big\{\mathbf{x} = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m : \ x_i \geq 0, \text{ for any } i \text{ and } \sum_{i=1}^{m} x_i = 1\Big\}$$

be the $(m-1)$-dimensional simplex.

Let us consider a continuous mapping $\Psi\colon S^{m-1} \to S^{m-1}$. A mapping Ψ is called *stochastic* if $\Psi(S^{m-1}) \subset S^{m-1}$.

The trajectory $\{\mathbf{x}^{(n)}\}, n = 0, 1, 2, \dots$ of Ψ for an initial value $\mathbf{x}^{(0)} \in S^{m-1}$ is defined by

$$\mathbf{x}^{(n+1)} = \Psi\big(\mathbf{x}^{(n)}\big) = \Psi^{n+1}\big(\mathbf{x}^{(0)}\big), \quad n = 0, 1, 2, \dots$$

Denote by $\omega_\Psi\big(\mathbf{x}^{(0)}\big)$ the set of limit points of the trajectory $\{\mathbf{x}^{(n)}\}_{n=0}^{\infty}$.

A mapping Ψ is called *regular* if the limit $\lim\limits_{n\to\infty} \Psi^n(\mathbf{x})$ for an initial value $\mathbf{x} \in S^{m-1}$ exists. A mapping Ψ is said to be *ergodic* if the limit

$$M_\Psi(\mathbf{x}) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Psi^k(\mathbf{x})$$

exists for any $\mathbf{x} \in S^{m-1}$.

ISSN 0944-6532 / \$ 2.50

A *quadratic stochastic operator* (QSO) is a mapping $V : S^{m-1} \to S^{m-1}$ of the simplex into itself, of the form $V(\mathbf{x}) = \mathbf{x}' \in S^{m-1}$, where

$$V : x'_k = \sum_{i,j\in E} p_{ij,k} x_i x_j, \quad k \in E,$$

and the coefficients $p_{ij,k}$ satisfy

$$p_{ij,k} = p_{ji,k} \geq 0, \quad \sum_{k=1}^{m} p_{ij,k} = 1, \quad i, j, k \in E.$$

Such operators frequently arise in many models of mathematical genetics, namely theory of heredity (see e.g. [1, 3, 4, 5, 7, 9, 11, 13, 16, 17, 22, 23, 26]). The main problem in mathematical biology consists in the study of the asymptotical behaviour of the trajectories for a given QSO. In other words, the main task is the description of the set $\omega_V\big(\mathbf{x}^{(0)}\big)$ for any initial point $\mathbf{x}^{(0)} \in S^{m-1}$ for a given QSO V. This problem is an open problem even in two-dimensional case.

A QSO is called a *Volterra* operator if $p_{ij,k} = 0$ for any $k \notin \{i, j\}, \quad i, j, k \in E$. The asymptotic behaviour of trajectories Volterra QSOs was analysed in [5, 6] using the theory of Lyapunov functions and tournaments.

On the basis of numerical calculations, Ulam conjectured that any QSO is ergodic [24]. But in 1977 Zakharevich [25] considered the following QSO on S^2,

$$x'_1 = x_1^2 + 2x_1x_2, \quad x'_2 = x_2^2 + 2x_2x_3, \quad x'_3 = x_3^2 + 2x_1x_3, \tag{1}$$

and showed that it is a non-ergodic transformation, that is, he proved that Ulam's conjecture is false in general. Later, a sufficient condition for a QSO defined on S^2 to be a non-ergodic transformation was established in [4], that is, Zakharevich's result was generalized to a class of Volterra QSOs defined on S^2. In [3], we have shown the correlation between non-ergodicity of Volterra QSOs and rock-paper-scissors games, so the QSO (1) can be reinterpreted in terms of game theory as a rock-paper-scissors game. In [15] the random dynamics of Volterra QSOs is studied. For a recent review on the theory of quadratic stochastic operators see [7].

A *cubic stochastic operator* (CSO), mapping of the simplex S^{m-1} into itself, is of the form

$$W : x'_l = \sum_{i,j,k\in E} p_{ijk,l} x_i x_j x_k, \quad l \in E, \tag{2}$$

where $p_{ijk,l}$ are the coefficients of heredity such that

$$p_{ijk,l} \geq 0, \quad \sum_{l\in E} p_{ijk,l} = 1, \tag{3}$$

and the coefficients $p_{ijk,l}$ do not change for any permutation of i, j and k.

Let $p_{ijk,l} = p_{ij,l}$ for any $i, j, k, l = 1, \ldots, m$, then (2) has the form

$$x'_l = \sum_{i,j,k\in E} p_{ijk,l} x_i x_j x_k = \sum_{k\in E} x_k \sum_{i,j\in E} p_{ij,l} x_i x_j = \sum_{i,j\in E} p_{ij,l} x_i x_j, \quad l \in E,$$

that is, in this case, the cubic stochastic operator defined by (2) and (3) coincides with a quadratic stochastic operator defined on the finite-dimensional simplex.

In the present paper we consider certain families of discrete-time dynamical systems generated by CSOs. Namely CSOs depending on the parameter θ. For more motivations of such investigations see e.g. [8, 10, 12, 20, 21] and references therein. In [20], a notion of cubic stochastic operator is introduced and investigated. For a recent review on the convex combinations of QSOs and on the convex combinations of CSOs see e.g. [13, 14]. In [18] a non-ergodic Lotka-Volterra operator is considered.

The paper is organized as follows. In Section 2 we recall the definitions and known results. In Section 3 we consider convex combinations of a QSO and a CSO. We also show that there is a critical value θ^* such that for $\theta < \theta^*$ (resp. $\theta > \theta^*$) any Volterra CSO of this family is regular (resp. non-ergodic) operator. In Section 4 we consider convex combinations of another Volterra CSOs and we prove that there are two critical values of the parameter such that for $\theta \in (1/14, 1/2)$ (resp. $\theta \in [0, 1/14) \cup (1/2, 1]$) any Volterra CSO of this family is a regular (resp. non-ergodic) operator.

2. Preliminaries and known results

In this section we recall some necessary notions and known results. A point $\mathbf{x} \in S^{m-1}$ is called a fixed point of a stochastic operator Ψ if $\Psi(\mathbf{x}) = \mathbf{x}$. Let

$$D\Psi(\mathbf{x}^*) = \big(\partial\Psi_i/\partial x_j(\mathbf{x}^*)\big)_{i,j=1}^{m}$$

be the Jacobi matrix of the operator Ψ at the point $\mathbf{x}^*$.

A fixed point $\mathbf{x}^*$ is called *hyperbolic* if its Jacobi matrix $D\Psi(\mathbf{x}^*)$ has no eigenvalues 1 in absolute value. A hyperbolic fixed point $\mathbf{x}^*$ is called *attracting* (resp. *repelling*) if all the eigenvalues of the Jacobi matrix $D\Psi(\mathbf{x}^*)$ are less (resp. greater) than 1 in absolute value; it is called a saddle if some of eigenvalues of $D\Psi(\mathbf{x}^*)$ are less than 1 in absolute value and other eigenvalues are greater than 1 in absolute value (see [2]).

Let $$S := S^2 = \{\mathbf{x} \in \mathbb{R}^3 : x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_1 + x_2 + x_3 = 1\}.$$

Define $\operatorname{int} S = \{\mathbf{x} \in S : x_1x_2x_3 > 0\}$ and $\partial S = S \setminus \operatorname{int} S$.

Consider the following QSOs defined on S:

$$V_1 : \begin{cases} x_1' = x_1\left(1 + x_2 - x_3\right), \\ x_2' = x_2\left(1 + x_3 - x_1\right), \\ x_3' = x_3\left(1 + x_1 - x_2\right), \end{cases} \qquad V_2 : \begin{cases} x_1' = x_1\left(1 + x_3 - x_2\right), \\ x_2' = x_2\left(1 + x_1 - x_3\right), \\ x_3' = x_3\left(1 + x_2 - x_1\right). \end{cases} \tag{4}$$

Theorem 2.1. [4] *The following statements hold for the QSOs V_1 and V_2 of* (4):

(i) *$Fix(V_1) = Fix(V_2) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{c}\}$;*

(ii) *the vertexes $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are saddle points and the center $\mathbf{c}$ is a repelling point;*

(iii) *the function $\varphi(\mathbf{x}) = x_1x_2x_3$ is a decreasing Lyapunov function;*

(iv) *$\omega_{V_1}\big(\mathbf{x}^{(0)}\big) \subset \partial S$ is an infinite compact set for any $\mathbf{x}^{(0)} \in int S \setminus \{\mathbf{c}\}$;*

(v) *$\omega_{V_2}\big(\mathbf{x}^{(0)}\big) \subset \partial S$ is an infinite compact set for any $\mathbf{x}^{(0)} \in int S \setminus \{\mathbf{c}\}$;*

(vi) *for all $\mathbf{x}^{(0)} \in int S \setminus \{\mathbf{c}\}$ and for the operators V_1 and V_2 the limit of averages $M_{V_1}(\mathbf{x})$ and $M_{V_2}(\mathbf{x})$ do not exist.*

Consider the following CSOs defined on S

$$W_1: \begin{cases} x_1' = x_1\left(1 + x_1x_2 - x_3^2\right), \\ x_2' = x_2\left(1 + x_2x_3 - x_1^2\right), \\ x_3' = x_3\left(1 + x_1x_3 - x_2^2\right), \end{cases} \qquad W_2: \begin{cases} x_1' = x_1\left(1 + x_3^2 - x_1x_2\right), \\ x_2' = x_2\left(1 + x_1^2 - x_2x_3\right), \\ x_3' = x_3\left(1 + x_2^2 - x_1x_3\right). \end{cases} \tag{5}$$

Theorem 2.2. [14] *The following statements hold for the CSOs W_1 and W_2 of* (5):

(i) $Fix(W_1) = Fix(W_2) = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{c}\}$;

(ii) *the vertexes $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are non-hyperbolic points. The center $\mathbf{c}$ is a repelling point for the CSO W_1 and it is an attracting point for the CSO W_2;*

(iii) *the function $\varphi(\mathbf{x}) = x_1x_2x_3$ is a Lyapunov function;*

(iv) $\omega_{W_1}\left(\mathbf{x}^{(0)}\right) \subset \partial S$ *is an infinite compact set for any* $\mathbf{x}^{(0)} \in int S \setminus \{\mathbf{c}\}$;

(v) *for all $\mathbf{x}^{(0)} \in int S \setminus \{\mathbf{c}\}$ and for the operator W_1 the limit of averages $M_{W_1}(\mathbf{x})$ doesn't exist;*

(vi) *the set of limit points* $\omega_{W_2}\left(\mathbf{x}^{(0)}\right) = \{\mathbf{c}\}$ *for any* $\mathbf{x}^{(0)} \in int S \setminus \{\mathbf{c}\}$.

By Theorem 2.1 and Theorem 2.2 it follows that the QSOs V_1, V_2 and the CSO W_1 are non-ergodic transformations and the CSO W_2 is a regular transformation.

3. On convex combinations of QSOs and CSOs

In this section we consider on S the following family of Volterra CSOs defined by

$$\theta V_2 + (1-\theta)W_2 = V_\theta: \begin{cases} x_1' = x_1\Big(1 + \theta\big(x_3 - x_2\big) + (1-\theta)\big(x_3^2 - x_1x_2\big)\Big), \\ x_2' = x_2\Big(1 + \theta\big(x_1 - x_3\big) + (1-\theta)\big(x_1^2 - x_2x_3\big)\Big), \\ x_3' = x_3\Big(1 + \theta\big(x_2 - x_1\big) + (1-\theta)\big(x_2^2 - x_1x_3\big)\Big), \end{cases} \tag{6}$$

where $\theta \in [0,1]$, that is convex combination of a non-ergodic QSO V_2 and a regular CSO W_2. Note that an evolution operator $V_\theta : S \to S$ of the population which is a discrete analogy of the Kolmogorov model of three interacting populations.

Let $\theta^* = \sqrt{6} - 2 \approx 0.4495$. It is easy to verify that the faces $\Gamma_{\{1,2\}}, \Gamma_{\{2,3\}}, \Gamma_{\{1,3\}}$ are invariant sets with respect to V_θ. The points

$$\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3,\ \mathbf{c} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

are the fixed points of the operator (6). If $0 \le \theta \le \theta^*$ then the center $\mathbf{c}$ is an attracting point and it is a repelling fixed point when $\theta^* < \theta \le 1$. The vertices $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are saddle points.

Consider the dynamics on the invariant face $\Gamma_{\{1,2\}}$. On this face the CSO (6) can be written as follows

$$f_\theta(x) = x((1-\theta)x^2 + (2\theta - 1)x + (1-\theta)), \quad x = x_1 \in [0,1].$$

One can easily verify that the function $f_\theta(x)$ has the fixed points $x=0$, $x=1$ and the fixed point $x=0$ is an attracting point and $x=1$ is a repelling point. Further for any $0<x<1$ one has that

$$\lim_{n\to\infty} f_\theta^n(x)=0. \tag{7}$$

Consequently, we have $\lim\limits_{n\to\infty} V_\theta\big(\mathbf{x}^{(0)}\big)=\mathbf{e}_2$ for any $\mathbf{x}^{(0)}\in\Gamma_{\{1,2\}}$.

Other faces can be considered in similar manner.

Theorem 3.1. *For the Volterra CSO* (6) *the following statements hold:*

(i) *The function $\varphi(\mathbf{x})=x_1x_2x_3$ is a Lyapunov function;*

(ii) *If $\theta\in[0,\theta^*]$ then $\omega_{V_\theta}(\mathbf{x}^0)=\{\mathbf{c}\}$ for any $\mathbf{x}^0\in\operatorname{int} S$;*

(iii) *If $\theta\in(\theta^*,1]$ then $\omega_{V_\theta}(\mathbf{x}^0)$ is an infinite subset of ∂S for any $\mathbf{x}^0\in\operatorname{int} S^2\setminus\{\mathbf{c}\}$.*

Proof. (i) Evidently the function $\varphi(\mathbf{x})=x_1x_2x_3$ is continuous on the simplex S. It is easy to see that

$$\varphi(\mathbf{x})=0 \ \text{ iff } \ \mathbf{x}\in\partial S \ \text{ and } \ \max_{\mathbf{x}\in S}\varphi(\mathbf{x})=\frac{1}{27} \ \text{ iff } \ \mathbf{x}=C.$$

From (6) one has $\varphi(\mathbf{x}')=\varphi(\mathbf{x})\psi_\theta(\mathbf{x})$, where

$$\begin{aligned}\psi_\theta(\mathbf{x})=&\Big(1+\theta\big(x_3-x_2\big)+(1-\theta)\big(x_3^2-x_1x_2\big)\Big)\\ &\cdot\Big(1+\theta\big(x_1-x_3\big)+(1-\theta)\big(x_1^2-x_2x_3\big)\Big)\\ &\cdot\Big(1+\theta\big(x_2-x_1\big)+(1-\theta)\big(x_2^2-x_1x_3\big)\Big).\end{aligned}\tag{8}$$

Using $x_3=1-x_1-x_2$ from (8) one has that

$$\begin{aligned}\psi_\theta(x_1,x_2)=&\Big(1+\theta\big(1-x_1-2x_2\big)+(1-\theta)\big((1-x_1-x_2)^2-x_1x_2\big)\Big)\\ &\cdot\Big(1+\theta\big(2x_1+x_2-1\big)+(1-\theta)\big(x_1^2-x_1(1-x_1-x_2)\big)\Big)\\ &\cdot\Big(1+\theta\big(x_2-x_1\big)+(1-\theta)\big(x_2^2-x_1(1-x_1-x_2)\big)\Big),\end{aligned}$$

where $(x_1,x_2)\in S=\{(x_1,x_2): x_1,x_2\geq 0, x_1+x_2\in[0;1]\}$ and x_1,x_2 are the first two coordinates of a point lying in the simplex S.

A simple but tedious calculation leads to

$$\min_{(x_1,x_2)\in S}\psi_\theta(x_1,x_2)=1 \ \text{ iff } \ x_1=x_2=\frac{1}{3} \ \text{ and } \ \theta\in[0,\theta^*], \tag{9}$$

$$\max_{(x_1,x_2)\in S}\psi_\theta(x_1,x_2)=1 \ \text{ iff } \ x_1=x_2=\frac{1}{3} \ \text{ and } \ \theta\in(\theta^*,1]. \tag{10}$$

Thus if $\theta\in(\theta^*,1]$ then $\psi_\theta(\mathbf{x})\leq 1$ and $\varphi(\mathbf{x}')\leq\varphi(\mathbf{x})$. With other words, the function $\varphi(\mathbf{x})=x_1x_2x_3$ is a decreasing Lyapunov function. If $\theta\in[0,\theta^*]$ then $\psi_\theta(\mathbf{x})\geq 1$ and $\varphi(\mathbf{x}')\geq\varphi(\mathbf{x})$, that is, the function $\varphi(\mathbf{x})=x_1x_2x_3$ is an increasing Lyapunov function.

(ii) Suppose that $\theta \in [0, \theta^*]$. Then the function $\varphi(\mathbf{x})$ is an increasing Lyapunov function. Therefore due to (9) we have $\psi_\theta(\mathbf{x}) > 1$ for any $\mathbf{x} \in \operatorname{int} S \setminus \{\mathbf{c}\}$. Consequently it follows that $\varphi(V_\theta(\mathbf{x})) > \varphi(\mathbf{x})$ whenever $\mathbf{x} \neq \mathbf{c}$.

So $\{\varphi(\mathbf{x}^{(n)})\}$, $n = 0, 1, \dots$ is a strictly increasing bounded sequence. Therefore there exits $\lim_{n\to\infty} \varphi(\mathbf{x}^{(n)}) = 1/27$. Since $\max_{\mathbf{x}\in S} \varphi(\mathbf{x}) = 1/27$ iff $\mathbf{x} = \mathbf{c}$ we obtain that $\lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{c}$.

(iii) Suppose that $\theta \in (\theta^*, 1]$. Then the function $\varphi(\mathbf{x})$ is a decreasing Lyapunov function. Consequently, we have

$$\varphi(\mathbf{x}^{(n+1)}) \le \varphi(\mathbf{x}^{(n)})\psi_\theta(\mathbf{x}^{(n)}) \le \varphi(\mathbf{x}^{(n)}), \quad n = 0, 1, \dots,$$

that is there exists $\lim_{n\to\infty} \varphi(\mathbf{x}^{(n)}) = \xi$. Since $\mathbf{x}^0 \neq \mathbf{c}$ it follows that

$$0 \le \xi \le \varphi(\mathbf{x}^0) < \varphi(\mathbf{c}) = \frac{1}{27}.$$

We claim that $\xi = 0$. Suppose that $\xi > 0$. Then due to

$$1 = \lim_{n\to\infty} \frac{\varphi(\mathbf{x}^{(n+1)})}{\varphi(\mathbf{x}^{(n)})} = \lim_{n\to\infty} \psi_\theta(\mathbf{x}^{(n)}),$$

it follows that $\psi_\theta(\mathbf{x}^{(n)})$ converges to its upper bound. Then the expressions

$$\begin{aligned}
&\theta\Big(x_3^{(n)} - x_2^{(n)}\Big) + (1-\theta)\Big(\big(x_3^{(n)}\big)^2 - x_1^{(n)}x_2^{(n)}\Big)\\
&\theta\Big(x_1^{(n)} - x_3^{(n)}\Big) + (1-\theta)\Big(\big(x_1^{(n)}\big)^2 - x_2^{(n)}x_3^{(n)}\Big)\\
&\theta\Big(x_2^{(n)} - x_1^{(n)}\Big) + (1-\theta)\Big(\big(x_2^{(n)}\big)^2 - x_1^{(n)}x_3^{(n)}\Big)
\end{aligned}$$

converge to zero, consequently their sum

$$(1-\theta)\Big((x_1^{(n)})^2 + (x_2^{(n)})^2 + (x_3^{(n)})^2 - x_1^{(n)}x_2^{(n)} - x_1^{(n)}x_3^{(n)} - x_2^{(n)}x_3^{(n)}\Big) =$$

$$= \frac{1-\theta}{2}\Big((x_1^{(n)} - x_2^{(n)})^2 + (x_1^{(n)} - x_3^{(n)})^2 + (x_2^{(n)} - x_3^{(n)})^2\Big)$$

also converge to zero from which it follows

$$\lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{c}.$$

But this is impossible, because the fixed point $\mathbf{c}$ of the operator (6) is a repelling point when $\theta^* < \theta \le 1$. Thus $\xi = 0$, that is $\omega_{V_\theta}(\mathbf{x}^0) \subset \partial S$.

We note that any trajectory of V_θ cannot converge to a vertex. Suppose on the contrary that $\mathbf{x}^{(n)} \to \mathbf{e}_1$, $n \to \infty$, that is, $x_1^{(n)} \to 1$, so $x_2^{(n)} \to 0$, $x_3^{(n)} \to 0$, $n \to \infty$. Then using $\theta\Big(x_1^{(n)} - x_3^{(n)}\Big) + (1-\theta)\Big(\big(x_1^{(n)}\big)^2 - x_2^{(n)}x_3^{(n)}\Big) \to 1$ it follows that there is a number n_0 such that $\theta\Big(x_1^{(n)} - x_3^{(n)}\Big) + (1-\theta)\Big(\big(x_1^{(n)}\big)^2 - x_2^{(n)}x_3^{(n)}\Big) > 0$ for all $n > n_0$.

Consequently from (6) we obtain that $x_2^{(n+1)} > x_2^{(n)} > 0$ for any $n > n_0$ which would prevent $x_2^{(n)} \to 0$ when $n \to \infty$.

Thus $\omega_{V_\theta}(\mathbf{x}^0) \neq \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. So the limit set contains an interior point of some edge. Since $V_\theta(\omega_{V_\theta}(\mathbf{x}^0)) = \omega_{V_\theta}(\mathbf{x}^0)$ it follows that $\{\mathbf{y}^{(n)}\}_{n=0}^{\infty} \subset \omega_{V_\theta}(\mathbf{x}^0)$ for any $\mathbf{y} \in \omega_{V_\theta}(\mathbf{x}^0) \setminus \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. This trajectory is an infinite set, e.g. the restriction of (6) to the invariant face $\Gamma_{\{1,2\}}$ has the form $f_\theta(x_1)$ due to (7) it is follow that if $0 < x_1 < 1$ then for any sufficiently large n we have $0 < x_1^{(n+1)} < x_1^{(n)} < 1$. Thus $\omega_{V_\theta}(\mathbf{x}^0)$ is an infinite set, and the proof of the theorem is complete. □

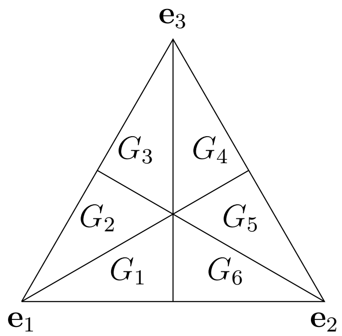

Figure 3.1: The partition of S[17]

Consider the partition of the simplex S by their medians M_{12}, M_{13}, M_{23} into six parts and their closures(see Figure 3.1):

$$G_1 = \{\mathbf{x} \in S : x_1 \geq x_2 \geq x_3\},\ G_2 = \{\mathbf{x} \in S : x_1 \geq x_3 \geq x_2\},$$

$$G_3 = \{\mathbf{x} \in S : x_3 \geq x_1 \geq x_2\},\ G_4 = \{\mathbf{x} \in S : x_3 \geq x_2 \geq x_1\},$$

$$G_5 = \{\mathbf{x} \in S : x_2 \geq x_3 \geq x_1\},\ G_6 = \{\mathbf{x} \in S : x_2 \geq x_1 \geq x_3\}.$$

Proposition 3.2. *Let $\theta > \theta^*$ then in a long run time, the trajectory of V_θ goes around the periodic path $G_1 \to G_6 \to G_5 \to G_4 \to G_3 \to G_2 \to G_1$.*

Proof. Let $\theta > \theta^*$ and $\mathbf{x} \in \operatorname{int} S \setminus \{\mathbf{c}\}$ be an initial point. By Theorem 3.1 for any sufficiently small $\varepsilon > 0$ there is a natural n_0 such that for all $n > n_0$

$$\mathbf{x}^{(n)} \in S_\varepsilon = \{\mathbf{x} \in S : \min\{x_1, x_2, x_3\} < \varepsilon\}.$$

Suppose $\mathbf{x}^{(n)} \in G_1 \cap S_\varepsilon$, that is, $x_3^{(n)}$ is sufficiently small. Then from (6) one easily concludes that $x_1^{(n+1)} \leq x_1^{(n)}$, $x_2^{(n+1)} \geq x_2^{(n)}$, $x_3^{(n+1)} \leq x_3^{(n)}$ and also $x_1^{(n+1)} \geq x_3^{(n+1)}$, $x_2^{(n+1)} \geq x_3^{(n+1)}$. Therefore either $x_1^{(n+1)} \geq x_2^{(n+1)} \geq x_3^{(n+1)}$ or $x_1^{(n+1)} \geq x_1^{(n+1)} \geq x_3^{(n+1)}$, that is, either $\mathbf{x}^{(n+1)} \in G_1$ or $\mathbf{x}^{(n+1)} \in G_6$. Similarly one has if $\mathbf{x}^{(n)} \in G_6$ then either $\mathbf{x}^{(n+1)} \in G_6$ or $\mathbf{x}^{(n+1)} \in G_5$, etc. Hence, the proposition is proved. □

Denote by U_0 a neighborhood of the center $\mathbf{c}$ and by U_1, U_2, U_3 neighborhoods of the vertices $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ respectively, defined by

$$U_1 = (G_1 \cup G_2) \setminus U_0, \quad U_2 = (G_5 \cup G_6) \setminus U_0, \quad U_3 = (G_3 \cup G_4) \setminus U_0.$$

Let us choose a neighborhood U_0 such that $U_0 \subset \operatorname{int} S$ and the neighborhoods U_1, U_2, U_3 are convex and they satisfy $\overline{U}_1 \cap \overline{U}_2 \cap \overline{U}_3 = \emptyset$ (e.g., see Figure 3.2).

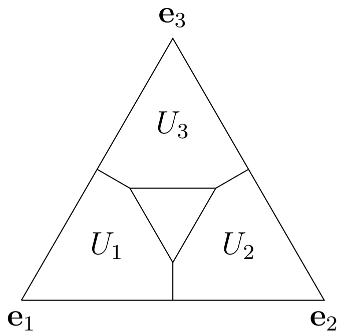

Figure 3.2: The neighborhoods of fixed points [17]

By Proposition 3.2 the trajectory moves from the neighborhood of a vertex to the neighborhood of another vertex, staying for some time $T(n)$ in this neighborhood and then moving to the neighborhood of a third vertex, etc. We would like to estimate the staying time of the trajectory in a neighborhood of a vertex.

Proposition 3.3. *Let $\theta > \theta^*$ and let $\mathbf{x} \in G_1$, $\mathbf{x}^{(s)} \in U_2$ for any $s = 1, \ldots, n$ and $\mathbf{x}^{(n+1)} \in G_4$, then*

$$n > A \log(B/\varphi(\mathbf{x}')),$$

where A, B some constants and $\mathbf{x}' = V_\theta(\mathbf{x})$.

Proof. Let $\mathbf{x} \in G_1$, $\mathbf{x}^{(s)} \in U_2$ for any $s = 1, \ldots, n$ and $\mathbf{x}^{(n+1)} \in G_4$. Then for $\mathbf{x} \in G_1$ from (6) we obtain

$$\frac{x_3^{(n+1)}}{x_3'} = \prod_{k=1}^{n} \Big(1 + \theta\big(x_2^{(k)} - x_1^{(k)}\big) + (1-\theta)\Big(\big(x_2^{(k)}\big)^2 - x_1^{(k)} x_3^{(k)}\Big)\Big) < 2^n$$

For $\mathbf{x} \in G_1$, $\mathbf{x}' \in U_2$ and $\mathbf{x}^{(n+1)} \in G_4$ one has

$$2^n > \frac{x_3^{(n+1)}}{x_3'} \geq \frac{1}{3x_3'} = \frac{x_1' x_2'}{3\varphi(\mathbf{x}')}$$

One can verify that

$$\begin{aligned} x_1' &= x_1\big(1 + \theta(x_3 - x_2) + (1-\theta)\big(x_3^2 - x_1 x_2\big)\big) \\ &= x_1(x_1^2 + 3(1-\theta)x_2^2 + 2x_3^2 + x_1x_2 + 3\theta x_1 x_3 + 2x_2x_3) \geq x_1^3 \\ &= \frac{x_2(x_2^2 + 11x_2^2)}{12} \geq \frac{x_2(2x_1^2 + x_2^2 + 3(1-\theta)x_3^2 + 3\theta x_1 x_2 + 2x_1x_3 + x_2x_3)}{12} \\ &= \frac{x_2\big(1 + \theta(x_1 - x_3) + (1-\theta)\big(x_1^2 - x_2x_3\big)\big)}{12} = \frac{x_2'}{12}, \end{aligned}$$

and using $\varphi(\mathbf{x}') \leq 1$ we obtain

$$2^n > \frac{x_1' x_2'}{3\varphi(\mathbf{x}')} \geq \frac{(x_2')^2}{36\varphi(\mathbf{x}')} \geq \frac{1}{324\varphi(\mathbf{x}')},$$

Thus

$$n > \log_2(1/(324\varphi(\mathbf{x}'))) > A \log(B/\varphi(\mathbf{x}')).$$

Moreover for the point $\mathbf{x}^{(n)}$ from the last inequalities we have

$$2^{T(n)}\varphi(\mathbf{x}^{(n+1)}) > \frac{1}{324}. \tag{11}$$

The proposition is proved. $\square$

Remark 3.4. The other parts of the cycle can be considered in a similar manner.

Proposition 3.5. *Let $\theta > \theta^*$ and let U be one of neighborhoods U_i, $i = 1, 2, 3$ and $\mathbf{x} \in \operatorname{int} S \setminus \{\mathbf{c}\}$. Let $\{u_i\}_{i=1}^{\infty}$, $\{v_i\}_{i=1}^{\infty}$ be the sequences of natural numbers such that $\mathbf{x}^{(u_i)} \notin U$, $\mathbf{x}^{(u_i+t)} \in U$, for $t = 1, 2, \ldots, v_i$, and $\mathbf{x}^{(u_i+v_i+1)} \notin U$, then there is a positive constant K such that $u_i > Kv_i$.*

Proof. By Theorem 3.1 and Proposition 3.2 it follows that $u_i \to \infty$ and

$$\varphi(\mathbf{x}') < \tau\varphi(\mathbf{x}), \quad \text{for any } \ \mathbf{x} \in S \setminus U_0,$$

where $\tau = \max\limits_{\mathbf{x}\in S\setminus U_0} \psi(\mathbf{x}) < 1$. Using Proposition 3.3 one has

$$u_i > A\log\left(\frac{B}{\varphi(\mathbf{x}^{(u_i)})}\right) > A'\log\left(\frac{B}{\tau^{u_i}\varphi(\mathbf{x}^0)}\right) > Kv_i. \qquad \square$$

From Propositions 3.3 and 3.5, it follows that the time spent in any neighborhood of any vertex tends to infinity and the average time spent in the exterior of the union of sufficiently small neighborhoods of all vertices is equal to 0.

Denote by U a neighborhood of the vertex set.

Theorem 3.6. *If $\theta > \theta^*$ then for any $\mathbf{x} \in \operatorname{int} S \setminus \{\mathbf{c}\}$ and for the Volterra CSO V_θ (6) the limit of averages M_{V_θ} doesn't exist.*

Proof. Suppose on the contrary, let $\mathbf{x} \in \operatorname{int} S \setminus \{\mathbf{c}\}$ be an initial point and the limit of averages M_{V_θ} exits.

Due to Theorem 3.1 it follows $\omega_{V_\theta}(\mathbf{x}^0) \subset \partial S$ and from Proposition 3.3 one has that most of the time the trajectory belong to U and only finite points of the trajectory belong to ∂S, otherwise it converges to one of the vertices $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Therefore if we denote by n_1 the time spent of $\mathbf{x}^{(n)}$ in U and n_2 is the remaining time, then we obtain $\lim\limits_{n\to\infty} \frac{n_1}{n} = 1$, where $n = n_1 + n_2$.

Hence for any $\theta > 0$ there exists a neighborhood U of a vertex set with $\varphi(\mathbf{x}') \le \theta\varphi(\mathbf{x})$ for $\mathbf{x} \in U$. Consequently $\varphi(\mathbf{x}^{(n)}) \le \theta^{n_1}\varphi(\mathbf{x})$ and using $\lim\limits_{n\to\infty} \frac{n_1}{n} = 1$ one obtains $\limsup\limits_{n\to\infty} \sqrt[n]{\varphi(\mathbf{x}^{(n)})} \le \theta$. Due to the fact that θ is arbitrary we obtain

$$\lim_{n\to\infty} \sqrt[n]{\varphi(\mathbf{x}^{(n)})} = 0.$$

Using (11), we have $2^{T(n_1)}\varphi(\mathbf{x}^{(n_1+1)}) > \dfrac{1}{324}$ and $\dfrac{T(n_1)}{n} \to \infty$ as $n \to \infty$.

But then the average $(\mathbf{x}+\mathbf{x}'+\cdots+\mathbf{x}^{(n)})/(n+1)$ at the time the point leaves U_1 is arbitrarily close to the closure $\overline{U}_1$. Consequently, if the limit of averages M_{W_θ} exists, it lies in $\overline{U}_1$. By the same argument it lies in $\overline{U}_2$ and $\overline{U}_3$ which contradicts to $\overline{U}_1\cap\overline{U}_2\cap\overline{U}_3=\emptyset$. The theorem is proved. $\square$

Remark 3.7. By Theorem 3.1 it follows that if $0\le\theta\le\theta^*$ then $\omega_{V_\theta}\big(\mathbf{x}\big)=\{\mathbf{c}\}$ for any initial point $\mathbf{x}\in\operatorname{int}S\setminus\{\mathbf{c}\}$. Also by Theorem 3.1 it follows that if $\theta>\theta^*$ then for any initial point $\mathbf{x}\in\operatorname{int}S\setminus\{\mathbf{c}\}$ any trajectory of the Volterra CSO V_θ diverges. Moreover by Theorem 3.6 it follows that average of a trajectory of the Volterra CSO V_θ also diverges. Besides, note that the Volterra CSO V_θ is different from known non-ergodic transformations (see [14], [19]).

One can consider the CSO $\widetilde{V}_\theta=\theta V_1+(1-\theta)W_1,\quad \theta\in[0,1]$, that is convex combination of a non-ergodic QSO V_1 and a non-ergodic CSO W_1. Using the above techniques and repeating similar calculations one can prove the following

Theorem 3.8. *For any $0\le\theta\le1$ and for any $\mathbf{x}\in\operatorname{int}S\setminus\{\mathbf{c}\}$ and for the Volterra CSO $\widetilde{V}_\theta$ the limit of averages $M_{\widetilde{V}_\theta}$ doesn't exist.*

Remark 3.9. Theorem 3.8 shows that convex combination of a non-ergodic QSO and a non-ergodic CSO, is also a non-ergodic transformation.

4. On convex combinations of CSOs

We consider the following family of CSOs defined on S

$$W_\theta:\begin{cases} x_1'=x_1\Big(1+\frac{2\theta-1}{2}\big(x_2+x_1x_2-x_3-x_3^2\big)\Big),\\ x_2'=x_2\Big(1+\frac{2\theta-1}{2}\big(x_3+x_2x_3-x_1-x_1^2\big)\Big),\\ x_3'=x_3\Big(1+\frac{2\theta-1}{2}\big(x_1+x_1x_3-x_2-x_2^2\big)\Big)\end{cases}\tag{12}$$

where $\theta\in[0,1]$.

It is easy to verify that $W_\theta=\theta P+(1-\theta)Q$, $\theta\in[0,1]$, where the operators P and Q have the form

$$P=\frac{1}{2}V_1+\frac{1}{2}W_1=\begin{cases} x_1'=x_1\Big(1+\frac{1}{2}\big(x_2+x_1x_2-x_3-x_3^2\big)\Big),\\ x_2'=x_2\Big(1+\frac{1}{2}\big(x_3+x_2x_3-x_1-x_1^2\big)\Big),\\ x_3'=x_3\Big(1+\frac{1}{2}\big(x_1+x_1x_3-x_2-x_2^2\big)\Big),\end{cases}$$

$$Q=\frac{1}{2}V_2+\frac{1}{2}W_2=\begin{cases} x_1'=x_1\Big(1+\frac{1}{2}\big(x_3+x_3^2-x_2-x_1x_2\big)\Big),\\ x_2'=x_2\Big(1+\frac{1}{2}\big(x_1+x_1^2-x_3-x_2x_3\big)\Big),\\ x_3'=x_3\Big(1+\frac{1}{2}\big(x_2+x_2^2-x_1-x_1x_3\big)\Big).\end{cases}$$

Note that for any $\theta\in[0,1]$ the operator W_θ is a Volterra CSO and also it can be seen as a discrete version of Kolmogorov equations for three interacting species.

Since the operator W_θ is the convex combination of the operators P and Q from the results of [13], [14] and the references therein it is natural to expect that also there is a critical value of θ and depending to this value the trajectories are different. Note that if $\theta = 1/2$ the W_θ is the identity map.

The faces $\Gamma_{ij} = \{\mathbf{x} \in S : x_k = 0; k \notin \{i,j\}\}, \quad i \neq j \in \{1,2,3\}$ are invariant sets with respect to W_θ. The vertexes $\mathbf{e}_1 = (1,0,0), \mathbf{e}_2 = (0,1,0), \mathbf{e}_3 = (0,0,1)$ and the center $\mathbf{c} = (1/3, 1/3, 1/3)$ are the fixed points of the operator W_θ.

Proposition 4.1. *If $\theta \neq 1/2$ then the vertexes $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are saddles and they are non-hyperbolic when $\theta = 1/2$. The center*

$$\mathbf{c} \quad \textit{is a/an} \quad \begin{cases} \textit{attracting point}, & \textit{if } \theta \in (1/14, 1/2), \\ \textit{non-hyperbolic point}, & \textit{if } \theta \in \{1/14, 1/2\} \\ \textit{repelling point}, & \textit{if } \theta \in [0, 1/14) \cup (1/2, 1]. \end{cases}$$

Proof. We rewrite the operator W_θ in the following form

$$\begin{aligned} x_1' &= x_1\Big(1 + \frac{2\theta - 1}{2}\big(3x_1 + 4x_2 - 2 - x_1^2 - x_2^2 - x_1x_2\big)\Big), \\ x_2' &= x_2\Big(1 + \frac{2\theta - 1}{2}\big(1 - 2x_1 - x_1x_2 - x_1^2 - x_2^2\big)\Big), \end{aligned} \tag{13}$$

where $(x_1, x_2) \in \tilde{S} = \{(x_1, x_2) : x_1, x_2 \geq 0, x_1 + x_2 \in [0;1]\}$ and x_1, x_2 are the first two coordinates of a point lying in the simplex S.

(i) It is easy to verify that the Jacobian of the operator (13) at the vertex $\mathbf{e}_1$ has the eigenvalues $\mu_1 = 2(1-\theta), \mu_2 = (2\theta + 1)/2$. Since $\theta \in [0,1]$ it follows that the vertex $\mathbf{e}_1$ is a saddle point when $\theta = 1/2$ and otherwise it is a saddle point. Other vertexes can be considered in a similar manner.

(ii) After simple algebra one has that the Jacobian of the operator (13) at the center $\mathbf{c}$ has the eigenvalues

$$\mu_{1,2} = \frac{11 + 2\theta \pm 3\sqrt{3}(2\theta - 1)i}{12} \quad \text{and} \quad |\mu_{1,2}| = \frac{\sqrt{28\theta^2 - 16\theta + 37}}{6}.$$

The analysis of polynomial $28\theta^2 - 16\theta + 37, \quad \theta \in [0,1]$ gives that the center

$$\mathbf{c} \quad \text{is a/an} \quad \begin{cases} \text{attracting point}, & \text{if } \theta \in (1/14, 1/2), \\ \text{non-hyperbolic point}, & \text{if } \theta \in \{1/14, 1/2\} \\ \text{repelling point}, & \text{if } \theta \in [0, 1/14) \cup (1/2, 1]. \end{cases} \qquad \square$$

Proposition 4.2. *If $\theta \in [0, 1/14) \cup (1/2, 1]$ (resp. $\theta \in (1/14, 1/2)$) then the function $\varphi(\mathbf{x}) = x_1x_2x_3$ is a decreasing (resp. an increasing) Lyapunov function for the operator W_θ.*

Proof. It is easy to see that the function $\varphi(\mathbf{x}) = x_1x_2x_3$ is continuous on S.

One has $\varphi(W_\theta(\mathbf{x})) = \varphi(\mathbf{x})\psi(\mathbf{x})$, where

$$\psi(\mathbf{x}) = \Big(1 + \frac{2\theta-1}{2}(x_2 + x_1x_2 - x_3 - x_3^2)\Big)\Big(1 + \frac{2\theta-1}{2}(x_3 + x_2x_3 - x_1 - x_1^2)\Big)$$
$$\Big(1 + \frac{2\theta-1}{2}(x_1 + x_1x_3 - x_2 - x_2^2)\Big).$$

The case $\theta = 1/2$ is obvious.

Let $\theta > 1/2$ then using AM-GM relation one has

$$\psi(\mathbf{x}) \le \left(1 + \frac{(2\theta-1)(x_1x_2 + x_2x_3 + x_1x_3 - x_1^2 - x_2^2 - x_3^2)}{6}\right)^3$$
$$= \left(1 - \frac{(2\theta-1)\big((x_1-x_2)^2 + (x_2-x_3)^2 + (x_3-x_1)^2\big)}{12}\right)^3 \le 1.$$

Let $0 \le \theta < 1/14$ then using equality $x_3 = 1 - x_1 - x_2$ one has that $\psi(\mathbf{x})$ on the set $\tilde{S}$ has the form

$$\psi(\mathbf{x}) = \Big(1 + \frac{2\theta-1}{2}(x_2 + x_1x_2 - (1-x_1-x_2) - (1-x_1-x_2)^2)\Big)$$
$$\cdot \Big(1 + \frac{2\theta-1}{2}((1-x_1-x_2) + x_2(1-x_1-x_2) - x_1 - x_1^2)\Big)$$
$$\cdot \Big(1 + \frac{2\theta-1}{2}(x_1 + x_1(1-x_1-x_2) - x_2 - x_2^2)\Big).$$

A simple but tedious calculation gives that $(1/3, 1/3) \in \tilde{S}$ is the unique solution of the system

$$\frac{\partial\psi(\mathbf{x})}{\partial x_1} = 0, \quad \frac{\partial\psi(\mathbf{x})}{\partial x_2} = 0$$

and it follows that

$$\max_{\mathbf{x}\in S}\psi(\mathbf{x}) = 1 \ \text{ iff } \ x_1 = x_2 = \frac{1}{3} \ \text{ and } \ \theta \in \Big[0, \frac{1}{14}\Big) \cup \Big(\frac{1}{2}, 1\Big]. \tag{14}$$

$$\min_{\mathbf{x}\in S}\psi(\mathbf{x}) = 1 \ \text{ iff } \ x_1 = x_2 = \frac{1}{3} \ \text{ and } \ \theta \in \Big(\frac{1}{14}, \frac{1}{2}\Big) \tag{15}$$

Therefore if either $0 \le \theta < 1/14$ or $1/2 < \theta \le 1$ then by (14) it follows that $\psi(\mathbf{x}) \le 1$ and $\varphi(W_\theta(\mathbf{x})) \le \varphi(\mathbf{x})$ for all $\mathbf{x} \in S$.

Let $1/14 < \theta < 1/2$. Then by (15) we get $\psi(\mathbf{x}) \ge 1$ and $\varphi(W_\theta(\mathbf{x})) \ge \varphi(\mathbf{x})$ for all $\mathbf{x} \in S$.

Thus the function $\varphi(\mathbf{x}) = x_1x_2x_3$ is a decreasing (resp. an increasing) Lyapunov function for the operator W_θ when $\theta \in [0, 1/14) \cup (1/2, 1]$ (resp. $\theta \in (1/14, 1/2)$). $\square$

Theorem 4.3. *For the operator W_θ the following statements are true:*

(i) *if $\theta > 1/2$ then $\omega_{W_\theta}(\mathbf{x}^{(0)}) = \{\mathbf{e}_i\}$ for any $\mathbf{x}^{(0)} \in \Gamma_{ij}$, $\{ij\} \in \big\{\{12\}, \{23\}, \{31\}\big\}$;*

(ii) *if $\theta < 1/2$ then $\omega_{W_\theta}(\mathbf{x}^{(0)}) = \{\mathbf{e}_j\}$ for any $\mathbf{x}^{(0)} \in \Gamma_{ij}$, $\{ij\} \in \big\{\{12\}, \{23\}, \{31\}\big\}$.*

Proof. Let $\theta > 1/2$ and $\mathbf{x}^{(0)} \in \Gamma_{12}$.Using $x_2 = 1 - x_1$ we can rewrite the operator W_θ in the form

$$W_\theta(x_1) = x_1\Big(1 + \frac{2\theta - 1}{2}\big(1 - x_1^2\big)\Big).$$

It is easy to verify that the function $W_\theta(x_1)$ is monotone increasing in $(0,1)$ and it has $x_1 = 1$ attracting and $x_1 = 0$ repelling fixed points and one easily has that for a $x_1 \in (0,1)$ the trajectory $W_\theta^n(x_1) \to 1$ when $n \to \infty$.

Let $\theta < 1/2$ then using the similar method it can be shown that the trajectory $W_\theta^n(x_1) \to 0$ when $n \to \infty$ for any $\mathbf{x}^{(0)} \in \Gamma_{12}$.

Other faces Γ_{ij}, $\{ij\} \in \big\{\{23\}, \{31\}\big\}$ can be considered in a similar manner. $\square$

Theorem 4.4. (i) *Let $1/14 < \theta < 1/2$. Then we have $\omega_{W_\theta}\big(\mathbf{x}^{(0)}\big) = \{\mathbf{c}\}$ for any $\mathbf{x}^{(0)} \in int S \setminus \{\mathbf{c}\}$;*

(ii) *Let $\theta \in [0, 1/14) \cup (1/2, 1]$ then the set $\omega_{W_\theta}\big(\mathbf{x}^{(0)}\big) \subset \partial S$ is an infinite compact set for any $\mathbf{x}^{(0)} \in int S \setminus \{\mathbf{c}\}$.*

Proof. It is evident that $\varphi(\mathbf{x}) = 0$ iff $\mathbf{x} \in \partial S$ and $\max_{\mathbf{x}\in S} \varphi(\mathbf{x}) = 1/27$ iff $\mathbf{x} = \mathbf{c}$.

(i) Let $1/14 < \theta < 1/2$ then by Proposition 4.2 the function $\varphi(\mathbf{x}) = x_1x_2x_3$ is an increasing Lyapunov function. Besides, due to the fact that $\min_{\mathbf{x}\in S} \psi(\mathbf{x}) = 1$ if and only if $\mathbf{x} = \mathbf{c}$ we have $\psi(\mathbf{x}) > 1$ for any $\mathbf{x} \in \operatorname{int} S \setminus \{\mathbf{c}\}$. Consequently it follows that $\varphi(W_\theta(\mathbf{x})) > \varphi(\mathbf{x})$ whenever $\mathbf{x} \neq \mathbf{c}$.

So $\{\varphi(\mathbf{x}^{(n)})\}$, $n = 0, 1, \ldots$ is a strict increasing bounded sequence. Therefore there exits $\lim_{n\to\infty} \varphi(\mathbf{x}^{(n)}) = 1/27$. Since $\max_{\mathbf{x}\in S} \varphi(\mathbf{x}) = 1/27$ iff $\mathbf{x} = \mathbf{c}$ we obtain $\lim_{n\to\infty} \mathbf{x}^{(n)}) = \mathbf{c}$.

(ii) Let $\theta \in [0, 1/14) \cup (1/2, 1]$ then by Proposition 4.2 the function $\varphi(\mathbf{x}) = x_1x_2x_3$ is a decreasing Lyapunov function. So using $\psi(\mathbf{x}) \leq 1$ and $\varphi(W_\theta(\mathbf{x})) \leq \varphi(\mathbf{x})$ one has that for any $\mathbf{x} \in \operatorname{int} S \setminus \{\mathbf{c}\}$ the sequence $\{\varphi(\mathbf{x}^{(n)})\}$, $n = 0, 1, \ldots$ is decreasing and bounded. Therefore there exits $\lim_{n\to\infty} \varphi(\mathbf{x}^{(n)}) = \tau$.

We claim that $\tau = 0$. Suppose on the contrary that $\tau > 0$ then we get

$$1 = \lim_{n\to\infty} \frac{\varphi(\mathbf{x}^{(n+1)})}{\varphi(\mathbf{x}^{(n)})} = \lim_{n\to\infty} \frac{\varphi(\mathbf{x}^{(n)})\psi(\mathbf{x}^{(n)})}{\varphi(\mathbf{x}^{(n)})} = \lim_{n\to\infty} \psi(\mathbf{x}^{(n)}).$$

Therefore it follows that $\psi(\mathbf{x}^{(n)})$ converges to its upper bound. So the expressions

$$x_2^{(n)} + x_1^{(n)}x_2^{(n)} - x_3^{(n)} - \big(x_3^{(n)}\big)^2, \quad x_3^{(n)} + x_2^{(n)}x_3^{(n)} - x_1^{(n)} - \big(x_1^{(n)}\big)^2,$$

$$x_1^{(n)} + x_1^{(n)}x_3^{(n)} - x_2^{(n)} - \big(x_2^{(n)}\big)^2$$

converge to zero, consequently their sum

$$x_1^{(n)}x_2^{(n)} + x_2^{(n)}x_3^{(n)} + x_1^{(n)}x_3^{(n)} - \big(x_1^{(n)}\big)^2 - \big(x_2^{(n)}\big)^2 - \big(x_3^{(n)}\big)^2$$

also converge to zero, that is, we obtain

$$\lim_{n\to\infty} x_1^{(n)} = \lim_{n\to\infty} x_2^{(n)} = \lim_{n\to\infty} x_3^{(n)} = \frac{1}{3} \Rightarrow \lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{c}.$$

But this is impossible, because it contradicts that the center $\mathbf{c}$ is a repelling point. Thus $\tau = 0$, that is $\omega_{W_\theta}(\mathbf{x}^{(0)}) \subset \partial S$.

We claim that any trajectory of W_θ cannot converge to a vertex. Indeed, suppose on the contrary that $\mathbf{x}^{(n)} \to \mathbf{e}_1$, $n \to \infty$, that is $x_1^{(n)} \to 1$, so $x_2^{(n)} \to 0$, $x_3^{(n)} \to 0$, $n \to \infty$. It is easy to see that if $\theta > 1/2$ then from $x_1^{(n)} + x_1^{(n)} x_3^{(n)} - x_2^{(n)} - \left(x_2^{(n)}\right)^2 \to 1$ it follows that there is n_0 number such that $x_1^{(n)} + x_1^{(n)} x_3^{(n)} - x_2^{(n)} - \left(x_2^{(n)}\right)^2 > 0$ holds for all $n > n_0$. Consequently from (12) one has $x_3^{(n+1)} > x_3^{(n)} > 0$ for any $n > n_0$ which would prevent $x_3^{(n)} \to 0$ when $n \to \infty$.

Similarly if $\theta < 1/2$ then using $x_3^{(n)} + x_2^{(n)} x_3^{(n)} - x_1^{(n)} - \left(x_1^{(n)}\right)^2 \to -2$ it follows that there is a number n_0 such that $x_3^{(n)} + x_2^{(n)} x_3^{(n)} - x_1^{(n)} - \left(x_1^{(n)}\right)^2 < 0$ holds for all $n > n_0$. Therefore from (12) we obtain $x_2^{(n+1)} > x_2^{(n)} > 0$ for any $n > n_0$ which would prevent $x_2^{(n)} \to 0$ when $n \to \infty$.

Evidently $\omega_{W_\theta}(\mathbf{x}^{(0)}) \neq \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Since $W_\theta\big(\omega_{W_\theta}(\mathbf{x}^{(0)})\big) = \omega_{W_\theta}(\mathbf{x}^{(0)})$ it follows that $\{\mathbf{y}^{(n)}\}_{n=0}^{\infty} \subset \omega_{W_\theta}(\mathbf{x}^{(0)})$ for any $\mathbf{y}^{(0)} \in \omega_{W_\theta}(\mathbf{x}^{(0)}) \setminus \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. This trajectory is an infinite set, e.g. the restriction of W_θ to the invariant face Γ_{12} has the form $x_1' = x_1\big(1 + (2\theta - 1)/2(1 - x_2^2)\big)$ which implies if $\theta > 1/2$ then $0 < x_1 < x_1' < 1$ for any $0 < x_1 < 1$ and if $\theta < 1/2$ then $0 < x_1' < x_1 < 1$ for any $0 < x_1 < 1$. Thus $\omega_{W_\theta}(\mathbf{x}^{(0)})$ is an infinite set. The proof of the proposition is complete. □

Proposition 4.5. *If $\theta \in (1/2, 1]$ (resp. $\theta \in [0, 1/14) \cup (1/14, 1/2)$) then in a long run time, the trajectory $\{\mathbf{x}^{(n)}\}_{n=0}^{\infty}$ goes around the periodic path*

$$G_1 \to G_2 \to G_3 \to G_4 \to G_5 \to G_6 \to G_1$$

(resp. $G_1 \to G_6 \to G_5 \to G_4 \to G_3 \to G_2 \to G_1$).

Proof. Let $\mathbf{x} \in \operatorname{int} S \setminus \{\mathbf{c}\}$ be an initial point and $\theta \in (1/2, 1]$. Suppose $\mathbf{x} \in G_1$ then one easily has that $x_1' \geq x_1$, $x_2' \leq x_2$ and also $x_1' \geq x_2'$, $x_1' \geq x_3'$. Therefore either $x_1' \geq x_2' \geq x_3'$ or $x_1' \geq x_3' \geq x_2'$, that is either $\mathbf{x}' \in G_1$ or $\mathbf{x}' \in G_2$. Similarly one has if $\mathbf{x} \in G_2$ then either $\mathbf{x}' \in G_2$ or $\mathbf{x}' \in G_3$, etc.

The case $\theta \in [0, 1/14)$ can be considered similarly, and the proof is complete. □

Denote by U_0 a neighborhood the center $\mathbf{c}$ and the neighborhoods U_1, U_2, U_3 of the vertices $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ respectively, and we define them as

$$U_1 = (G_1 \cup G_2) \setminus U_0, \quad U_2 = (G_5 \cup G_6) \setminus U_0, \quad U_3 = (G_3 \cup G_4) \setminus U_0.$$

Let us choose a neighborhood U_0 such that $U_0 \subset \operatorname{int} S$ and the neighborhoods U_1, U_2, U_3 are convex and they satisfy $\overline{U}_1 \cap \overline{U}_2 \cap \overline{U}_3 = \emptyset$. By Proposition 4.5 the trajectory moves from the neighborhood of a vertex to the neighborhood of another vertex, staying for some time $T(n)$ in this neighborhood and then moving to the neighborhood of a third vertex, etc. We would like to estimate the staying time of the trajectory in a neighborhood of a vertex. Let U be one of neighborhoods U_1, U_2, U_3.

Proposition 4.6. *Let $\theta \in [0, 1/14) \cup (1/2, 1]$, $\mathbf{x} \notin U$, $\mathbf{x}^{(s)} \in U$ for any $s = 1, \dots, n$ and $\mathbf{x}^{(n+1)} \notin U$, then*

$$n > A \log(B/\varphi(\mathbf{x}')),$$

where A, B are some constants and $\mathbf{x}' = W_\theta(\mathbf{x})$.

Proof. Let $\theta > 1/2$. Without loss of generality we can assume that $\mathbf{x} \in G_6$, $\mathbf{x}^{(s)} \in U_1$ for any $s = 1, \dots, n$ and $\mathbf{x}^{(n+1)} \in G_3$. Then for $\mathbf{x} \in G_6$ we obtain

$$\frac{x_3^{(n+1)}}{x_3'} = \prod_{k=1}^{n} \Big(1 + \frac{2\theta - 1}{2}\big(x_1^{(k)} + x_1^{(k)} x_3^{(k)} - x_2^{(k)} - \big(x_2^{(k)}\big)^2\big)\Big) < 2^n.$$

For $\mathbf{x} \in G_6$, $\mathbf{x}' \in U_1$ and $\mathbf{x}^{(n+1)} \in G_3$ one has

$$2^n > \frac{x_3^{(n+1)}}{x_3'} \geq \frac{1}{3x_3'} = \frac{x_1' x_2'}{3\varphi(\mathbf{x}')}.$$

From the form of W_θ it follows that

$$\begin{aligned}
x_2' &= x_2\Big(2(1-\theta)x_1^2 + x_2^2 + \frac{2\theta+1}{2}x_3^2 + \frac{5-2\theta}{2}x_1x_2 + 2x_1x_3 + (2\theta+1)x_2x_3\Big) \\
&\geq x_2^3 = \frac{x_2(x_2^2 + 13x_2^2)}{14} \geq \frac{x_1}{14}\big(x_1^2 + 3x_2^2 + 2x_3^2 + 3x_1x_2 + 3x_1x_3 + 2x_2x_3\big) \\
&\geq \frac{x_1}{14}\Big(x_1^2 + \frac{2\theta+1}{2}x_2^2 + 2(1-\theta)x_3^2 + (2\theta+1)x_1x_2 + \frac{5-2\theta}{2}x_1x_3 + 2x_2x_3\Big) \\
&\geq \frac{x_1'}{14},
\end{aligned}$$

and using $\varphi(\mathbf{x}')) \leq 1$ we obtain

$$2^n > \frac{x_1' x_2'}{\varphi(\mathbf{x}')} \geq \frac{(x_1')^2}{42\varphi(\mathbf{x}')} \geq \frac{1}{378\varphi(\mathbf{x}')}.$$

Thus

$$n > \log_2 \frac{1}{378\varphi(\mathbf{x}')} > A \log \frac{B}{\varphi(\mathbf{x}')}.$$

Moreover for the point $\mathbf{x}^{(n)}$ from the last inequalities we have

$$2^{T(n)}\varphi\big(\mathbf{x}^{(n+1)}\big) > \frac{1}{378}. \tag{16}$$

The case $\theta \in [0, 1/14)$ can be considered in a similar manner, and the proof is complete. □

Proposition 4.7. *Let $\theta \in [0, 1/14) \cup (1/2, 1]$ and let U be one of neighborhoods U_i, $i = 1, 2, 3$ and $\mathbf{x} \in \operatorname{int} S \setminus \{\mathbf{c}\}$. Let $\{u_i\}_{i=1}^\infty$, $\{v_i\}_{i=1}^\infty$ be the sequences of natural numbers such that $\mathbf{x}^{(u_i)} \notin U$, $\mathbf{x}^{(u_i+t)} \in U$ for $t = 1, 2, \dots, v_i$, and $\mathbf{x}^{(u_i+v_i+1)} \notin U$, then there is a positive constant K such that $u_i > Kv_i$.*

Proof. Let $\theta > 1/2$. It follows from Propositions 4.2 and 4.5 that $u_i \to \infty$ and

$$\varphi(\mathbf{x}') < \tau\varphi(\mathbf{x}), \quad \text{for any} \quad \mathbf{x} \in S \setminus U_0,$$

where $\tau = \max\limits_{\mathbf{x}\in S\setminus U_0} \varphi(\mathbf{x}) < 1$. Using Proposition 4.6 one has

$$u_i > A \log\left(\frac{B}{\varphi\big(\mathbf{x}^{(u_i)}\big)}\right) > A' \log\left(\frac{B}{\tau^{u_i}\varphi\big(\mathbf{x}^{(0)}\big)}\right) > Kv_i.$$

The case $\theta \in [0, 1/14)$ can be considered similarly. □

From Propositions 4.5, 4.6, 4.7, it follows that the time spent in any neighborhood of any vertex tends to infinity and the average time spent in the exterior of the union of sufficiently small neighborhoods of all vertices is equal to 0.

Denote by U a neighborhood of the vertex set.

Theorem 4.8. *If $\theta \in [0, 1/14) \cup (1/2, 1]$ then for the operator W_θ the limit of averages M_{W_θ} doesn't exist for all $\mathbf{x} \in int S \setminus \{\mathbf{c}\}$.*

Proof. Let $\theta > 1/2$. Suppose on the contrary, let $\mathbf{x} \in \operatorname{int} S \setminus \{\mathbf{c}\}$ be an initial point and the limit of averages exits.

Due to Proposition 4.2 it follows that $\omega_{W_\theta}\big(\mathbf{x}^{(0)}\big) \subset \partial S$ and from Proposition 4.6 one has that most of the time the trajectory belong to U and only finite points of the trajectory belong to ∂S, otherwise it converges to one of the vertices $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Therefore if we denote by n_1 the time spent of $\mathbf{x}^{(n)}$ in U and n_2 is the remaining time, then we obtain $\lim_{n\to\infty} n_1/n = 1$, where $n = n_1 + n_2$. Hence for any $\tau > 0$ there exists a neighborhood U of a vertex set such that $\varphi(\mathbf{x}') \le \tau\varphi(\mathbf{x})$ for any $\mathbf{x} \in U$. Consequently $\varphi\big(\mathbf{x}^{(n)}\big) \le \tau^{n_1}\varphi(\mathbf{x})$ and using

$$\lim_{n\to\infty} n_1/n = 1 \quad \text{and} \quad \lim_{n\to\infty} \sup \sqrt[n]{\varphi\big(\mathbf{x}^{(n)}\big)} \le \tau$$

and due to the fact that τ is arbitrary we obtain

$$\lim_{n\to\infty} \sqrt[n]{\varphi\big(\mathbf{x}^{(n)}\big)} = 0.$$

Using (16), one has $$2^{T(n_1)}\varphi\big(\mathbf{x}^{(n_1+1)}\big) > \frac{1}{378}$$

and $T(n_1)/n \to \infty$ as $n \to \infty$. But then the average

$$\Big(\mathbf{x} + W_\theta(\mathbf{x}) + \cdots + W_\theta^{n-1}(\mathbf{x})\Big)/n$$

at the time the point leaves U_1 is arbitrarily close to the closure $\overline{U}_1$. Consequently, if the limit of averages M_{W_θ} exists, it lies in $\overline{U}_1$. By the same argument it lies in $\overline{U}_2$ and $\overline{U}_3$ which contradicts to $\overline{U}_1 \cap \overline{U}_2 \cap \overline{U}_3 = \emptyset$. The theorem is proved.

The case $\theta \in [0, 1/14)$ can be considered in a similar manner, and the proof is complete. □

Conclusions

Note that the Volterra CSOs V_θ and W_θ are different from known non-ergodic CSOs (see [14], [19]). We shall call by regularity (resp. non-ergodicity) interval an interval of the parameter θ where an operator is a regular (resp. a non-ergodic) transformation. By Theorem 2.2 we have that critic value of the parameter is $\theta = 1/2$ and the (Lebesgue) measures of regularity and non-ergodicity intervals of the convex combination of CSOs W_1 and W_2 are equal. In the third section we studied the trajectory of convex combination of a non-ergodic QSO V_2 and regular CSO W_2. By the assertion of Theorem 3.1 the critic value of the parameter is $\theta^* = \sqrt{6}-2$, that is, as naturally expected the critic values of the parameter also depend on the degree of the given operators. Consequently, by Theorem 3.6 it follows that the (Lebesgue) measures of regularity and non-ergodicity intervals of the operator V_θ are not equal. In the last section we considered a non-ergodic CSO P which is the equiprobable combination of a non-ergodic QSO V_1 and a non-ergodic CSO W_1 and regular CSO Q which is the equiprobable combination of a non-ergodic QSO V_2 and a regular CSO W_2. We also studied CSO W_θ which is the convex combination of the operators P and Q. Despite to that we took equiprobable combinations of the operators due to Theorems 4.4 and 4.8 it follows that there are two critic values of the parameter $\theta^* = 1/14, \theta^{**} = 1/2$ and the measures regularity and non-ergodicity intervals of the operator W_θ are different values. Namely, the measures of the regularity and non-ergodicity intervals are equal to 3/4 and 4/7, respectively.

References

[1] S. Bernstein: *Solution of a mathematical problem connected with the theory of heredity*, Ann. Math. Stat. 13/1 (1942) 53–61.

[2] R. L. Devaney: *An Introduction to Chaotic Dynamical Systems*, reprint of the second 1989 edition, Studies in Nonlinearity, Westview Press, Boulder (2003).

[3] N. N. Ganikhodjaev, R. N. Ganikhodjaev, U. U. Jamilov (Zhamilov): *Quadratic stochastic operators and zero-sum game dynamics*, Ergodic Theory Dyn. Syst. 35/5 (2015) 1443–1473.

[4] N. N. Ganikhodzhaev, D. V. Zanin: *On a necessary condition for the ergodicity of quadratic operators defined on the two-dimensional simplex*, Russ. Math. Surv. 59/3 (2004) 571–572.

[5] R. N. Ganikhodzhaev: *Quadratic stochastic operators, Lyapunov functions and tournaments*, Sb. Math. 76/2 (1993) 489–506.

[6] R. N. Ganikhodzhaev: *Map of fixed points and Lyapunov functions for one class of discrete dynamical systems*, Math. Notes 56/5 (1994) 1125–1131.

[7] R. Ganikhodzhaev, F. Mukhamedov, U. Rozikov: *Quadratic stochastic operators and processes: results and open problems*, Inf. Dim. Analysis Quant. Prob. Rel. Topics 14/2 (2011) 279–335.

[8] A. J. Homburg, U. U. Jamilov, M. Scheutzow: *Asymptotics for a class of iterated random cubic operators*, Nonlinearity 32 (2019) 3646–3660.

[9] U. U. Jamilov: *On a family of strictly non-Volterra quadratic stochastic operators*, J. Phys. Conf. Ser. 697 (2016), art. no. 012013.

[10] U. U. Jamilov, A. Yu. Khamraev, M. Ladra: *On a Volterra cubic stochastic operator*, Bull. Math. Biol. 80/2 (2018) 319–334.

[11] U. U. Jamilov, M. Ladra: *Non-ergodicity of uniform quadratic stochastic operators*, Qual. Theory Dyn. Syst. 15/1 (2016) 257–271.

[12] U. U. Jamilov, M. Ladra: *On identically distributed non-Volterra cubic stochastic operator*, J. Appl. Nonlin. Dyn. 6/1 (2017) 79–90.

[13] U. U. Jamilov, M. Ladra, *On a family of non-Volterra quadratic operators acting on a simplex*, Qual. Theo. Dyn. Sys. 19/3 (2020), art. no. 95.

[14] U. U. Jamilov, A. Reinfelds: *A family of Volterra cubic stochastic operators*, J. Convex Analysis 28/1 (2021) 19–30.

[15] U. U. Jamilov, M. Scheutzow, M. Wilke-Berenguer: *On the random dynamics of Volterra quadratic operators*, Ergodic Theory Dyn. Syst. 37/1 (2017) 228–243.

[16] H. Kesten: *Quadratic transformations: A model for population growth I*, Adv. Appl. Probab. 2 (1970) 1–82.

[17] Y. I. Lyubich: *Mathematical Structures in Population Genetics*, Biomathematics 22, Springer, Berlin (1992).

[18] F. M. Mukhamedov, U. U. Jamilov, A. T. Pirnapasov: *On non-ergodic uniform Lotka-Volterra operators*, Math. Notes. 105 (2019) 258–264.

[19] F. M. Mukhamedov, C. H. Pah, A. Rosli: *On non-ergodic Volterra cubic stochastic operators*, Qual. Theory Dyn. Syst. 18 (2019) 1225–1235.

[20] U. A. Rozikov, A. Yu. Khamraev: *On cubic operators defined on finite-dimensional simplices*, Ukrainian Math. J. 56/10 (2004) 1699–1711.

[21] U. A. Rozikov, A. Yu. Khamraev: *On construction and a class of non-Volterra cubic stochastic operators*, Nonlinear Dyn. Syst. Theory 14/1 (2014) 92–100.

[22] U. A. Rozikov, U. Zhamilov: *F-quadratic stochastic operators*, Math. Notes 83/3-4 (2008) 554–559.

[23] U. A. Rozikov, U. U. Zhamilov: *Volterra quadratic stochastic operators of a two-sex population*, Ukrainian Math. J. 63/7 (2011) 1136–1153.

[24] S. M. Ulam: *A Collection of Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics 8, Interscience Publishers, New York (1960).

[25] M. I. Zakharevich: *On the behaviour of trajectories and the ergodic hypothesis for quadratic mappings of a simplex*, Russ. Math. Surveys 33/6 (1978) 265–266.

[26] U. U. Zhamilov, U. A. Rozikov: *On the dynamics of strictly non-Volterra quadratic stochastic operators on the two-dimensional simplex*, Sb. Math. 200/9 (2009) 1339–1351.