

Representations of Multimeasures via the Multivalued Bartle-Dunford-Schwartz Integral

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An integral for a scalar function with respect to a multimeasure N taking its values in a locally convex space is introduced. The definition is independent of the selections of N and is related to a functional version of the Bartle-Dunford-Schwartz integral with respect to a vector measure presented by Lewis. Its properties are studied together with its application to Radon-Nikodým theorems in order to represent as an integrable derivative the ratio of two general multimeasures or two d_H -multimeasures; equivalent conditions are provided in both cases.

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1. Introduction

The theory of the Bartle-Dunford-Schwartz-integral (BDS-integral) of a scalar function with respect to a vector valued measure was introduced in 1955 by Bartle, Dunford and Schwartz [1], and subsequently extensively studied by several authors ([14, 19, 20, 21, 23, 27, 29, 30]).

In the 1970s Lewis [21] proposed an equivalent functional version of the integral (see also [19, 20, 27]). The literature concerning the Bartle-Dunford-Schwartz integration is rather wide so we quote here only those papers which are close to the topic of our work and two books ([20, 27]).

In [23] the second author proved the existence of the Radon-Nikodým derivative of a vector valued measure ν with respect to a vector valued measure κ by means of the BDS-integral, under suitable hypotheses on the measures ν and κ . From this article then came out [4], where the derivative belongs to a suitable space and that of [12], where a Radon-Nikodým theorem was given based on a construction of Maynard type.

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In [18] Kandilakis defined an integral of a scalar function with respect to a multimeasure N , whose values were weakly compact and convex subsets of a Banach space. This integral is constructed using the selections of N . Contrary to the classical BDS-integral, the Kandilakis' integral is only sublinear with respect to the integrable functions.

It is our aim to define an integral with respect to a multimeasure N , independent of the selections of N . To achieve it we take into account the support functions of N . Our approach is a consequence of Kandilakis' calculations. The considered multimeasures N are very general, in fact they can take nonempty closed, convex values in an arbitrary locally convex space X . Sometimes we will assume quasi-completeness of X (closed bounded sets are complete).

The theory of multifunctions and multimeasures is an interesting field of research since it has applications in various applied sciences. In particular, recently, interval-valued multifunctions and multimeasures have been applied also to signal and image processing, see for example [10] and the references therein.

In Section 2 we study properties of the integral. Moreover, we compare the Aumann-BDS integral and the new one: if the multimeasure N possesses selections and the scalar integrable function is bounded, then the Aumann-BDS integrability implies the N -integrability. To obtain the main results we require the existence of control measures for the multimeasures under investigation and in Section 3 we give conditions guaranteeing the existence of such controls. In particular in Theorem 3.5 a characterization of its existence through the countable chain condition (ccc) is provided, while in Theorem 3.1 we describe the class of Banach spaces with all $cb(X)$ -multimeasures being d_H -multimeasures (and so admitting control measures). In Theorem 3.6 a class of spaces is given in which every $c(X)$ -valued multimeasures has (ccc). We quote also very recent characterization of Rodriguez [28] of Banach spaces with all multimeasures admitting control measures.

As an application of the new integral, we study the problem of existence of the Radon-Nikodým derivative of a given multimeasure M with respect to a multimeasure N . In the classical measure theory, the Radon-Nikodým theorem states in concise conditions, namely absolute continuity, domination and subordination, how a measure can be factorized by another measure through a density function.

In Section 4 we first consider arbitrary multimeasures. In such a case we are able to characterize the existence of the Radon-Nikodým derivative of M with respect to N (see Theorem 4.4) by means of the notions of uniform scalar absolute continuity, uniform scalar domination and uniform scalar subordination.

In Section 5 multimeasures which are countably additive in the Hausdorff metric are studied. In such a case the multimeasures take values in nonempty, bounded, closed, convex subsets of a locally convex space. The differentiation of M with respect to N is in general not equivalent to differentiation of its Rådström embedding $j \circ M$ with respect to $j \circ N$. The reason is that one has to take into account also integration with respect to the measure $j \circ (-N)$. In the case of multimeasures, we have in general $j \circ (-N) \neq -j \circ N$; moreover, since the proofs given are based on support functions which obviously lost the additivity, the methods of our proofs are not a mere repetition of the vector case. The main result is Theorem 5.6 where we

find conditions (the strong uniform scalar absolute continuity, the strong uniform scalar domination and the strong uniform scalar subordination) guaranteeing the existence of the Radon-Nikodým derivative both of M with respect to N and of their Rådström embeddings.

At last in Section 6 we provide some examples of multimeasures that can be represented by another multimeasure through a multivalued Bartle-Dunford-Schwartz density.

2. Preliminaries

Throughout (Ω, Σ) is a measurable space, the real numbers are denoted by $\mathbb{R}$ and $\mathbb{R}_0^+$ denotes the non-negative reals. If $\nu : \Sigma \rightarrow (-\infty, +\infty]$ is a measure, then $|\nu|$ denotes its variation. Σ_E is the family of all Σ -measurable subsets of E . Let X be a locally convex linear topological space (shortly, locally convex space) and let X' be its conjugate space. Given a subset S of X , we write $\text{co}(S)$, $\text{aco}(S)$ and $\text{span}(S)$ to denote, respectively, the convex, absolutely convex and linear hull of S .

The symbol $c(X)$ denotes the collection of all nonempty closed convex subsets of X and $cb(X)$, $cwk(X)$, $ck(X)$ denote, respectively, the family of all bounded and the family of all (weakly) compact members of $c(X)$. For every $C \in c(X)$ the *support function of C* is denoted by $s(\cdot, C)$ and defined on X' by $s(x', C) = \sup\{\langle x', x \rangle : x \in C\}$, for each $x' \in X'$. The symbol $\oplus$ denotes the closure of the Minkowski addition.

We say that $M : \Sigma \rightarrow c(X)$ is a *multimeasure* if for every $x' \in X'$ the set function $s(x', M(\cdot)) : \Sigma \rightarrow (-\infty, +\infty]$ is a σ -finite measure. The σ -finiteness of each $s(x', M)$ seems to be the weakest possible assumption, otherwise we meet problems connected with the RN-theorem. Simply if some $\nu := s(x', M)$ is not σ -finite but is absolutely continuous with respect to a finite measure μ , then there is a set Ω_0 such that ν is σ -finite on Ω_0 and takes only values $0, +\infty$ on Ω^c . If $\nu(E) = \int_E f d\mu$, then f must take infinite values and we are not interested in such a situation.

Given a multimeasure $M : \Sigma \rightarrow c(X)$ we denote by $\mathcal{N}(M) := \{E \in \Sigma : M(E) = \{0\}\}$ the family of null sets. A multimeasure $M : \Sigma \rightarrow c(X)$ is said to be σ -bounded if there is a sequence $(\Omega_n)_n$ of elements of Σ such that $\Omega \setminus \bigcup_n \Omega_n \in \mathcal{N}(M)$ and M is $cb(X)$ -valued on each algebra Σ_{Ω_n} .

A multimeasure M is called *pointless* if its restriction to no set $E \in \Sigma \setminus \mathcal{N}(M)$ is a vector measure. Each multimeasure determined by a function (see [7] for the definition) that is not scalarly equivalent to zero function is pointless. Less trivial examples can be deduced from [24, Example 1.11] if one assumes that the function r appearing there is strictly positive. Two multimeasures $M, N : \Sigma \rightarrow c(X)$ are *consistent* if there exists $H \in \Sigma$ such that M and N are pointless on H and vector measures on H^c .

If M is a $cb(X)$ -valued multimeasure, then for each $x' \in X'$ the measure $s(x', M)$ is finite. If $A \in \Sigma$, then $M|_A$ is the multimeasure defined on Σ_A by $(M|_A)(E) := M(A \cap E)$. A multimeasure $M : \Sigma \rightarrow c(X)$ is called *positive*, if $0 \in M(E)$ for each $E \in \Sigma$.

If X is a Banach space and a multimeasure $M : \Sigma \rightarrow cb(X)$ is countably additive in the Hausdorff metric d_H , then it is called a d_H -multimeasure.

In the following if Z is any metric space, we use the symbol B_Z to denote its closed unit ball.

A well-known and helpful tool for studying d_H -multimeasures is the Rådström embedding $j : cb(X) \rightarrow \ell_\infty(B_{X'})$, defined by $j(A) := s(\cdot, A)$, (see, for example [2, Theorem 3.2.9 and Theorem 3.2.4(1)] or [9, Theorem II-19]). It is known that $B_{X'}$ can be embedded into $\ell'_\infty(B_{X'})$ by the mapping $x' \rightarrow e_{x'}$, where $\langle e_{x'}, h \rangle = h(x')$, for each $h \in \ell_\infty(B_{X'})$. Moreover, the range of $B_{X'}$ is a norming subset of $\ell'_\infty(B_{X'})$. The embedding j satisfies the following properties:

- (P1) $j(\alpha A \oplus \beta C) = \alpha j(A) + \beta j(C)$ for every $A, C \in cb(X)$, $\alpha, \beta \in \mathbb{R}^+$;
- (P2) $d_H(A, C) = \|j(A) - j(C)\|_\infty$, $A, C \in cb(X)$;
- (P3) $j(cb(X))$ is a closed cone in the space $\ell_\infty(B_{X'})$ equipped with the norm of the uniform convergence.

Observe that instead of $\ell_\infty(B_{X'})$, we may use $C_B(B_{X'}, \tau_{(X', X)})$ (where $\tau_{(X', X)}$ is the Mackey topology) for weakly compact sets, $C(B_{X'}, \sigma(X', X))$ in case of compact sets and $C_B(B_{X'}, \|\cdot\|_\infty)$ in case of closed bounded sets. But as we do not apply any special properties of space in which we embed $cb(X)$, we will stay with

$$\ell_\infty(B_{X'}). \quad (1)$$

We denote by $R(M_E)$ the range of the multimeasure M in $c(X)$, restricted to measurable subsets of E : $R(M_E) := \{M(F) : F \in \Sigma_E\} \subset c(X)$. Moreover, we have $\mathcal{R}(M_E) := \{z \in X : \exists F \in \Sigma_E, z \in M(F)\} \subset X$.

We say that a multimeasure $M : \Sigma \rightarrow c(X)$ is *absolutely continuous* with respect to a multimeasure $N : \Sigma \rightarrow c(X)$ (we write then $M \ll N$), if $\mathcal{N}(N) \subset \mathcal{N}(M)$. If $M \ll N$ and $N \ll M$, then the multimeasures are called equivalent.

We say that a non negative measure μ is a *control measure* for a multimeasure $M : \Sigma \rightarrow c(X)$ if μ is a finite measure and for each $E \in \Sigma$ the condition $\mu(E) = 0$ yields $M(E) = \{0\}$. If μ is a control measure for $M : \Sigma \rightarrow c(X)$, then M is σ -bounded if and only if it is locally bounded, i.e. for each set $E \notin \mathcal{N}(\mu)$ there exists a subset F of E of positive μ -measure such that M is $cb(X)$ -valued on Σ_F . It is a direct consequence of [1, Theorem 1.4] that each d_H -multimeasure has a control measure.

If $M : \Sigma \rightarrow c(X)$ is a multimeasure, then $\mathcal{S}_M$ denotes the family of all countably additive X -valued selections of M .

Definition 2.1. (1) Let $n : \Sigma \rightarrow X$ be a vector measure. A measurable function $f : \Omega \rightarrow \mathbb{R}$ is called *Bartle-Dunford-Schwartz (BDS) integrable with respect to n* , if for every $x' \in X'$ f is $\langle x', n \rangle$ -integrable and for each $E \in \Sigma$ there exists a point $\nu(E) \in X$ such that

$$\langle x', \nu(E) \rangle = \int_E f d\langle x', n \rangle, \quad \text{for every } x' \in X'.$$

Such an approach to the Bartle-Dunford-Schwartz integral was suggested by Lewis [21].

(2) Let $N : \Sigma \rightarrow c(X)$ be a multimeasure. A measurable function $f : \Omega \rightarrow \mathbb{R}$ is called *Aumann-Bartle-Dunford-Schwartz (Aumann-BDS)-integrable with respect to N* if $\mathcal{S}_N \neq \emptyset$ and f is integrable in the sense of Bartle-Dunford-Schwartz with

respect to all members of $\mathcal{S}_N$. The integral on a set $E \in \Sigma$ is then defined by the formula

$$({}_s) \int_E f dN := \overline{\left\{ ({}_{BDS}) \int_E f dn : n \in \mathcal{S}_N \right\}}.$$

The above definition was suggested by Kandilakis [18] for $cwk(X)$ -valued multimeasures and a Banach space. In that case the set in the braces was closed and the additional closure is superfluous. The symbol (s) used here indicates that the integral is constructed using the selections of N .

Remark 2.2. (1) A multimeasure $N : \Sigma \rightarrow c(X)$ is called *rich* if we have $N(A) = \overline{\{n(A), n \in \mathcal{S}_N\}}$. By a result of Costé, quoted in [16, Theorem 7.9], this is fulfilled for multimeasures which take as their values weakly compact convex subsets of a Banach space X , or $cb(X)$ -valued when X is a Banach space possessing the RNP ([11, Théorème 1]).

(2) If N is a $cwk(X)$ valued multimeasure of bounded variation and X is a Banach space, then $\mathcal{S}_N \neq \emptyset$ (see for example [15, 17]). Moreover, by [18, Theorem 3.2], we have $({}_s) \int_E f dN \in cwk(X)$.

(3) If f, g are Aumann-BDS-integrable, then

$$({}_s) \int_E (f + g) dN \subseteq ({}_s) \int_E f dN \oplus ({}_s) \int_E g dN.$$

The example of $g = -f$ shows that the equality fails in general. In particular, the integral is not additive.

(4) We know from [18, Theorem 3.3] that in case of a multimeasure N with $cwk(X)$ values in a Banach space X and bounded f , we have for every $x' \in X'$ and $E \in \Sigma$

$$\begin{aligned} s\left(x', ({}_s) \int_E f dN\right) &= \int_E f^+ ds(x', N) + \int_E f^- ds(-x', N) \\ &= \int_E f^+ ds(x', N) + \int_E f^- ds(x', -N) \\ &\stackrel{\text{in general}}{\neq} \int_E f^+ ds(x', N) - \int_E f^- ds(x', N) \\ &= \int_E f ds(x', N). \end{aligned} \quad (2)$$

But it follows from that proof that equality (2) holds true in each case when $\mathcal{S}_N \neq \emptyset$, that is even when X is a locally convex space and N is not weakly compactly valued. In particular, if $f \geq 0$, then

$$s\left(x', ({}_s) \int_E f dN\right) = \int_E f ds(x', N) \quad (3)$$

and the integral is additive for non-negative functions. Moreover, it follows from (2) that the integral is a multimeasure, for every Aumann-BDS-integrable f . $\square$

As noticed in [30, Remark 3.2], if f is negative, then equality (3) may fail.

In fact, according to (2), we then have

$$\begin{aligned} s\left(x', (s) \int_E (-1) dN\right) &= \int_E 1 ds(-x', N) = s(-x', N(E)) \\ &\stackrel{\text{in general}}{\neq} -s(x', N(E)) = \int_E -1 ds(x', N). \end{aligned}$$

Now we would like to define an integral with respect to a multimeasure that would be independent of selections of the multimeasure but would be consistent with earlier definitions via selections. We know already ([18, 30]) that if X is a Banach space and a non-negative $\theta : \Omega \rightarrow \mathbb{R}$ is integrable with respect to $N : \Sigma \rightarrow cwk(X)$, then for every $E \in \Sigma$ there exists $W_E \in cwk(X)$ such that for every $x' \in X'$ holds true the equality $s(x', W_E) = \int_E \theta ds(x', N)$.

We take this property as the definition of the integral of a non-negative function with respect to an arbitrary multimeasure $N : \Sigma \rightarrow c(X)$. Our approach is close to Lewis' [21] functional equivalent definition of an integral with respect to a vector measure.

It is worth to remember that the Lewis definition in [21] was also considered by Kluvánek in [19] (see also [27]). We call the integral that we are going to define a *multivalued-Bartle-Dunford-Schwartz integral* since Bartle, Dunford and Schwartz were the first who considered such a kind of integration in the vector case.

Definition 2.3. Let $N : \Sigma \rightarrow c(X)$ be a multimeasure. If $f : \Omega \rightarrow \mathbb{R}$ is a non-negative measurable function, we say that f is *multivalued-Bartle-Dunford-Schwartz integrable with respect to N* (shortly *BDS_m-integrable with respect to N*) in $c(X)$ ($cb(X), cwk(X), ck(X)$), if for every $E \in \Sigma$ there exists $C_E \in c(X)$, ($cb(X), cwk(X), ck(X)$) such that for every $x' \in X'$

$$s(x', C_E) = \int_E f ds(x', N). \quad (4)$$

We define $C_E := \int_E f dN$.

We say that a measurable $f : \Omega \rightarrow \mathbb{R}$ is *multivalued-Bartle-Dunford-Schwartz integrable with respect to N* in $c(X)$ ($cb(X), cwk(X), ck(X)$), if f^+ and f^- are *BDS_m-integrable with respect to N* in $c(X)$ ($cb(X), cwk(X), ck(X)$). Then, for every $E \in \Sigma$, we define the integral of f with respect to N as

$$\int_E f dN := \int_E f^+ dN \oplus \int_E f^- d(-N).$$

The above definition is consistent with the property described in [18, Theorem 3.3]. (There is a missprint in [18, Theorem 3.3], the sign minus should be replaced by plus.) Equivalently, $f : \Omega \rightarrow \mathbb{R}$ is *BDS_m-integrable with respect to N* in $c(X)$, ($cb(X), cwk(X), ck(X)$), if for each $E \in \Sigma$ there exists $M_f(E) \in c(X)$ ($cb(X), cwk(X), ck(X)$) such that for every $x' \in X'$

$$s(x', M_f(E)) = \int_E f^+ ds(x', N) + \int_E f^- ds(x', -N) \quad (5)$$

and the right hand side of (5) makes sense. We then write $\int_E f dN := M_f(E)$.

The above definition is consistent with the property described in [18, Theorem 3.3]. Since $s(x', \pm N) : \Sigma \rightarrow (-\infty, +\infty]$ are σ -finite measures for every x' then, by (5), the set functions $s(x', M_f)$ are σ -finite measures with values in $(-\infty, +\infty]$. Consequently, M_f is a multimeasure. Let us notice that if N is a vector measure (say $N = \nu$) then the right hand side of (5) looks as follows:

$$s(x', M_f(E)) = \int_E f d\langle x', \nu \rangle.$$

One can easily check that the integral on the right hand side has to be finite and M is a vector measure (see Lemma 2.7).

Remark 2.4. (1) Assume that $\mathcal{S}_N \neq \emptyset$ (see for example Remark 2.2). Observe that if f is a bounded, measurable, Aumann-BDS-integrable function whose integral belongs to $cb(X)$ then f is BDS_m -integrable with respect to N and

$$(s) \int_E f dN = M_f(E).$$

(2) If f is a BDS_m -integrable with respect to N then $M_f(A \cup B) = M_f(A) \oplus M_f(B)$ for every $A, B \in \Sigma$ with $A \cap B = \emptyset$. In fact, for every $x' \in X'$, we have:

$$\begin{aligned} s\left(x', \int_{A \cup B} f dN\right) &= \int_{\Omega} (f\chi_{A \cup B})^+ ds(x', N) + \int_{\Omega} (f\chi_{A \cup B})^- ds(x', -N) \\ &= \int_{\Omega} (f^+ \chi_A + f^+ \chi_B) ds(x', N) + \int_{\Omega} (f^- \chi_A + f^- \chi_B) ds(x', -N) \\ &= s\left(x', \int_A f dN\right) + s\left(x', \int_B f dN\right) = s(x', M_f(A) \oplus M_f(B)). \end{aligned}$$

So, for every $x' \in X'$, we have

$$s(x', M_f(A \cup B)) = s(x', M_f(A)) + s(x', M_f(B)) = s(x', M_f(A) \oplus M_f(B)).$$

Remark 2.5. Let X be a Banach space and $N : \Sigma \rightarrow cb(X)$ be a d_H -multimeasure. If f is a measurable function such that there exists a sequence of simple functions $(f_n)_n$ which pointwise converges to f and the sequences $(\int_E f_n^\pm dN)_n$ are Cauchy in $(cb(X), d_H)$, then f is BDS_m integrable with respect to N .

Notice first that if $f = \sum_{i=1}^n a_i 1_{E_i}$ is measurable with non-negative $a_i, i = 1, \dots, n$, and N is $cb(X)$ -valued, then

$$\int_E f dN = \bigoplus_{i=1}^n a_i N(E \cap E_i)$$

for every $E \in \Sigma$, since for every $x' \in X'$ the support function is additive with respect to the Minkowski addition. In the general case, if a sequence $(f_n)_n$ of simple functions converges pointwise to f , then $(f_n^\pm)_n$ converges to $f^\pm$. We consider first f^+ . $(f_n^+)_n$ converges pointwise to f^+ and for every $x' \in X'$ and for every $E \in \Sigma$

$$\left(\int_E f_n^+ ds(x', N) \right)_n = \left(s(x', \int_E f_n^+ dN) \right)_n.$$

Since $(\int_E f_n^+ dN)_n$ is Cauchy in $(cb(X), d_H)$, by the completeness of the hyperspace for every $E \in \Sigma$ there exists $M^+(E) \in cb(X)$ such that

$$\lim_{n \rightarrow \infty} d_H \left(\int_E f_n^+ dN, M^+(E) \right) = 0.$$

We apply now [21, Lemma 2.3] to f_n^+, f^+ and $s(x', N)$ and we obtain that f^+ is integrable with respect to $s(x', N)$ and

$$\lim_{n \rightarrow \infty} \int_E f_n^+ ds(x', N) = \int_E f^+ ds(x', N), \quad \text{uniformly with respect to } E \in \Sigma.$$

Since $d_H \left(\int_E f_n^+ dN, M^+(E) \right) = \sup_{x' \in B_{X'}} \left| s(x', \int_E f_n^+ dN) - s(x', M^+(E)) \right|$ we have

$$s(x', M^+(E)) = \int_E f^+ ds(x', N).$$

For the negative part f^- we apply the same construction using $s(x', -N)$; in this way we obtain analogously $M^-(E)$. Finally, considering $M^+(E) \oplus M^-(E)$ we obtain

$$s(x', M^+(E) \oplus M^-(E)) = \int_E f^+ ds(x', N) + \int_E f^- ds(x', -N).$$

Similarly, if f is a BDS_m -integrable function with respect to a d_H -multimeasure N , then there exists a sequence $(f_n)_n$ of simple functions that is pointwise convergent to f and the sequence $(\int_E f_n dN)_n$ is Cauchy in $(cb(X), d_H)$. See Remark 5.3 for the proof.

Proposition 2.6. *If N is a positive multimeasure, then the BDS_m -integral with respect to N is a sublinear function of its integrands.*

Proof. Indeed, assume that f, g are BDS_m -integrable with respect to N and that $a, b \geq 0$. Then,

$$\begin{aligned} s \left(x', \int_E (af + bg) dN \right) &= \int_E (af + bg)^+ ds(x', N) + \int_E (af + bg)^- ds(-x', N) \\ &\leq \int_E (af^+ + bg^+) ds(x', N) + \int_E (af^- + bg^-) ds(-x', N) \\ &= a \left[\int_E f^+ ds(x', N) + \int_E f^- ds(-x', N) \right] \\ &\quad + b \left[\int_E g^+ ds(x', N) + \int_E g^- ds(-x', N) \right] \\ &= as \left(x', \int_E f dN \right) + bs \left(x', \int_E g dN \right). \end{aligned} \quad \square$$

The following lemma is essential for our further investigation.

Lemma 2.7. *Let $M, N : \Sigma \rightarrow c(X)$ be two multimeasures possessing control measure μ and such that $M(E) = \int_E \theta dN$, for every $E \in \Sigma$. Then M is pointless if and only if N is pointless. Equivalently, M is a vector measure if and only if N is a vector measure.*

Proof. Assume that $N(E) = \{\kappa(E)\}$ for every $E \in \Sigma$ and $\kappa : \Sigma \rightarrow X$ is a vector measure. Then, we have for each $E \in \Sigma$ and each $x' \in X'$

$$\begin{aligned} s(x', M(E)) &= \int_E \theta^+ ds(x', N) + \int_E \theta^- ds(x', -N) = \int_E \theta^+ d\langle x', \kappa \rangle + \int_E \theta^- d\langle x', -\kappa \rangle \\ &= \int_E \theta^+ d\langle x', \kappa \rangle - \int_E \theta^- d\langle x', \kappa \rangle = \int_E \theta d\langle x', \kappa \rangle. \end{aligned} \quad (6)$$

By our assumption the integral $\int_E \theta d\langle x', \kappa \rangle$ exists and has to be finite. If not, then the equality $\int_E \theta d\langle x', \kappa \rangle = +\infty$ yields $s(-x', M(E)) = -\infty$, what is impossible. Thus, M has only bounded sets as its values. In such a case the expression on the right hand side of (6) is a linear function on X' . Hence the same holds true for $x' \rightarrow s(x', M(E))$. But that means that M is a vector measure.

A similar situation takes place if $M(E) := \{\nu(E)\}$, $E \in \Sigma$, where ν is a vector measure. By the assumption, we have

$$\forall E \in \Sigma, \quad \forall x' \in X', \quad \langle x', \nu(E) \rangle = \int_E \theta^+ ds(x', N) + \int_E \theta^- ds(x', -N).$$

Let $A := \{\omega : \theta(\omega) > 0\}$. By the classical Radon-Nikodym theorem there exists a function $f_{x'}$ such that

$$\forall E \in \Sigma_A, \quad \forall x' \in X', \quad s(x', N(E)) = \int_E f_{x'} d\mu. \quad (7)$$

$$\text{If } E \in \Sigma_A, \text{ then } \quad \forall E \in \Sigma_A, \quad \forall x' \in X', \quad \langle x', \nu(E) \rangle = \int_E \theta^+ ds(x', N).$$

Since for every $E \in \Sigma_A$, the function $x' \rightarrow \langle x', \nu(E) \rangle$ is linear, the same holds true for $x' \rightarrow \int_E \theta^+ ds(x', N)$. Consequently, if $a, b \in \mathbb{R}$ and $x', y' \in X'$, then

$$\begin{aligned} \int_E \theta^+ f_{ax'+by'} d\mu &= \int_E \theta^+ ds(ax' + by', N) = \\ &= \int_E \theta^+ d[a \cdot s(x', N) + b \cdot s(y', N)] = \int_E \theta^+ [af_{x'} + bf_{y'}] d\mu. \end{aligned}$$

As $E \in \Sigma_A$ is arbitrary and $\theta^+|_A > 0$, we obtain the equality

$$f_{ax'+by'} = af_{x'} + bf_{y'} \quad \mu - a.e.$$

Then $x' \rightarrow s(x', N(E))$ is linear by (7) and this proves that N is a vector measure on Σ_A . Similarly for A^c . $\square$

Proposition 2.8. *Let $M, N : \Sigma \rightarrow c(X)$ be two multimeasures and assume that N is pointless. If $\theta : \Omega \rightarrow \mathbb{R}$ is a measurable function such that*

$$\forall E \in \Sigma \quad \forall x' \in X' \quad s(x', M(E)) = \int_E \theta ds(x', N), \quad (8)$$

then θ is non negative N -almost everywhere.

Proof. Let $\theta = \theta^+ - \theta^-$ and let $H := \{\omega \in \Omega : \theta(\omega)^- > 0\} \in \Sigma$. Then, we obtain for every $E \in \Sigma_H$ the equality

$$s(x', M(E)) = \int_E -\theta^- ds(x', N).$$

It is enough to prove $N(H) = \{0\}$. We suppose, by contradiction, that $N(H) \neq \{0\}$. Then $x' \rightarrow s(x', M(E))$ is sublinear for each $E \in \Sigma_H$ and also $x' \rightarrow -s(x', M(E))$ is sublinear, because $-s(x', M(E)) = \int_E \theta^- ds(x', N)$. Hence $x' \rightarrow s(x', M(E))$ is linear. Therefore

$$0 = s(0, M(E)) = s(x' - x', M(E)) = s(x', M(E)) + s(-x', M(E))$$

and so $s(-x', M(E)) = -s(x', M(E)) \neq \pm\infty$ implying directly the linearity of $x' \rightarrow s(x', M(E))$. But that is possible only if $M(E)$ is one point set. This however forces N to be a vector measure on Σ_H , what contradicts the pointlessness of N . $\square$

Remark 2.9. It follows from Proposition 2.8 that an integral defined by the equality (8) does not present the proper approach to integrability with respect to a pointless multimeasure, since only non-negative functions could be integrable.

If in Proposition 2.8 N restricted to an element $F \notin \mathcal{N}(N)$ is a vector measure, then M restricted to F is a vector measure (see Lemma 2.7) and $\theta|_F$ is not necessarily non-negative.

3. Control measures

Our aim is to determine when a multimeasure M can be seen as an integral of a scalar function with respect to a given multimeasure N . We obtain our results under the assumption of the existence of control measures for the considered multimeasures. In the case of d_H -multimeasures control measures always exist (see [1]), but in the case of an arbitrary multimeasure this is not obvious. The subsequent theorem describes completely the class of Banach spaces where every multimeasure is a d_H -multimeasure.

Theorem 3.1. *Every $cb(X)$ -valued multimeasure is a d_H -multimeasure if and only if X does not contain any isomorphic copy of c_0 .*

Proof. If $c_0 \not\subseteq X$ isomorphically, then the assertion is proved in [8, Proposition 4.1]. If c_0 can be isomorphically embedded into X , then [25, Example 3.6 and 3.8] are two examples of $cb(c_0)$ -valued multimeasures which are not d_H -multimeasures. $\square$

The proofs below provide a characterisation of multimeasures possessing a control measure in the language of the countable chain condition. The proofs are related to those given in [22], where vector measures were under consideration.

Definition 3.2. A multimeasure $M : \Sigma \rightarrow c(X)$ satisfies the *countable chain condition* (ccc) if each family of pairwise disjoint not M -null sets is at most countable.

Lemma 3.3. Assume that $M, N : \Sigma \rightarrow c(X)$ are two multimeasures with $M \ll N$. If N satisfies (ccc), then for every $A \in \Sigma \setminus \mathcal{N}(M)$ there exists $B \in \Sigma \setminus \mathcal{N}(M)$ such that $B \subset A$ and $N \ll M$ on Σ_B .

Proof. Assume that there is a set $A \in \Sigma \setminus \mathcal{N}(M)$ such that for every $B \in \Sigma_A \setminus \mathcal{N}(M)$ there exists $D \in [\mathcal{N}(M) \cap B] \setminus \mathcal{N}(N)$. The lemma of Kuratowski-Zorn and (ccc) give the existence of at most countable maximal family $\{D_n\}_n$ of pairwise disjoint sets $D_n \in [\mathcal{N}(M) \cap A] \setminus \mathcal{N}(N)$.

One can easily check that $\bigcup_n D_n \in \mathcal{N}(M)$. Since we have $A \notin \mathcal{N}(M)$, we conclude that $A \setminus \bigcup_n D_n \notin \mathcal{N}(M)$. But this contradicts the maximality of $\{D_n\}_n$. $\square$

Lemma 3.4. If $N : \Sigma \rightarrow c(X)$ is a multimeasure satisfying (ccc), then there exists at most countable family $\{x'_n : n \in \mathbb{N}\} \subset X'$ satisfying the equality

$$\bigcap_n \mathcal{N}[s(x'_n, N)] = \bigcap_{x' \in X'} \mathcal{N}[s(x', N)]. \quad (9)$$

Proof. Lemma 3.3 implies that for every $x' \in X'$ there exists $D_{x'} \in \Sigma \setminus \mathcal{N}[s(x', N)]$ such that $N \ll s(x', N)$ on $\Sigma_{D_{x'}}$. The lemma of Kuratowski-Zorn and (ccc) guarantee existence of at most countable maximal family $\{D_n\}_n$ of disjoint sets D_n corresponding to measures $s(x'_n, N)$.

Let $D = \bigcup_n D_n$. If $A \in \Sigma$ and $A \cap D = \emptyset$, then clearly $A \in \mathcal{N}[s(x', N)]$, for every $x' \in B_{X'}$.

Let now A be an $s(x'_n, N)$ -null set, for every n . We have $A \cap D_n \in \mathcal{N}(N)$ for all n and so $A \cap D_n \in \mathcal{N}[s(x', N)]$ for every x' . Consequently, $A \cap D \in \mathcal{N}[s(x', N)]$ for every x' . Hence, $A = (A \setminus D) \cup (A \cap D) \in \mathcal{N}[s(x', N)]$, proving (9). $\square$

Theorem 3.5. A multimeasure $N : \Sigma \rightarrow c(X)$ has a finite control measure if and only if it satisfies (ccc). Then, there exists a control measure that is equivalent to N .

Proof. It is obvious that the existence of a control measure yields (ccc) of N . So assume that N satisfies (ccc). For each $x' \in X'$ let $\nu_{x'} : \Sigma \rightarrow [0, +\infty)$ be a measure equivalent to $s(x', N)$. By Lemma 3.4 there exist $x'_n \in X', n \in \mathbb{N}$, such that

$$\mathcal{N}(N) = \bigcap_{x' \in X'} \mathcal{N}[s(x', N)] = \bigcap_n \mathcal{N}[s(x'_n, N)] = \bigcap_n \mathcal{N}(\nu_{x'_n}).$$

$$\text{The measure} \quad \mu(E) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\nu_{x'_n}(E)}{1 + \nu_{x'_n}(\Omega)} \quad (10)$$

is the required control measure for N that is equivalent to N . $\square$

Theorem 3.6. If X' is weak'-separable, then every multimeasure $M : \Sigma \rightarrow c(X)$ satisfies (ccc).

Proof. Let $\{x'_n : n \in \mathbb{N}\}$ be weak'-dense in X' . Suppose there exists a multimeasure $M : \Sigma \rightarrow c(X)$ and an uncountable family $\{B_\alpha \in \Sigma : \alpha \in \mathbb{A}\}$ of pairwise disjoint

sets with $M(B_\alpha) \neq \{0\}$. Without loss of generality we may assume that the family is ordered by the ordinals less than ω_1 . Due to the countability of the weak'-dense set, there exists $\beta < \omega_1$ such that $s(x'_n, M(B_\alpha)) = 0$, for every $\alpha > \beta$ and every $n \in \mathbb{N}$. Hence, if $\alpha > \beta$ and $x \in B_\alpha$, then $\langle x'_n, x \rangle \leq 0$ for every n . It follows that $\langle x', x \rangle \leq 0$ for every $x' \in X'$, if $x \in B_\alpha$ and $\alpha > \beta$. This is of course impossible for $x \neq 0$. $\square$

The following recent result describes Banach spaces with all $cb(X)$ -multimeasures possessing control measures.

Theorem 3.7. [28] *If X is a Banach space not containing any isomorphic copy of $c_0(\omega_1)$, then each multimeasure $M : \Sigma \rightarrow cb(X)$ admits a control measure.*

4. Arbitrary multimeasures

Now we start by examining the problem when a multimeasure M can be represented as an integral of a scalar function with respect to a given multimeasure N , for suitable multisubmeasures this problem was also faced in [5] for the Gould integral. In the case of vector measures with values in a locally convex space a Radon-Nikodým theorem for the Bartle-Dunford-Schwartz integral was obtained by Musiał in [23]. If Y is a locally convex space and $\nu, \kappa : \Sigma \rightarrow Y$ are vector measures, then the following definitions were formulated in [23]:

- (usac) ν is *uniformly scalarly absolutely continuous* (usac) with respect to κ , if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $y' \in Y'$ and each $E \in \Sigma$, the inequality $|y'\kappa|(E) < \delta$ yields $|y'\nu|(E) < \varepsilon$. We denote it by $\nu \ll \kappa$.
- (usd) ν is *uniformly scalarly dominated* (usd) by κ , if there exists $b \in \mathbb{R}^+$ such that $\forall y' \in Y', \forall E \in \Sigma$ it is $|y'\nu|(E) \leq b|y'\kappa|(E)$.
- (sub) ν is *subordinated* to κ , if there exists $d \in \mathbb{R}^+$ such that for every $E \in \Sigma$

$$\nu(E) \in d \overline{aco} \mathcal{R}(\kappa_E).$$

We say that a vector measure $\nu : \Sigma \rightarrow Y$ has *locally* a property with respect to a vector measure $\kappa : \Sigma \rightarrow Y$, if for each $E \in \Sigma \setminus \mathcal{N}(\kappa)$ there exists $F \subset E$ with $F \in \Sigma \setminus \mathcal{N}(\kappa)$ such that ν has this property with respect to κ on the set F .

In order to find proper conditions guaranteeing the differentiation of an arbitrary $cb(X)$ -valued multimeasure M with respect to N , we must adapt properly the definitions given for vector measures.

Definition 4.1. Given two multimeasures $M, N : \Sigma \rightarrow cb(X)$ we say that:

- (usac) M is *uniformly scalarly absolutely continuous* with respect to N (usac or $M \ll N$), if there exists $A \in \Sigma$ such that

$$\begin{aligned} \forall \varepsilon > 0, \exists \delta > 0 : \quad & \forall \alpha, \beta \in \mathbb{R}, \forall x', y' \in X', \forall E \in \Sigma \quad (11) \\ & [|\alpha s(x', N) + \beta s(y', N)|(E \cap A) + |\alpha s(x', -N) + \beta s(y', -N)|(E \cap A^c) \leq \delta] \\ & \Rightarrow |\alpha s(x', M) + \beta s(y', M)|(E) \leq \varepsilon; \end{aligned}$$

- (usd) M is *uniformly scalarly dominated* (usd) by a multimeasure N , if there exist $c \in \mathbb{R}$ and $A \in \Sigma$ such that $\forall \alpha, \beta \in \mathbb{R}, \forall x', y' \in X', \forall E \in \Sigma$

$$\begin{aligned} & |\alpha s(x', M) + \beta s(y', M)|(E) \leq \\ & \leq c[|\alpha s(x', N) + \beta s(y', N)|(E \cap A) + c|\alpha s(x', -N) + \beta s(y', -N)|(E \cap A^c)]; \end{aligned}$$

(uss) M is *uniformly scalarly subordinated* (*uss*) to N if there exist $d \in \mathbb{R}^+$ and $A \in \Sigma$ such that $\forall \alpha, \beta \in \mathbb{R}, \forall x', y' \in X', \forall E \in \Sigma$

$$\begin{aligned} & \alpha s(x', M(E)) + \beta s(y', M(E)) \in \\ & \in d \overline{aco}\{\alpha s(x', N(F \cap A)) + \beta s(y', N(F \cap A))\} : F \in \Sigma_E \} + \\ & + d \overline{aco}\{\alpha s(x', -N(F \cap A^c)) + \beta s(y', -N(F \cap A^c))\} : F \in \Sigma_E \}. \end{aligned}$$

Remark 4.2. If M and N are vector measures then the uniform scalar absolute continuity and the uniform scalar domination of M with respect to N coincide respectively with the corresponding definitions given for vector measures, therefore we use the same notation for the two notions.

In case of *uss* the definition looks different with respect to *sub* and so we decided to change the name. They actually are equivalent if we assume that X is a quasi-complete locally convex space. This is due to the fact that for every $E \in \Sigma$ and for every $x' \in X'$

$$\overline{aco}\{\langle x', \mathcal{R}(\kappa_E) \rangle\} = \langle x', \overline{aco}(\mathcal{R}(\kappa_E)) \rangle \quad (12)$$

with

$$\langle x', \mathcal{R}(\kappa_E) \rangle := \{\langle x', w \rangle : w \in \mathcal{R}(\kappa_E)\}.$$

Here the inclusion $\supset$ is a consequence of the continuity of $x' \in X'$:

$$\langle x', \overline{aco}(\mathcal{R}(\kappa_E)) \rangle \subset \overline{\langle x', \overline{aco}(\mathcal{R}(\kappa_E)) \rangle} = \overline{aco}\{\langle x', \mathcal{R}(\kappa_E) \rangle\}.$$

While for the reverse inclusion, we can observe that the set $\overline{aco} \mathcal{R}(\kappa_E)$ is symmetric and weakly compact, due to [20, Theorem IV.6.1].

So we have $\langle x', \overline{aco} \mathcal{R}(\kappa_E) \rangle = [-\alpha, \alpha]$. The inclusion $\mathcal{R}(\kappa_E) \subset \overline{aco} \mathcal{R}(\kappa_E)$ yields $\langle x', \mathcal{R}(\kappa_E) \rangle \subset [-\alpha, \alpha]$ and so $\overline{aco} \langle x', \mathcal{R}(\kappa_E) \rangle \subset [-\alpha, \alpha]$.

Proposition 4.3. For arbitrary multimeasures $M, N : \Sigma \rightarrow cb(X)$ the properties *usac* and *usd* are equivalent and *uss* implies each of them.

Proof. (usd) $\Rightarrow$ (usac) is obvious.

(usac) $\Rightarrow$ (usd): Let $A \in \Sigma, \varepsilon > 0$ and $\delta > 0$ be such that

$$|\alpha s(x', N) + \beta s(y', N)|(E) < \delta \text{ implies } |\alpha s(x', M) + \beta s(y', M)|(E) < \varepsilon.$$

If $|\alpha s(x', N) + \beta s(y', N)|(E) = 0$, then $M \lll N$ yields $|\alpha s(x', M) + \beta s(y', M)|(E) < \varepsilon$ for every $\varepsilon > 0$. So *usd* follows.

Suppose now that $|\alpha s(x', N) + \beta s(y', N)|(E) > 0$. Let

$$\hat{x} := \delta x' [2|\alpha s(x', N) + \beta s(y', N)|(E)]^{-1}, \quad \hat{y} := \delta y' [2|\alpha s(x', N) + \beta s(y', N)|(E)]^{-1}.$$

Then $|\alpha s(\hat{x}, N) + \beta s(\hat{y}, N)|(E) < \delta$ and consequently $|\alpha s(\hat{x}, M) + \beta s(\hat{y}, M)|(E) \leq \varepsilon$. It follows that, if we take $c = 2\varepsilon/\delta$, then

$$|\alpha s(x', M) + \beta s(y', M)|(E) \leq c |\alpha s(x', N) + \beta s(y', N)|(E).$$

In a similar way one obtains the required inequalities for very $E \in \Sigma_{A^c}$. That proves *usd*.

(uss) $\Rightarrow$ (usd): By definition, for every $F \in \Sigma_E$, we have

$$\begin{aligned} \alpha s(x', M(E)) + \beta s(y', M(E)) &\leq \\ &\leq \sup d \overline{ac\overline{o}}\{[\alpha s(x', N(F \cap A)) + \beta s(y', N(F \cap A))]\} + \\ &\quad + \sup d \overline{ac\overline{o}}\{[\alpha s(x', -N(F \cap A^c)) + \beta s(y', -N(F \cap A^c))]\} \leq \\ &\leq d|\alpha s(x', N) + \beta s(y', N)|(E \cap A) + \\ &\quad + d|\alpha s(x', -N) + \beta s(y', -N)|(E \cap A^c). \end{aligned}$$

So the assertion follows. $\square$

Theorem 4.4. *Let $M, N : \Sigma \rightarrow cb(X)$ be two consistent multimeasures possessing control measures. Then the following are equivalent:*

(RN1) *There exists a bounded measurable function $\theta : \Omega \rightarrow \mathbb{R}$ such that for every $E \in \Sigma$ we have*

$$M(E) = \int_E \theta dN; \quad (13)$$

(4.4.a) *M is uniformly scalarly dominated by N ;*

(4.4.b) *M is uniformly scalarly absolutely continuous with respect to N ;*

(4.4.c) *M is uniformly scalarly subordinated to N .*

Proof. Let μ be a finite control measure for N . Without loss of generality, we may assume that μ is also a control measure for M . Let $H \in \Sigma$ be a set as in the definition of the consistency for M and N . We divide the proof into the pointless and vector parts.

(Pointless part) We assume for simplicity that $H = \Omega$.

(4.4.a) $\Rightarrow$ (RN1) By the classical Radon-Nikodým theorem, for every $x' \in X'$, there exist two measurable real functions $f_{x'}$ and $g_{x'}$ such that, $\forall E \in \Sigma$,

$$s(x', M(E)) = \int_E f_{x'} d\mu \quad \text{and} \quad s(x', N(E)) = \int_E g_{x'} d\mu. \quad (14)$$

Part 1: Let A be the set satisfying the definition of uniform scalar domination of M with respect to N . If $E \in \Sigma_A$ and $\alpha, \beta \in \mathbb{R}$, then

$$\begin{aligned} \int_E |\alpha f_{x'} + \beta f_{y'}| d\mu &= |\alpha s(x', M) + \beta s(y', M)|(E) \leq c|\alpha s(x', N) + \beta s(y', N)|(E) \\ &= c \int_E |\alpha g_{x'} + \beta g_{y'}| d\mu. \end{aligned}$$

So $|\alpha f_{x'} + \beta f_{y'}| \leq c|\alpha g_{x'} + \beta g_{y'}|$ μ -a.e. on A . Hence (see also [23, Lemma]),

$$\frac{f_{x'}}{cg_{x'}} = \frac{f_{y'}}{cg_{y'}} \quad \mu - \text{a.e. on the set} \quad \{\omega \in \Omega : g_{x'}(\omega)g_{y'}(\omega) \neq 0\} \cap A.$$

According to [31, Theorem 7.35.2] (see also [23, Lemma] for a short proof) there exists a measurable $\theta_1 : A \rightarrow [-1, 1]$ such that $f_{x'} = c\theta_1 g_{x'}$ μ -a.e. on the set $\{\omega \in A : g_{x'}(\omega)g_{y'}(\omega) \neq 0\}$ for each $x' \in X'$ separately. The equality on the set $\{\omega \in A : g_{x'}(\omega) = 0\}$ is obvious (notice that $|f_{x'}| \leq c|g_{x'}|$ a.e. on A).

So it follows that

$$s(x', M(E)) = \int_E c \theta_1 ds(x', N)$$

for each $E \in \Sigma_A$ and $x' \in X'$. We can prove that $\theta_1 \geq 0$ N -a.e. on A , thanks to Proposition 2.8.

Part 2: Let now $F \in \Sigma_{A^c}$. For every $\alpha, \beta \in \mathbb{R}$ and for every $x', y' \in X'$ we have

$$\begin{aligned} \int_F |\alpha f_{x'} + \beta f_{y'}| d\mu &= |\alpha s(x', M) + \beta s(y', M)|(F) \\ &\leq c |\alpha s(x', -N) + \beta s(y', -N)|(F) = c \int_F |\alpha g_{-x'} + \beta g_{-y'}| d\mu. \end{aligned}$$

A similar calculation gives the equality

$$\frac{f_{x'}}{cg_{-x'}} = \frac{f_{y'}}{cg_{-y'}} \quad \mu - \text{a.e. on the set } \{\omega \in A^c : g_{-x'}(\omega)g_{-y'}(\omega) \neq 0\}$$

and then

$$s(x', M(F)) = \int_F c \theta_2 ds(x', -N).$$

As before, $\theta_2 : A^c \rightarrow \mathbb{R}$ is non-negative N -a.e.. Thus, setting $\theta = c\theta_1 - c\theta_2$, we obtain for every $E \in \Sigma$

$$\begin{aligned} s(x', M(E)) &= s(x', M(E \cap A)) + s(x', M(E \cap A^c)) = \\ &= \int_E \theta^+ ds(x', N) + \int_E \theta^- ds(x', -N), \end{aligned}$$

what means that θ is a Radon-Nikodým derivative of M with respect to N .

(RN1) $\Rightarrow$ (4.4.c) We set $A := \{\omega \in \Omega : \theta(\omega) \geq 0\}$, and $d > \sup_{\omega \in \Omega} |\theta(\omega)|$. Since θ is BDS_m -integrable with respect to N , by Definition 2.3 we have for each $E \in \Sigma$ the equality

$$\alpha s(x', M(E \cap A)) + \beta s(y', M(E \cap A)) = \int_{E \cap A} \theta^+ d[\alpha s(x', N) + \beta s(y', N)].$$

Let x', y', α, β be fixed and assume that B, C generate the Hahn decomposition of $\alpha s(x', N) + \beta s(y', N)$. Then,

$$\begin{aligned} &\alpha s(x', M(E \cap A \cap B)) + \beta s(y', M(E \cap A \cap B)) \\ &\in \overline{co}[\theta^+(E \cap A \cap B)] \cdot [\alpha s(x', N(E \cap A \cap B)) + \beta s(y', N(E \cap A \cap B))]. \end{aligned}$$

Similarly,

$$\begin{aligned} &\alpha s(x', M(E \cap A \cap C)) + \beta s(y', M(E \cap A \cap C)) \\ &\in \overline{co}[\theta^+(E \cap A \cap C)] \cdot [\alpha s(x', N(E \cap A \cap C)) + \beta s(y', N(E \cap A \cap C))]. \end{aligned}$$

All together this yields

$$\begin{aligned} &\alpha s(x', M(E \cap A)) + \beta s(y', M(E \cap A)) \\ &\in \overline{co}[\theta^+(E \cap A)] \cdot ([\alpha s(x', N(E \cap A \cap B)) + \beta s(y', N(E \cap A \cap B))] \\ &\quad + [\alpha s(x', N(E \cap A \cap C)) + \beta s(y', N(E \cap A \cap C))]) \end{aligned}$$

$$\begin{aligned}
&= \overline{co}[\theta^+(E \cap A)] \cdot [\alpha s(x', N(E \cap A)) + \beta s(y', N(E \cap A))] \\
&\subset d\overline{aco}\{\alpha s(x', N(F \cap A)) + \beta s(y', N(F \cap A)) : F \in \Sigma_E\}.
\end{aligned}$$

In the same way one obtains the inclusion

$$\begin{aligned}
&\alpha s(x', M(E \cap A^c)) + \beta s(y', M(E \cap A^c)) \\
&\subset d\overline{aco}\{\alpha s(x', -N(F \cap A^c)) + \beta s(y', -N(F \cap A^c)) : F \in \Sigma_E\}.
\end{aligned}$$

The remaining equivalences follow from Proposition 4.3.

(Vector part) We assume now that $H^c = \Omega$. In virtue of Remark 4.2, the equivalence of the four conditions is a consequence of [23, Theorem 1]. $\square$

Similarly to the vector case we say that a multimeasure $M : \Sigma \rightarrow cb(X)$ has *locally* a property with respect to a multimeasure $N : \Sigma \rightarrow cb(X)$, if for each $E \in \Sigma \setminus \mathcal{N}(N)$ there exists $F \subset E$ with $F \in \Sigma \setminus \mathcal{N}(N)$ such that M has the property with respect to N on the set F .

Using the local properties we obtain:

Theorem 4.5. *Let $M, N : \Sigma \rightarrow c(X)$ be two σ -bounded consistent multimeasures possessing control measures. Then the following are equivalent:*

(RN2) *There exists a measurable function $\theta : \Omega \rightarrow \mathbb{R}$ such that*

$$\forall E \in \Sigma \quad M(E) = \int_E \theta dN;$$

(4.5.a) *M is locally uniformly scalarly dominated by N ;*

(4.5.b) *M is locally uniformly scalarly absolutely continuous with respect to N ;*

(4.5.c) *M is locally uniformly scalarly subordinated to N .*

Proof. Assume that $\Omega = \bigcup_n \Omega_n \cup B$, where Ω_n -s are pairwise disjoint, $B \in \mathcal{N}(\mu)$ and the restriction of M and N to each Ω_n is $cb(X)$ -valued. First we assume that M, N are $cb(X)$ -valued on Σ .

Analogously to Theorem 4.4 we have to divide the proof into the pointless and vector parts. The vector part follows immediately from [23, Theorem 2], so we prove here only the pointless part.

The equivalences **(4.5.a)** $\Leftrightarrow$ **(4.5.b)** $\Leftrightarrow$ **(4.5.c)** follow from the corresponding equivalences in Theorem 4.4.

Also **(RN2)** $\Rightarrow$ **(4.5.a)** is obvious (it is enough to take for each $E \in \Sigma \setminus \mathcal{N}(N)$ a set $F \subset E$ such that $F \in \Sigma \setminus \mathcal{N}(N)$ and θ is bounded on F).

Now we are going to prove that **(4.5.a)** $\Rightarrow$ **(RN2)**.

Let μ be a control measure for N . By definition there exist $E \in \Sigma \setminus \mathcal{N}(\mu)$ such that M is scalarly dominated on E by N . We denote by

$$\begin{aligned}
\mathcal{H}_1 &:= \{E \in \Sigma \setminus \mathcal{N}(\mu) : \exists A_E \in \Sigma_E \quad \forall F \in \Sigma_E, \forall x', y' \in X', \forall \alpha, \beta \in \mathbb{R}, \\
&\quad |\alpha s(x', M) + \beta s(y', M)|(F) \\
&\leq |\alpha s(x', N) + \beta s(y', N)|(F \cap A_E) + |\alpha s(x', -N) + \beta s(y', -N)|(F \setminus A_E)\}.
\end{aligned}$$

By the completeness of the algebra $\Sigma/\mathcal{N}(\mu)$ there is $E_1 \in \Sigma$ such that its equivalence class is the least upper bound of $\mathcal{H}_1$ in $\Sigma/\mathcal{N}(\mu)$. If $\mathcal{H}_1$ is empty we choose $E_1 = \emptyset$.

Then we consider the class $\mathcal{H}_2$ of all sets $E \in \Sigma_{\Omega \setminus E_1} \setminus \mathcal{N}(\mu)$ in which we have the scalar domination with respect to $c = 2$ and we choose analogously E_2 .

After n steps we received in this way sets $E_1, \dots, E_n$ and $A_{E_1} \subset E_1, \dots, A_{E_n} \subset E_n$ such that for each $k \leq n$ the inequality

$$\begin{aligned} & |\alpha s(x', M) + \beta s(y', M)|(F) \\ & \leq k \{ |\alpha s(x', N) + \beta s(y', N)|(F \cap A_{E_k}) + |\alpha s(x', -N) + \beta s(y', -N)|(F \setminus A_{E_k}) \} \end{aligned}$$

holds true for all $F \in \Sigma_{E_k}$, for all $\alpha, \beta \in \mathbb{R}$ and for all $x', y' \in X'$. Then we construct E_{n+1} such that its equivalence class is the least upper bound of

$$\begin{aligned} \mathcal{H}_{n+1} &:= \{ E \in \Sigma_{\Omega \setminus \bigcup_{k=1}^n E_k} \setminus \mathcal{N}(\lambda) : \exists A_E \in \Sigma_E \forall F \in \Sigma_E, \forall x', y' \in X', \forall \alpha, \beta \in \mathbb{R} \\ & \quad |\alpha s(x', M) + \beta s(y', M)|(F) \leq (n+1) |\alpha s(x', N) + \beta s(y', N)|(F \cap A_E) \\ & \quad + (n+1) |\alpha s(x', -N) + \beta s(y', -N)|(F \setminus A_E) \}. \end{aligned}$$

in $\Sigma/\mathcal{N}(\mu)$. A few first sets E_k may be of measure zero but then we meet the first set E_k of positive measure. In this way we obtain a sequence (possibly finite) (E_n) of sets with $E_n \in \mathcal{H}_n$.

Without loss of generality we may assume that the sets cover all Ω . Now we apply Theorem 4.4 to each set E_n and we get a bounded measurable function θ_n such that

$$\forall E \in \Sigma_{E_n}, \forall x' \in X', \quad s(x', M(E)) = \int_E \theta_n^+ ds(x', N) + \int_E \theta_n^- ds(x', -N). \quad (15)$$

Let $\theta = \sum_{n=1}^{\infty} \theta_n \chi_{E_n}$ and let $x' \in X'$ be fixed. Then, $\theta^+ = \sum_{n=1}^{\infty} \theta_n^+ \chi_{E_n}$ and $\theta^- = \sum_{n=1}^{\infty} \theta_n^- \chi_{E_n}$. By (15) we have for every $E \in \Sigma$

$$\begin{aligned} s(x', M(E)) &= \sum_{n=1}^{\infty} s(x', M(E \cap E_n)) = \\ &= \sum_{n=1}^{\infty} \left(\int_{E \cap E_n} \theta_n^+ ds(x', N) + \int_{E \cap E_n} \theta_n^- ds(x', -N) \right). \quad (16) \end{aligned}$$

Let $\Omega^+ := \{\omega \in \Omega : \theta(\omega) \geq 0\}$, $\Omega^- := \Omega \setminus \Omega^+$. Let $P'_1, P'_2, Q'_1, Q'_2 \in \Sigma$ be two Hahn decomposition for $s(x', N), s(x', -N)$ respectively. We have

$$\begin{aligned} s(x', M(E \cap P'_1 \cap \Omega^+)) &= \sum_{n=1}^{\infty} s(x', M(E \cap E_n \cap P'_1 \cap \Omega^+)) = \\ &= \sum_{n=1}^{\infty} \int_{E \cap E_n \cap P'_1 \cap \Omega^+} \theta_n^+ ds(x', N) \\ s(x', M(E \cap P'_2 \cap \Omega^+)) &= \sum_{n=1}^{\infty} s(x', M(E \cap E_n \cap P'_2 \cap \Omega^+)) = \\ &= \sum_{n=1}^{\infty} \int_{E \cap E_n \cap P'_2 \cap \Omega^+} \theta_n^+ ds(x', N) \end{aligned}$$

$$\begin{aligned}
s(x', M(E \cap Q'_1 \cap \Omega^-)) &= \sum_{n=1}^{\infty} s(x', M(E \cap E_n \cap Q'_1 \cap \Omega^-)) = \\
&= \sum_{n=1}^{\infty} \int_{E \cap E_n \cap P'_1 \cap \Omega^-} \theta_n^- ds(x', -N) \\
s(x', M(E \cap Q'_2 \cap \Omega^-)) &= \sum_{n=1}^{\infty} s(x', M(E \cap E_n \cap Q'_2 \cap \Omega^-)) = \\
&= \sum_{n=1}^{\infty} \int_{E \cap E_n \cap P'_2 \cap \Omega^-} \theta_n^- ds(x', -N).
\end{aligned}$$

Each of the series closing the above equalities is convergent with all its terms of the same sign. Hence, they are absolutely convergent. It follows that for all $E \in \Sigma$

$$\begin{aligned}
s(x', M(E)) &= \sum_{n=1}^{\infty} s(x', M(E \cap E_n)) = \int_{E \cap P'_1} \theta^+ ds(x', N) + \int_{E \cap P'_2} \theta^+ ds(x', N) + \\
&\quad + \int_{E \cap Q'_1} \theta^- ds(x', -N) + \int_{E \cap Q'_2} \theta^- ds(x', -N) = \\
&= \int_E \theta^+ ds(x', N) + \int_E \theta^- ds(x', -N).
\end{aligned}$$

Let us consider now the general case. We know already that for each $n \in \mathbb{N}$ there exists a measurable function $\xi_n : \Omega_n \rightarrow \mathbb{R}$ such that

$$\forall E \in \Sigma_{\Omega_n} \forall x' \in X' \quad s(x', M(E)) = \int_E \xi_n^+ ds(x', N) + \int_E \xi_n^- ds(x', -N). \quad (17)$$

Then, we follow the proof presented after formula (15). We have to remember only that we have always $s(x', M(E)) > -\infty$ and so each series appearing in the proof is either divergent to $+\infty$ or convergent. $\square$

Remark 4.6. It follows from Lemma 2.7 that some introductory assumptions in Theorem 4.4 concerning M and N are necessary. If M is pointless but N is not, then M cannot be represented by a BDS_m -integral with respect to N . To construct an example let X be a quasi-complete locally convex space and $N := \nu : \Sigma \rightarrow X$ be a non-atomic vector measure.

If M is defined by the formula $M(E) := \overline{co} \mathcal{R}(\nu_E)$ (or $M(E) := \overline{aco} \mathcal{R}(\nu_E)$), then $\overline{aco} \mathcal{R}(\nu_E)$ is a weakly compact set (see [20, Theorem IV.6.1]) and the conditions *uss*, *usd* and *usac* are fulfilled by M and N . Indeed the formula (12) in Remark 4.2, shows that

$$\alpha s(x', M(E)) + \beta s(x', M(E)) \in d \overline{aco} \{ \langle x', \mathcal{R}(\nu_E) \rangle \},$$

whenever $E \in \Sigma$ and $x' \in X'$ are arbitrary. It remains to prove that

$$M(E) := \overline{co} \mathcal{R}(\nu_E)$$

is a multimeasure.

M is clearly finitely additive: if A, B are disjoint, then

$$\begin{aligned} M(A \cup B) &= \overline{co} \mathcal{R}(\nu_{A \cup B}) = \overline{co} (\mathcal{R}(\nu_A) + \mathcal{R}(\nu_B)) = \overline{co} \mathcal{R}(\nu_A) \oplus \overline{co} \mathcal{R}(\nu_B) \\ &= M(A) \oplus M(B). \end{aligned}$$

It follows that for every $x' \in X'$, $s(x', M) : \Sigma \rightarrow \mathbb{R}$ is finitely additive. [3, Proposition 3.8] yields the countable additivity of each $s(x', M)$ and so M is a multimeasure.

Remark 4.7. A particular example of a quasi-complete locally convex space is a conjugate Banach space endowed with the weak* topology $\sigma(X', X)$. Let $cw^*k(X')$ denote the family of all non-empty weak*-compact and convex subsets of X' .

$M : \Sigma \rightarrow cw^*k(X')$ is called a *weak*-multimeasure* if $s(x, M(\cdot))$ is a measure, for every $x \in X$. It is proved in [26, Theorem 3.4] that X does not contain any isomorphic copy of ℓ^∞ if and only if each weak*-multimeasure $M : \Sigma \rightarrow cw^*k(X')$ is a d_H -multimeasure. The Radon-Nikodým theorem for two weak*-multimeasures formulates exactly as Theorem 4.5, one should only remember that now the conjugate to $(X', \sigma(X', X))$ is the space X itself.

5. d_H -multimeasures and their Rådström embeddings

Throughout this section we assume that X is a Banach space and we consider now the case of d_H -multimeasures. Let us recall that if a Banach space X does not contain any isomorphic copy of c_0 , then each $cb(X)$ -valued multimeasure is a d_H -multimeasure (see [8, 6]). Examples of $cb(c_0)$ -valued measures that are not d_H -measures can be found in [25, Example 3.6, Example 3.8]. When we write $\sum_{m=1}^n a_m x'_m \in \text{span} B_{X'}$, we always mean that $x'_m \in B_{X'}$ and $a_m \in \mathbb{R}$, for every $m \leq n$. Let now $M, N : \Sigma \rightarrow cb(X)$ be two d_H -multimeasures. We remember that

$$\|s(\cdot, M(E))\|_\infty = \sup_{x' \in B_{X'}} |s(x', M(E))|$$

and, according to (1), we set $Y := \ell_\infty(B_{X'})$. By [23, Theorem 1], for $\nu := j \circ M$, $\kappa = j \circ N$ we have

Theorem 5.1. *If $M, N : \Sigma \rightarrow cb(X)$ are two d_H -multimeasures, then the following are equivalent:*

(RN3) *There exists a bounded measurable function (measurable function) $\theta : \Omega \rightarrow \mathbb{R}$ such that for all $E \in \Sigma$ and $y' \in \ell'_\infty(B_{X'})$*

$$\langle y', j \circ M(E) \rangle = \int_E \theta d\langle y', j \circ N \rangle;$$

(5.1.a) *$j \circ M$ is (locally) uniformly scalarly absolutely continuous with respect to $j \circ N$;*

(5.1.b) *$j \circ M$ is (locally) uniformly scalarly dominated by $j \circ N$;*

(5.1.c) *$j \circ M$ is (locally) subordinated to $j \circ N$.*

We observe that even if M and N are d_H -multimeasures, the differentiation of M with respect to N is in general not equivalent to differentiation of $j \circ M$ with re-

spect to $j \circ N$. If θ is the Radon-Nikodým derivative of M with respect to N , the representation

$$j \circ M(E) = (BDS) \int_E \theta d(j \circ N) \quad \text{for every } E \in \Sigma$$

may fail (see the subsequent Example 6.4). In fact one has to take into account also integration with respect to the measure $j \circ (-N)$. If N is a vector measure, then $j \circ (-N) = -j \circ N$ and formally the measure $j \circ (-N)$ is absent in the calculations and the integral with respect to N coincides with its BDS-integral.

Note that in the case of multimeasures that are not vector measures, we have in general $j \circ (-N) \neq -j \circ N$ and the presence of the measure $j \circ (-N)$ becomes visible. The next theorem shows the shape of $j \circ N$ provided M has the Radon-Nikodým derivative (in the sense of the integral investigated in this paper) with respect to N .

Theorem 5.2. *Let $M, N : \Sigma \rightarrow cb(X)$ be two d_H -multimeasures and $\theta : \Omega \rightarrow \mathbb{R}$ be a measurable function. Then $M(E) = \int_E \theta dN$ for every $E \in \Sigma$ if and only if for all $E \in \Sigma$ and for all $y' \in \ell'_\infty(B_{X'})$*

$$\langle y', j \circ M(E) \rangle = \int_E \theta^+ d\langle y', j \circ N \rangle + \int_E \theta^- d\langle y', j \circ (-N) \rangle. \quad (18)$$

$$\text{Equivalently, } j \circ M(E) = (BDS) \int_E \theta^+ d(j \circ N) + (BDS) \int_E \theta^- d(j \circ (-N)).$$

Proof. “ $\Rightarrow$ ”

Step A: We assume first that θ is bounded. Let $M(E) = \int_E \theta dN$ for every $E \in \Sigma$. According to (5) we have then

$$\begin{aligned} \langle e_{x'}, j \circ M(E) \rangle &= s(x', M(E)) = \int_E \theta^+ ds(x', N) + \int_E \theta^- ds(-x', N) \\ &= \int_E \theta^+ d\langle e_{x'}, j \circ N \rangle + \int_E \theta^- d\langle e_{-x'}, j \circ N \rangle = \\ &= \int_E \theta^+ d\langle e_{x'}, j \circ N \rangle + \int_E \theta^- d\langle e_{x'}, j \circ (-N) \rangle. \end{aligned}$$

Since θ is bounded, it is $j \circ N$ integrable (as a BDS-integral) and so there are measures $\kappa_1, \kappa_2 : \Sigma \rightarrow \ell_\infty(B_{X'})$ such that

$$\langle y', \kappa_1(A) \rangle = \int_E \theta^+ d\langle y', j \circ N \rangle \quad \text{whenever } y' \in \ell'_\infty(B_{X'})$$

$$\text{and} \quad \langle y', \kappa_2(E) \rangle = \int_E \theta^- d\langle y', j \circ (-N) \rangle \quad \text{whenever } y' \in \ell'_\infty(B_{X'}).$$

Hence, we obtain the equality

$$\langle e_{x'}, j \circ M(E) \rangle = \langle e_{x'}, \kappa_1(E) \rangle + \langle e_{x'}, \kappa_2(E) \rangle.$$

But $\{e_{x'} : x' \in B_{X'}\}$ is norming and the measures $j \circ M, \kappa_1$ and κ_2 have weakly relatively compact ranges. It follows that $\forall y' \in \ell'_\infty(B_{X'}), \forall E \in \Sigma$

$$\langle y', j \circ M(E) \rangle = \int_E \theta^+ d\langle y', j \circ N \rangle + \int_E \theta^- d\langle y', j \circ (-N) \rangle. \quad (19)$$

Assume now that the equality (19) is fulfilled. Then, $\forall x' \in B_{X'}, \forall E \in \Sigma$

$$\begin{aligned} s(x', M(E)) &= \langle e_{x'}, j \circ M(E) \rangle = \int_E \theta^+ d\langle e_{x'}, j \circ N \rangle + \int_E \theta^- d\langle e_{x'}, j \circ (-N) \rangle \\ &= \int_E \theta^+ ds(x', N) + \int_E \theta^- ds(x', -N) = s(x', \int_E \theta dN). \end{aligned}$$

Step B: Suppose now that $\theta \geq 0$ is arbitrary and $M(E) = \int_E \theta dN$ for every $E \in \Sigma$. That means that for every $x' \in X'$ and $E \in \Sigma$ we have

$$s(x', M(E)) = \int_E \theta ds(x', N).$$

θ can be represented as $\theta = \sum_n \theta \chi_{E_n}$, where the sets E_n are pairwise disjoint and each $\theta \chi_{E_n}$ is bounded. According to Step A, we have for all $y' \in \ell'_\infty(B_{X'})$

$$\langle y', j \circ M(E \cap E_n) \rangle = \int_{E \cap E_n} \theta d\langle y', j \circ N \rangle, \quad n \in \mathbb{N}.$$

Since for each $y' \in \ell'_\infty(B_{X'})$ the set function $\langle y', j \circ M \rangle$ is a measure, we obtain the equality $\sum_n \langle y', j \circ M(E \cap E_n) \rangle = \langle y', j \circ M(E) \rangle$. Since for each $y' \in \ell'_\infty(B_{X'})$ the set function $\langle y', j \circ N \rangle$ is a measure, we have also

$$\sum_n \int_{E \cap E_n} \theta d\langle y', j \circ N \rangle = \int_E \theta d\langle y', j \circ N \rangle.$$

Step C: θ is arbitrary and $M(E) = \int_E \theta dN$ for every $E \in \Sigma$. The proof is obvious.

“ $\Leftarrow$ ”: In (18) one should substitute $e_{x'}$ instead of y' . $\square$

Remark 5.3. Let f be a BDS_m -integrable function and assume that N is $cb(X)$ -valued d_H -multimeasure. Then there exists a sequence $(f_n)_n$ of simple functions that is pointwise convergent to f and the sequence $(\int_E f_n dN)_n$ is Cauchy in $(cb(X), d_H)$. Indeed, let $M(E) = \int_E f dN, E \in \Sigma$. According to Theorem 5.2 f^+ and f^- are $j \circ N$ integrable as (BDS) integrals) and we have for all $E \in \Sigma$, with $E \subset \text{supp } f^+$

$$j \circ M(E) = (BDS) \int_E f^+ d(j \circ N).$$

By [1, Definition 2.5] there exists a sequence of pointwise convergent to f^+ simple functions $h_n : \text{supp } f^+ \rightarrow [0, \infty)$ such that the sequence $((BDS) \int_E h_n d(j \circ N))_n$ is Cauchy in $\ell^\infty(B_{X'})$. It follows from Theorem 5.2 that the sequence $(\int_E h_n dN)_n$ is Cauchy in d_H . We repeat the procedure with f^- obtaining a sequence $(g_n)_n$ of simple functions. Setting $f_n := h_n - g_n$ we find the required sequence.

By Theorem 5.2 and Proposition 2.8 we get

Corollary 5.4. *Let $M, N : \Sigma \rightarrow cb(X)$ be two consistent d_H -multimeasures. Moreover let $\theta : \Omega \rightarrow \mathbb{R}$ be a measurable function.*

- (1) *If $j \circ M(E) = (BDS) \int_E \theta d(j \circ N)$ for every $E \in \Sigma$, then $\theta \geq 0$ N -a.e. and for all $E \in \Sigma$ and all $x' \in X'$ $s(x', M(E)) = \int_E \theta ds(x', N)$.*
- (2) *And conversely, if θ is non-negative and $s(x', M(E)) = \int_E \theta ds(x', N)$ for every $E \in \Sigma$, then $j \circ M(E) = (BDS) \int_E \theta d(j \circ N)$ for every $E \in \Sigma$.*

It is our aim to obtain a Radon-Nikodým theorem for Rådström embeddings of multimeasures in terms of the multimeasures themselves, not their Rådström embeddings. To achieve it we modify the definitions of *usac*, *usd* and *uss* in the following way:

Definition 5.5. Given two multimeasures $M, N : \Sigma \rightarrow cb(X)$ we say that:

(s-usac) A multimeasure $M : \Sigma \rightarrow cb(X)$ is *strongly uniformly scalarly absolutely continuous (s-usac) with respect to a multimeasure $N : \Sigma \rightarrow cb(X)$* , if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $\sum_{m=1}^n a_m x'_m \in \text{span} B_{X'}$ and each $E \in \Sigma$, if $|\sum_{m=1}^n a_m s(x'_m, N)|(E) \leq \delta$, then

$$\left| \sum_{m=1}^n a_m s(x'_m, M) \right|(E) \leq \varepsilon.$$

We denote it by $M \lll_s N$.

(s-usd) A multimeasure $M : \Sigma \rightarrow cb(X)$ is *strongly uniformly scalarly dominated (s-usd) by a multimeasure $N : \Sigma \rightarrow cb(X)$* , if there exists a positive $d \in \mathbb{R}$ such that for every $E \in \Sigma$ and every $\sum_{m=1}^n a_m x'_m \in \text{span} B_{X'}$, one has

$$\left| \sum_{m=1}^n a_m s(x'_m, M) \right|(E) \leq d \left| \sum_{m=1}^n a_m s(x'_m, N) \right|(E).$$

(s-uss) A multimeasure M is called *strongly uniformly scalarly subordinated to N (s-uss)*, if there exists a positive number $d \in \mathbb{R}$ such that for every $E \in \Sigma$ and every $\sum_{m=1}^n a_m x'_m \in \text{span} B_{X'}$, one has

$$\sum_{m=1}^n a_m s(x'_m, M(E)) \in d \overline{aco} \left\{ \sum_{m=1}^n a_m s(x'_m, N(F)) : F \in \Sigma_E \right\}. \quad (20)$$

Using these strong versions of uniform scalar absolute continuity, uniform scalar domination and uniform scalar subordination we are able to prove the following result:

Theorem 5.6. *Let $M, N : \Sigma \rightarrow cb(X)$ be two consistent d_H -multimeasures. Then the following are equivalent*

(RN3) *There exists a bounded measurable function (measurable function) $\theta : \Omega \rightarrow \mathbb{R}$ such that for all $E \in \Sigma$ and $y' \in \ell'_\infty(B_{X'})$*

$$\langle y', j \circ M(E) \rangle = \int_E \theta d\langle y', j \circ N \rangle;$$

(5.6.a) *M is (locally) strongly uniformly scalarly dominated by N ;*

(5.6.b) *M is (locally) strongly uniformly scalarly absolutely continuous with respect to N ;*

(5.6.c) *M is (locally) strongly uniformly scalarly subordinated to N .*

Proof. (RN3) $\Rightarrow$ (5.6.c)

By Theorem 5.1 the condition (RN3) implies that $j \circ M$ is s -uss to $j \circ N$. Assume that $d > 0$ is such that

$$j \circ M(E) \in d \overline{\text{aco}}\{j \circ N(F) : F \in \Sigma_E\} \in cb(\ell_\infty(B_{X'})) \quad \text{for every } E \in \Sigma.$$

This means that for every $y' \in \ell'_\infty(B_{X'})$, we have

$$\begin{aligned} \langle y', j \circ M(E) \rangle &\leq d \sup\{\langle y', z \rangle : z \in \overline{\text{aco}}\{j \circ N(F) : F \in \Sigma_E\}\} \\ &= d \sup s(y', \text{aco}\{j \circ N(F) : F \in \Sigma_E\}) \\ &\leq d \max\{s(y', \text{co}\{j \circ N(F) : F \in \Sigma_E\}), s(y', \text{co}\{-j \circ N(F) : F \in \Sigma_E\})\} \\ &= d \max\{s(y', \{j \circ N(F) : F \in \Sigma_E\}), s(y', \{-j \circ N(F) : F \in \Sigma_E\})\}. \end{aligned}$$

So, if $\{x'_1, \dots, x'_n\} \subset B_{X'}$ and $a_i \in \mathbb{R}$, $i = 1, \dots, n$, then

$$\begin{aligned} \sum_{i=1}^n a_i s(x'_i, M(E)) &= \sum_{i=1}^n a_i \langle e_{x'_i}, j \circ M(E) \rangle = \left\langle \sum_{i=1}^n a_i e_{x'_i}, j \circ M(E) \right\rangle \\ &\leq d \max \left\{ s \left(\sum_{i=1}^n a_i e_{x'_i}, \{j \circ N(F) : F \in \Sigma_E\} \right), \right. \\ &\quad \left. s \left(\sum_{i=1}^n a_i e_{x'_i}, \{-j \circ N(F) : F \in \Sigma_E\} \right) \right\} \\ &= d \max \left\{ \sup_{F \in \Sigma_E} \left\langle \sum_{i=1}^n a_i e_{x'_i}, j \circ N(F) \right\rangle, \sup_{F \in \Sigma_E} \left\langle \sum_{i=1}^n a_i e_{x'_i}, -j \circ N(F) \right\rangle \right\} \\ &= d \max \left\{ \sup_{F \in \Sigma_E} \left[\sum_{i=1}^n a_i \langle e_{x'_i}, j \circ N(F) \rangle \right], \sup_{F \in \Sigma_E} \left[- \sum_{i=1}^n a_i \langle e_{x'_i}, j \circ N(F) \rangle \right] \right\} \\ &= d \max \left\{ \sup_{F \in \Sigma_E} \left[\sum_{i=1}^n a_i s(x'_i, N(F)) \right], \sup_{F \in \Sigma_E} \left[- \sum_{i=1}^n a_i s(x'_i, N(F)) \right] \right\} \\ &= d \max \left\{ \sup_{F \in \Sigma_E} \left[\sum_{i=1}^n a_i s(x'_i, N(F)) \right], - \inf_{F \in \Sigma_E} \left[\sum_{i=1}^n a_i s(x'_i, N(F)) \right] \right\} \\ &= d \sup \overline{\text{aco}} \left\{ \sum_{i=1}^n a_i s(x'_i, N(F)) : F \in \Sigma_E \right\}. \end{aligned}$$

Since we have also

$$- \sum_{i=1}^n a_i s(x'_i, M(E)) \leq d \sup \overline{\text{aco}} \left\{ \sum_{i=1}^n a_i s(x'_i, N(F)) : F \in \Sigma_E \right\},$$

we deduce that

$$\sum_{i=1}^n a_i s(x'_i, M(E)) \in d \overline{\text{aco}} \left\{ \sum_{i=1}^n a_i s(x'_i, N(F)) : F \in \Sigma_E \right\}.$$

(5.6.c) $\Rightarrow$ (5.6.a) By the assumption we have

$$\begin{aligned} \pm \sum_{i=1}^n a_i s(x'_i, M(E)) &\leq d \sup \overline{\text{aco}} \left\{ \sum_{i=1}^n a_i s(x'_i, N(F)) : F \in \Sigma_E \right\} \\ &\leq d \sup \overline{\text{aco}} \left\{ \left| \sum_{i=1}^n a_i s(x'_i, N) \right| (F) : F \in \Sigma_E \right\} \leq d \left| \sum_{i=1}^n a_i s(x'_i, N) \right| (E). \end{aligned}$$

Hence, for every set $E \in \Sigma$ we have

$$\left| \sum_{i=1}^n a_i s(x'_i, M)(E) \right| \leq d \left| \sum_{i=1}^n a_i s(x'_i, N) \right| (E)$$

and the standard calculation gives the required inequality for the variations.

(5.6.a) $\Rightarrow$ (5.6.b) is obvious.

(5.6.b) $\Rightarrow$ (RN3)

Assume that for each $\varepsilon > 0$ there exists $\varepsilon/2 > \delta > 0$ such that for each $E \in \Sigma$ and each $\sum_{m=1}^n a_m x'_m \in \text{span } B_{X'}$ we have

$$\left| \sum_{m=1}^n a_m s(x'_m, N) \right| (E) \leq \delta, \Rightarrow \left| \sum_{m=1}^n a_m s(x'_m, M) \right| (E) \leq \varepsilon/2.$$

It follows that

$$\left| \left\langle \sum_{m=1}^n a_m e_{x'_m}, j \circ N \right\rangle \right| (E) \leq \delta \Rightarrow \left| \left\langle \sum_{m=1}^n a_m e_{x'_m}, j \circ M \right\rangle \right| (E) \leq \varepsilon/2. \quad (21)$$

Let $E \in \Sigma$ and $y' \in \ell'_\infty(B_{X'})$ be such that $|\langle y', j \circ N \rangle|(E) \leq \delta/2$. Then, let $E_{y'}^+ \cup E_{y'}^- = E$ be the Hahn decomposition of E with respect to $\langle y', j \circ N \rangle$.

Since $\{e_{x'} : x' \in B_{X'}\}$ is norming for $\ell_\infty(B_{X'})$ there exists $z' \in \text{span}\{e_{x'} : x' \in B_{X'}\}$ such that

$$|\langle y' - z', j \circ N(E_{y'}^+) \rangle| < \delta/4 \quad \text{and} \quad |\langle y' - z', j \circ N(E_{y'}^-) \rangle| < \delta/4$$

$$\text{and} \quad |\langle y' - z', j \circ M(E_{y'}^+) \rangle| < \delta/4 \quad \text{and} \quad |\langle y' - z', j \circ M(E_{y'}^-) \rangle| < \delta/4.$$

We then have

$$\begin{aligned} |\langle z', j \circ N \rangle(E)| &= |\langle z', j \circ N(E_{y'}^+) \rangle + \langle z', j \circ N(E_{y'}^-) \rangle| \\ &= [\langle z', j \circ N(E_{y'}^+) \rangle - \langle y', j \circ N(E_{y'}^+) \rangle] + [\langle z', j \circ N(E_{y'}^-) \rangle - \langle y', j \circ N(E_{y'}^-) \rangle] \\ &\quad + [\langle y', j \circ N(E_{y'}^+) \rangle + \langle y', j \circ N(E_{y'}^-) \rangle] < \delta. \end{aligned}$$

By assumption we have $|\langle z', j \circ M \rangle|(E) \leq \varepsilon/2$. Now we follow the reverse way:

$$\begin{aligned} |\langle y', j \circ M \rangle|(E) &= \langle y', j \circ M(E_{y'}^+) \rangle + \langle y', j \circ M(E_{y'}^-) \rangle \\ &= [\langle y', j \circ M(E_{y'}^+) \rangle - \langle z', j \circ M(E_{y'}^+) \rangle] + [\langle y', j \circ M(E_{y'}^-) \rangle - \langle z', j \circ M(E_{y'}^-) \rangle] \\ &\quad + [\langle z', j \circ M(E_{y'}^+) \rangle + \langle z', j \circ M(E_{y'}^-) \rangle] < \delta + \varepsilon/2 < \varepsilon. \end{aligned}$$

That means that $j \circ M$ is s -usac with respect to $j \circ N$ and so, in virtue of Theorem 5.1 $j \circ M$ has the derivative.

The proof of the local versions is analogous to that given in Theorem 4.5 where the construction is given for the general case of an arbitrary multimeasure. $\square$

Remark 5.7. Theorems 5.1 and 5.6 allow us to observe that $j \circ M \lll j \circ N$ if and only if M is s -usac with respect to N . Analogously $j \circ M$ is usd by $j \circ N$ if and only if M is s -usd by N .

Remark 5.8. It turns out that the condition (RN3) implies the following one (that coincides with the sub condition in case of vector measures):

(sub) $\forall E \in \Sigma \ M(E) \subset d\overline{\text{aco}}[\mathcal{R}(N_E) \cup \mathcal{R}(-N_E)]$.

In fact, by Theorem 5.1 $j \circ M$ is subordinated to $j \circ N$. Assume that $d > 0$ is such that, for every $E \in \Sigma$,

$$j \circ M(E) \in d\overline{\text{aco}}\{j \circ N(F) : F \in \Sigma_E\} \in cb(\ell_\infty(B_{X'})).$$

This means that for every $y' \in \ell'_\infty(B_{X'})$, we have

$$\begin{aligned} \langle y', j \circ M(E) \rangle &\leq d s(y', \overline{\text{aco}}\{j \circ N(F) : F \in \Sigma_E\}) = d s(y', \text{aco}\{j \circ N(F) : F \in \Sigma_E\}) \\ &\leq d \max\{s(y', \text{co}\{j \circ N(F) : F \in \Sigma_E\}), s(y', \text{co}\{-j \circ N(F) : F \in \Sigma_E\})\} \\ &= d \max\{s(y', \{j \circ N(F) : F \in \Sigma_E\}), s(y', \{-j \circ N(F) : F \in \Sigma_E\})\}. \end{aligned}$$

In particular, if $x' \in B_{X'}$, then

$$\begin{aligned} s(x', M(E)) &= \langle e_{x'}, j \circ M(E) \rangle \leq d s(e_{x'}, \text{aco}\{j \circ N(F) : F \in \Sigma_E\}) \\ &= d \max\{s(e_{x'}, \{j \circ N(F) : F \in \Sigma_E\}), s(e_{x'}, \{-j \circ N(F) : F \in \Sigma_E\})\} \\ &= d \max\{s(e_{x'}, \{j \circ N(F) : F \in \Sigma_E\}), s\{\langle e_{x'}, -j \circ N(F) \rangle : F \in \Sigma_E\}\} \\ &= d \max\{s(x', N(F)) : F \in \Sigma_E\}, s\{\langle x', -N(F) \rangle : F \in \Sigma_E\}\} \\ &= d \max\{s(x', \mathcal{R}(N_E)), s(x', \mathcal{R}(-N_E))\} \leq d s(x', \mathcal{R}(N_E) \cup \mathcal{R}(-N_E)) \\ &\leq d s(x', \overline{\text{aco}}[\mathcal{R}(N_E) \cup \mathcal{R}(-N_E)]) = s(x', d\overline{\text{aco}}[\mathcal{R}(N_E) \cup \mathcal{R}(-N_E)]). \end{aligned}$$

In particular, if $x \in M(E)$, then $\langle x', x \rangle \leq s(x', d\overline{\text{aco}}[\mathcal{R}(N_E) \cup \mathcal{R}(-N_E)])$. In virtue of the Hahn-Banach theorem $x \in d\overline{\text{aco}}[\mathcal{R}(N_E) \cup \mathcal{R}(-N_E)]$. Thus,

$$M(E) \subset d\overline{\text{aco}}[\mathcal{R}(N_E) \cup \mathcal{R}(-N_E)].$$

The example given in Remark 4.6 proves that in case of arbitrary multimeasures the condition *sub* is sometimes essentially weaker than (RN3). The current example shows that also in case of pointless multimeasures the *sub* does not guarantee the existence of the Radon-Nikodým derivative. It is enough to take a pointless multimeasure $N : \Sigma \rightarrow cwk(X)$ and define $M : \Sigma \rightarrow cwk(X)$ by

$$M(E) := \overline{\text{aco}}[\mathcal{R}(N_E) \cup \mathcal{R}(-N_E)].$$

The next theorem provides a Radon-Nikodým representation without invoking to the Rådström embedding.

Theorem 5.9. *Let $M, N : \Sigma \rightarrow cb(X)$ be two consistent d_H -multimeasures. If M is s -usac (s -usd or s -uss) with respect to N , then there exists a non-negative, measurable, bounded, BDS_m -integrable with respect to N function $\theta : \Omega \rightarrow \mathbb{R}$ such that*

$$M(E) = \int_E \theta dN, \quad \forall E \in \Sigma.$$

Proof. By Theorem 5.6 the s -usac (s -usd, s -uss) condition on M and N is equivalent to property (RN3). Then, for every $x' \in B_{X'}$, it is

$$\langle e_{x'}, j \circ M(E) \rangle = \int_E \theta d\langle e_{x'}, j \circ N \rangle$$

which is equivalent to $s(x', M(E)) = \int_E \theta ds(x', N)$.

Then, by Proposition 2.8, θ is non-negative N -a.e. Now, by (RN3) and Theorem 5.2, we have that θ is BDS_m -integrable with respect to N and

$$M(E) = \int_E \theta dN. \quad \square$$

Corollary 5.10. *Let $M, N : \Sigma \rightarrow cb(X)$ satisfy the hypotheses of Theorem 5.9. If $j \circ M$ can be represented as a BDS -integral with respect to $j \circ N$ with a non-negative bounded density θ , then also M can be represented as the BDS_m -integral of θ with respect to N .*

6. Examples

Example 6.1. Let μ be a finite measure on (Ω, Σ) and $N(E) := [0, \mu(E)]$ be an interval valued multimeasure (for results and applications of this kind of multimeasure see for example [10, 13] and the references therein). We can observe that if f is a scalar, bounded measurable function, then f is BDS_m -integrable with respect to N and

$$M_f(E) = \left[- \int_E f^- d\mu, \int_E f^+ d\mu \right].$$

Example 6.2. Let let N be as in Example 6.1 and let now $M(E) := [0, \nu(E)]$ where ν is a finite measure on (Ω, Σ) , equivalent with μ . Let $\nu(E) = \int_E \theta d\mu$ for all $E \in \Sigma$. Then there exists a partition $\Omega = \bigcup_n \Omega_n$ such that $0 \leq \theta \leq n$ μ -a.e. on Ω_n .

Moreover, let $\xi : \mathbb{R} \rightarrow \mathbb{R}$ be defined by: $\xi(a) = a$ if $a > 0$, otherwise $\xi(a) = 0$. So, for every $a \in \mathbb{R}$ $s(a, N(E)) = \xi(a)\mu(E)$. If $\alpha, \beta \in \mathbb{R}$ and $x' = a, y' = b \in \mathbb{R}$ then

$$|\alpha s(a, N) + \beta s(b, N)|(E) = |\alpha \xi(a) + \beta \xi(b)|\mu(E)$$

and

$$|\alpha s(a, M) + \beta s(b, M)|(E) = |\alpha \xi(a) + \beta \xi(b)|\nu(E).$$

It follows that if $E \in \Sigma_{\Omega_n}$, then

$$\begin{aligned} |\alpha s(a, M) + \beta s(b, M)|(E) &= |\alpha s(a, N) + \beta s(b, N)|(E) \cdot \frac{\nu(E)}{\mu(E)} \\ &\leq n |\alpha s(a, N) + \beta s(b, N)|(E). \end{aligned}$$

Therefore the multimeasure M is locally *usd* with respect to the multimeasure N and can be represented as a BDS_m -integral with respect to N . One can easily check that $M(E) = \int_E \theta dN$.

Example 6.3. Assume that X is a Banach space and consider $([0, 1], \mathcal{L}, \lambda)$, where λ is the Lebesgue measure and $\mathcal{L}$ is the family of all Lebesgue measurable subsets of $[0, 1]$. Let $f, g : [0, 1] \rightarrow X$ be two Pettis integrable functions and define $\Gamma(t) := \text{co}\{0, f(t)\}$, $\Delta(t) := \text{co}\{0, g(t)\}$ as the multifunctions determined by f and g respectively. They are $ck(X)$ -valued and Pettis integrable (see [7, Propositions 2.3 and 2.5]). We observe that, for every $x' \in X'$,

$$s(x', \Gamma(t)) = \langle x', f \rangle^+(t), \quad s(x', \Delta(t)) = \langle x', g \rangle^+(t). \quad (22)$$

If $M, N : \mathcal{L} \rightarrow cwk(X)$ are the indefinite multivalued Pettis integrals of f, g , then, $\forall x' \in X', \forall E \in \mathcal{L}$,

$$s(x', M(E)) = \int_E \langle x', f \rangle^+ d\lambda, \quad \& \quad s(x', N(E)) = \int_E \langle x', g \rangle^+ d\lambda. \quad (23)$$

Assume that there exists a measurable scalar function θ which is BDS_m -integrable with respect to N and

$$M(E) = \int_E \theta dN, \quad \forall E \in \mathcal{L}.$$

Since M, N are positive, θ is non-negative. It is a consequence of Definition 2.3 that

$$s(x', M(E)) = \int_E \theta ds(x', N) \quad \text{for all } x' \in X' \text{ and } E \in \mathcal{L}.$$

Due to (23) we have

$$\int_E \langle x', f \rangle^+ d\lambda = \int_E \theta ds(x', N) = \int_E \theta \langle x', g \rangle^+ d\lambda \quad \text{for all } x' \in X' \text{ and } E \in \mathcal{L}.$$

It follows that for each $x' \in X'$ we have $\langle x', f \rangle = \theta \langle x', g \rangle$ μ -a.e.

Example 6.4. Assume that X is a Banach space and that μ is a non-trivial atomless finite measure on (Ω, Σ) . Let $f_1, f_2 : \Omega \rightarrow X$ be scalarly integrable functions and $r_1, r_2 : \Omega \rightarrow (0, \infty)$ be μ -integrable functions. Following [24] or [7, Example 2.13], we define for $i = 1, 2$ the multifunctions $\Gamma_i : \Omega \rightarrow cb(X)$ by the formulae $\Gamma_i(\omega) = B(f_i(\omega), r_i(\omega))$, where $B(x, \delta)$ is the closed ball with its center in x and of radius δ . Then $s(x', \Gamma_i(\omega)) = \langle x', f_i(\omega) \rangle + r_i(\omega)\|x'\|$ and so each Γ_i is scalarly integrable. If f_1, f_2 are Pettis integrable, then each Γ_i is Pettis integrable in $cb(X)$. Moreover,

$$(P) \int_E \Gamma_i d\mu = B\left((P) \int_E f_i d\mu, \int_E r_i d\mu\right) \quad i = 1, 2.$$

Let $M(E) := (P) \int_E \Gamma_1 d\mu$ and $N(E) := (P) \int_E \Gamma_2 d\mu$. One can easily check that

M and N are d_H -measures. Suppose that $M(E) = \int_E \theta dN$, $E \in \Sigma$, i.e.

$$\forall x' \in X', \forall E \in \Sigma, \quad s(x', M(E)) = \int_E \theta^+ ds(x', N) + \int_E \theta^- ds(x', -N).$$

That yields the equality

$$\forall x' \in X', \forall E \in \Sigma, \quad \int_E \langle x', f_1 \rangle d\mu + \|x'\| \int_E r_1 d\mu = \int_E \theta \langle x', f_2 \rangle d\mu + \|x'\| \int_E r_2 |\theta| d\mu.$$

But the sets $\{\int_E f_i d\mu : E \in \Sigma\}$, $i = 1, 2$ are relatively weakly compact and so there exists $0 \neq x'_0 \in X'$ vanishing on these sets. It follows that $r_1 = r_2 |\theta|$ μ -a.e. Appealing to [24, Lemma] we find that for each $x' \in X'$ one has $\langle x', f_1 \rangle = \theta \langle x', f_2 \rangle$ μ -a.e. (the exceptional set depends on x') i.e. f_1 is scalarly equivalent to $\theta \cdot f_2$.

One can easily check that also the reverse implication holds true: if there exists a measurable function θ such that f_1 is scalarly equivalent to θf_2 and $r_1 = r_2 |\theta|$ μ -a.e., then $M = \int \theta dN$.

A similar calculation shows that $j \circ M$ can be represented as a *BDS*-integral with respect to $j \circ N$ if and only if the above θ is non-negative.

If we assume only that r_1 and r_2 are only positive and measurable, then we are in the local version of the RN-theorem 5.6.

Example 6.5. If we assume in Example 6.4 that $X = Z'$ and f_1, f_2 are only Gelfand integrable then Γ_1, Γ_2 are weak* multimeasures (see [24] for definitions). Applying now Remark 4.7, we see that

$$\forall z \in Z, \forall E \in \Sigma, \quad s(z, M(E)) = \int_E \theta^+ ds(z, N) + \int_E \theta^- ds(z, -N)$$

if and only there is θ such that $r_1 = r_2 \cdot |\theta|$ μ -a.e. and f_1 is weak* scalarly equivalent to $\theta \cdot f_2$.

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