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New Semi-Norm of Semi-Hilbertian Space Operators and its Application

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We introduce a new semi-norm of operators on a semi-Hilbertian space, which generalizes the A-numerical radius and A-operator semi-norm. We study the basic properties of this semi-norm, including upper and lower bounds for it. As an application of this new semi-norm, we obtain upper bounds for the A-numerical radius of semi-Hilbertian space operators, which are sharper than the earlier ones.

Keywords: A-numerical radius, A-operator semi-norm, semi-Hilbertian space, inequality.

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1. Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with usual inner product $\langle .,. \rangle$ and $\|.\|$ be the corresponding norm on $\mathcal{H}$, induced by the usual inner product $\langle .,. \rangle$. Throughout this paper, O is the *zero operator* on $\mathcal{H}$ and I is the *identity operator* on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, $R(T)$ and $N(T)$ denote the *range space* of T and *null space* of T, respectively. We denote, the *norm closure* of $R(T)$ in $\mathcal{H}$ by $\overline{R(T)}$ and the *orthogonal projection* onto $\overline{R(T)}$ by P_T. Let A be a self-adjoint operator in $\mathcal{B}(\mathcal{H})$. Then A is called *positive* if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$. Let $\mathcal{B}(\mathcal{H})^+$ denote the cone of all positive operators on $\mathcal{B}(\mathcal{H})$, i.e.,

$$\mathcal{B}(\mathcal{H})^+ = \{A \in \mathcal{B}(\mathcal{H}) \ : \ \langle Ax, x \rangle \geq 0 \ \forall x \in \mathcal{H}\}.$$

Henceforth we reserve the symbol A for a positive operator on $\mathcal{H}$, i.e., $A \in \mathcal{B}(\mathcal{H})^+$. A functional $\langle .,. \rangle_A \ : \ \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$ defined by $\langle x, y \rangle_A = \langle Ax, y \rangle$ for all $x, y \in \mathcal{H}$, is a *semi-inner product* on $\mathcal{H}$ and correspondingly $(\mathcal{H}, \langle .,. \rangle_A)$ is a semi-inner product space. The semi-norm induced by the semi-inner product $\langle .,. \rangle_A$ is denoted by $\|.\|_A$, i.e., $\|x\|_A = \sqrt{\langle x, x \rangle_A} = \|A^{1/2}x\|$ for all $x \in \mathcal{H}$. Clearly, $\|x\|_A = 0$ if and only if $x \in N(A)$ and therefore, $\|.\|_A$ is a norm if and only if A is one-one. Also, the semi-normed space $(\mathcal{H}, \|.\|_A)$ is *complete* if $R(A)$ is closed in $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, the A-*operator semi-norm* of T is given by

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$$\|T\|_A = \sup\left\{ \frac{\|Tx\|_A}{\|x\|_A} \ : \ x \in \overline{R(A)}, x \neq 0 \right\}.$$

Now, $\|T\|_A < \infty$ if there exists a constant $c > 0$ such that $\|Tx\|_A \leq c\|x\|_A$ for all $x \in \mathcal{H}$ and in this case, T is called *A-bounded operator*. For $T \in \mathcal{B}(\mathcal{H})$, the *$A$-minimum modulus* of T, denoted by $m_A(T)$ and is defined by

$$m_A(T) = \inf\left\{ \frac{\|Tx\|_A}{\|x\|_A} \ : \ x \in \overline{R(A)}, x \neq 0 \right\}.$$

For $T \in \mathcal{B}(\mathcal{H})$, an operator $R \in \mathcal{B}(\mathcal{H})$ is an *A-adjoint* of T if $\langle Tx, y\rangle_A = \langle x, Ry\rangle_A$ for every $x, y \in \mathcal{H}$. For any $T \in \mathcal{B}(\mathcal{H})$, the existence of an A-adjoint of T is not guaranteed. In fact, an operator $T \in \mathcal{B}(\mathcal{H})$ may admit none, one or many A-adjoint. By Douglas' Theorem ([12]) we have that for an operator $T \in \mathcal{B}(\mathcal{H})$ if $R(T^*A) \subseteq R(A)$, then T admits an A-adjoint. Let $\mathcal{B}_A(\mathcal{H})$ denote the collection of all operators in $\mathcal{B}(\mathcal{H})$ which admit A-adjoint, i.e.,

$$\mathcal{B}_A(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : R(T^*A) \subseteq R(A)\}.$$

For $T \in \mathcal{B}_A(\mathcal{H})$, the operator equation $AX = T^*A$ has a unique solution $T^{\sharp_A}$, satisfying $R(T^{\sharp_A}) \subseteq \overline{R(A)}$. Here T^* is the adjoint of T. Clearly, for $T \in \mathcal{B}_A(\mathcal{H})$, $AT^{\sharp_A} = T^*A$. Note that $T^{\sharp_A} = A^{\dagger}T^*A$, where $A^{\dagger}$ is the Moore-Penrose inverse of A, (see [16]). Again, by applying Douglas' theorem, we have, $\mathcal{B}_{A^{1/2}}(\mathcal{H})$ is the set of all operators in $\mathcal{B}(\mathcal{H})$ which admit $A^{1/2}$ adjoint, i.e., $R(T^*A^{1/2}) \subseteq R(A^{1/2})$. It is well-known that

$$\mathcal{B}_{A^{1/2}}(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) \ : \ \exists\, c > 0 \ \text{ such that } \ \|Tx\|_A \leq c\|x\|_A \ \ \forall x \in \mathcal{H}\}.$$

Therefore, the operators in $\mathcal{B}_{A^{1/2}}(\mathcal{H})$ are called *A-bounded operators*. Notice that $\mathcal{B}_A(\mathcal{H})$ and $\mathcal{B}_{A^{1/2}}(\mathcal{H})$ are sub-algebras of $\mathcal{B}(\mathcal{H})$. Moreover,

$$\mathcal{B}_A(\mathcal{H}) \subseteq \mathcal{B}_{A^{1/2}}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$$

and these inclusions become equalities if A is one-one and has closed range. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *A-self-adjoint* if AT is self-adjoint, i.e., $AT = T^*A$. If T is A-self-adjoint, then $T \in \mathcal{B}_A(\mathcal{H})$ and in general, $T \neq T^{\sharp_A}$. For $T \in \mathcal{B}_A(\mathcal{H})$, $T = T^{\sharp_A}$ if and only if T is A-self-adjoint and $R(T) \subseteq \overline{R(A)}$. For $T, S \in \mathcal{B}_A(\mathcal{H})$, we have $((T^{\sharp_A})^{\sharp_A})^{\sharp_A} = T^{\sharp_A}$, $(T^{\sharp_A})^{\sharp_A} = P_ATP_A$, and $(TS)^{\sharp_A} = S^{\sharp_A}T^{\sharp_A}$. Also, for $T, S \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$, we have $\|TS\|_A \leq \|T\|_A\|S\|_A$ and $\|Tx\|_A \leq \|T\|_A\|x\|_A$ for all $x \in \mathcal{H}$. Clearly, $T^{\sharp_A}T$ and $TT^{\sharp_A}$ are A-self-adjoint operators and satisfy

$$\|TT^{\sharp_A}\|_A = \|T^{\sharp_A}T\|_A = \|T^{\sharp_A}\|_A^2 = \|T\|_A^2.$$

For $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$, the *$A$-numerical range* and the *A-numerical radius* of T, denoted by $W_A(T)$ and $w_A(T)$, are defined respectively as

$$W_A(T) = \{\langle Tx, x\rangle_A \ : \ x \in \mathcal{H}, \|x\|_A = 1\}.$$

and
$$w_A(T) = \sup\{|\langle Tx, x\rangle_A| \ : \ x \in \mathcal{H}, \|x\|_A = 1\}.$$

Similar as the A-numerical radius, the A-Crawford number of $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$, denoted as $c_A(T)$ and is defined as

$$c_A(T) = \inf\{|\langle Tx, x\rangle_A| \ : \ x \in \mathcal{H}, \|x\|_A = 1\}.$$

It is well known that the A-numerical radius and the A-operator semi-norm are equivalent semi-norm on $\mathcal{B}_{A^{1/2}}(\mathcal{H})$, satisfying that for every $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$,

$$\frac{1}{2}\|T\|_A \leq w_A(T) \leq \|T\|_A.$$

The numerical radius inequalities and the A-numerical radius inequalities have been studied by many mathematicians over the years; we refer the reader to [5, 9, 10, 15, 19] for some classical numerical radius inequalities and [7, 11, 14, 17, 20] for some A-numerical radius inequalities. Motivated by the new norm [18], for each $\alpha \in [0,1]$, we introduce A_α-semi-norm on $\mathcal{B}_{A^{1/2}}(\mathcal{H})$, defined as follows:

$$\|T\|_{A_\alpha} \;=\; \sup\left\{\sqrt{\alpha|\langle Tx, x\rangle_A|^2 + (1-\alpha)\|Tx\|_A^2} \;:\; x \in \mathcal{H}, \|x\|_A = 1\right\}.$$

If $\alpha = 1$, then $\|T\|_{A_\alpha} = w_A(T)$ and if $\alpha = 0$, then $\|T\|_{A_\alpha} = \|T\|_A$. Considering $A = I$ we get, recently introduced the α-norm on $\mathcal{B}(\mathcal{H})$, (see in [6, 18]).

In this article, we study the basic properties of this semi-norm. We also develop upper and lower bounds for A_α-semi-norm. Applying these inequalities we obtain upper bounds for the A-numerical radius which improve and generalize the existing ones.

2. Main results

We begin this section with the following proposition, proof of which follows easily from the definition of A_α-semi-norm.

Proposition 2.1. $\| \,.\, \|_{A_\alpha}$ *defines a semi-norm on* $\mathcal{B}_{A^{1/2}}(\mathcal{H})$. *This semi-norm is equivalent to the A-numerical radius and the A-operator semi-norm, satisfying the following inequalities:*

(i) $w_A(T) \leq \|T\|_{A_\alpha} \leq \sqrt{(4-3\alpha)}\,w_A(T)$,

(ii) $\max\left\{\frac{1}{2}, \sqrt{1-\alpha}\right\}\|T\|_A \leq \|T\|_{A_\alpha} \leq \|T\|_A$.

Next we prove the following equivalent conditions.

Proposition 2.2. *Let* $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$ *and let* $\alpha \neq 0, 1$. *Then the following conditions are equivalent:*

(i) $\|T\|_{A_\alpha} = \sqrt{\alpha w_A^2(T) + (1-\alpha)\|T\|_A^2}$.

(ii) *There exists a sequence* $\{x_n\}$ *in* $\mathcal{H}$ *with* $\|x_n\|_A = 1$ *such that* $\lim_{n\to\infty}\|Tx_n\|_A = \|T\|_A$ *and* $\lim_{n\to\infty}|\langle Tx_n, x_n\rangle_A| = w_A(T)$.

Proof. (i) $\Rightarrow$ (ii): Clearly from the definition of $\|T\|_{A_\alpha}$ it follows that there exists a sequence $\{x_n\}$ in $\mathcal{H}$ with $\|x_n\|_A = 1$ such that

$$\|T\|_{A_\alpha} = \lim_{n\to\infty}\sqrt{\alpha|\langle Tx_n, x_n\rangle_A|^2 + (1-\alpha)\|Tx_n\|_A^2}.$$

Now, $\{|\langle Tx_n, x_n\rangle_A|\}$ and $\{\|Tx_n\|_A\}$ are both bounded sequences of real numbers and so there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{|\langle Tx_{n_k}, x_{n_k}\rangle_A|\}$ and $\{\|Tx_{n_k}\|_A\}$ are convergent.

Therefore, we get

$$\begin{aligned}
\alpha w_A^2(T) + (1-\alpha)\|T\|_A^2 &= \|T\|_{A_\alpha}^2 \\
&= \lim_{k\to\infty} \left\{\alpha|\langle Tx_{n_k}, x_{n_k}\rangle_A|^2 + (1-\alpha)\|Tx_{n_k}\|_A^2\right\} \\
&= \alpha \lim_{k\to\infty} |\langle Tx_{n_k}, x_{n_k}\rangle_A|^2 + (1-\alpha) \lim_{k\to\infty} \|Tx_{n_k}\|_A^2 \\
&\leq \alpha w_A^2(T) + (1-\alpha)\|T\|_A^2.
\end{aligned}$$

This implies that $\lim_{k\to\infty} \|Tx_{n_k}\|_A = \|T\|_A$ and $\lim_{k\to\infty} |\langle Tx_{n_k}, x_{n_k}\rangle_A| = w_A(T)$.

(ii) $\Rightarrow$ (i): From the definition of A_α-semi-norm we have

$$\begin{aligned}
\|T\|_{A_\alpha} &= \sup_{\|x\|_A=1} \sqrt{\alpha|\langle Tx, x\rangle_A|^2 + (1-\alpha)\|Tx\|_A^2} \\
&\geq \lim_{n\to\infty} \sqrt{\alpha|\langle Tx_n, x_n\rangle_A|^2 + (1-\alpha)\|Tx_n\|_A^2} \\
&= \sqrt{\alpha w_A^2(T) + (1-\alpha)\|T\|_A^2}.
\end{aligned}$$

This completes the proof. □

Next, we study an important basic property for the A_α-semi-norm. First we recall that an operator $U \in \mathcal{B}_A(\mathcal{H})$ is said to be A-unitary if $\|Ux\|_A = \|U^{\sharp_A}x\|_A = \|x\|_A$, for all $x \in \mathcal{H}$. It was shown in [1] that an operator $U \in \mathcal{B}_A(\mathcal{H})$ is A-unitary if and only if $U^{\sharp_A}U = (U^{\sharp_A})^{\sharp_A}U^{\sharp_A} = P_A$.

Proposition 2.3. *If $T \in \mathcal{B}_A(\mathcal{H})$, then*

$$\|UTU^{\sharp_A}\|_{A_\alpha} = \|T\|_{A_\alpha} \text{ for every } A\text{-unitary operator } U.$$

Proof. For $S \in \mathcal{B}_A(\mathcal{H})$, we consider a set L_S as follows:

$$L_S = \left\{\left(\sqrt{\alpha}\langle Sx, x\rangle_A, \sqrt{1-\alpha}\|Sx\|_A\right) : x \in \mathcal{H}, \|x\|_A = 1\right\}.$$

Let $(\lambda, \mu) \in L_{UTU^{\sharp_A}}$. Then there exists $x \in \mathcal{H}$ with $\|x\|_A = 1$ such that we have $\lambda = \sqrt{\alpha}\langle UTU^{\sharp_A}x, x\rangle_A$ and $\mu = \sqrt{1-\alpha}\|UTU^{\sharp_A}x\|_A$. It is easy to verify that we have $\lambda = \sqrt{\alpha}\langle TU^{\sharp_A}x, U^{\sharp_A}x\rangle_A$ and $\mu = \sqrt{1-\alpha}\|TU^{\sharp_A}x\|_A$. Since $\|U^{\sharp_A}x\|_A = \|x\|_A$ for all $x \in \mathcal{H}$, so $(\lambda, \mu) \in L_T$. Therefore, we have $L_{UTU^{\sharp_A}} \subseteq L_T$.

Now, let $(\beta, \gamma) \in L_T$. Then, there exist $x \in \mathcal{H}$ with $\|x\|_A = 1$ such that we have $\beta = \sqrt{\alpha}\langle Tx, x\rangle_A$ and $\gamma = \sqrt{1-\alpha}\|Tx\|_A$, i.e., $\gamma = \sqrt{(1-\alpha)\langle UTx, UTx\rangle_A}$. Let $x = P_Ax + y$ where $y \in N(A)$. Since $N(A)$ is invariant subspace under T, i.e., $T(N(A)) \subseteq N(A)$, so by using this argument we get,

$$\beta = \sqrt{\alpha}\langle TP_Ax, P_Ax\rangle_A \text{ and } \gamma = \sqrt{(1-\alpha)\langle UTP_Ax, UTP_Ax\rangle_A}.$$

Since $U^{\sharp_A}U = P_A$, so we have, $\beta = \sqrt{\alpha}\langle UTU^{\sharp_A}Ux, Ux\rangle_A$ and

$$\gamma = \sqrt{(1-\alpha)\langle UTU^{\sharp_A}Ux, UTU^{\sharp_A}Ux\rangle_A} = \sqrt{1-\alpha}\|UTU^{\sharp_A}Ux\|_A.$$

This implies $\|Ux\|_A = \|x\|_A$ for all $x \in \mathcal{H}$, so $(\beta, \gamma) \in L_{UTU^{\sharp_A}}$. Therefore, we have $L_T \subseteq L_{UTU^{\sharp_A}}$. Hence, we conclude that $L_T = L_{UTU^{\sharp_A}}$. This completes the proof. □

In the following theorem we obtain a lower bound for the A_α-semi-norm.

Theorem 2.4. *Let $T \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\|T\|_{A_\alpha}^2 \geq \max\Big\{\alpha w_A^2(T) + (1-\alpha)c_A(T^{\sharp_A}T), \alpha c_A^2(T) + (1-\alpha)\|T\|_A^2, \\ 2\sqrt{\alpha(1-\alpha)}w_A(T)\sqrt{c_A(T^{\sharp_A}T)}, 2\sqrt{\alpha(1-\alpha)}c_A(T)\|T\|_A\Big\}.$$

Proof. Let $x \in \mathcal{H}$ with $\|x\|_A = 1$. Then we get,

$$\begin{aligned}\|T\|_{A_\alpha}^2 &\geq \alpha|\langle Tx, x\rangle_A|^2 + (1-\alpha)\|Tx\|_A^2 = \alpha|\langle Tx, x\rangle_A|^2 + (1-\alpha)\langle T^{\sharp_A}Tx, x\rangle_A \\ &\geq \alpha|\langle Tx, x\rangle_A|^2 + (1-\alpha)c_A(T^{\sharp_A}T).\end{aligned}$$

Taking supremum over all x in $\mathcal{H}$ with $\|x\|_A = 1$, we get that

$$\|T\|_{A_\alpha}^2 \geq \alpha w_A^2(T) + (1-\alpha)c_A(T^{\sharp_A}T). \tag{1}$$

Proceeding similarly we also get

$$\|T\|_{A_\alpha}^2 \geq \alpha c_A^2(T) + (1-\alpha)\|T\|_A^2. \tag{2}$$

Now, we also have

$$\begin{aligned}\|T\|_{A_\alpha}^2 &\geq \alpha|\langle Tx, x\rangle_A|^2 + (1-\alpha)\|Tx\|_A^2 \geq 2\sqrt{\alpha(1-\alpha)}|\langle Tx, x\rangle_A|\|Tx\|_A \\ &\geq 2\sqrt{\alpha(1-\alpha)}|\langle Tx, x\rangle_A|\sqrt{c_A(T^{\sharp_A}T)}.\end{aligned}$$

Taking supremum over all x in $\mathcal{H}$ with $\|x\|_A = 1$, we get

$$\|T\|_{A_\alpha}^2 \geq 2\sqrt{\alpha(1-\alpha)}w_A(T)\sqrt{c_A(T^{\sharp_A}T)}. \tag{3}$$

Similarly,
$$\|T\|_{A_\alpha}^2 \geq 2\sqrt{\alpha(1-\alpha)}c_A(T)\|T\|_A. \tag{4}$$

Combining (1), (2), (3) and (4) we get our desired inequality. □

Next, we obtain upper bounds for the A_α-semi-norm of the product of two operators.

Theorem 2.5. *Let $T, S \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$ and $\alpha \neq 1$. Then*

$$\|TS\|_{A_\alpha} \leq \min\left\{\frac{2}{\sqrt{1-\alpha}}, \frac{1}{(1-\alpha)}, 4\right\}\|T\|_{A_\alpha}\|S\|_{A_\alpha}.$$

Proof. From the definition of A_α-semi-norm we get,

$$\begin{aligned}\|TS\|_{A_\alpha}^2 &= \sup_{\|x\|_A=1}\left\{\alpha|\langle TSx, x\rangle_A|^2 + \beta\|TSx\|_A^2\right\} \leq \alpha w_A^2(TS) + (1-\alpha)\|TS\|_A^2 \\ &\leq \alpha\|TS\|_A^2 + (1-\alpha)\|TS\|_A^2 \leq \|T\|_A^2\|S\|_A^2 \\ &\leq 4w_A^2(T)\|S\|_A^2, && \text{as } \|T\|_A \leq 2w_A(T) \\ &\leq 4w_A^2(T)\,(1/(1-\alpha))\,\|S\|_{A_\alpha}^2, && \text{by Proposition 2.1} \\ &\leq \frac{4}{1-\alpha}\|T\|_{A_\alpha}^2\|S\|_{A_\alpha}^2, && \text{by Proposition 2.1.}\end{aligned}$$

Thus,
$$\|TS\|_{A_\alpha} \leq \sqrt{\frac{4}{1-\alpha}}\, \|T\|_{A_\alpha} \|S\|_{A_\alpha}. \tag{5}$$

Proceeding as above, and using $\sqrt{1-\alpha}\,\|T\|_A \leq \|T\|_{A_\alpha}$ and $\sqrt{1-\alpha}\,\|S\|_A \leq \|S\|_{A_\alpha}$

we get,
$$\|TS\|_{A_\alpha} \leq \sqrt{\frac{1}{(1-\alpha)^2}}\, \|T\|_{A_\alpha} \|S\|_{A_\alpha}. \tag{6}$$

Finally, using $\|T\|_A \leq 2w_A(T)$ and $\|S\|_A \leq 2w_A(S)$, and by Proposition 2.1 we get,
$$\|TS\|_{A_\alpha} \leq 4\, \|T\|_{A_\alpha} \|S\|_{A_\alpha}. \tag{7}$$

Combining (5),(6) and (7) we get the required inequality. □

To prove next theorem we need the following lemma.

Lemma 2.6. *Let $T, S \in \mathcal{B}_A(\mathcal{H})$. Then the following inequalities hold:*
(i) *If $TS = ST$, then $w_A(TS) \leq 2w_A(T)w_A(S)$.*
(ii) *If $(TS)^{\sharp_A} = T^{\sharp_A}S$, then $w_A(TS) \leq w_A(T)\,\|S\|_A$.*

Proof. (i) Without loss of generality we assume that $w_A(T) = 1$ and $w_A(S) = 1$. Now, we have
$$\begin{aligned}
w_A(TS) &= w_A\Big(\frac{1}{4}\left((T+S)^2 - (T-S)^2\right)\Big)\\
&\leq \frac{1}{4}\left(w_A((T+S)^2) + w_A((T-S)^2)\right)\\
&\leq \frac{1}{4}\left(w_A^2(T+S) + w_A^2(T-S)\right), \text{ as } w_A(T^n) \leq w_A^n(T) \ \ \forall n \in \mathbb{N}\\
&\leq \frac{1}{4}\left((w_A(T)+w_A(S))^2 + (w_A(T)+w_A(S))^2\right) = 2.
\end{aligned}$$

This completes the proof of (i). For (ii) see [20, Cor. 3.3]. □

Theorem 2.7. *Let $T, S \in \mathcal{B}_A(\mathcal{H})$. Then the following inequalities hold:*
(i) If $TS = ST$ and $\alpha \neq 1$, then
$$\|TS\|_{A_\alpha} \leq \sqrt{\left(4\alpha + \frac{1}{1-\alpha}\right)}\, \|T\|_{A_\alpha} \|S\|_{A_\alpha}.$$

(ii) If $(TS)^{\sharp_A} = T^{\sharp_A}S$ and $\alpha \neq 1$, then
$$\|TS\|_{A_\alpha} \leq \sqrt{1+\alpha}\, \min\left\{2, \frac{1}{\sqrt{1-\alpha}}\right\} \|T\|_{A_\alpha} \|S\|_{A_\alpha}.$$

Proof. (i) From the definition of A_α-semi-norm we get,
$$\begin{aligned}
\|TS\|_{A_\alpha}^2 &= \sup_{\|x\|_A=1} \left\{\alpha|\langle TSx, x\rangle_A|^2 + (1-\alpha)\|TSx\|_A^2\right\}\\
&\leq \alpha w_A^2(TS) + (1-\alpha)\,\|TS\|_A^2\\
&\leq 4\alpha w_A^2(T)w_A^2(S) + (1-\alpha)\,\|T\|_A^2\,\|S\|_A^2, \quad \text{by Lemma 2.6(i)}\\
&\leq \left(4\alpha + \frac{1}{1-\alpha}\right) \|T\|_{A_\alpha}^2 \|S\|_{A_\alpha}^2, \qquad \text{by Proposition 2.1.}
\end{aligned}$$

Therefore, $$\|TS\|_{A_\alpha} \leq \sqrt{\left(4\alpha + \frac{1}{1-\alpha}\right)} \|T\|_{A_\alpha} \|S\|_{A_\alpha}.$$

(ii) Using the definition of A_α-semi-norm we get,

$$\begin{aligned}
\|TS\|^2_{A_\alpha} &= \sup_{\|x\|_A=1} \left\{\alpha|\langle TSx, x\rangle_A|^2 + (1-\alpha)\|TSx\|^2_A\right\} \\
&\leq \alpha w_A^2(TS) + (1-\alpha)\|TS\|^2_A \\
&\leq \alpha w_A^2(T)\|S\|^2_A + (1-\alpha)\|T\|^2_A\|S\|^2_A, && \text{by Lemma 2.6(ii)} \\
&\leq \alpha\|T\|^2_{A_\alpha}\|S\|^2_A + \|T\|^2_{A_\alpha}\|S\|^2_A, && \text{by Proposition 2.1} \\
&= (\alpha+1)\|T\|^2_{A_\alpha}\|S\|^2_A \\
&\leq (\alpha+1)\min\left\{2, \frac{1}{\sqrt{1-\alpha}}\right\}^2 \|T\|^2_{A_\alpha}\|S\|^2_{A_\alpha}, && \text{by Proposition 2.1.}
\end{aligned}$$

Therefore, $$\|TS\|_{A_\alpha} \leq \sqrt{(\alpha+1)}\min\left\{2, \frac{1}{\sqrt{1-\alpha}}\right\}\|T\|_{A_\alpha}\|S\|_{A_\alpha}.$$ □

Next we need the following two notations: For $T \in \mathcal{B}_A(\mathcal{H})$, let $\Re_A(T) = \frac{1}{2}(T + T^{\sharp_A})$ and $\Im_A(T) = \frac{1}{2\mathrm{i}}(T - T^{\sharp_A})$. Then, T can be expressed as $T = \Re_A(T) + \mathrm{i}\Im_\mathrm{A}(\mathrm{T})$. By using this decomposition we prove the following inequalities.

Theorem 2.8. *Let $T \in \mathcal{B}_A(\mathcal{H})$. Then the following inequalities hold:*

(i) $$\|T\|^2_{A_\alpha} \geq \max\left\{\frac{1}{2}\left\|\frac{\alpha}{4}(T^{\sharp_A}T + TT^{\sharp_A}) + (1-\alpha)T^{\sharp_A}T\right\|_A, \frac{1}{3}\left\|\frac{\alpha}{2}(T^{\sharp_A}T + TT^{\sharp_A}) + (1-\alpha)T^{\sharp_A}T\right\|_A\right\},$$

(ii) $$\|T\|^2_{A_\alpha} \leq \left\|\frac{\alpha}{2}(T^{\sharp_A}T + TT^{\sharp_A}) + (1-\alpha)T^{\sharp_A}T\right\|_A,$$

(iii) $$w_A^2(T) \leq \min_\alpha \left\|\frac{\alpha}{2}(T^{\sharp_A}T + TT^{\sharp_A}) + (1-\alpha)T^{\sharp_A}T\right\|_A \leq \frac{1}{2}\left\|T^{\sharp_A}T + TT^{\sharp_A}\right\|_A.$$

Proof. We observe that $(\Re_A(T^{\sharp_A}))^{\sharp_A} = \Re_A(T^{\sharp_A})$ and $(\Im_A(T^{\sharp_A}))^{\sharp_A} = \Im_A(T^{\sharp_A})$, and

$$\Re_A^2(T^{\sharp_A}) + \Im_A^2(T^{\sharp_A}) = \frac{T^{\sharp_A}(T^{\sharp_A})^{\sharp_A} + (T^{\sharp_A})^{\sharp_A}T^{\sharp_A}}{2} = \left(\frac{T^{\sharp_A}T + TT^{\sharp_A}}{2}\right)^{\sharp_A}.$$

Let us consider $x \in \mathcal{H}$ with $\|x\|_A = 1$. Then, we have

$$\begin{aligned}
|\langle Tx, x\rangle_A|^2 = |\langle x, Tx\rangle_A|^2 &= |\langle T^{\sharp_A}x, x\rangle_A|^2 \\
&= |\langle\left(\Re_A(T^{\sharp_A}) + i\Im_A(T^{\sharp_A})\right)x, x\rangle_A|^2 \\
&= |\langle\Re_A(T^{\sharp_A})x, x\rangle_A|^2 + |\langle\Im_A(T^{\sharp_A})x, x\rangle_A|^2 \\
&\geq \frac{1}{2}|\langle(\Re_A(T^{\sharp_A}) \pm \Im_A(T^{\sharp_A}))x, x\rangle_A|^2.
\end{aligned}$$

Taking supremum over all x in $\mathcal{H}$ with $\|x\|_A = 1$, we get

$$\begin{aligned}
&\sup_{\|x\|_A=1} \alpha|\langle Tx, x\rangle_A|^2 \\
&\geq \sup_{\|x\|_A=1} \frac{\alpha}{2}|\langle(\Re_A(T^{\sharp_A}) \pm \Im_A(T^{\sharp_A}))x, x\rangle_A|^2 \\
&= \frac{\alpha}{2}\left\|\Re_A(T^{\sharp_A}) \pm \Im_A(T^{\sharp_A})\right\|_A^2 \\
&= \frac{\alpha}{2}\left\|(\Re_A(T^{\sharp_A}) \pm \Im_A(T^{\sharp_A}))^2\right\|_A, \text{ since } \Re_A(T^{\sharp_A}) \pm \Im_A(T^{\sharp_A}) \ \ \textit{is A-self-adjoint} \\
&= \sup_{\|x\|_A=1} \langle\frac{\alpha}{2}(\Re_A(T^{\sharp_A}) \pm \Im_A(T^{\sharp_A}))^2 x, x\rangle_A \\
&\geq \sup_{\|x\|_A=1} \langle\frac{\alpha}{4}\left((\Re_A(T^{\sharp_A}) + \Im_A(T^{\sharp_A}))^2 + (\Re_A(T^{\sharp_A}) - \Im_A(T^{\sharp_A}))^2\right)x, x\rangle_A \\
&\geq \sup_{\|x\|_A=1} \langle\frac{\alpha}{2}(\Re_A^2(T^{\sharp_A}) + \Im_A^2(T^{\sharp_A})^2)x, x\rangle_A.
\end{aligned}$$

Hence, $$\|T\|_{A_\alpha}^2 \geq \sup_{\|x\|_A=1} \langle\frac{\alpha}{2}(\Re_A^2(T^{\sharp_A}) + \Im_A^2(T^{\sharp_A}))x, x\rangle_A.$$

We also have $\|T\|_{A_\alpha}^2 \geq \sup_{\|x\|_A=1} \langle(1-\alpha)T^{\sharp_A}Tx, x\rangle_A$. Therefore,

$$\begin{aligned}
2\|T\|_{A_\alpha}^2 &\geq \sup_{\|x\|_A=1} \left\langle\left(\frac{\alpha}{2}(\Re_A^2(T^{\sharp_A}) + \Im_A^2(T^{\sharp_A})) + (1-\alpha)T^{\sharp_A}T\right)x, x\right\rangle_A \\
&= \left\|\frac{\alpha}{2}(\Re_A^2(T^{\sharp_A}) + \Im_A^2(T^{\sharp_A})) + (1-\alpha)T^{\sharp_A}T\right\|_A \\
&= \left\|\frac{\alpha}{2}\left(\frac{T^{\sharp_A}T + TT^{\sharp_A}}{2}\right)^{\sharp_A} + (1-\alpha)(T^{\sharp_A}T)^{\sharp_A}\right\|_A \\
&= \left\|\frac{\alpha}{4}(T^{\sharp_A}T + TT^{\sharp_A}) + (1-\alpha)T^{\sharp_A}T\right\|_A.
\end{aligned}$$

Hence, $$\|T\|_{A_\alpha}^2 \geq \frac{1}{2}\left\|\frac{\alpha}{4}(T^{\sharp_A}T + TT^{\sharp_A}) + (1-\alpha)T^{\sharp_A}T\right\|_A. \tag{8}$$

Again, by using similar arguments as above, we have that

$$\begin{aligned}
3\|T\|_{A_\alpha}^2 &\geq \sup_{\|x\|_A=1} \left\langle\left(\alpha(\Re_A^2(T^{\sharp_A}) + \Im_A^2(T^{\sharp_A})) + (1-\alpha)T^{\sharp_A}T\right)x, x\right\rangle_A \\
&= \left\|\frac{\alpha}{2}(T^{\sharp_A}T + TT^{\sharp_A}) + (1-\alpha)T^{\sharp_A}T\right\|_A.
\end{aligned}$$

Hence, $$\|T\|_{A_\alpha}^2 \geq \frac{1}{3}\left\|\frac{\alpha}{2}(T^{\sharp_A}T + TT^{\sharp_A}) + (1-\alpha)T^{\sharp_A}T\right\|_A. \tag{9}$$

Combining (8) and (9) we get inequality (i). Now we prove inequality (ii). We have

$$\begin{aligned}
|\langle Tx, x\rangle_A|^2 &= |\langle\Re_A(T^{\sharp_A})x, x\rangle_A|^2 + |\langle\Im_A(T^{\sharp_A})x, x\rangle_A|^2 \\
&\leq \|\Re_A(T^{\sharp_A})x\|_A^2 + \|\Im_A(T^{\sharp_A})x\|_A^2 \\
&= \langle\Re_A^2(T^{\sharp_A})x, x\rangle_A + \langle\Im_A^2(T^{\sharp_A})x, x\rangle_A \\
&= \langle(\Re_A^2(T^{\sharp_A}) + \Im_A^2(T^{\sharp_A}))x, x\rangle_A.
\end{aligned}$$

So,
$$\begin{aligned}
&\alpha|\langle Tx, x\rangle_A|^2 + (1-\alpha)\|Tx\|_A^2\\
&\leq \langle \alpha(\Re_A^2(T^{\sharp_A}) + \Im_A^2(T^{\sharp_A}))x, x\rangle_A + \langle (1-\alpha)T^{\sharp_A}Tx, x\rangle_A\\
&= \left\langle \left(\alpha(\Re_A^2(T^{\sharp_A}) + \Im_A^2(T^{\sharp_A})) + (1-\alpha)T^{\sharp_A}T\right)x, x\right\rangle_A\\
&\leq \left\|\alpha(\Re_A^2(T^{\sharp_A}) + \Im_A^2(T^{\sharp_A})) + (1-\alpha)T^{\sharp_A}T\right\|_A\\
&= \left\|\alpha\left(\frac{T^{\sharp_A}T + TT^{\sharp_A}}{2}\right)^{\sharp_A} + (1-\alpha)(T^{\sharp_A}T)^{\sharp_A}\right\|_A\\
&= \left\|\frac{\alpha}{2}(T^{\sharp_A}T + TT^{\sharp_A}) + (1-\alpha)T^{\sharp_A}T\right\|_A.
\end{aligned}$$

Therefore, by taking supremum over all $x \in \mathcal{H}$ with $\|x\|_A = 1$, we get the inequality (ii). Now, using Proposition 2.1 in the inequality (ii), we get

$$w_A^2(T) \leq \left\|\frac{\alpha}{2}(T^{\sharp_A}T + TT^{\sharp_A}) + (1-\alpha)T^{\sharp_A}T\right\|_A.$$

This holds for all α, so considering minimum over α, we get that

$$w_A^2(T) \leq \min_{\alpha}\left\|\frac{\alpha}{2}(T^{\sharp_A}T + TT^{\sharp_A}) + (1-\alpha)T^{\sharp_A}T\right\|_A.$$

This is the first inequality in (iii). The second inequality in (iii) follows from the case $\alpha = 1$. □

Remark 2.9. Zamani in [20, Theorem 2.10] proved that for $T \in \mathcal{B}_A(\mathcal{H})$,

$$w_A^2(T) \leq \frac{1}{2}\|TT^{\sharp_A} + T^{\sharp_A}T\|_A.$$

Clearly, the first inequality obtained in Theorem 2.8(iii) improves on [20, Theorem 2.10]. Here we consider an example to show proper improvement. Let

$$T = \begin{pmatrix} 0 & 0 & 0\\ 2 & 0 & 0\\ 0 & 1 & 0\end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}.$$

Then, simple calculations show that

$$\left\|\frac{\alpha}{2}(T^{\sharp_A}T + TT^{\sharp_A}) + (1-\alpha)T^{\sharp_A}T\right\|_A = \frac{23}{8} \text{ for } \alpha = \frac{7}{8}$$

and
$$\frac{1}{2}\left\|T^{\sharp_A}T + TT^{\sharp_A}\right\|_A = 3.$$

Hence, for the matrices T and A, we have

$$\min_{\alpha}\left\|\frac{\alpha}{2}(T^{\sharp_A}T + TT^{\sharp_A}) + (1-\alpha)T^{\sharp_A}T\right\|_A < \frac{1}{2}\left\|T^{\sharp_A}T + TT^{\sharp_A}\right\|_A.$$

To prove the next theorem we first recall the following lemma.

Lemma 2.10. ([4, Lemma 2.11]) *Let $a, b, e \in \mathcal{H}$ with $\|e\|_A = 1$. Then,*

$$|\langle a, e\rangle_A \langle e, b\rangle_A| \leq \frac{1}{2} \left(|\langle a, b\rangle_A| + \|a\|_A \|b\|_A\right).$$

Theorem 2.11. *Let $T \in \mathcal{B}_A(\mathcal{H})$. Then the following inequalities hold:*

(i) $$\|T\|^2_{A_\alpha} \leq \frac{\alpha}{2} w_A(T^2) + \left\| \frac{\alpha}{4}(T^{\sharp_A}T + TT^{\sharp_A}) + (1-\alpha)T^{\sharp_A}T \right\|_A,$$

(ii) $$\begin{aligned} w_A^2(T) &\leq \min_{\alpha} \left\{ \frac{\alpha}{2} w_A(T^2) + \left\| \frac{\alpha}{4}(T^{\sharp_A}T + TT^{\sharp_A}) + (1-\alpha)T^{\sharp_A}T \right\|_A \right\} \\ &\leq \frac{1}{2} w_A(T^2) + \frac{1}{4} \|T^{\sharp_A}T + TT^{\sharp_A}\|_A. \end{aligned}$$

Proof. Let $x \in \mathcal{H}$ with $\|x\|_A = 1$. Taking $a = Tx$, $b = T^{\sharp_A}x$ and $e = x$ in Lemma 2.10, and using the arithmetic-geometric mean inequality, we get

$$\begin{aligned} |\langle Tx, x\rangle_A|^2 &\leq \frac{1}{2} \left(|\langle T^2x, x\rangle_A| + \|Tx\|_A \|T^{\sharp_A}x\|_A \right) \\ &\leq \frac{1}{2} |\langle T^2x, x\rangle_A| + \frac{1}{2} \langle T^{\sharp_A}Tx, x\rangle_A^{1/2} \langle TT^{\sharp_A}x, x\rangle_A^{1/2} \\ &\leq \frac{1}{2} |\langle T^2x, x\rangle_A| + \frac{1}{4} \left(\langle T^{\sharp_A}Tx, x\rangle_A + \langle TT^{\sharp_A}x, x\rangle_A \right) \\ &= \frac{1}{2} |\langle T^2x, x\rangle_A| + \frac{1}{4} \langle (T^{\sharp_A}T + TT^{\sharp_A})x, x\rangle_A. \end{aligned}$$

Therefore,

$$\begin{aligned} &\alpha |\langle Tx, x\rangle_A|^2 + (1-\alpha)\|Tx\|_A^2 \\ &\leq \frac{\alpha}{2} |\langle T^2x, x\rangle_A| + \left\langle \left(\frac{\alpha}{4}(T^{\sharp_A}T + TT^{\sharp_A}) + (1-\alpha)T^{\sharp_A}T \right) x, x \right\rangle_A \\ &\leq \frac{\alpha}{2} w_A(T^2) + \left\| \frac{\alpha}{4}(T^{\sharp_A}T + TT^{\sharp_A}) + (1-\alpha)T^{\sharp_A}T \right\|_A. \end{aligned}$$

Taking supremum over all x in $\mathcal{H}$ with $\|x\|_A = 1$, we get

$$\|T\|^2_{A_\alpha} \leq \frac{\alpha}{2} w_A(T^2) + \left\| \frac{\alpha}{4}(T^{\sharp_A}T + TT^{\sharp_A}) + (1-\alpha)T^{\sharp_A}T \right\|_A.$$

This is the inequality in (i). Now using Proposition 2.1 in (i), we get

$$w_A^2(T) \leq \frac{\alpha}{2} w_A(T^2) + \left\| \frac{\alpha}{4}(T^{\sharp_A}T + TT^{\sharp_A}) + (1-\alpha)T^{\sharp_A}T \right\|_A,$$

This holds for all α, so taking infimum over all α, we get the first inequality in (ii). The remaining inequality follows from the case $\alpha = 1$. $\square$

It should be mentioned here that the inequality,

$$w_A^2(T) \leq \tfrac{1}{2} w_A(T^2) + \tfrac{1}{4} \|T^{\sharp_A}T + TT^{\sharp_A}\|_A$$

is known and was given in [20, Th. 2.11]. Clearly, the first inequality in Theorem 2.11 (ii) is sharper than the above mentioned inequality. To show that the inequality in Theorem 2.11 (ii) is strictly sharper, we consider the following example.

Example 2.12. Let $T = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Then, simple calculations show that

$$\frac{\alpha}{2} w_A(T^2) + \left\| \frac{\alpha}{4}(T^{\sharp_A}T + TT^{\sharp_A}) + (1-\alpha)T^{\sharp_A}T \right\|_A = \frac{6\sqrt{2}+20}{13} \approx 2.19117549033$$

for $\alpha = \frac{12}{13}$ and $\frac{1}{2} w_A(T^2) + \frac{1}{4}\|T^{\sharp_A}T + TT^{\sharp_A}\|_A = \frac{3+\sqrt{2}}{2} \approx 2.20710678119.$

Hence, for the matrices T and A, the inequality obtained in Theorem 2.11(ii) is sharper than that obtained in [20, Theorem 2.11].

Next we note that for each $x \in \mathcal{H}$ with $\|x\|_A = 1$, we have

$$\begin{aligned} \alpha|\langle Tx, x\rangle_A|^2 + (1-\alpha)\|Tx\|_A^2 &\leq (1-\alpha)\|Tx\|_A^2 + \alpha\|T^{\sharp_A}x\|_A^2 \\ &= (1-\alpha)\langle Tx, Tx\rangle_A + \alpha\langle T^{\sharp_A}x, T^{\sharp_A}x\rangle_A \\ &= (1-\alpha)\langle T^{\sharp_A}Tx, x\rangle_A + \alpha\langle TT^{\sharp_A}x, x\rangle_A \\ &\leq \|(1-\alpha)T^{\sharp_A}T + \alpha TT^{\sharp_A}\|_A. \end{aligned}$$

This leads us to the inequality

$$\|T\|_{A_\alpha}^2 \leq \|(1-\alpha)T^{\sharp_A}T + \alpha TT^{\sharp_A}\|_A, \tag{10}$$

and by using Proposition 2.1 (i) we also get,

$$w_A^2(T) \leq \min_{\alpha} \|\alpha T^{\sharp_A}T + (1-\alpha)TT^{\sharp_A}\|_A. \tag{11}$$

Here we note that the inequality,

$$w_A^2(T) \leq \min_{\mu\in[0,1]} \|\mu T^{\sharp_A}T + (1-\mu)TT^{\sharp_A}\|_A,$$

recently obtained in [8, Theorem 1], is the same as the one obtained in (11).

We end this section with the observation that by applying the A_α-semi-norm of operators in $\mathcal{B}_A(\mathcal{H})$ we always get the A-numerical radius inequalities of operators in $\mathcal{B}_A(\mathcal{H})$, which improve on the existing ones.

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