Journal of Convex Analysis
Volume 29 (2022), No. 4, 1083–1117

Existence of Positive Solutions for a Critical Nonlocal Elliptic System

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Received: August 14, 2021
Accepted: February 27, 2022

We establish the existence of positive solution to the critical nonlocal elliptic system

$$(S) \qquad \begin{cases} (-\Delta)_p^s u + a(x)|u|^{p-2}u + c(x)|v|^{p-2}v = \frac{1}{p_s^*}K_u(u,v) \ \text{ in } \ \mathbb{R}^N, \\ (-\Delta)_p^s v + c(x)|u|^{p-2}u + b(x)|v|^{p-2}v = \frac{1}{p_s^*}K_v(u,v) \ \text{ in } \ \mathbb{R}^N, \\ u, v > 0 \text{ in } \mathbb{R}^N, \ u, v \in D^{s,p}(\mathbb{R}^N), \ N > ps, \ s \in (0,1). \end{cases}$$

Here $(-\Delta)_p^s$ denotes the fractional p-Laplacian, a, b and c are suitable functions and K is a p_s^*-homogeneous function, $p_s^* = (pN)/(N-ps)$, $N > ps$. One of the main tools is to apply the global compactness result for the associated energy functional similar to that due to M. Struwe [*A global compactness result for elliptic boundary value problems involving limiting nonliarities*, Math. Zeitschrift 187/4 (1984) 511–517] combined with some information on a limit system of (S) with $a = b = c = 0$, the concentration compactness due to P. L. Lions [*The concentration-compactness principle in the calculus of variations. I: The limit case*, Rev. Mat. Iberoamericana 1/1 (1985) 145–201] and the Brouwer degree theory.

Keywords: Variational critical system, fractional equations, global compactness result, Brouwer degree theory.
2010 Mathematics Subject Classification: Primary 35J50, 35R11; secondary 58E05, 47H11.

1. Introduction

In this paper we show the existence of positive solution to the critical nonlocal elliptic system of type

$$(\mathrm{S}) \qquad \begin{cases} (-\Delta)_p^s u + a(x)|u|^{p-2}u + c(x)|v|^{p-2}v = \frac{1}{p_s^*}K_u(u,v) \ \text{ in } \ \mathbb{R}^N, \\ (-\Delta)_p^s v + c(x)|u|^{p-2}u + b(x)|v|^{p-2}v = \frac{1}{p_s^*}K_v(u,v) \ \text{ in } \ \mathbb{R}^N, \\ u, v > 0 \text{ in } \mathbb{R}^N, \ u, v \in D^{s,p}(\mathbb{R}^N), \ N > ps, \ s \in (0,1), \end{cases}$$

* G. M. F. received research grants from CNPq/Brazil.
† O. H. M. received research grants from CNPq/Brazil Proc 307061/2018-3 and FAPESP Proc 2019/24901-3.

ISSN 0944-6532 / $ 2.50

where $p_s^* = \frac{pN}{N-ps}$, $N > ps$, K, a, b c are continuous function, $(-\Delta)_p^s$ is the fractional p-Laplacian, which is defined, up to a normalization constant, as

$$(-\Delta)_p^s \phi(x) = 2 \lim_{\epsilon \to 0^+} \int\limits_{\mathbb{R}^N \setminus \{B_\epsilon(x)\}} \frac{|\phi(x)-\phi(y)|^{p-2}(\phi(x)-\phi(y))}{|x-y|^{N+ps}}\, dy, \quad x \in \mathbb{R}^N,\ \phi \in C_0^\infty(\mathbb{R}^N),$$

and, by normalizing the constant,

$$\|w\|^p = \int\int_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^p}{|x-y|^{N+ps}} dxdy \equiv [w]_{s,p}^p,$$

and we denote by matrix notation

$$A(u,v) = \begin{bmatrix} a & c \\ c & b \end{bmatrix} \begin{bmatrix} |u|^{p-2}u \\ |v|^{p-2}v \end{bmatrix}.$$

Define the space $D^{s,p}(\mathbb{R}^N)$, as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm above $\|w\|$. Let $Y = D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$ be the Sobolev space with norm

$$\|z\|_Y = (\|u\|^p + \|v\|^p)^{\frac{1}{p}},$$

where $z = (u,v) \in Y$ and $\|\cdot\|$ is the usual norm in $D^{s,p}(\mathbb{R}^N)$.

Recently, great attention has been focussed on the study of nonlinear problems involving fractional Laplacian, in view of real-world applications. For instance, this type of operators arise in thin obstacle problems, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science and water waves, see Di Nezza, Palatucci and Valdinoci [14].

Let $\mathbb{R}_+^2 := [0,\infty) \times [0,\infty)$. Our main hypotheses on the function $K \in C^2(\mathbb{R}_+^2, \mathbb{R})$ can be stated as follows:

(K_0) K is p_s^*-homogeneous, that is,

$$K(\lambda\alpha, \lambda\beta) = \lambda^{p_s^*} K(\alpha,\beta) \quad \text{for each } \lambda > 0,\ (\alpha,\beta) \in \mathbb{R}_+^2;$$

(K_1) there exists $c_1 > 0$ such that

$$|K_\alpha(\alpha,\beta)| + |K_\beta(\alpha,\beta)| \le c_1 \left(\alpha^{p_s^*-1} + \beta^{p_s^*-1}\right) \quad \text{for each } (\alpha,\beta) \in \mathbb{R}_+^2;$$

(K_2) $K(\alpha,\beta) > 0$ for each $\alpha, \beta > 0$;

(K_3) $\nabla K(0,1) = \nabla K(1,0) = (0,0)$;

(K_4) $K_\alpha(\alpha,\beta), K_\beta(\alpha,\beta) \ge 0$ for each $(\alpha,\beta) \in \mathbb{R}_+^2$;

(K_5) the 1-homogeneous function $G : \mathbb{R}_+^2 \to \mathbb{R}$ given by $G(\alpha^{p_s^*}, \beta^{p_s^*}) := K(\alpha,\beta)$ is concave.

Notice that by homogeneity condition (K_0) we obtain

$$K(\alpha,\beta)=\frac{1}{p_s^*}\alpha K_\alpha(\alpha,\beta)+\frac{1}{p_s^*}\beta K_\beta(\alpha,\beta). \tag{1}$$

To state our main result we need some previous definitions and notations. Let us denote by $\widetilde{S}_{K,s}$ the best constant of the immersion

$$D^{s,p}(\mathbb{R}^N)\times D^{s,p}(\mathbb{R}^N)\hookrightarrow L^{p_s^*}(\mathbb{R}^N)\times L^{p_s^*}(\mathbb{R}^N),$$

that is,
$$\widetilde{S}_{K,s}:=\inf_{u,v\in D^{s,p}(\mathbb{R}^N),u,v\neq 0}\frac{(\|u\|^p+\|v\|^p)}{\left(\int_{\mathbb{R}^N}K(u,v)dx\right)^{p/p_s^*}}.$$

From now on, even we do not know all the minimizers of $\widetilde{S}_{K,s}$, more exactly, of the minimizers of

$$S=\inf_{u\in D^{s,p}(\mathbb{R}^N),u\neq 0}\frac{\|u\|^p}{\left(\int_{\mathbb{R}^N}|u|^{p_s^*}dx\right)^{p/p_s^*}},$$

in virtue of [5](see also [20]), we consider the function $\Phi_{\delta,x_0}\in D^{s,p}(\mathbb{R}^N)$ given by

$$\Phi_{\delta,x_0}(x)=C\left(\frac{\delta^{p'}}{\delta^{p'}+|x-x_0|^{p'}}\right)^{(N-ps)/p},\quad x,x_0\in\mathbb{R}^N,\ p'=\frac{p}{p-1},\ \text{and}\quad \delta>0, \tag{2}$$

where C is a positive constant. The functions Φ_{δ,x_0} are positive radial solutions of

$$(P_\infty)\qquad \begin{cases} (-\Delta)_p^s u=|u|^{p_s^*-2}u \ \text{ in } \ \mathbb{R}^N,\\ u>0 \ \text{ in } \ \mathbb{R}^N,\\ u\in D^{s,p}(\mathbb{R}^N),\quad N>ps.\end{cases}$$

That is, $u=\Phi_{\delta,x_0}$ satisfies for all $v\in D^{s,p}(\mathbb{R}^N)$,

$$\int\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}dx\,dy=\int_{\mathbb{R}^N}|u|^{p_s^*-2}uvdx.$$

Moreover, it verifies
$$\|\Phi_{\delta,x_0}\|^p=S\quad\text{and}\quad|\Phi_{\delta,x_0}|_{p_s^*}=1. \tag{3}$$

For $p=2$, the existence of a minimizer (see also [24] for the local case, [9] for uniqueness) is obtained in [12]. We will use the following notation

$$<L(u),v>=\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}$$

and
$$\int\int_{\mathbb{R}^{2N}}<L(u),u>dxdx=\|u\|^p.$$

By [13, Lemma 1], (see [15, Lemma 5.1]), there exist $\alpha_o,\beta_o>0$ such that $\widetilde{S}_{K,s}$ is attained by $(\alpha_o\Phi_{\delta,y},\beta_o\Phi_{\delta,y})$, which is the unique solution (see [21, 4]), at least for $p=2$.

Moreover,
$$M_K \widetilde{S}_{K,s} = S, \tag{4}$$
where $M_K = \max\limits_{\alpha^p+\beta^p=1} K(\alpha,\beta)^{p/p_s^*} = K(\alpha_o,\beta_o)^{p/p_s^*}$.

The hypotheses on the functions $a, b, c : \mathbb{R}^N \mapsto \mathbb{R}^+$ are given by:

(A_1) The functions a, b, c are positive in a same set of positive measure.

(A_2) $a, b, c \in L^q(\mathbb{R}^N)$ for all $q \in [p_1, p_2]$ with $1 < p_1 < \dfrac{N}{ps} < p_2$ and $p_2 < \dfrac{N(p-1)}{p^2 s - N}$ if $N < p^2 s$.

(A_3) $\alpha_o^p |a|_{L^{N/ps}(\mathbb{R}^N)} + \beta_o^p |b|_{L^{N/ps}(\mathbb{R}^N)} + \alpha_o^{p-1}\beta_o |c|_{L^{N/ps}(\mathbb{R}^N)} + \alpha_o \beta_o^{p-1} |c|_{L^{N/ps}(\mathbb{R}^N)} < \widetilde{S}_{K,s}(p^{ps/N} - 1)$.

Since we are going to use the variational method, we consider the C^1 functional $I : D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N) \mapsto \mathbb{R}$ associated with system (S) given by

$$I(u,v) = \frac{1}{p}(\|u\|^p + \|v\|^p) + \frac{1}{p}\int_{\mathbb{R}^N} (A(u,v),(u,v))_{\mathbb{R}^2} dx - \frac{1}{p_s^*}\int_{\mathbb{R}^N} K(u,v)dx,$$

whose Fréchet derivative is given, for all $(\varphi,\psi) \in D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$, by

$$I'(u,v)(\varphi,\psi) = \int\int_{\mathbb{R}^{2N}} (< L(u), \varphi > + < L(v), \psi >) dxdy$$
$$+ \int_{\mathbb{R}^N} (A(u,v),(\varphi,\psi))_{\mathbb{R}^2}\, dx - \frac{1}{p_s^*}\int_{\mathbb{R}^N} K_u(u,v)\varphi dx + \frac{1}{p_s^*}\int_{\mathbb{R}^N} K_v(u,v)\psi dx.$$

The critical points are precisely the weak solutions, that is, we can say that the function $(u,v) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \times \mathbb{R}$ is a positive weak solution of (S) if $u, v > 0$ in $D^{s,p}(\mathbb{R}^N)$ and for all $\varphi, \psi \in D^{s,p}(\mathbb{R}^N)$ we get $I'(u,v)(\varphi,\psi) = 0$.

Using the above notation we are able to state our main result.

Theorem 1.1. *Assume that* $(A_1)-(A_3)$ *and* $(K_0)-(K_5)$ *hold. Then,* (S) *has a positive solution* $(u_0,v_0) \in D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$ *with*

$$\frac{s}{N}\widetilde{S}_{K,s}^{N/ps} < I(u_0,v_0) < \frac{ps}{N}\widetilde{S}_{K,s}^{N/ps}.$$

The paper is organized as follows. In Section 2 we study the limit system associated to (S). In Section 3 we give the complete descriptions for the Palais-Smale (PS) sequences for the functional I. The proof of the main result is in Section 4.

In the celebrated paper [3], Benci and Cerami studied the following semilinear elliptic problem

$$(BC) \qquad \begin{cases} -\Delta u + a(x)u = u^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N), \quad u \geq 0, \quad N \geq 3, \end{cases}$$

where

(a_1) $a(x) \geq 0$ and $a(x) \geq a_0 > 0$, for all $x \in \mathbb{R}^N$ in a neighborhood of a point $\bar{x}$.

(a_2) $a \in L^q(\mathbb{R}^N)$ for all $q \in [p_1, p_2]$ with $1 < p_1 < \frac{N}{2} < p_2$ with $p_2 < \dfrac{N}{4-N}$ if $N = 3$.

(a_3) $|a|_{L^{N/2}(\mathbb{R}^N)} < S(2^{2/N} - 1)$.

They used the properties of the solutions of a limit problem given by (BC) with $a = 0$, the version to $\mathbb{R}^N$ of Struwe's Global Compactness result [23], Lions's Concentration and Compactness result [19] and arguments of Brouwer degree theory.

We would also like to mention that the same arguments used in [3] were also used by Cerami and Passasseo in [7] with Neumann boundary conditions in a half-space $\mathbb{R}^N_+$ and by Alves in [1] with p-Laplacian operator. As far as the extension to the p-Laplacian operator is concerned, some technical difficulties as the lack of linearity and homogeneity must be faced. The version of bi-Laplacian operator was studied by Alves and do Ó in [2].

A multiplicity result involving category theory was studied in [8] by Chabrowski and Yang. More recently, in [26] Xie, Ma and Xu proved a version for [3] considering the Kirchhoff operator. Correia and Figueiredo show the same result of [3] considering the fractional Laplacian. A version for Choquard equation was proved by Gao, da Silva, Yang, and Zhou in [17] and a version for Schrödinger-Poisson system was studied by Cerami and Molle in [6]. In [10], Chen, Wei and Yan showed existence of infinitely many non-radial solutions, whose energy can be made arbitrarily large with a radial. In [22] Penga, Wang and Yan showed the existence of infinitely many non-radial solutions with a partially radial.

A natural, still open question is to know whether Benci and Cerami's results is true in the system of equations case. Recently, Figueiredo and Silva in [16] answered positively the question for the local case. In this paper, we are going study for the fractional p-Laplacian systems. We recall that, the extension to local or nonlocal systems involves some technical difficulties which are overcome with some refined estimates, as can be seen in Lemma 3.1, Theorem 3.2 and Subsection 4.2. More precisely, in these results we give the complete descriptions for the Palais-Smale (PS) sequences of the corresponding energy functionals and by using these descriptions, the existence results of solutions are obtained. Moreover, the main feature of the system is a "double" lack of compactness due to the unboundedness of the domain and the presence of the critical Sobolev exponent. The solutions are sought by means of variational methods, although the functional related to the problem does not satisfy the Palais-Smale compactness condition.

2. Limit problem

In this section we study the limit problem given by

$$(S_\infty) \qquad \begin{cases} (-\Delta)^s_p u = \dfrac{1}{p^*_s} K_u(u,v) \ \text{ in } \ \mathbb{R}^N, \\ (-\Delta)^s_p v = \dfrac{1}{p^*_s} K_v(u,v) \ \text{ in } \ \mathbb{R}^N, \\ \ u, v > 0 \ \text{ in } \ \mathbb{R}^N, \ u, v \in D^{s,p}(\mathbb{R}^N), \ N > ps, \end{cases}$$

which the functional associated $I_\infty : D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N) \mapsto \mathbb{R}$ is given by

$$I_\infty(u,v) = \frac{1}{p}(\|u\|^p + \|v\|^p) - \frac{1}{p^*_s}\int_{\mathbb{R}^N} K(u,v)dx.$$

Lemma 2.1. *Let (u_n, v_n) be sequence $(PS)_c$ for I_∞. Then*

(*i*) *The sequence (u_n, v_n) is bounded in $D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$.*

(*ii*) *If $u_n \rightharpoonup u$ in $D^{s,p}(\mathbb{R}^N)$ and $v_n \rightharpoonup v$ in $D^{s,p}(\mathbb{R}^N)$, then $I'_\infty(u, v) = 0$.*

(*iii*) *If $c \in (-\infty, \frac{s}{N}\widetilde{S}_{K,s}^{N/ps})$, then I_∞ satisfies the $(PS)_c$ condition, i.e, up to a subsequence, $(u_n, v_n) \to (u, v)$ in $D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$.*

Proof. Let (u_n, v_n) be a $(PS)_c$ sequence for I_∞, that is, we have $I_\infty(u_n, v_n) \to c$ and $I'_\infty(u_n, v_n) \to 0$.

(i) The boundedness follows by using (1) and estimating

$$I_\infty(u_n v_n) - \frac{1}{p_s^*} I'_\infty(u_n, v_n)(u_n, v_n).$$

(ii) follows by a density argument.

(iii) Consider $w_n = u_n - u$ and $z_n = v_n - v$. Note that applying [18, Lemma 4.6] also from [13, Lemma 8], we have

$$\|w_n\|^p + \|u\|^p + \|z_n\|^p + \|v\|^p - \frac{1}{p_s^*}\int_{\mathbb{R}^N} K_u(w_n, z_n)w_n dx$$
$$-\frac{1}{p_s^*}\int_{\mathbb{R}^N} K_v(w_n z_n)z_n dx - \frac{1}{p_s^*}\int_{\mathbb{R}^N} K_u(u, v)u dx - \frac{1}{p_s^*}\int_{\mathbb{R}^N} K_v(u, v)v dx = o_n(1).$$

Usingitem (ii) and (1) we obtain

$$\|w_n\|^p + \|z_n\|^p - \int_{\mathbb{R}^N} K(w_n, z_n)dx = o_n(1).$$

Up to a subsequence, we conclude that there exists $\rho \geq 0$ such that

$$0 \leq \rho = \lim_{n\to\infty}\Big[\|w_n\|^p + \|z_n\|^p\Big] = \lim_{n\to\infty}\int_{\mathbb{R}^N} K(w_n, z_n)dx.$$

Suppose, by contradiction, that $\rho > 0$. From the inequality

$$\widetilde{S}_{K,s}\left(\int_{\mathbb{R}^N} K(w_n, z_n)dx\right)^{p/p_s^*} \leq \|w_n\|^p + \|z_n\|^p,$$

we get
$$\rho \geq \widetilde{S}_{K,s}\rho^{p/p_s^*} \Rightarrow \rho \geq \widetilde{S}_{K,s}^{N/ps}. \tag{5}$$

Since
$$I_\infty(u, v) = \left(\frac{1}{p} - \frac{1}{p_s^*}\right)[\|u\|^p + \|v\|^p] = \frac{s}{N}[\|u\|^p + \|v\|^p] \geq 0$$

and
$$c = \frac{s}{N}[\|w_n\|^p + \|z_n\|^p] + I_\infty(u, v) + o_n(1), \tag{6}$$

we conclude
$$\begin{aligned} c &= \frac{s}{N}[\|w_n\|^p + \|z_n\|^p] + I_\infty(u, v) + o_n(1) \\ &\geq \frac{s}{N}[\|w_n\|^p + \|z_n\|^p] + o_n(1) = \frac{s}{N}\rho \geq \frac{s}{N}\widetilde{S}_{K,s}^{N/ps}, \end{aligned}$$

which is a contradiction. Hence $\rho = 0$ and

$$\|w_n\|^p = \|u_n - u\|^p \to 0 \text{ and } \|z_n\|^p = \|v_n - v\|^p \to 0. \qquad \square$$

3. A compactness result

Now, we establish the following lemma which will be useful to prove a compactness result.

Lemma 3.1. *Let (u_n, v_n) be a $(PS)_c$ sequence for the functional I_∞ with $u_n \rightharpoonup 0$, $v_n \rightharpoonup 0$ and $u_n \nrightarrow 0$, $v_n \nrightarrow 0$. Then, there are sequences $(R_n) \subset \mathbb{R}$, $(x_n) \subset \mathbb{R}^N$ and $(\Upsilon_0, \Upsilon_1) \in D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$ nontrivial solution of (P_∞) and a sequence (τ_n, ζ_n) which is a $(PS)_{\tilde{c}}$ for the I_∞ such that, up to a subsequence of (u_n, v_n),*

$$\tau_n(x) = u_n(x) - R_n^{(N-ps)/p}\Upsilon_0(R_n(x - x_n)) + o_n(1)$$

and
$$\zeta_n(x) = u_n(x) - R_n^{(N-ps)/p}\Upsilon_1(R_n(x - x_n)) + o_n(1).$$

Proof. Let $(u_n, v_n) \subset D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$ be a $(PS)_c$ sequence for the functional I_∞, i.e,

$$I_\infty(u_n, v_n) \to c \quad \text{and} \quad I'_\infty(u_n, v_n) \to 0. \tag{7}$$

From Lemma 2.1(i), we get that (u_n, v_n) is bounded in $D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$. Since $u_n \rightharpoonup 0$, $v_n \rightharpoonup 0$ and $u_n \nrightarrow 0$, $v_n \nrightarrow 0$ it follows from Lemma 2.1(iii) that

$$c \geq \frac{s}{N}\widetilde{S}_{K,s}^{N/ps}.$$

Note that from (1) we obtain

$$c + o_n(1) = I_\infty(u_n, v_n) - \frac{1}{p_s^*}I'_\infty(u_n, v_n)(u_n, v_n) = \frac{s}{N}(\|u_n\|^p + \|v_n\|^p),$$

which implies
$$\lim_{n\to\infty}(\|u_n\|^p + \|v_n\|^p) \geq \widetilde{S}_{K,s}^{N/ps}. \tag{8}$$

Let θ be a number such that $B_2(0)$ is covered by θ balls of radius 1, $(R_n) \subset \mathbb{R}$, $(x_n) \subset \mathbb{R}^N$ such that

$$\begin{aligned}
&\sup_{y\in\mathbb{R}^N}\int\int_{B^2_{R_n^{-1}}(y)}[< L(u_n), u_n > + < L(v_n), v_n >]dxdy \\
&= \int\int_{B^2_{R_n^{-1}}(x_n)}[< L(u_n), u_n > + < L(v_n), v_n >]dxdy = \frac{\widetilde{S}_{K,s}^{N/ps}}{p\theta}
\end{aligned}$$

and the function

$$(w_n(x), z_n(x)) = \Big(R_n^{(ps-N)/p}u_n\Big(\frac{x}{R_n} + x_n\Big), R_n^{(ps-N)/p}v_n\Big(\frac{x}{R_n} + x_n\Big)\Big).$$

Using a change of variable, we can prove that

$$\begin{aligned}
&\int\int_{B_1^2(0)}[< L(w_n), w_n > + < L(z_n), z_n >]dxdy = \frac{\widetilde{S}_{K,s}^{N/ps}}{p\theta} \\
&= \sup_{y\in\mathbb{R}^N}\int\int_{B_1^2(y)}[< L(w_n), w_n > + < L(z_n), z_n >]dxdy.
\end{aligned}$$

Now, for each $(\Phi_1, \Phi_2) \in D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$, we define

$$(\tilde{\Phi}_{1,n}, \tilde{\Phi}_{2,n})(x) = (R_n^{(N-ps)/p}\Phi_1(R_n(x - x_n)), R_n^{(N-ps)/p}\Phi_2(R_n(x - x_n)))$$

which satisfies
$$\int\int_{\mathbb{R}^{2N}}[< L(u_n),\tilde{\Phi}_{1,n} > + < L(v_n),\tilde{\Phi}_{2,n} >]dxdy$$
$$= \int\int_{\mathbb{R}^{2N}}[< L(w_n),\Phi_1 > + < L(z_n),\Phi_2 >]dxdy \tag{9}$$

and
$$\int_{\mathbb{R}^N}[K_u(u_n,v_n)\tilde{\Phi}_{1,n} + K_v(u_n,v_n)\tilde{\Phi}_{2,n}]dx$$
$$= \int_{\mathbb{R}^N}[K_w(w_n,z_n)\Phi_1 + K_z(w_n,z_n)\Phi_2]dx, \tag{10}$$

where we conclude that

$$I_\infty(w_n,z_n) \to c \quad\text{and}\quad I'_\infty(w_n,z_n) \to 0. \tag{11}$$

From Lemma 2.1, there exists $(\Upsilon_0,\Upsilon_1) \in D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$ such that, up to a subsequence, $(w_n,z_n) \rightharpoonup (\Upsilon_0,\Upsilon_1)$ in $D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$ and $I'_\infty(\Upsilon_0,\Upsilon_1) = 0$.

As a consequence of [13, Lemma 6], we get

$$\int_{\mathbb{R}^N} K(w_n,z_n)\phi dx \to \int_{\mathbb{R}^N} K(\Upsilon_0,\Upsilon_1)\phi dx + \sum_{j\in J}\phi(x_j)\nu_j, \quad \forall\phi \in C_0^\infty(\mathbb{R}^N) \tag{12}$$

and
$$< L(w_n),w_n > + < L(z_n),z_n > \rightharpoonup \mu + \sigma$$
$$\geq |\nabla\Upsilon_0|^p + |\nabla\Upsilon_1|^p + \sum_{j\in J}\phi(x_j)\mu_j + \sum_{j\in J}\phi(x_j)\sigma_j,$$

$\forall\phi \in C_0^\infty(\mathbb{R}^N)$, for some $\{x_j\}_{j\in J} \subset \mathbb{R}^N$ and for some $\{\nu_j\}_{j\in J}$, $\{\mu_j\}_{j\in J}$, $\{\sigma_j\}_{j\in J} \subset \mathbb{R}^+$.
Since $\widetilde{S}_K \nu_j^{p/p_s^*} \leq \mu_j + \sigma_j$, we can conclude that J is finite.
From now on, we denote by $J = \{1,2,...,m\}$ and $\Gamma \subset \mathbb{R}^N$ the set given by

$$\Gamma = \{x_j \in \{x_j\}_{j\in J}; |x_j| > 1\}, \quad (x_j \text{ given by } (12)).$$

We are going to show that $(\Upsilon_0,\Upsilon_1) \neq (0,0)$. Suppose, by contradiction, that $(\Upsilon_0,\Upsilon_1) = (0,0)$. Then, by (12) we have

$$\int_{\mathbb{R}^N} K(w_n,z_n)\phi dx \to 0, \quad \forall\phi \in C_0^\infty(\mathbb{R}^N \setminus \{x_1,x_2,...,x_m\}). \tag{13}$$

Since $(\phi_{1,n},\phi_{2,n}) = (\phi w_n,\phi z_n)$ with $\phi \in C_0^\infty(\mathbb{R}^N \setminus \{x_1,x_2,...,x_m\})$ is bounded, we obtain $I'_\infty(w_n,z_n)(\phi_{1,n},\phi_{2,n}) = o_n(1)$, that is,

$$\int\int_{\mathbb{R}^{2N}}[< L(w_n),\phi_{1,n} > + < L(z_n),\phi_{2,n} >]dxdy$$
$$-\frac{1}{p_s^*}\int_{\mathbb{R}^N}[K_w(w_n,z_n)\phi_{1,n} + K_z(w_n,z_n)\phi_{2,n}]dx = o_n(1).$$

Using the definition of $(\phi_{1,n}, \phi_{2,n})$ and (1) , we have

$$\int\int_{\mathbb{R}^{2N}} [< L(w_n), w_n > + < L(z_n), z_n >]\phi(x)dxdy$$
$$+\int\int_{\mathbb{R}^{2N}} [w_n(y)< L(w_n), \phi > + z_n(y) < L(z_n), \phi >]dxdy - \int_{\mathbb{R}^N} K(w_n, z_n)\phi dx = o_n(1).$$

Using the Hölder inequality we get

$$\begin{aligned}&\int\int_{\mathbb{R}^{2N}} [< L(w_n), w_n > + < L(z_n), z_n >]\phi(x)dxdy\\ &\leq \|w_n\|^p\Big(\int\int_{\mathbb{R}^{2N}} |w_n|^p \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+ps}}dxdy\Big)^{1/2}\\ &\quad + \|z_n|\|^p\Big(\int\int_{\mathbb{R}^{2N}} |z_n|^p \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+ps}}dxdy\Big)^{1/p} + \int_{\mathbb{R}^N} K(w_n, z_n)\phi dx + o_n(1).\end{aligned}$$

Since there exists $R > 0$ such that $supp\phi \subset B_R(0)$, we have

$$\begin{aligned}&\int\int_{\mathbb{R}^{2N}} [< L(w_n), w_n > + < L(z_n), z_n >]\phi(x)dxdy\\ &\leq C\|w_n\|^p\Big(\int\int_{B_R^2(0)} |w_n|^p \frac{1}{|x-y|^{N+ps}}dxdy\Big)^{1/2}\\ &\quad + C\|z_n|\|^p\Big(\int\int_{B_R^2(0)} |z_n|^p \frac{1}{|x-y|^{N+ps}}dxdy\Big)^{1/p} + \int_{\mathbb{R}^N} K(w_n, z_n)\phi dx + o_n(1).\end{aligned}$$

Since (w_n, z_n) is bounded in $D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$, from compact embedding and (13), we obtain

$$w_n, z_n \to 0, \text{ a.e., as } n \to \infty, \text{ and verifying } |w_n|, |z_n| \leq 1, \ \forall n.$$

Now, for fixed R and $N > sp$, we have $\dfrac{1}{|x-y|^{N+ps}} \in L^1(B_R^2(0))$. Then by the dominated convergence theorem we get

$$\int\int_{B_R^2(0)} |z_n|^p \frac{1}{|x-y|^{N+ps}}dxdy \to 0, \int\int_{B_R^2(0)} |w_n|^p \frac{1}{|x-y|^{N+ps}}dxdy \to 0,$$

and, for all $\phi \in C_0^\infty(\mathbb{R}^N \setminus \{x_1, x_2, ..., x_m\})$,

$$\int\int_{\mathbb{R}^{2N}} [< L(w_n), w_n > + < L(z_n), z_n >]\phi(x)dxdy \to 0, \tag{14}$$

Let $\rho \in \mathbb{R}$ be a number that satisfies $0 < \rho < \min\{dist(\Gamma, \bar{B}_1(0)), 1)\}$. We show that

$$\int\int_{(B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0))^2} [< L(w_n), w_n > + < L(z_n), z_n] > \phi(x)dxdy \to 0. \tag{15}$$

We consider $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \phi(x) \leq 1$ and $\phi(x) = 1$ if $x \in B_{1+\rho}(0)$. If $\tilde{\phi} = \phi|_{\mathbb{R}^N \setminus \{x_1,\dots,x_m\}}$, follows by (14) that

$$\int\int_{\mathbb{R}^{2N}} [< L(w_n), w_n > + < L(z_n), z_n >]\tilde{\phi} dxdy \to 0.$$

Since

$$\begin{aligned} 0 &\leq \int\int_{(B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0))^2} [< L(w_n), w_n > + < L(z_n), z_n >] dxdy \\ &\leq \int\int_{(B_{1+\rho}(0))^2} [< L(w_n), w_n > + < L(z_n), z_n >] dxdy \\ &= \int\int_{B^2_{1+\rho}(0)} [< L(w_n), w_n > + < L(z_n), z_n >]\tilde{\phi} dxdy \\ &\leq \int\int_{\mathbb{R}^{2N}} [< L(w_n), w_n > + < L(z_n), z_n >]\tilde{\phi} dxdy, \end{aligned}$$

we have that (15) is true.

Let $\Psi \in C_0^\infty(\mathbb{R}^N)$ be such that $0 \leq \Psi(x) \leq 1$ for all $x \in \mathbb{R}^N$ and

$$\Psi(x) = \begin{cases} 1, & x \in B_{1+\frac{\rho}{3}}(0), \\ 0, & x \in B^c_{1+\frac{2\rho}{3}}(0). \end{cases}$$

Consider the sequence $(\Psi_{1,n},\Psi_{2,n})$ given by $(\Psi_{1,n},\Psi_{2,n})(x) = (\Psi(x)w_n(x),\Psi(x)z_n(x))$. Note that, by Hölder's inequality, we have

$$\begin{aligned} &\int\int_{(B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0))^2} [< L(\Psi_{1,n}), \Psi_{1,n} > + < L(\Psi_{2,n}), \Psi_{2,n} >] dxdy \\ &\leq \int\int_{[B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)]^2} |\Psi|(x)(< L(w_n), w_n > + < L(z_n), z_n >)) dxdy \\ &+ \int\int_{[(B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)]^2} (|w_n(y)|^p + |z_n(y)|^p) < L(\Psi), \Psi >) dxdy. \end{aligned}$$

From (15), as well as, in its proof, we obtain

$$\int\int_{((B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0))^2} [< L(\Psi_{1,n}), \Psi_{1,n} > + < L(\Psi_{2,n}, \Psi_{2,n} >)] dxdy \to 0. \tag{16}$$

Since $(\Psi_{1,n}, \Psi_{2,n})$ is bounded in $D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$, we derive that

$$\begin{aligned} &\int\int_{(B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0))^2} < L(w_n), \Psi_{1,n} > dxdy + \int\int_{(B_{1+\frac{\rho}{3}}(0))^2} < L(w_n), \Psi_{1,n} > dxdy \\ &+ \int\int_{(B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0))^2} < L(z_n), \Psi_{2,n} > dxdy + \int\int_{(B_{1+\frac{\rho}{3}}(0))^2} < L(z_n), \Psi_{2,n} > dxdy \end{aligned}$$

$$-\frac{1}{p_s^*}\int_{(B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0))}\Psi_{1,n}K_w(w_n,z_n)dx-\frac{1}{p_s^*}\int_{(B_{1+\frac{\rho}{3}}(0))}\Psi_{1,n}K_w(w_n,z_n)dx \tag{17}$$
$$-\frac{1}{p_s^*}\int_{(B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0))}\Psi_{2,n}K_z(w_n,z_n)dx-\frac{1}{p_s^*}\int_{(B_{1+\frac{\rho}{3}}(0))}\Psi_{2,n}K_z(w_n,z_n)dx=o_n(1).$$

From the definition of Ψ, we get

$$\iint_{(B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0))^2}(< L(w_n),\Psi_{1,n} > + < L(z_n),\Psi_{2,n} >)dxdy \to 0 \quad \text{as} \quad n\to\infty. \tag{18}$$

Moreover, from a direct calculation we obtain

$$\frac{1}{p_s^*}\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)}(\Psi_{1,n}K_w(w_n,z_n)+\Psi_{2,n}K_z(w_n,z_n))dx=o_n(1). \tag{19}$$

From (17), (18) and (19) we conclude

$$\int\int_{(B_{1+\frac{\rho}{3}}(0))^2}(< L(\Psi_{1,n}),\Psi_{1,n} > + < L(\Psi_{2,n}),\Psi_{2,n} > dxdy \tag{20}$$
$$-\frac{1}{p_s^*}\int_{B_{1+\frac{\rho}{3}}(0)}\Psi_{1,n}K_w(\Psi_{1,n},\Psi_{2,n})dx\frac{1}{p_s^*}\int_{B_{1+\frac{\rho}{3}}(0)}\Psi_{2,n}K_z(\Psi_{1,n},\Psi_{2,n})dx=o_n(1).$$

Note that

$$\begin{aligned}
&\int\int_{\mathbb{R}^{2N}}[< L(\Psi_{1,n}),\Psi_{1,n} > + < L(\Psi_{2,n}),\Psi_{1,n} >]dxdy\\
&=\int\int_{B_{1+\rho}(0))^2}[< L(\Psi_{1,n}),\Psi_{1,n} > + < L(\Psi_{2,n}),\Psi_{1,n} >]dxdy\\
&=\left(\iint_{(B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0))^2}+\iint_{(B_{1+\frac{\rho}{3}}(0))^2}\right)[< L(\Psi_{1,n}),\Psi_{1,n} > + < L(\Psi_{2,n}),\Psi_{1,n} >]dxdy\\
&=o_n(1)+\int\int_{(B_{1+\frac{\rho}{3}}(0))^2}[< L(\Psi_{1,n}),\Psi_{1,n} > + < L(\Psi_{2,n}),\Psi_{1,n} >]dxdy
\end{aligned}$$

and using (1), we get

$$\begin{aligned}
\int_{\mathbb{R}^N}K(\Psi_{1,n},\Psi_{2,n})dx&=\int_{B_{1+\rho}(0)}K(\Psi_{1,n},\Psi_{2,n})dx\\
&=\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)}K(\Psi_{1,n},\Psi_{2,n})dx+\int_{B_{1+\frac{\rho}{3}}(0)}K(\Psi_{1,n},\Psi_{2,n})dx
\end{aligned}$$

we conclude that

$$\int\int_{\mathbb{R}^{2N}}[< L(\Psi_{1,n}),\Psi_{1,n})+ < L(\Psi_{2,n}),\Psi_{2,n} > dxdy-\int_{\mathbb{R}^N}K(\Psi_{1,n},\Psi_{2,n})dx=o_n(1).$$

From the definition of $\widetilde{S}_{K,s}$, we have

$$\begin{aligned}
&\|\Psi_{1,n}\|^p + \|\Psi_{2,n}\|^p \left[1 - \left(\frac{1}{\widetilde{S}_{K,s}^{p_s^*/p}}\right)[\|\Psi_{1,n}\|^p + \|\Psi_{2,n}\|^p]^{p_s^*-2}\right] \\
&= [\|\Psi_{1,n}\|^p + \|\Psi_{2,n}\|^p] - \frac{1}{\widetilde{S}_{K,s}^{p_s^*/p}}[\|\Psi_{1,n}\|^p + \|\Psi_{2,n}\|^p]^{p_s^*} \\
&\leq \|\Psi_{1,n}\|^p + \|\Psi_{2,n}\|^p] - \int_{\mathbb{R}^N} K(\Psi_{1,n}, \Psi_{2,n})dx = o_n(1). \qquad (21)
\end{aligned}$$

Note that

$$\begin{aligned}
&\|\Psi_{1,n}\|^p + \|\Psi_{2,n}\|^p \\
&= (\int\int_{(B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0))^2} + \int\int_{(B_{1+\frac{\rho}{3}}(0))^2})[< L(\Psi_{1,n}), \Psi_{1,n}) + < L(\Psi_{2,n}), \Psi_{2,n} >]dxdy \\
&= o_n(1) + \int\int_{(B_{1+\frac{\rho}{3}}(0))^2}[< L(\Psi_{1,n}), \Psi_{1,n}) + < L(\Psi_{2,n}), \Psi_{2,n} >]dxdy.
\end{aligned}$$

Since $\Phi_{1,n} = w_n$, $\Phi_{2,n} = z_n$ in $B_{1+\frac{\rho}{3}(0)}$ and that $B_{1+\frac{\rho}{3}(0)} \subset B_2(0)$, we obtain

$$\|\Psi_{1,n}\|^p + \|\Psi_{2,n}\|^p \leq o_n(1) + \int\int_{B_2^2(0)}[< L(\Psi_{1,n}), \Psi_{1,n}) + < L(\Psi_{2,n}), \Psi_{2,n} >]dxdy,$$

which implies

$$\begin{aligned}
\|\Psi_{1,n}\|^p + \|\Psi_{2,n}\|^p &\leq o_n(1) + \int\int_{\bigcup_{k=1}^L B_1^2(y_k)}[< L(w_n), w_n > + < L(z_n), z_n >]dxdy \\
&\leq o_n(1) + L\sup_{y\in\mathbb{R}^N}\int\int_{B_1^2(y)}[< L(w_n), w_n > + < L(z_n), z_n >]dxdy \\
&\leq o_n(1) + \frac{\widetilde{S}_{K,s}^{N/ps}}{p}.
\end{aligned}$$

Then, this implies

$$\left(\|\Psi_{1,n}\|^p + \|\Psi_{2,n}\|^p\right)^{(p_s^*-p)/p} \leq o_n(1) + \left(\frac{\widetilde{S}_{K,s}^{N/p^2s}}{p^{1/p}}\right)^{p_s^*-p}. \qquad (22)$$

Using (21) and (22), we have that

$$\begin{aligned}
&[\|\Psi_{1,n}\|^p + \|\Psi_{2,n}\|^p]\left[1 + o_n(1) - \frac{1}{\widetilde{S}_{K,s}^{p_s^*/ps}}\left(\frac{\widetilde{S}_{K,s}^{N/p^2s}}{p^{1/p}}\right)^{p_s^*-p}\right] \\
\leq\ &[\|\Psi_{1,n}\|^p + \|\Psi_{2,n}\|^p]\left[1 - \frac{1}{\widetilde{S}_{K,s}^{p_s^*/p}}[\|\Psi_{1,n}\|^p + \|\Psi_{2,n}\|^p]^{p_s^*-p}\right] = o_n(1).
\end{aligned}$$

But notice that $\dfrac{N}{p^2s}(p_s^*-p) - \dfrac{p_s^*}{ps} = 0$ implies

$$\|\Phi_n\|^p\left[1 - \left(\frac{1}{p}\right)^{(p_s^*-p)/p}\right] \leq o_n(1),$$

from which we conclude that $(\Phi_{1,n}, \Phi_{1,n}) \to (0,0)$ in $D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$.

Since $w_n = \Phi_{1,n}$, $z_n = \Phi_{2,n}$ in $B_1(0)$, we obtain

$$0 \leq \int\int_{B_1^2(0)} [< L(w_n), w_n > + < L(z_n), z_n >] dxdy \ \leq \|\Psi_{1,n}\|^p + \|\Psi_{2,n}\|^p,$$

which implies

$$\int\int_{B_1^2(0)} [< L(w_n), w_n > + < L(z_n), z_n >] dxdy \ \to 0 \ \text{ as } \ n \to \infty.$$

But this last convergence it is a contradiction with

$$\int\int_{B_1^2(0)} [< L(w_n), w_n > + < L(z_n), z_n >] dxdy = \frac{\widetilde{S}_{K,s}^{N/ps}}{2\theta}, \quad \forall n \in \mathbb{N}.$$

Hence we have $(\Upsilon_0, \Upsilon_1) \neq (0,0)$. Now we are going to show that there is (τ_n, ζ_n) in $D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$ such that (τ_n, ζ_n) is a $(PS)_{\tilde{c}}$ sequence for I_∞ satisfying

$$\tau_n(x) = u_n(x) - R_n^{(N-ps)/p}\Upsilon_0(R_n(x-x_n)) + o_n(1),$$

$$\zeta_n(x) = v_n(x) - R_n^{(N-ps)/p}\Upsilon_1(R_n(x-x_n)) + o_n(1),$$

for some subsequence of (u_n, v_n) that still denote by (u_n, v_n). For this, we consider $\psi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \psi(x) \leq 1$ for all $x \in \mathbb{R}^N$ and

$$\psi(x) = \begin{cases} 1, & \text{if } \ x \in B_1(0), \\ 0, & \text{if } \ x \in B_2^c(0) \end{cases}$$

and consider a sequence (τ_n, ζ_n) defined by

$$\tau_n(x) = u_n(x) - R_n^{(N-ps)/p}\Upsilon_0(R_n(x-x_n))\psi(\bar{R}_n(x-x_n)), \tag{23}$$

$$\zeta_n(x) = v_n(x) - R_n^{(N-ps)/p}\Upsilon_1(R_n(x-x_n))\psi(\bar{R}_n(x-x_n)), \tag{24}$$

where $(\bar{R}_n)$ satisfies $\tilde{R}_n = \dfrac{R_n}{\bar{R}_n} \to \infty$. From (23) and (24), we obtain

$$R_n^{(ps-N)/p}\tau_n(x) = R_n^{(ps-N)/p}u_n(x) - \Upsilon_0(R_n(x-x_n))\psi(\bar{R}_n(x-x_n))$$

and $$R_n^{(ps-N)/p}\zeta_n(x) = R_n^{(ps-N)/p}v_n(x) - \Upsilon_1(R_n(x-x_n))\psi(\bar{R}_n(x-x_n)).$$

Making change of variable, we conclude

$$R_n^{(ps-N)/p}\tau_n\left(\frac{z}{R_n} + x_n\right) = R_n^{(ps-N)/p}u_n\left(\frac{z}{R_n} + x_n\right) - \Upsilon_0\psi\left(\frac{z}{\tilde{R}_n}\right)$$

and $$R_n^{(ps-N)/p}\zeta_n\left(\frac{z}{R_n} + x_n\right) = R_n^{(ps-N)/p}v_n\left(\frac{z}{R_n} + x_n\right) - \Upsilon_1\psi\left(\frac{z}{\tilde{R}_n}\right).$$

Now we define $$\widetilde{\tau}_n = R_n^{(ps-N)/p}\tau_n\left(\frac{z}{R_n} + x_n\right)$$

and
$$\widetilde{\zeta}_n = R_n^{(ps-N)/p}\zeta_n\left(\frac{z}{R_n}+x_n\right).$$

Since
$$w_n(x) = R_n^{(ps-N)/p}u_n\left(\frac{x}{R_n}+x_n\right)$$

and
$$z_n(x) = R_n^{(ps-N)/p}v_n\left(\frac{x}{R_n}+x_n\right),$$

we get,
$$\widetilde{\tau}_n(z) = w_n(z) - \Upsilon_0(z)\psi\left(\frac{z}{\tilde{R}_n}\right) \tag{25}$$

and
$$\widetilde{\zeta}_n(z) = \zeta_n(z) - \Upsilon_1(z)\psi\left(\frac{z}{\tilde{R}_n}\right). \tag{26}$$

If
$$\psi_n(z) = \psi\left(\frac{z}{\tilde{R}_n}\right) \tag{27}$$

we have that
$$\psi_n(z) = \begin{cases} 1, & \text{if} \quad z \in B_{\tilde{R}_n}(0), \\ 0, & \text{if} \quad z \in B^c_{2\tilde{R}_n}(0). \end{cases} \tag{28}$$

From (25), (26) and (27), we derive that

$$\widetilde{\tau}_n(z) = w_n(z) - \Upsilon_0(z)\psi_n(z) \quad \text{and} \quad \widetilde{\zeta}_n(z) = z_n(z) - \Upsilon_1(z)\psi_n(z).$$

Since $\tilde{R}_n \to \infty$, it is not difficult to show that $\Upsilon_i\psi_n \to \Upsilon_i$ in $D^{s,p}(\mathbb{R}^N)$, $i = 0, 1$.

Then
$$\widetilde{\tau}_n(z) = w_n(z) - \Upsilon_0(z) + o_n(1) \tag{29}$$

and
$$\widetilde{\zeta}_n(z) = z_n(z) - \Upsilon_1(z) + o_n(1). \tag{30}$$

To finish the proof, it is enough to show that (τ_n, ζ_n) is a $(PS)_{\tilde{c}}$ sequence for I_∞. Note that making a change of variable we get

$$I_\infty(\tau_n, \zeta_n) = I_\infty(\widetilde{\tau}_n, \widetilde{\zeta}_n)$$

Using (29) and (30) and applying [18, Lemma 4.6], [13, Lemma 8] and (11), we have

$$I_\infty(\tau_n, \zeta_n) \;\; = \;\; I_\infty(w_n, z_n) - I_\infty(\Upsilon_0, \Upsilon_1) + o_n(1) = \widetilde{c} + o_n(1),$$

where $\widetilde{c} = c - I_\infty(\Upsilon_0, \Upsilon_1)$. Now, since

$$0 \le \|I'_\infty(\tau_n, \zeta_n)\|_{D'} \le \|I'_\infty(\widetilde{\tau}_n, \widetilde{\zeta}_n)\|_{D'},$$

it is sufficient to prove that $\|I'_\infty(\widetilde{\tau}_n, \widetilde{\zeta}_n)\|_{D'} \to 0$ which is equivalent to show that

$$\|I'_\infty(\widetilde{\tau}_n, \widetilde{\zeta}_n) - I'_\infty(w_n, z_n) + I'_\infty(\Upsilon_0, \Upsilon_1)\|_{D'} \to 0. \tag{31}$$

But the last convergence is a direct consequence of [13, Lemma 8]. □

The next result is a version for a gradient system in $\mathbb{R}^N$ of the result due to Struwe that can be found in [23].

Theorem 3.2. (A global compactness result) *Let (u_n, v_n) be a $(PS)_c$ sequence for I with $u_n \rightharpoonup u_0$ in $D^{s,p}(\mathbb{R}^N)$ and $v_n \rightharpoonup v_0$ in $D^{s,p}(\mathbb{R}^N)$. Then, up to a subsequence, (u_n, v_n) satisfies either,*

(a) $(u_n, v_n) \to (u_0, v_0)$ *in* $D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$ *or,*

(b) *there exists $k \in \mathbb{N}$ and nontrivial solutions $(z_0^1, \zeta_0^1), (z_0^2, \zeta_0^2), ..., (z_0^k, \zeta_0^k)$ for the system (S_∞), such that*

$$\|u_n\|^p + \|v_n\|^p \to \|u_0\|^p + \|v_0\|^p + \sum_{j=1}^{k} [\|z_0^j\|^p + \|\zeta_0^j\|^p]$$

and
$$I(u_n, v_n) \to I(u_0, v_0) + \sum_{j=1}^{k} I_\infty(z_0^j, \zeta_0^j).$$

Proof. From the weak convergence and a density argument, we can conclude that (u_0, v_0) is a critical point of I. Suppose that $u_n \nrightarrow u_0$, $v_n \nrightarrow v_0$ in $D^{s,p}(\mathbb{R}^N)$ and let $(w_n^1, z_n^1) \subset D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$ be the sequence given by $w_n^1 = u_n - u_0$ and $z_n^1 = v_n - v_0$. Then, $w_n^1 \rightharpoonup 0$, $z_n^1 \rightharpoonup 0$ in $D^{s,p}(\mathbb{R}^N)$ and $w_n^1 \nrightarrow 0$, $z_n^1 \nrightarrow 0$ in $D^{s,p}(\mathbb{R}^N)$. Applying [18, Lemma 4.6] and [13, Lemma 8], we obtain

$$I_\infty(w_n^1, z_n^1) = I(u_n, v_n) - I(u_0, v_0) + o_n(1) \tag{32}$$

and
$$I'_\infty(w_n^1, z_n^1) = I'(u_n, v_n) - I'(u_0, v_0) + o_n(1). \tag{33}$$

Then, we conclude from (32) and (33) that (w_n^1, z_n^1) is a $(PS)_{c_1}$ sequence for I_∞. Hence, by Lemma 3.1, there are sequences $(R_{n,1}) \subset \mathbb{R}$, $(x_{n,1}) \subset \mathbb{R}^N$, a nontrivial solution $(z_0^1, \zeta_0^1) \in D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$ for the system (P_∞) and a $(PS)_{c_2}$ sequence $(w_n^2, z_n^2) \subset D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$ for I_∞ such that

$$w_n^2(x) = w_n^1(x) - R_{n,1}^{(N-ps)/p} z_0^1(R_{n,1}(x - x_{n,1})) + o_n(1)$$

and
$$z_n^2(x) = z_n^1(x) - R_{n,1}^{(N-ps)/p} \zeta_0^1(R_{n,1}(x - x_{n,1})) + o_n(1).$$

If we define
$$\Phi_n^1(x) = R_{n,1}^{(ps-N)/p} w_n^1\Big(\frac{x}{R_{n,1}} + x_{n,1}\Big),$$
$$\Psi_n^1(x) = R_{n,1}^{(ps-N)/p} z_n^1\Big(\frac{x}{R_{n,1}} + x_{n,1}\Big)$$

and
$$\widetilde{w}_n^2(x) = R_{n,1}^{(ps-N)/p} w_n^2\Big(\frac{x}{R_{n,1}} + x_{n,1}\Big),$$
$$\widetilde{z}_n^2(x) = R_{n,1}^{(ps-N)/p} z_n^2\Big(\frac{x}{R_{n,1}} + x_{n,1}\Big),$$

we get
$$\widetilde{w}_n^2(x) = \Phi_n^1(x) - z_0^1(x) + o_n(1),$$
$$\widetilde{z}_n^2(x) = \Psi_n^1(x) - \zeta_0^1(x) + o_n(1)$$

and

$$\|\Phi_n^1\| = \|w_n^1\|, \quad \|\Psi_n^1\| = \|z_n^1\| \quad \text{and} \quad \int_{\mathbb{R}^N} K(\Phi_n^1, \Psi_n^1)dx = \int_{\mathbb{R}^N} K(w_n^1, z_n^1)dx. \tag{34}$$

Hence,
$$I_\infty(\Phi_n^1, \Psi_n^1) = I_\infty(w_n^1, z_n^1) \tag{35}$$
and
$$I'_\infty(\Phi_n^1, \Psi_n^1) \to 0 \ \text{ in } \ (D^{s,p}(\mathbb{R}^N))'. \tag{36}$$

From (35), (36) and from item (a) by lemma 2.1, we have that (Φ_n^1, Ψ_n^1) is a bounded sequence in $D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$ and, up to a subsequence,
$$\Phi_n^1 \rightharpoonup z_0^1, \quad \Psi_n^1 \rightharpoonup \zeta_0^1 \ \text{ in } \ D^{s,p}(\mathbb{R}^N). \tag{37}$$

Applying [18, Lemma 4.6] and [13, Lemma 8] again, we obtain
$$\begin{aligned} I_\infty(\widetilde{w}_n^2, \widetilde{z}_n^2) &= I_\infty(\Phi_n^1, \Psi_n^1) - I_\infty(z_0^1, \zeta_0^1) + o_n(1) \\ &= I(u_n, v_n) - I(u_0, v_0) - I_\infty(z_0^1, \zeta_0^1) + o_n(1). \end{aligned} \tag{38}$$
and
$$I'_\infty(\widetilde{w}_n^2, \widetilde{z}_n^2) = I'_\infty(\Phi_n^1, \Psi_n^1) - I'_\infty(z_0^1, \zeta_0^1) + o_n(1). \tag{39}$$

If $\widetilde{w}_n^2, \widetilde{z}_n^2 \to 0$ in $D^{s,p}(\mathbb{R}^N)$, the proof is over for $k = 1$, because in this case, we have
$$\|u_n\|^p + \|v_n\|^p \to \|u_0\|^p + \|v_0\|^p + \|z_0^1\|^p + \zeta_0^1\|^p.$$

Moreover, from continuity of I_∞, we get
$$I(u_n, v_n) \to I(u_0, v_0) + I_\infty(z_0^1, \zeta_0^1).$$

If $\widetilde{w}_n^2 \nrightarrow 0, \widetilde{z}_n^2 \nrightarrow 0$ in $D^{s,p}(\mathbb{R}^N)$, using (3) and (37) that $\widetilde{w}_n^2, \widetilde{z}_n^2 \rightharpoonup 0$ $D^{s,p}(\mathbb{R}^N)$, by (38) and (39), we conclude that $(\widetilde{w}_n^2, \widetilde{z}_n^2)$ is a $(PS)_{c_2}$ sequence for I_∞.

From Lemma 3.1 we conclude that there are sequences $(R_{n,2}) \subset \mathbb{R}$, $(x_{n,2}) \subset \mathbb{R}^N$, nontrivial solutions $(z_0^2, \zeta_0^2) \in D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$ of (S_∞) and a $(PS)_{c_3}$ sequence $(w_n^3, z_n^3) \subset D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$ for I_∞ such that
$$w_n^3(x) = \widetilde{w}_n^2(x) - R_{n,2}^{(N-ps)/p} z_0^2(R_{n,2}(x - x_{n,2})) + o_n(1),$$
$$z_n^3(x) = \widetilde{z}_n^2(x) - R_{n,2}^{(N-ps)/p} \zeta_0^2(R_{n,2}(x - x_{n,2})) + o_n(1).$$

If
$$\Phi_n^2(x) = R_{n,2}^{(ps-N)/p} \widetilde{w}_n^2\Big(\frac{x}{R_{n,2}} + x_{n,2}\Big),$$
$$\Psi_n^2(x) = R_{n,2}^{(ps-N)/p} \widetilde{z}_n^2\Big(\frac{x}{R_{n,2}} + x_{n,2}\Big)$$
and
$$\widetilde{w}_n^3(x) = R_{n,2}^{(ps-N)/p} w_n^3\Big(\frac{x}{R_{n,2}} + x_{n,2}\Big), \tag{40}$$
$$\widetilde{z}_n^3(x) = R_{n,2}^{(ps-N)/p} z_n^3\Big(\frac{x}{R_{n,2}} + x_{n,2}\Big), \tag{41}$$
we have that
$$\widetilde{w}_n^3(x) = \Phi_n^2(x) - z_0^2(x) + o_n(1), \tag{42}$$
$$\widetilde{z}_n^3(x) = \Psi_n^2(x) - \zeta_0^2(x) + o_n(1). \tag{43}$$

Arguing as before, we conclude

$$\|\widetilde{w}_n^3\|^p + \|\widetilde{z}_n^3\|^p = \tag{44}$$

$$= \|u_n\|^p + \|v_n\|^p - \|u_0\|^p - \|v_0\|^p - \|z_0^1\|^p - \|\zeta_0^1\|^p - \|z_0^2\|^p - \|\zeta_0^2\|^p + o_n(1),$$

$$I_\infty(\widetilde{w}_n^3, \widetilde{z}_n^3) = I(u_n, v_n) - I(u_0, v_0) - I_\infty(z_0^1, \zeta_0^1) - I_\infty(z_0^2, \zeta_0^2) + o_n(1) \tag{45}$$

and
$$I'_\infty(\widetilde{w}_n^3, \widetilde{z}_n^3) = I'_\infty(\Phi_n^2, \Psi_n^2) - I'_\infty(z_0^2, \zeta_0^2) + o_n(1). \tag{46}$$

If $\tilde{w}_n^3, \tilde{z}_n^3 \to 0$ in $D^{s,p}(\mathbb{R}^N)$, the proof is over with $k = 2$, because $\|\tilde{w}_n^3\|^2 \to 0$, $\|\tilde{z}_n^3\|^2 \to 0$ and from (44), we have

$$\|u_n\|^p + \|v_n\|^p \to \|u_0\|^p + \|v_0\|^p + \sum_{j=1}^{2}[\|z_0^j\|^p + \|\zeta_0^j\|^p].$$

Moreover, from the continuity of I_∞, we have $I_\infty(\tilde{z}_n^3) \to 0$. Now using (45) we get

$$I(u_n, v_n) \to I(u_0, v_0) + \sum_{j=1}^{2} I_\infty(z_0^j, \zeta_0^j).$$

If $\widetilde{w}_n^3, \widetilde{z}_n^3 \nrightarrow 0$ in $D^{s,p}(\mathbb{R}^N)$, we can repeat the same arguments before and we can find $(z_0^1, \zeta_0^1), (z_0^2, \zeta_0^2), ..., (z_0^{k-1}, \zeta_0^{k-1})$ nontrivial solutions for the system (S_∞) satisfying

$$\|\widetilde{w}_n^k\|^p + \|\widetilde{z}_n^k\|^p = \|u_n\|^p + \|v_n\|^p - \|u_0\|^p - \|v_0\|^p - \sum_{j=1}^{k-1}[\|z_0^j\|^p - \|\zeta_0^j\|^p] + o_n(1), \tag{47}$$

and
$$I_\infty(\widetilde{z}_n^k, \widetilde{z}_n^k) = I(u_n, v_n) - I(u_0, v_0) - \sum_{j=1}^{k-1} I_\infty(z_0^j, \zeta_0^j) + o_n(1). \tag{48}$$

From the definition of $\widetilde{S}_{K,s}$, we conclude that

$$\Big(\int_{\mathbb{R}^N} K(z_0^j, \zeta_0^j)dx\Big)^{2/2^*} \widetilde{S}_{K,s} \leq \|z_0^j\|^p + \|\zeta_0^j\|^p, \quad j = 1, 2, ..., k-1. \tag{49}$$

Since (z_0^j, ζ_0^j) is nontrivial solution of (S_∞), for all $j = 1, 2, ..., k-1$, we get

$$\|z_0^j\|^p + \|\zeta_0^j\|^p = \int_{\mathbb{R}^N} K(z_0^j, \zeta_0^j)dx.$$

Hence,
$$-\|z_0^j\|^p - \|\zeta_0^j\|^p \leq -\widetilde{S}_{K,s}^{N/2}, \quad j = 1, 2, ..., k-1. \tag{50}$$

From (47) and (50), we have

$$\|\widetilde{w}_n^k\|^p + \|\widetilde{z}_n^k\|^p = \|u_n\|^p + \|v_n\|^p - \|u_0\|^p - \|v_0\|^p - \sum_{j=1}^{k-1}[\|z_0^j\|^p + \|\zeta_0^j\|^p] + o_n(1)$$

$$\leq \|u_n\|^p + \|v_n\|^p - \|u_0\|^p - \|v_0\|^p - (k-1)\widetilde{S}_{K,s}^{N/2} + o_n(1). \tag{51}$$

Since (u_n, v_n) is bounded in $D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$, for k sufficient large, we conclude that $\widetilde{w}_n^k, \;\; \widetilde{z}_n^k \to 0$ in $D^{s,p}(\mathbb{R}^N)$ and the proof is complete. □

Corollary 3.3. *Let (u_n, v_n) be a $(PS)_c$ sequence for I with $c \in (0, \frac{s}{N}\widetilde{S}_{K,s}^{N/ps})$. Then, up to a subsequence, (u_n, v_n) strong converges in $D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$.*

Proof. We have that (u_n, v_n) is bounded in $D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$,

$$u_n \rightharpoonup u_0, \quad v_n \rightharpoonup v_0 \quad \text{in} \quad D^{s,p}(\mathbb{R}^N)$$

and by a density argument $I'(u_0, v_0) = 0$. Suppose, by contradiction, that

$$u_n \nrightarrow u_0, \quad v_n \nrightarrow v_0 \quad \text{in} \quad D^{s,p}(\mathbb{R}^N).$$

From Theorem 3.2, there are $k \in \mathbb{N}$ and nontrivial solutions $(z_0^1, \zeta_0^1), (z_0^2, \zeta_0^2), ..., (z_0^k, \zeta_0^k)$ of the system (S_∞) such that,

$$\|u_n\|^p + \|v_n\|^p \to \|u_0\|^p + \|v_0\|^p + \sum_{j=1}^{k} [|z_0^j\|^p + |\zeta_0^j\|^p]$$

and
$$I(u_n, v_n) \to I(u_0, v_0) + \sum_{j=1}^{k} I_\infty(z_0^j, \zeta_0^j).$$

Note that by (1) we have

$$\begin{aligned}
I(u_0, v_0) &= \frac{1}{p}\|u_0\|^p + \frac{1}{p}\|v_0\|^p + \frac{1}{p}\int_{\mathbb{R}^N} (A(u_0, v_0), (u_0, v_0))_{\mathbb{R}^2} dx - \frac{1}{p_s^*}\int_{\mathbb{R}^N} K(u_0, v_0) dx \\
&= \frac{1}{p}\|u_0\|^p + \frac{1}{p}\|v_0\|^p + \frac{1}{p}\left(\int_{\mathbb{R}^N} K(u_0, v_0) dx - \|u_0\|^p - \|v_0\|^p\right) - \frac{1}{p_s^*}\int_{\mathbb{R}^N} K(u_0, v_0) dx \\
&= \frac{s}{N}\int_{\mathbb{R}^N} K(u_0, v_0) dx \geq 0.
\end{aligned}$$

Then, $c = I(u_0, v_0) + \sum_{j=1}^{k} I_\infty(z_0^j, \zeta_0^j) \geq \sum_{j=1}^{k} I_\infty(z_0^j, \zeta_0^j) \geq \frac{ks}{N}\widetilde{S}_{K,s}^{N/ps} \geq \frac{s}{N}\widetilde{S}_{K,s}^{N/ps}$, which is a contradiction to $c \in \left(0, \frac{s}{N}\widetilde{S}_{K,s}^{N/ps}\right)$. □

Corollary 3.4. *The functional $I : D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N) \to \mathbb{R}$ satisfies the Palais-Smale condition in $\left(\frac{s}{N}\widetilde{S}_{K,s}^{N/ps}, \frac{2s}{N}\widetilde{S}_{K,s}^{N/ps}\right)$.*

Proof. Let (u_n, v_n) be a sequence in $D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$ that satisfies

$$I(u_n, v_n) \to c \quad \text{and} \quad I'(u_n, v_n) \to 0.$$

Since (u_n, v_n) is bounded in $D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$, up to a subsequence, we have

$$u_n \rightharpoonup u_0, \quad v_n \rightharpoonup v_0 \quad \text{in} \quad D^{s,p}(\mathbb{R}^N).$$

Moreover, $I(u_0, v_0) \geq 0$. Suppose, by contradiction, that

$$u_n \nrightarrow u_0, \quad v_n \nrightarrow v_0 \quad \text{in} \quad D^{s,p}(\mathbb{R}^N).$$

From Theorem 3.2, there are $k \in \mathbb{N}$ and nontrivial solutions (z_0^1, ζ_0^1), (z_0^2, ζ_0^2), $\dots$, (z_0^k, ζ_0^k) of the system (S_∞) such that

$$\|u_n\|^p + \|v_n\|^p \to \|u_0\|^p + \|v_0\|^p + \sum_{j=1}^{k} [\|z_0^j\|^p + \|\zeta_0^j\|^p]$$

and
$$I(u_n, v_n) \to I(u_0, v_0) + \sum_{j=1}^{k} I_\infty(z_0^j, \zeta_0^j) = c.$$

Since $I(u_0, v_0) \geq 0$, then $k = 1$ and z_0^1, ζ_0^1 cannot change of the sign. Hence,

$$c = I(u_0, v_0) + I_\infty(z_0^1, \zeta_0^1) = I(u_0, v_0) + \frac{s}{N} \widetilde{S}_{K,s}^{N/ps}.$$

From the definition of $\widetilde{S}_K$, $I'(u_0, v_0)(u_0, v_0) = 0$ and

$$I(u_0, v_0) = \frac{1}{N} \int_{\mathbb{R}^N} K(u_0, v_0) dx,$$

we have,
$$\frac{2s}{N} \widetilde{S}_{K,s}^{N/ps} \leq I(u_0, v_0) + \frac{s}{N} \widetilde{S}_{K,s}^{N/ps} = c,$$

which is a contradiction to $c \in \left(\frac{s}{N} \widetilde{S}_{K,s}^{N/ps}, \frac{2s}{N} \widetilde{S}_{K,s}^{N/ps} \right)$. □

Corollary 3.5. *Let $(u_n, v_n) \subset D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$ be a $(PS)_c$ sequence for I with $c \in (\frac{ks}{N} \widetilde{S}_{K,s}^{N/ps}, \frac{(k+1)s}{N} \widetilde{S}_{K,s}^{N/ps})$, where $k \in \mathbb{N}$. Then, the weak limit (u_0, v_0) of (u_n, v_n) is not trivial.*

Proof. Suppose, by contradiction, that $u_0, v_0 \equiv 0$. Since $c > 0$, then $u_n, v_n \nrightarrow 0$ in $D^{s,p}(\mathbb{R}^N)$. From Theorem 3.2, up to subsequence, we get

$$\|u_n\|^p + \|v_n\|^p \to \|u_0\|^p + \|v_0\|^p + \sum_{j=1}^{k} [\|z_0^j\|^p + \|\zeta_0^j\|^p] = \sum_{j=1}^{k} [\|z_0^j\|^p + \|\zeta_0^j\|^p]$$

and $I(u_n, v_n) \to I(u_0, v_0) + \sum_{j=1}^{k} I_\infty(z_0^j, \zeta_0^j) = \sum_{j=1}^{k} I_\infty(z_0^j, \zeta_0^j) = c \geq \dfrac{(k+1)s}{N} \widetilde{S}_{K,s}^{N/ps}$,

which is a contradiction to $c \in \left(\frac{k}{N} \widetilde{S}_{K,s}^{N/ps}, \frac{(k+1)s}{N} \widetilde{S}_{K,s}^{N/ps} \right)$. □

From now on we consider the functional $f : D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$f(u, v) = \|u\|^p + \|v\|^p + \int_{\mathbb{R}^N} (A(u, v), (u, v))_{\mathbb{R}^2} dx$$

and the manifold $\mathcal{M} \subset D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$ given by

$$\mathcal{M} = \left\{ (u, v) \in D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} K(u, v) dx = 1 \right\}.$$

The next results are direct consequence of the corollaries above.

Lemma 3.6. *Let $(u_n, v_n) \subset \mathcal{M}$ be a sequence that satisfies*

$$f(u_n, v_n) \to c \quad \text{and} \quad f'|_{\mathcal{M}}(u_n, v_n) \to 0.$$

Then, the sequence $(w_n, z_n) \subset D^{s,p}(\mathbb{R}^N)$ satisfies the following limits:

$$I(w_n, z_n) \to \frac{s}{N} c^{N/ps} \quad \text{and} \quad I'(w_n, z_n) \to 0,$$

where $(w_n, z_n) = (c^{(N-ps)/p^2 s} u_n, c^{(N-ps)/p^2 s} v_n)$.

Lemma 3.7. *Suppose we have a sequence $(u_n, v_n) \subset \mathcal{M}$ and $c \in (\widetilde{S}_{K,s}, p^{ps/N} \widetilde{S}_{K,s})$ such that*

$$f(u_n, v_n) \to c \quad \text{and} \quad f'|_{\mathcal{M}}(u_n, v_n) \to 0.$$

Then, up to a subsequence, $u_n \to u$, $v_n \to v$ in $D^{s,p}(\mathbb{R}^N)$, for some $u, v \in D^{s,p}(\mathbb{R}^N)$.

Corollary 3.8. *Suppose we have a sequence $(u_n, v_n) \subset \mathcal{M}$ and $c \in (\widetilde{S}_{K,s}, p^{ps/N} \widetilde{S}_{K,s})$ such that*

$$f(u_n, v_n) \to c \quad \text{and} \quad f'(u_n, v_n) \to 0.$$

Then I has a critical point $(u_0, v_0) \in D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$ with $I(u_0, v_0) = \frac{s}{N} c^{N/ps}$.

4. Existence of a positive solution to (S)

Now we recall some properties on the function Φ_{δ,x_0} given by in (2). Note that

$$(\Phi_{\delta,x_0}, \Phi_{\delta,x_0}) \in \Sigma = \left\{ (u, v) \in D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N); u, v \geq 0 \right\}. \tag{52}$$

Moreover, making a change of variable we can prove that

$$\Phi_{\delta,x_0} \in L^q(\mathbb{R}^N) \quad \text{for} \quad q \in \left(\frac{N}{N-ps}, p_s^* \right], \quad \forall \delta > 0 \text{ and } \forall x_0 \in \mathbb{R}^N. \tag{53}$$

The proof of the next result can be seen in [1, Lemma 4].

Lemma 4.1. *For each $x_0 \in \mathbb{R}^N$, we have*

(i) $\|\Phi_{\delta,x_0}\|_{H^{1,\infty}(\mathbb{R}^N)} \to 0$ *as* $\delta \to +\infty$,

(ii) $|\Phi_{\delta,x_0}|_q \to 0$ *as* $\delta \to 0$, $\forall q \in \left(\frac{N}{N-ps}, p_s^* \right)$,

(iii) $|\Phi_{\delta,x_0}|_q \to +\infty$ *as* $\delta \to +\infty$, $\forall q \in \left(\frac{N}{N-ps}, p_s^* \right)$.

The proof of next result can be consulted in [1, Lemma 5].

Lemma 4.2. *For each $\varepsilon > 0$, we have*

$$\int\int_{\mathbb{R}^{2N} \setminus B_\varepsilon^2} < L(\Phi_{\delta,0}), \Phi_{\delta,0} > dxdy \to 0 \quad \text{as} \quad \delta \to 0.$$

4.1. Technical lemmas

Lemma 4.3. *Suppose that $a, b, c \in L^q(\mathbb{R}^N)$, $\forall q \in [p_1, p_2]$, where $1 < p_1 < \frac{N}{ps} < p_2$ with $p_2 < \frac{3(p-1)}{p^2 s - 3}$ if $N = 3$. Then, for each $\varepsilon > 0$, there are $\underline{\delta} = \underline{\delta}(\varepsilon) > 0$ and $\bar{\delta} = \bar{\delta}(\varepsilon) > 0$ such that*

$$\sup_{y \in \mathbb{R}^N} f(\alpha_o \Phi_{\delta,x_0}, \beta_o \Phi_{\delta,x_0}) < \widetilde{S}_{K,s} + \varepsilon, \quad \delta \in (0, \underline{\delta}] \cup [\bar{\delta}, \infty).$$

Proof. Consider $x_0 \in \mathbb{R}^N$, $q \in \left(\frac{N}{ps}, p_2\right]$ and $t \in (1, +\infty)$ with $\frac{1}{q} + \frac{1}{t} = 1$. A direct calculations leads to

$$\frac{N}{N - ps} < pt < p_s^*. \tag{54}$$

Since $\Phi_{\delta,x_0} \in L^d(\mathbb{R}^N)$, $\forall d \in \left(\frac{N}{N-ps}, p_s^*\right)$, we get $|\Phi_{\delta,x_0}|^p \in L^t(\mathbb{R}^N)$. Then, using the Hölder inequality and change of variable, we obtain

$$\int_{\mathbb{R}^N} a(x)|\Phi_{\delta,x_0}|^p dx \le |a|_q |\Phi_{\delta,0}|_{pt}^p, \quad \forall y \in \mathbb{R}^N,$$

$$\int_{\mathbb{R}^N} b(x)|\Phi_{\delta,x_0}|^p dx \le |b|_q |\Phi_{\delta,0}|_{pt}^p, \quad \forall y \in \mathbb{R}^N,$$

and

$$\int_{\mathbb{R}^N} c(x)|\Phi_{\delta,x_0}|^p dx \le |c|_q |\Phi_{\delta,0}|_{pt}^p, \quad \forall y \in \mathbb{R}^N.$$

From item (ii) of Lemma 4.1, given $\varepsilon > 0$, there exists $\underline{\delta} = \underline{\delta}(\varepsilon) > 0$ such that

$$\sup_{y \in \mathbb{R}^N} f(\alpha_o \Phi_{\delta,x_0}, \beta_o \Phi_{\delta,x_0}) \le \widetilde{S}_{K,s} + \frac{4\varepsilon}{5} < \widetilde{S}_{K,s} + \varepsilon, \quad \forall \delta \in (0, \underline{\delta}].$$

Suppose that $q \in \left[p_1, \frac{N}{ps}\right)$ with $t \in (1, +\infty)$ and $\frac{1}{q} + \frac{1}{t} = 1$. Note that $pt - p_s^* > 0$ and for $\delta > 1$,

$$|\Phi_{\delta,x_0}| \in L^\infty(\mathbb{R}^N) \tag{55}$$

and $|\Phi_{\delta,x_0}|^{p_s^*} \in L^1(\mathbb{R}^N)$. Then, $|\Phi_{\delta,x_0}|^p \in L^t(\mathbb{R}^N)$. Using the Hölder inequality with q and t, we get

$$\begin{aligned}
&\alpha_o^p \int_{\mathbb{R}^N} a(x)|\Phi_{\delta,x_0}|^p dx \le \alpha_o^p |a|_q \left(\int_{\mathbb{R}^N} |\Phi_{\delta,0}|^{pt} dz \right)^{1/t} \\
&= \alpha_o^p |a|_q \left(\int_{\mathbb{R}^N} |\Phi_{\delta,0}|^{p_s^*} |\Phi_{\delta,0}|^{pt - p_s^*} dz \right)^{1/t} \\
&\le \alpha_o^p |a|_q |\Phi_{\delta,0}|_\infty^{(pt-p_s^*)/t} \left(\int_{\mathbb{R}^N} |\Phi_{\delta,0}|^{p_s^*} dz \right)^{1/t} \le \alpha_o^p |a|_q |\Phi_{\delta,0}|_\infty^{(pt-p_s^*)/t} \\
&\le \alpha_o^p |a|_q C^{(pt-p_s^*)/t} \delta^{\left(\frac{p'(N-ps)}{p} - \frac{N-ps}{p-1}\right)((pt-p_s^*)/t)}, \quad \forall x_0 \in \mathbb{R}^N.
\end{aligned}$$

Then, given $\varepsilon > 0$, there is $\bar{\delta} = \bar{\delta}(\varepsilon) > 1$ such that

$$\delta^{(\frac{p'(N-ps)}{p} - \frac{N-ps}{p-1})((pt-p_s^*)/t)} < \frac{\varepsilon}{5\alpha_o^p |a|_q C^{(pt-p_s^*)/t}}, \quad \forall \delta \in [\bar{\delta}, \infty).$$

Arguing as the same way, we have for all $x_0 \in \mathbb{R}^N$

$$\beta_o^p \int_{\mathbb{R}^N} b(x)|\Phi_{\delta,x_0}|^p dx \leq \beta_o^p |b|_q C^{(pt-p_s^*)/t} \delta^{(\frac{p'(N-ps)}{p} - \frac{N-ps}{p-1})((pt-p_s^*)/t)},$$

$$\alpha_o \beta_o^{p-1} \int_{\mathbb{R}^N} c(x)|\Phi_{\delta,x_0}|^p dx \leq \alpha_o \beta_o^{p-1} |c|_q C^{(pt-p_s^*)/t} \delta^{(\frac{p'(N-ps)}{p} - \frac{N-ps}{p-1})((pt-p_s^*)/t)},$$

and

$$\alpha_o^{p-1} \beta_o \int_{\mathbb{R}^N} c(x)|\Phi_{\delta,x_0}|^p dx \leq \alpha_o^{p-1} \beta_o |c|_q C^{(pt-p_s^*)/t} \delta^{(\frac{p'(N-ps)}{p} - \frac{N-ps}{p-1})((pt-p_s^*)/t)}.$$

Then

$$\begin{aligned} f(\alpha_o\Phi_{\delta,x_0}, \beta_o\Phi_{\delta,x_0}) &= \widetilde{S}_{K,s} + \alpha_o^p \int_{\mathbb{R}^N} a(x)|\Phi_{\delta,x_0}|^p dx + \beta_o^p \int_{\mathbb{R}^N} b(x)|\Phi_{\delta,x_0}|^p dx \\ &\quad + \alpha_o^{p-1}\beta_o \int_{\mathbb{R}^N} c(x)|\Phi_{\delta,x_0}|^p dx + \alpha_o\beta_o^{p-1} \int_{\mathbb{R}^N} c(x)|\Phi_{\delta,x_0}|^p dx \\ &\leq \widetilde{S}_{K,s} + \alpha_o^p \sup_{y\in\mathbb{R}^N} \int_{\mathbb{R}^N} a(x)|\Phi_{\delta,x_0}|^p dx + \beta_o^p \sup_{y\in\mathbb{R}^N} \int_{\mathbb{R}^N} b(x)|\Phi_{\delta,x_0}|^p dx \\ &\quad + \alpha_o^{p-1}\beta_o \sup_{y\in\mathbb{R}^N} \int_{\mathbb{R}^N} c(x)|\Phi_{\delta,x_0}|^p dx + \alpha_o\beta_o^{p-1} \sup_{y\in\mathbb{R}^N} \int_{\mathbb{R}^N} c(x)|\Phi_{\delta,x_0}|^p dx \\ &\leq \widetilde{S}_{K,s} + \frac{4\varepsilon}{5} < \widetilde{S}_{K,s} + \varepsilon, \quad \forall x_0 \in \mathbb{R}^N \quad \text{and} \quad \forall \delta \in [\bar{\delta}, \infty). \end{aligned}$$ □

Lemma 4.4. *Suppose that (A_3) is true. Then,*

$$\sup_{\substack{x_0\in\mathbb{R}^N \\ \delta\in(0,+\infty)}} f(\alpha_o\Phi_{\delta,x_0}, \beta_o\Phi_{\delta,x_0}) < p^{ps/N}\widetilde{S}_{K,s}.$$

Proof. Using the Hölder inequality with N/ps and $N/(N-ps)$, we get

$$\begin{aligned} &f(\alpha_o\Phi_{\delta,x_0}, \beta_o\Phi_{\delta,x_0}) \\ &\leq \widetilde{S}_K + \alpha_o^p |a|_{L^{N/ps}(\mathbb{R}^N)} + \beta_o^p |b|_{L^{N/ps}(\mathbb{R}^N)} + \alpha_o^{p-1}\beta_o |c|_{L^{N/ps}(\mathbb{R}^N)} + \alpha_o\beta_o^{p-1} |c|_{L^{N/ps}(\mathbb{R}^N)}. \end{aligned}$$

From (A_3) we conclude

$$\sup_{\substack{x_0\in\mathbb{R}^N \\ \delta\in(0,\infty)}} f(\alpha_o\Phi_{\delta,x_0}, \beta_o\Phi_{\delta,x_0}) < \widetilde{S}_{K,s} + \widetilde{S}_{K,s}(p^{ps/N} - 1) = p^{ps/N}\widetilde{S}_{K,s}.$$ □

Consider now the function $\xi(x) = \begin{cases} 0, & \text{if } |x| < 1, \\ 1, & \text{if } |x| \geq 1, \end{cases}$ and define the function $g : D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N) \to \mathbb{R}^{N+1}$ by

$$g(u,v) = \frac{1}{\widetilde{S}_{K,s}} \int\int_{\mathbb{R}^{2N}} \left(\frac{x}{|x|}, \xi(x)\right)[\alpha_o^p < L(u), u > +\beta_o^p < L(v), v >]dxdy$$

$$= (h(u,v), \gamma(u,v)),$$

where
$$h(u,v) = \frac{1}{\widetilde{S}_{K,s}} \int\int_{\mathbb{R}^{2N}} \frac{x}{|x|}[\alpha_o^p < L(u), u > +\beta_o^p < L(v), v >]dxdy$$

and
$$\gamma(u,v) = \frac{1}{\widetilde{S}_{K,s}} \int\int_{\mathbb{R}^{2N}} \xi(x)[\alpha_o^p < L(u), u > +\beta_o^p < L(v), v >]dxdy.$$

Lemma 4.5. *If* $|x_0| \geq \frac{1}{2}$, *then*

$$h(\Phi_{\delta,x_0}, \Phi_{\delta,x_0}) = \frac{x_0}{|x_0|} + o_\delta(1) \quad \text{when} \quad \delta \to 0.$$

Proof. Given $\varepsilon > 0$, from Lemma 4.2, there is $\hat{\delta} > 0$ such that

$$\int\int_{\mathbb{R}^{2N}\setminus B_\varepsilon^2(x_0)} < L(\Phi_{\delta,x_0}), \Phi_{\delta,x_0} > dxdy = \int\int_{\mathbb{R}^{2N}\setminus B_\varepsilon^2(0)} < L(\Phi_{\delta,x_0}), \Phi_{\delta,x_0} > dzdy < \varepsilon,$$

for all $\delta \in (0, \hat{\delta})$. Then,

$$\left| h(\Phi_{\delta,x_0}, \Phi_{\delta,x_0}) - \frac{\alpha_o^p + \beta_0^p}{\widetilde{S}_{K,s}} \int\int_{B_\varepsilon^2(x_0)} \frac{x}{|x|} < L(\Phi_{\delta,x_0}), \Phi_{\delta,x_0} > dxdy \right| \tag{56}$$
$$\leq \frac{\alpha_o^p + \beta_0^p}{\widetilde{S}_{K,s}} \int\int_{\mathbb{R}^{2N}\setminus B_\varepsilon^2(x_0)} < L(\Phi_{\delta,x_0}), \Phi_{\delta,x_0} > dxdy < \varepsilon, \qquad \forall \delta \in (0, \hat{\delta}).$$

Note that

$$\left| \frac{x_0}{|x_0|} - \frac{\alpha_o^p + \beta_0^p}{\widetilde{S}_{K,s}} \int\int_{B_\varepsilon^2(x_0)} \frac{x}{|x|} < L(\Phi_{\delta,x_0}), \Phi_{\delta,x_0} > dxdy \right| < 4\varepsilon + \varepsilon = \widetilde{C}\varepsilon, \tag{57}$$

for all $\delta \in (0, \hat{\delta})$. From (56) and (57), we have for all $\delta \in (0, \hat{\delta})$

$$\begin{aligned}
&\left| h(\Phi_{\delta,x_0}, \Phi_{\delta,x_0}) - \frac{x_0}{|x_0|} \right| \\
&= \left| h(\Phi_{\delta,x_0}, \Phi_{\delta,x_0}) - \frac{\alpha_o^p + \beta_0^p}{\widetilde{S}_{K,s}} \int\int_{B_\varepsilon^2(x_0)} \frac{x}{|x|} < L(\Phi_{\delta,x_0}), \Phi_{\delta,x_0} > dxdy - \frac{x_0}{|x_0|} \right. \\
&\qquad \left. + \frac{\alpha_o^p + \beta_0^p}{\widetilde{S}_{K,s}} \int\int_{B_\varepsilon^2(x_0)} \frac{x}{|x|} < L(\Phi_{\delta,x_0}), \Phi_{\delta,x_0} > dxdy \right| \\
&\leq \left| h(\Phi_{\delta,x_0}, \Phi_{\delta,x_0}) - \frac{\alpha_o^p + \beta_0^p}{\widetilde{S}_{K,s}} \int\int_{B_\varepsilon^2(x_0)} \frac{x}{|x|} < L(\Phi_{\delta,x_0}), \Phi_{\delta,x_0} > dxdy \right| \\
&\qquad + \left| \frac{\alpha_o^p + \beta_0^p}{\widetilde{S}_{K,s}} \int\int_{B_\varepsilon^2(x_0)} \frac{x}{|x|} < L(\Phi_{\delta,x_0}), \Phi_{\delta,x_0} > dxdy - \frac{x_0}{|x_0|} \right| \\
&< \varepsilon + \widetilde{C}\varepsilon = (1 + \widetilde{C})\varepsilon.
\end{aligned}$$

□

Lemma 4.6. *Suppose that $a, b, c \in L^q(\mathbb{R}^N)$, $\forall q \in [p_1, p_2]$, where $1 < p_1 < \frac{N}{ps} < p_2$ with $p_2 < \frac{3(p-1)}{p^2 s - 3}$ if $N = 3$. Then, for every $\delta > 0$, we have*

$$\lim_{|x_0| \to \infty} f(\alpha_o \Phi_{\delta,x_0}, \beta_o \Phi_{\delta,x_0}) = \widetilde{S}_{K,s}.$$

Proof. Since

$$f(\alpha_o \Phi_{\delta,x_0}, \beta_o \Phi_{\delta,x_0}) = \widetilde{S}_{K,s} + \alpha_o^p \int_{\mathbb{R}^N} a(x)|\Phi_{\delta,x_0}|^p dx + \beta_o^p \int_{\mathbb{R}^N} b(x)|\Phi_{\delta,x_0}|^p dx$$
$$+ \alpha_o^{p-1}\beta_o \int_{\mathbb{R}^N} c(x)|\Phi_{\delta,x_0}|^p dx + \alpha_o \beta_o^{p-1} \int_{\mathbb{R}^N} c(x)|\Phi_{\delta,x_0}|^p dx,$$

we need to prove that

$$\lim_{|x_0| \to \infty} \int_{\mathbb{R}^N} a(x)|\Phi_{\delta,x_0}|^p dx = 0, \quad \forall \delta > 0, \tag{58}$$

$$\lim_{|x_0| \to \infty} \int_{\mathbb{R}^N} b(x)|\Phi_{\delta,x_0}|^p dx = 0, \quad \forall \delta > 0, \tag{59}$$

and

$$\lim_{|x_0| \to \infty} \int_{\mathbb{R}^N} c(x)|\Phi_{\delta,x_0}|^p dx = 0, \quad \forall \delta > 0. \tag{60}$$

Note that given $\varepsilon > 0$, there is $k_0 > 0$ such that for all $\rho > k_0$

$$\left(\int_{\mathbb{R}^N \setminus B_\rho(0)} a(x)^{N/ps} dx \right)^{ps/N} < \varepsilon, \tag{61}$$

and

$$\left(\int_{\mathbb{R}^N \setminus B_\rho(x_0)} |\Phi_{\delta,x_0}|^{p_s^*} dx \right)^{1/p_s^*} = \left(\int_{\mathbb{R}^N \setminus B_\rho(0)} |\Phi_{\delta,0}|^{p_s^*} d\bar{x} \right)^{1/p_s^*} < \varepsilon. \tag{62}$$

Consider

$$k_0 < 2\rho < |x_0| \quad (\rho \text{ fixed}) \tag{63}$$

and note that

$$B_\rho(0) \cap B_\rho(x_0) = \emptyset. \tag{64}$$

Using the Hölder inequality with N/ps and $N/(N-ps)$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} a(x)|\Phi_{\delta,x_0}|^p dx \\
&\le \left(\int_{\mathbb{R}^N \setminus (B_\rho(0) \cup B_\rho(x_0))} a^{N/ps} dx \right)^{ps/N} \left(\int_{\mathbb{R}^N \setminus (B_\rho(0) \cup B_\rho(x_0))} |\Phi_{\delta,x_0}|^{p_s^*} dx \right)^{(N-ps)/N} \\
&\quad + \left(\int_{B_\rho(0)} a^{N/ps} dx \right)^{ps/N} \left(\int_{B_\rho(0)} |\Phi_{\delta,x_0}|^{p_s^*} dx \right)^{(N-ps)/N} \\
&\quad + \left(\int_{B_\rho(x_0)} a^{N/ps} dx \right)^{ps/N} \left(\int_{B_\rho(x_0)} |\Phi_{\delta,x_0}|^{p_s^*} dx \right)^{(N-ps)/N} \\
&\le \left(\int_{\mathbb{R}^N \setminus B_\rho(0)} a^{N/ps} dx \right)^{ps/N} \left(\int_{\mathbb{R}^N \setminus B_\rho(x_0)} |\Phi_{\delta,y}|^{p_s^*} dx \right)^{(N-ps)/N} \\
&\quad + \left(\int_{\mathbb{R}^N} a^{N/ps} dx \right)^{ps/N} \left(\int_{\mathbb{R}^N \setminus B_\rho(x_0)} |\Phi_{\delta,x_0}|^{p_s^*} dx \right)^{(N-ps)/N}
\end{aligned}$$

$$+\left(\int_{\mathbb{R}^N\setminus B_\rho(0)} a^{N/ps}dx\right)^{ps/N}\left(\int_{\mathbb{R}^N}|\Phi_{\delta,x_0}|^{p_0^*}dx\right)^{(N-ps)/N}$$

$$<\varepsilon\varepsilon^p+|a|_{N/ps}\varepsilon^p+\varepsilon.$$

Arguing along the same way as for the term (60) completes the proof. $\square$

Now we define the set $\Im=\left\{(u,v)\in\mathcal{M}; h(u,v)=\left(0,\frac{1}{2}\right)\right\}$ and note that from Lemma 4.2 and Lemma 4.1(i) there is $\delta_1>0$ such that $(\Phi_{\delta_1,0},\Phi_{\delta_1,0})\in\Im$.

Lemma 4.7. *The number* $c_0=\inf_{u\in\Im} f(u,v)$ *satisfies the inequality* $c_0>\widetilde{S}_{K,s}$.

Proof. Since $\Im\subset\mathcal{M}$, we have $\widetilde{S}_{K,s}\le c_0$.

Suppose, by contradiction, that $\widetilde{S}_{K,s}=c_0$. By Ekeland variational principle [25], there exists $(u_n,v_n)\subset D^{s,p}(\mathbb{R}^N)\times D^{s,p}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N}K(u_n,v_n)dx=1,\quad h(u_n,v_n)\to\left(0,\frac{1}{2}\right) \tag{65}$$

and

$$f(u_n,v_n)\to\widetilde{S}_{K,s},\quad f'|_{\mathcal{M}}(u_n,v_n)\to 0. \tag{66}$$

Then, (u_n,v_n) is bounded in $D^{s,p}(\mathbb{R}^N)\times D^{s,p}(\mathbb{R}^N)$ and, up to a subsequence, $u_n\rightharpoonup u_0$, $v_n\rightharpoonup v_0$ in $D^{s,p}(\mathbb{R}^N)$.

If $w_n=S^{(N-ps)/p^2s}u_n$, $z_n=S^{(N-ps)/p^2s}v_n$ and $w_0=S^{(N-ps)/p^2s}u_0$, $z_0=S^{(N-ps)/p^2s}v_0$, we have that $w_n\rightharpoonup w_0$, $z_n\rightharpoonup z_0$ in $D^{s,p}(\mathbb{R}^N)$. Moreover, from (66) and Lemma 3.6, we get

$$I(w_n,z_n)\to\frac{s}{N}\widetilde{S}_{K,s}^{N/ps}\quad\text{and}\quad I'(w_n,z_n)\to 0.$$

We are going to show that $(w_0,z_0)\equiv(0,0)$. Note that

$$u_n\nrightarrow u_0,\quad u_n\nrightarrow u_0\ \text{ in }\ D^{s,p}(\mathbb{R}^N), \tag{67}$$

since otherwise, $(u_0,v_0)\in\mathcal{M}$ implies $u_0\neq 0$, $v_0\neq 0$. Then,

$$\begin{aligned}\widetilde{S}_{K,s}\ &\le\ \frac{(\|u_0\|^p+\|v_0\|^p)}{\left(\int_{\mathbb{R}^N}K(u_0,v_0)dx\right)^{p/p_0^*}}=\|u_0\|^p+\|v_0\|^p\\ &<\ \|u_0\|^p+\|v_0\|^p+\int_{\mathbb{R}^N}(A(u_0,v_0),(u_0,v_0))_{\mathbb{R}^2}dx=\widetilde{S}_{K,s},\end{aligned}$$

which is absurd. Hence, $w_n\nrightarrow w_0$, $z_n\nrightarrow z_0$ in $D^{s,p}(\mathbb{R}^N)$ and, since (w_n,z_n) is a $(PS)_c$ sequence for I, by Theorem 3.2 we obtain that

$$I(w_n.z_n)\to I(w_0,z_0)+\sum_{j=1}^{k}I_\infty(z_0^j,\zeta_0^j)=\frac{s}{N}\widetilde{S}_{K,s}^{N/ps}.$$

Since $I'_\infty(z_0^j,\zeta_0^j)=0$, we have
$$I(w_0,z_0)=0, \tag{68}$$
$$k=1, \tag{69}$$
$$z_0^1,\zeta_0^1>0, \tag{70}$$
$$I(w_0,z_0)=\frac{s}{N}\int_{\mathbb{R}^N}K(w_0,z_0)dx$$

and from (68), we conclude that $w_0\equiv 0$ and $z_0\equiv 0$. Then, (w_n,z_n) is a $(PS)_c$ sequence for I such that $w_n\rightharpoonup 0$, $z_n\rightharpoonup 0$ and $w_n\nrightarrow 0$, $z_n\nrightarrow 0$. Note that

$$\int_{\mathbb{R}^N}a(x)|w_n|^p dx=o_n(1),\quad \int_{\mathbb{R}^N}b(x)|z_n|^p dx=o_n(1),$$
$$\int_{\mathbb{R}^N}c(x)|w_n|^{p-2}w_nz_ndx=o_n(1)\ \text{ and }\ \int_{\mathbb{R}^N}c(x)|z_n|^{p-2}z_nw_ndx=o_n(1).$$

Then,

$$\frac{s}{N}\widetilde{S}_K^{N/2}+o_n(1)=I(w_n,z_n)=I_\infty(w_n,z_n)+\int_{\mathbb{R}^N}a(x)|w_n|^p dx+\int_{\mathbb{R}^N}b(x)|z_n|^p dx \tag{71}$$
$$+\int_{\mathbb{R}^N}c(x)|w_n|^{p-2}w_nz_ndx+\int_{\mathbb{R}^N}c(x)|z_n|^{p-2}z_nw_ndx=I_\infty(v_n)+o_n(1)$$

and
$$\|I'_\infty(w_n,z_n)\|_{D'}\le\|I'(w_n,z_n)\|_{D'}+o_n(1). \tag{72}$$

From (71) and (72) we conclude that (w_n,z_n) is a $(PS)_c$ sequence for I_∞ and by Lemma 3.1, there are sequences $(R_n)\subset\mathbb{R}$, $(x_n)\subset\mathbb{R}^N$, a nontrivial solution (z_0^1,ζ_0^1) of (S_∞) and a sequence(Φ_n,Ψ_n) a $(PS)_c$ for I_∞ such that

$$w_n(x)=\Phi_n(x)+R_n^{(N-ps)/p}z_0^1(R_n(x-x_n))+o_n(1) \tag{73}$$

and
$$z_n(x)=\Psi_n(x)+R_n^{(N-ps)/p}\zeta_0^1(R_n(x-x_n))+o_n.$$

Note that if we define

$$\widetilde{\Phi}_n(x)=R_n^{(N-ps)/p}z_0^1(R_n(x-x_n)),\quad \widetilde{\Psi}_n(x)=R_n^{(N-ps)/p}\zeta_0^1(R_n(x-x_n)),$$

making change of variable, we have for all $n\in\mathbb{N}$,

$$I'_\infty(\widetilde{\Phi}_n,\widetilde{\Psi}_n)(\varphi,\psi)=I'_\infty(z_0^1,\zeta_0^1)(\varphi_n,\psi_n)=0,\quad \forall(\varphi,\psi)\in D^{s,p}(\mathbb{R}^N)\times D^{s,p}(\mathbb{R}^N),$$

i.e., $(\widetilde{\Phi}_n,\widetilde{\Psi}_n)$ is a solution of (S_∞), for all $n\in\mathbb{N}$.

Moreover, from the definition of $(\widetilde{\Phi}_n,\widetilde{\Psi}_n)$ and by (70), we get

$$\widetilde{\Phi}_n(x)=\widetilde{\Psi}_n(x)=C\left(\frac{\delta_n^{p'}}{\delta_n^{p'}+|x-(x_0)_n|^{p'}}\right)^{(N-ps)/p}.$$

By (73), we obtain

$$u_n(x)=\widehat{\Phi}_n(x)+\Phi_{\delta_n,(x_0)_n}(x)+o_n(1),\quad v_n(x)=\widehat{\Psi}_n(x)+\Phi_{\delta_n,(x_0)_n}(x)+o_n(1)$$

where
$$\widehat{\Phi}_n(x)=\frac{1}{\widetilde{S}_K^{(N-ps)/p^2s}}\Phi_n(x),\quad \widehat{\Psi}_n(x)=\frac{1}{\widetilde{S}_K^{(N-ps)/p^2s}}\Psi_n(x).$$

Using (69), we derive that $\Phi_n \to 0$, $\Psi_n \to 0$ in $D^{s,p}(\mathbb{R}^N)$, which implies that $\widehat{\Phi}_n \to 0$, $\widehat{\Psi}_n \to 0$ in $D^{s,p}(\mathbb{R}^N)$. From (65) we have

$$\left(0, \frac{1}{2}\right) + o_n(1) = h(u_n, v_n) = g(\widehat{\Phi}_n(x) + \Phi_{\delta_n,(x_0)_n}(x), \widehat{\Psi}_n(x) + \Phi_{\delta_n,(x_0)_n}(x)) + o_n(1))$$
$$= g(\Phi_{\delta_n,(x_0)_n}, \Phi_{\delta_n,(x_0)_n}),$$

which implies (i) $h(\Phi_{\delta_n,(x_0)_n}, \Phi_{\delta_n,(x_0)_n}) \to 0$

and (ii) $\gamma(\Phi_{\delta_n,(x_0)_n}, \Phi_{\delta_n,(x_0)_n}) \to \frac{1}{2}$.

Passing to a subsequence, one of these cases can occur.

(a) $\delta_n \to +\infty$ as $n \to +\infty$;

(b) $\delta_n \to \tilde{\delta} \neq 0$ as $n \to +\infty$;

(c) $\delta_n \to 0$ and $(x_0)_n \to \tilde{x}_0$ as $n \to +\infty$ with $|\tilde{x}_0| < \frac{1}{2}$;

(d) $\delta_n \to 0$ as $n \to +\infty$ and $|(x_0)_n| \geq \frac{1}{2}$ for n sufficient large.

Suppose that (a) is true. Then,

$$\gamma(\Phi_{\delta_n,(x_0)_n}, \Phi_{\delta_n,(x_0)_n}) = 1 - \frac{\alpha_o^p + \beta_0^p}{\widetilde{S}_{K,s}} \int\int_{B_1^2(0)} < L(\Phi_{\delta_n,(x_0)_n}), \Phi_{\delta_n,(x_0)_n} > dxdy,$$

which implies by Lemma 4.1,

$$|\gamma(\Phi_{\delta_n,(x_0)_n}, \Phi_{\delta_n,(x_0)_n}) - 1| = \frac{\alpha_o^p + \beta_0^p}{\widetilde{S}_{K,s}} \int\int_{B_1^2(0)} < L(\Phi_{\delta_n,(x_0)_n}), \Phi_{\delta_n,(x_0)_n} > dxdy$$
$$\leq \frac{\alpha_o^p + \beta_0^p}{\widetilde{S}_{K,s}} \int\int_{\mathbb{R}^{2N}} < L(\Phi_{\delta_n,(x_0)_n}), \Phi_{\delta_n,(x_0)_n} > dxdy = o_n(1),$$

which contradicts (ii).

Suppose that (b) is true. In this case we can suppose that $|(x_0)_n| \to +\infty$, because if $(x_0)_n \to \tilde{x}_0$, we can prove that

$$\Phi_{\delta_n,(x_0)_n} \to \Phi_{\tilde{\delta},\tilde{x}_0} \text{ in } D^{s,p}(\mathbb{R}^N).$$

Since $\widehat{\Phi}_n, \widehat{\Psi}_n \to 0$ in $D^{s,p}(\mathbb{R}^N)$ and

$$u_n = \widehat{\Phi}_n + \Phi_{\delta_n,(x_0)_n} + o_n(1), \;\; v_n = \widehat{\Psi}_n + \Phi_{\delta_n,(x_0)_n} + o_n(1),$$

we have that (u_n, v_n) converges in $D^{s,p}(\mathbb{R}^N) \times D^{s,p}(\mathbb{R}^N)$. But this is a contradiction to (67). Then,

$$\gamma(\Phi_{\delta_n,(x_0)_n}, \Phi_{\delta_n,(x_0)_n}) = \frac{\alpha_o^p + \beta_0^p}{\widetilde{S}_{K,s}} \int\int_{\mathbb{R}^{2N}} \xi(x) < L(\Phi_{\delta_n,(x_0)_n}), \Phi_{\delta_n,(x_0)_n} > dxdy$$
$$= \frac{\alpha_o^p + \beta_0^p}{\widetilde{S}_{K,s}} \int\int_{\mathbb{R}^{2N} \setminus B_1^2(0)} < L(\Phi_{\delta_n,(x_0)_n}), \Phi_{\delta_n,(x_0)_n} > dxdy$$
$$= 1 - \frac{\alpha_o^p + \beta_0^p}{\widetilde{S}_{K,s}} \int\int_{B_1^2(-(x_0)_n)} < L(\Phi_{\delta_n,0}), \Phi_{\delta_n,0} > d\bar{x}d\bar{y}. \tag{74}$$

From Lebesgue's Theorem we can prove that

$$\int\int_{B_1^2(-(x_0)_n)} < L(\Phi_{\delta_n,0}), \Phi_{\delta_n,0} > d\bar{x}d\bar{y} \to 0$$

and from (74), we obtain

$$\gamma(\Phi_{\delta_n,(x_0)_n}, \Phi_{\delta_n,(x_0)_n}) \to 1 \quad \text{as} \quad n \to +\infty,$$

which is a contradiction to (ii).

Suppose that (c) is true. We have

$$\begin{aligned}
\gamma(\Phi_{\delta_n,(x_0)_n}, \Phi_{\delta_n,(x_0)_n}) &= \frac{\alpha_o^p + \beta_0^p}{\widetilde{S}_{K,s}} \int\int_{\mathbb{R}^{2N}} \xi(x) < L(\Phi_{\delta_n,(x_0)_n}), \Phi_{\delta_n,(x_0)_n} > dxdy \\
&= \frac{\alpha_o^p + \beta_0^p}{\widetilde{S}_{K,s}} \int\int_{\mathbb{R}^{2N}\setminus B_1^2(0)} < L(\Phi_{\delta_n,(x_0)_n}), \Phi_{\delta_n,(x_0)_n} > dxdy \\
&= \frac{\alpha_o^p + \beta_0^p}{\widetilde{S}_{K,s}} \int\int_{\mathbb{R}^{2N}} < L(\Phi_{\delta_n,(x_0)_n}), \Phi_{\delta_n,(x_0)_n} > dxdy \qquad (75)\\
&\quad - \frac{\alpha_o^p + \beta_0^p}{\widetilde{S}_{K,s}} \int\int_{B_1^2(-(x_0)_n)} < L(\Phi_{\delta_n,0}), \Phi_{\delta_n,0} > d\bar{x}d\bar{y} \\
&= 1 - \frac{\alpha_o^p + \beta_0^p}{\widetilde{S}_{K,s}} \int\int_{B_1^2(-(x_0)_n)} < L(\Phi_{\delta_n,0}), \Phi_{\delta_n,0} > d\bar{x}d\bar{y}. \qquad (76)
\end{aligned}$$

Note that using Lebesgue's Theorem again, we can prove that

$$\lim_{n\to\infty} \frac{\alpha_o^p + \beta_0^p}{\widetilde{S}_{K,s}} \int\int_{B_1^2(-(x_0)_n)} < L(\Phi_{\delta_n,0}), \Phi_{\delta_n,0} > d\bar{x}d\bar{y} = 1.$$

Then, by (75) we have $\gamma(\Phi_{\delta_n,(x_0)_n}, \Phi_{\delta_n,(x_0)_n}) \to 0$, which is a contradiction to (ii).

Suppose that (d) is true. Since $|(x_0)_n| \geq \frac{1}{2}$ for n large, then $(x_0)_n \nrightarrow 0$ in $\mathbb{R}^N$. From Lemma 4.5, we get

$$h(\Phi_{\delta_n,(x_0)_n}, \Phi_{\delta_n,(x_0)_n}) = \frac{(x_0)_n}{|(x_0)_n|} + o_n(1).$$

Hence,

$$h(\Phi_{\delta_n,(x_0)_n}, \Phi_{\delta_n,(x_0)_n}) \nrightarrow 0,$$

which is a contradiction to (i). Hnece, we conclude that $\widetilde{S}_{K,s} < c_0$ and the proof is complete. $\square$

Lemma 4.8. *There is* $\delta_1 \in (0, 1/2)$ *such that*

(a) $f(\alpha_0\Phi_{\delta_1,x_0}, \beta_0\Phi_{\delta_1,x_0}) < \dfrac{\widetilde{S}_{K,s} + c_0}{2}, \quad \forall x_0 \in \mathbb{R}^N$;

(b) $\gamma(\Phi_{\delta_1,x_0}, \Phi_{\delta_1,x_0}) < \dfrac{1}{2}, \quad \forall x_0 \in \mathbb{R}^N$ *such that* $|x_0| < \dfrac{1}{2}$;

(c) $\left|h(\Phi_{\delta_1,x_0}, \Phi_{\delta_1,x_0}) - \dfrac{x_0}{|x_0|}\right| < \dfrac{1}{4}, \quad \forall x_0 \in \mathbb{R}^N$ *such that* $|x_0| \geq \dfrac{1}{2}$.

Proof. From Lemma 4.3, we can choose $\varepsilon = \dfrac{c_0 - \widetilde{S}_{K,s}}{2} > 0$, $\delta_2 < \min\{\underline{\delta}, 1/2\}$ and conclude that, for all $x_0 \in \mathbb{R}^N$,

$$f(\alpha_0\Phi_{\delta,x_0}, \beta_0\Phi_{\delta,x_0}) \leq \sup_{x_0\in\mathbb{R}^N} f(\alpha_0\Phi_{\delta,x_0}, \beta_0\Phi_{\delta,x_0}) < \widetilde{S}_{K,s} + \frac{c_0 - \widetilde{S}_{K,s}}{2} = \frac{\widetilde{S}_{K,s} + c_0}{2}, \tag{77}$$

Now by definition of ξ, we have

$$\gamma(\Phi_{\delta,x_0}, \Phi_{\delta,x_0}) = 1 - \frac{\alpha_o^p + \beta_0^p}{\widetilde{S}_{K,s}} \int\int_{B_1^2(-x_0)} < L(\Phi_{\delta,0}), \Phi_{\delta,0} > d\bar{x}d\bar{y}.$$

From the Lebesgue Theorem

$$\frac{\alpha_o^p + \beta_0^p}{\widetilde{S}_{K,s}} \int\int_{B_1^2(-x_0)} < L(\Phi_{\delta,0}), \Phi_{\delta,0} > d\bar{x}d\bar{y} = 1$$

and the proof of this item is complete.

Note that from Lemma 4.5, we conclude that

$$h(\Phi_{\delta,x_0}, \Phi_{\delta,x_0}) = \frac{x_0}{|x_0|} + o_\delta(1) \quad \text{as} \quad \delta \to 0, \quad \forall x_0 \in \mathbb{R}^N; \ |x_0| \geq \frac{1}{2}$$

and the proof is finished. □

Lemma 4.9. *There is $\delta_2 > 1$ such that*

(a) $\quad f(\alpha_0\Phi_{\delta_2,x_0}, \beta_0\Phi_{\delta_2,x_0}) < \dfrac{\widetilde{S}_{K,s} + c_0}{2}, \quad \forall x_0 \in \mathbb{R}^N,$

(b) $\quad \gamma(\Phi_{\delta_2,x_0}, \Phi_{\delta_2,x_0}) > \dfrac{1}{2}, \quad \forall x_0 \in \mathbb{R}^N.$

Proof. From Lemma 4.3, we can choose $\varepsilon = \dfrac{c_0 - \widetilde{S}_{K,s}}{2} > 0$, $\delta_3 > \max\{\bar{\delta}, 1\}$ we have

$$f(\alpha_0\Phi_{\delta,x_0}, \beta_0\Phi_{\delta,x_0}) \leq \sup_{x_0\in\mathbb{R}^N} f(\alpha_0\Phi_{\delta,x_0}, t_0\Phi_{\delta,x_0}) < \widetilde{S}_{K,s} + \frac{c_0 - \widetilde{S}_{K,s}}{2} = \frac{\widetilde{S}_K + c_0}{2}, \tag{78}$$

for all $x_0 \in \mathbb{R}^N$. Moreover, from definition of ξ and Lemma 4.1, we can conclude that

$$\gamma(\Phi_{\delta,x_0}, \Phi_{\delta,x_0}) \to 1 \quad \text{as} \quad \delta \to +\infty,$$

and the proof is complete. □

Lemma 4.10. *There is $R > 0$ such that*

(a) $\quad f(\alpha_0\Phi_{\delta,x_0}, \beta_0\Phi_{\delta,x_0}) < \dfrac{\widetilde{S}_{K,s} + c_0}{2}, \quad \forall x_0;\ |x_0| \geq R$ *and* $\delta \in [\delta_1, \delta_2]$,

(b) $\quad (h(\Phi_{\delta,x_0}, \Phi_{\delta,x_0}), x_0))_{\mathbb{R}^N} > 0 \ \ \forall x_0;\ |x_0| \geq R$ *and* $\delta \in [\delta_1, \delta_2]$.

Proof. The first item follows by Lemma 4.3 and the choose of $\varepsilon = \dfrac{c_0 - \widetilde{S}_{K,s}}{2} > 0$. The second item follows from the definition of h and Φ_{δ,x_0} and adaptations the same arguments explored in [3] $\square$

Consider the set

$$\mathcal{V} = \{(x_0,\delta) \in \mathbb{R}^{\mathbb{N}} \times (0,\infty); |x_0| < R \ \text{ and } \ \delta \in (\delta_1,\delta_2)\},$$

where δ_1, δ_2 and R are given by Lemmas 4.8, 4.9 and 4.10, respectively.
Let $Q : \mathbb{R}^N \times (0,+\infty) \to D^{s,p}(\mathbb{R}^N)$ be the continuous function given by

$$Q(x_0,\delta) = \Phi_{\delta,x_0}.$$

Consider now the sets

$$\Theta = \{(Q(x_0,\delta), Q(x_0,\delta)); (x_0,\delta) \in \overline{\mathcal{V}}\},$$

$$\mathcal{H} = \left\{\tilde{h} \in C(\Sigma \cap \mathcal{M});\ \tilde{h}(u,v) = (u,v)\, \forall (u,v) \in \Sigma \cap \mathcal{M};\ f(\alpha_o u, \beta_o v) < \frac{\widetilde{S}_{K,s} + c_0}{2}\right\}$$

and
$$\Gamma = \{\mathcal{A} \subset \Sigma \cap \mathcal{M}; \mathcal{A} = \tilde{h}(\Theta), \tilde{h} \in \mathcal{H}\}.$$

Note that $\Theta \subset \Sigma \cap \mathcal{M}$, $\Theta = Q(\overline{\mathcal{V}}) \times Q(\overline{\mathcal{V}})$ is compact and $\mathcal{H} \neq \emptyset$, because the identity function is in $\mathcal{H}$.

Lemma 4.11. *Let $\mathcal{F} : \overline{\mathcal{V}} \to \mathbb{R}^{N+1}$ be a function given by*

$$\mathcal{F}(x_0,\delta) = (g \circ (Q,Q))(x_0,\delta) = \frac{\alpha_o^p + \beta_o^p}{\widetilde{S}_{K,s}} \int\int_{\mathbb{R}^{2N}} \left(\frac{x}{|x|}, \xi(x)\right) < L(\Phi_{\delta,x_0}), \Phi_{\delta,x_0} > dxdy.$$

Then, $$d(\mathcal{F}, \mathcal{V}, (0,1/2)) = 1 \quad \textit{(Topological degree).}$$

Proof. Let $\mathcal{Z} : [0,1] \times \overline{\mathcal{V}} \to \mathbb{R}^{N+1}$ be the homotopy given by

$$\mathcal{Z}(\bar{t}, (x_0,\delta)) = \bar{t}\mathcal{F}(x_0,\delta) + (1-\bar{t}) I_{\overline{\mathcal{V}}}(x_0,\delta),$$

where $I_{\overline{\mathcal{V}}}$ is the identity operator. Using Lemma 4.8 and Lemma 4.9, we can show that $(0,1/2) \notin \mathcal{Z}([0,1] \times (\partial\mathcal{V}))$, i.e,

$$\bar{t}h(\Phi_{\delta,x_0}, \Phi_{\delta,x_0}) + (1-\bar{t})y \neq 0, \quad \forall \bar{t} \in [0,1] \ \text{ and } \ \forall (x_0,\delta) \in \partial\mathcal{V} \tag{79}$$

or
$$\bar{t}\gamma(\Phi_{\delta,x_0}, \Phi_{\delta,x_0}) + (1-\bar{t})\delta \neq \frac{1}{2}, \quad \forall \bar{t} \in [0,1] \ \text{ and } \ \forall (x_0,\delta) \in \partial\mathcal{V}. \tag{80}$$

Hence $(0,1/2) \notin \mathcal{Z}([0,1] \times \partial\mathcal{V})$. We conclude that $d(\mathcal{F}, \mathcal{V}, (0,1/2))$, $d(i_{\overline{\mathcal{V}}}, \mathcal{V}, (0,1/2))$ and $d(\mathcal{Z}(\bar{t},\cdot), \mathcal{V}, (0,1/2))$ are well defined and

$$d(\mathcal{F}, \mathcal{V}, (0,1/2)) = d(i_{\overline{\mathcal{V}}}, \mathcal{V}, (0,1/2)) = 1. \qquad \square$$

Lemma 4.12. *If $\mathcal{A} \in \Gamma$, then $\mathcal{A} \cap \Im \neq \emptyset$.*

Proof. It is sufficient to prove that for all $\tilde{h} \in \mathcal{H}$, there exists $(\bar{x}_0, \delta_0) \in \overline{\mathcal{V}}$ such that

$$(g \circ \mathcal{H} \circ (Q, Q))(\bar{x}_0, \delta_0) = \left(0, \frac{1}{2}\right).$$

Given $\tilde{h} \in \mathcal{H}$, let $\mathcal{F}_{\tilde{h}} : \overline{\mathcal{V}} \to \mathbb{R}^{N+1}$ be the continuous function given by

$$\mathcal{F}_{\tilde{h}}(x_0, \delta) = (g \circ \tilde{h} \circ (Q, Q))(x_0, \delta).$$

We are going to show that $\mathcal{F}_{\tilde{h}} = \mathcal{F}$ in $\partial\mathcal{V}$. Note that

$$\partial\mathcal{V} = \Pi_1 \cup \Pi_2 \cup \Pi_3, \tag{81}$$

where $\Pi_1 = \{(x_0, \delta_1); |x_0| \leq R\}, \quad \Pi_2 = \{(x_0, \delta_2); |x_0| \leq R\}$

and $\Pi_3 = \{(x_0, \delta); |x_0| = R \text{ and } \delta \in [\delta_1, \delta_2]\}.$

If $(x_0, \delta) \in \Pi_1$, then $(x_0, \delta) = (x_0, \delta_1)$ and by item (a) from Lemma 4.8, we have

$$\begin{aligned} f(\alpha_o Q(x_0, \delta), \beta_o Q(x_0, \delta)) &= f(\alpha_o Q(x_0, \delta_1), \beta_o Q(x_0, \delta_1)) \\ &= f(\alpha_0 \Phi_{\delta_1, x_0}, \beta_0 \Phi_{\delta_1, x_0}) < \frac{\widetilde{S}_{K,s} + c_0}{2}, \quad \forall (x_0, \delta) \in \Pi_1. \end{aligned} \tag{82}$$

If $(x_0, \delta) \in \Pi_2$, then $(x_0, \delta) = (x_0, \delta_2)$ and by item(a) from Lemma 4.9, we get

$$\begin{aligned} f(\alpha_o Q(x_0, \delta), \beta_o Q(x_0, \delta)) &= f(\alpha_o Q(x_0, \delta_2), \beta_o Q(x_0, \delta_2)) \\ &= f(\alpha_0 \Phi_{\delta_2, x_0}, \beta_0 \Phi_{\delta_2, x_0}) < \frac{\widetilde{S}_{K,s} + c_0}{2}, \quad \forall (x_0, \delta) \in \Pi_2. \end{aligned} \tag{83}$$

If $(x_0, \delta) \in \Pi_3$, then $|x_0| = R$ and $\delta \in [\delta_1, \delta_2]$ and by item (a) from Lemma 4.10, we obtain

$$f(\alpha_o Q(x_0, \delta), \beta_o Q(x_0, \delta)) = f(\alpha_0 \Phi_{\delta, x_0}, \beta_0 \Phi_{\delta, x_0}) < \frac{\widetilde{S}_{K,s} + c_0}{2}, \quad \forall (x_0, \delta) \in \Pi_3. \tag{84}$$

From (81), (82), (83) and (84) we conclude that

$$f(\alpha_o Q(x_0, \delta), \beta_o Q(x_0, \delta)) < \frac{\widetilde{S}_{K,s} + c_0}{2}, \quad \forall (x_0, \delta) \in \partial\mathcal{V}.$$

Hence,

$$\begin{aligned} \mathcal{F}_{\tilde{h}}(x_0, \delta) &= (g \circ \tilde{h} \circ (Q, Q))(x_0, \delta) = (g \circ \tilde{h})(Q(x_0, \delta), Q(x_0, \delta)) \\ &= g(\tilde{h}((Q(x_0, \delta), Q(x_0, \delta)))) = g((Q(x_0, \delta), Q(x_0, \delta))) \\ &= (g \circ (Q, Q))(x_0, \delta) = \mathcal{F}(x_0, \delta), \quad \forall (x_0, \delta) \in \partial\mathcal{V}. \end{aligned}$$

Since $(0, 1/2) \notin \mathcal{F}(\partial\mathcal{V})$, we have

$$d(\mathcal{F}, \mathcal{V}, (0, 1/2)) = d(\mathcal{F}_{\tilde{h}}, \mathcal{V}, (0, 1/2)).$$

From Lemma 4.11, we get

$$d(\mathcal{F}_{\tilde{h}}, \mathcal{V}, (0, 1/2)) = d(\mathcal{F}, \mathcal{V}, (0, 1/2)) = 1,$$

and there exists $(\bar{x}_0, \delta_0) \in \mathcal{V}$ such that

$$\mathcal{F}_{\tilde{h}}(\bar{x}_0, \delta_0) = (g \circ \tilde{h} \circ (Q, Q))(\bar{x}_0, \delta_0) = \left(0, \frac{1}{2}\right)$$

and the proof is complete. □

4.2. Proof of Theorem 1.1

Consider the number $c = \inf_{\mathcal{A}\in\Gamma} \max_{(u,v)\in\mathcal{A}} f(u,v)$ and for each $q \in \mathbb{R}$,

$$f^q = \{(u,v) \in \Sigma \cap \mathcal{M}; f(u,v) \leq q\}.$$

We are going to show that

$$\widetilde{S}_{K,s} < c < p^{ps/N}\widetilde{S}_{K,s}. \tag{85}$$

Note that

$$c = \inf_{\mathcal{A}\in\Gamma} \max_{(u,v)\in\mathcal{A}} f(u,v) \leq \max_{(u,v)\in\Theta} f(u,v) \leq \sup_{\substack{y\in\mathbb{R}^N \\ \delta\in(0,+\infty)}} f(\alpha_o\Phi_{\delta,x_0}, \beta_o\Phi_{\delta,x_0}) < p^{ps/N}\widetilde{S}_{K,s}.$$

On the other hand, from Lemma 4.12, we have

$$\begin{aligned} c_0 = \inf_{u\in\Im} f(u,v) \leq c &= \inf_{\mathcal{A}\in\Gamma} \max_{u\in\mathcal{A}} f(\alpha_o u, \beta_o v) \\ &\leq \sup_{\substack{x_0\in\mathbb{R}^N \\ \delta\in(0,+\infty)}} f(\alpha_o\Phi_{\delta,x_0}, \beta_o\Phi_{\delta,x_0}) < p^{ps/N}\widetilde{S}_{K,s}. \end{aligned} \tag{86}$$

From Lemma 4.7, we have that $\widetilde{S}_{K,s} < c_0$ and the proof is complete.

Using the definition of c, there exists $(u_n, v_n) \subset \Sigma \cap \mathcal{M}$ such that

$$f(u_n, v_n) \to c. \tag{87}$$

Suppose, by contradiction, that $f'|_{\mathcal{M}}(u_n, v_n) \nrightarrow 0$.

Then, there exists $(u_{nj}, v_{nj}) \subset (u_n, v_n)$ such that

$$\|f'|_{\mathcal{M}}(u_{nj}, v_{nj})\|_* \geq C > 0, \quad \forall j \in \mathbb{N}.$$

Using a Deformation Lemma [25], there exists a continuous mapping

$$\eta : [0,1] \times (\Sigma \cap \mathcal{M}) \to (\Sigma \cap \mathcal{M}), \ \ \varepsilon_0 > 0$$

such that

(1) $\eta(0, u, v) = (u, v)$;
(2) $\eta(t, u, v) = (u, v)$, $\forall (u,v) \in f^{c-\varepsilon_0} \cup \{(\Sigma \cap \mathcal{M}) \setminus f^{c+\varepsilon_0}\}$, $\forall t \in [0,1]$;
(3) $\eta(1, f^{c+\frac{\varepsilon_0}{2}}) \subset f^{c-\frac{\varepsilon_0}{2}}$.

From the definition of c, there exists $\tilde{\mathcal{A}} \in \Gamma$ such that

$$c \leq \max_{(u,v)\in\tilde{\mathcal{A}}} f(u,v) < c + \frac{\varepsilon_0}{2}, \quad \text{where } \ \tilde{\mathcal{A}} \subset f^{c+\frac{\varepsilon_0}{2}}. \tag{88}$$

Since $\tilde{\mathcal{A}} \in \Gamma$, we have $\tilde{\mathcal{A}} \subset (\Sigma \cap \mathcal{M})$ and there exists $\bar{h} \in \mathcal{H}$ such that

$$\bar{h}(\Theta) = \tilde{\mathcal{A}}. \tag{89}$$

From the definition of η, we have

$$\eta(1, \tilde{\mathcal{A}}) \subset (\Sigma \cap \mathcal{M}). \tag{90}$$

Let $\hat{h} : (\Sigma \cap \mathcal{M}) \to (\Sigma \cap \mathcal{M})$ be the function given by $\hat{h}(u,v) = \eta(1, \bar{h}(u,v))$ and note that $\hat{h} \in C(\Sigma \cap \mathcal{M})$. We are going to show that

$$f^{c+\varepsilon_0} \setminus f^{c-\varepsilon_0} \subset f^{p^{ps/N}\widetilde{S}_{K,s}} \setminus f^{(\widetilde{S}_{K,s}+c_0)/2}. \tag{91}$$

Considering $(u,v) \in f^{c+\varepsilon_0} \setminus f^{c-\varepsilon_0}$, we have

$$c - \varepsilon_0 < f(u,v) \leq c + \varepsilon_0$$

and by (85), for ε_0 sufficiently small, we get

$$c - \varepsilon_0 < f(u,v) \leq c + \varepsilon_0 < p^{ps/N}\widetilde{S}_{K,s}. \tag{92}$$

Now from Lemma 4.7 and (86), we obtain

$$\frac{\widetilde{S}_{K,s} + c_0}{2} < c_0 - \varepsilon_0 < c - \varepsilon_0 < p^{ps/N}\widetilde{S}_{K,s}$$

and
$$\frac{\widetilde{S}_{K,s} + c_0}{2} < c_0 - \varepsilon_0 \leq c - \varepsilon_0 < f(u,v), \tag{93}$$

which implies
$$(u,v) \in f^{p^{ps/N}\widetilde{S}_{K,s}} \setminus f^{(\widetilde{S}_{K,s}+c_0)/2}.$$

Consider $(u,v) \in (\Sigma \cap \mathcal{M})$ such that

$$f(u,v) < \frac{\widetilde{S}_{K,s} + c_0}{2}. \tag{94}$$

Then, $\bar{h}(u,v) = (u,v)$ and from (94), we have that $(u,v) \notin f^{p^{ps/N}\widetilde{S}_{K,s}} \setminus f^{(\widetilde{S}_{K,s}+c_0)/2}$ and by (91), we get

$$(u,v) \notin f^{c+\varepsilon_0} \setminus f^{c-\varepsilon_0}.$$

Then, $(u,v) \in f^{c-\varepsilon_0} \cup \{(\Sigma \cap \mathcal{M}) \setminus f^{c+\varepsilon_0}\}$ and from Deformation Lemma, we obtain $\eta(1,u,v) = (u,v)$. Hence,

$$\hat{h}(u,v) = \eta(1, \bar{h}(u,v)) = \eta(1,u,v) = (u,v)$$

from which we conclude that $\hat{h} \in \mathcal{H}$, which implies

$$\hat{h}(\Theta) = \eta(1, \bar{h}(\Theta)),$$

and from (89), we conclude that

$$\hat{h}(\Theta) = \eta(1, \bar{h}(\Theta)) = \eta(1, \tilde{\mathcal{A}}). \tag{95}$$

From (90), we have $\eta(1, \tilde{A}) \in \Gamma$, which implies

$$c = \inf_{\mathcal{A} \in \Gamma} \max_{u \in \mathcal{A}} f(u,v) \leq \max_{u \in \eta(1,\tilde{\mathcal{A}})} f(u,v). \tag{96}$$

From the Deformation Lemma again and by (88), we get

$$\eta(1, \tilde{\mathcal{A}}) \subset \eta(1, f^{c+\frac{\varepsilon_0}{2}}) \subset f^{c-\frac{\varepsilon_0}{2}}.$$

Then, $$f(u,v) \le c - \frac{\varepsilon_0}{2}, \quad \forall (u,v) \in \eta(1,\tilde{\mathcal{A}}),$$

which implies $$\max_{u \in \eta(1,\tilde{\mathcal{A}})} f(u,v) \le c - \frac{\varepsilon_0}{2}$$

and using (96), we conclude that

$$c \le \max_{u \in \eta(1,\tilde{\mathcal{A}})} f(u,v) \le c - \frac{\varepsilon_0}{2},$$

which is absurd. Then,

$$f(u_n, v_n) \to c \quad \text{and} \quad f'|_{\mathcal{M}}(u_n, v_n) \to 0$$

and from Lemma 3.7, up to a subsequence, $u_n \to \widetilde{u}_0$, $v_n \to \widetilde{v}_0$ in $D^{s,p}(\mathbb{R}^N)$, which implies that $\widetilde{u}_0, \widetilde{v}_0 \ge 0$,

$$f(\widetilde{u}_0, \widetilde{v}_0) = c \quad \text{and} \quad f'|_{\mathcal{M}}(\widetilde{u}_0, (\widetilde{v}_0) = 0$$

and from(85)

$$\widetilde{S}_{K,s} < f(\widetilde{u}_0, \widetilde{v}_0) < p^{ps/N} \widetilde{S}_{K,s}.$$

The positivity of $\widetilde{u}_0$ and $\widetilde{v}_0$ is a consequence of the classical maximum principle.

Acknowledgments. The authors would like to thank the referees for all insightful comments, which allow us to improve our original version.

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