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An Application of the Generalised James' Weak Compactness Theorem II

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We show that some recent results on weak compactness in Banach spaces may also be obtained from a generalised version of James' weak compactness theorem.

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This note is a sequel to the paper [4]. We shall begin with some basic notation and assumed terminology. In this paper all normed linear spaces will be over the field of real numbers (denoted $\mathbb{R}$). If $(X, \|\cdot\|)$ is a normed linear space, then $(X^*, \|\cdot\|^*)$ denotes its dual space, i.e., X^* is the set of all continuous linear functionals on X and $\|x^*\|^* = \sup_{x \in S_X} |x^*(x)|$, where $S_X := \{x \in X : \|x\| = 1\}$.

If X is a set and $f\colon X \to (-\infty, \infty]$ is a function, then $\mathrm{Dom}(f) := \{x \in X : f(x) < \infty\}$ and we say that f is a *proper* function if $\mathrm{Dom}(f) \neq \varnothing$.

We shall call a proper function $f : X \to (-\infty, \infty]$, defined on a vector space X, (over the real numbers) a *convex* function if, for each $x, y \in \mathrm{Dom}(f)$ and each $0 < \lambda < 1$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

and we shall call a function $f : (X, \tau) \to (-\infty, \infty]$ from a topological space (X, τ) *lower semicontinuous* if, for each $r \in \mathbb{R}$, $\{x \in X : f(x) \leq r\}$ is a closed subset of (X, τ). This is equivalent to saying that the *epigraph* of f,

$$\mathrm{epi}(f) := \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$$

is closed in $X \times \mathbb{R}$, (endowed with the product topology).

In addition to these definitions we need the notion of a subdifferential. Suppose that $f : X \to (-\infty, \infty]$ is a proper function on a normed linear space $(X, \|\cdot\|)$ and $x \in \mathrm{Dom}(f)$. Then we define the *subdifferential* $\partial f(x)$ by,

$$\partial f(x) := \{x^* \in X^* : x^*(y - x) \leq f(y) - f(x) \text{ for all } y \in \mathrm{Dom}(f)\}.$$

The primary purpose of this note is to show that some of the recent interesting results on James' weak compactness theorem that are contained in the papers [1, 2, 6] may also be obtained via the generalised James weak compactness theorem, [5, Theorem 4.13].

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To state the first theorem we need one more ingredient, that is, the notion of the Fenchel conjugate.

Let $(X, \|\cdot\|)$ be a normed linear space and let $f : X \to (-\infty, \infty]$ be a proper function, then the *Fenchel conjugate* of f is the function $f^* : X^* \to (-\infty, \infty]$ defined by,

$$f^*(x^*) := \sup_{x\in X}(x^*-f)(x) = \sup_{x\in \mathrm{Dom}(f)}(x^*-f)(x) = \sup_{(x,r)\in \mathrm{epi}(f)}(x^*, -1)(x, r), \text{ for all } x^* \in X^*.$$

Note that in the above equation $(x^*, -1)(x, r) := x^*(x) - r$. Note also that if we endow $X \times \mathbb{R}$ with the norm $\|(x, r)\|_1 := \|x\| + |r|$ for all $(x, r) \in X \times \mathbb{R}$, then $(x^*, -1) \in (X \times \mathbb{R})^*$. It is well-known and easily verified that f^* is convex and lower semicontinuous on $(X^*, \|\cdot\|^*)$, see [5, Theorem 5.2 part (i)].

For any x in a normed linear space $(X, \|\cdot\|)$ we shall define $\widehat{x} \in X^{**}$ by, $\widehat{x}(x^*) := x^*(x)$ for all $x^* \in X^*$. Then, $x \mapsto \widehat{x}$, is a linear isometric embedding of $(X, \|\cdot\|)$ into $(X^{**}, \|\cdot\|^{**})$. In particular, if $(X, \|\cdot\|)$ is a Banach space, then $\widehat{X}$ is a closed linear subspace of $(X^{**}, \|\cdot\|^{**})$. If A is a subset of a topological space, then $\mathrm{int}(A)$ denotes the *interior* of A, i.e., the largest open subset of A.

Theorem 1. ([4, Theorem 2.5]) *Let $f : X \to (-\infty, \infty]$ be a proper function defined on a Banach space $(X, \|\cdot\|)$ and let $g : X^* \to (-\infty, \infty]$ be a proper lower semicontinuous convex function such that*

(i) $\mathrm{Dom}(g)$ *is an open subset of* $\mathrm{Dom}(f^*)$ *and*

(ii) $\partial g(x^*) \cap \widehat{X} \neq \varnothing$ *for all* $x^* \in \mathrm{Dom}(g)$.

If we have $\partial f^*(x^*) \cap \widehat{X} \neq \varnothing$ for all $x^* \in \mathrm{Dom}(g)$, then for every $0 < \varepsilon$ and every $x_0^* \in \mathrm{int}(\mathrm{Dom}\, f^*)$, $\{x \in X : f^*(x_0^*) - \varepsilon \leq (x_0^* - f)(x)\}$ is relatively weakly compact. In particular, if $0 \in \mathrm{int}(\mathrm{Dom}\, f^*)$, then for every $\gamma \in \mathbb{R}$, $\{x \in X : f(x) \leq \gamma\}$ is relatively weakly compact.

Let us recall that a subset of a Banach space $(X, \|\cdot\|)$ is *relatively weakly compact* if its closure, in the weak topology of $(X, \|\cdot\|)$, is compact with respect to the weak topology of $(X, \|\cdot\|)$.

In consequence of Theorem 1 the question arises of what, if anything, can be said if $x_0^* \in \mathrm{Dom}(f^*) \setminus \mathrm{int}(\mathrm{Dom}(f^*))$.

We will say that a subset A of a Banach space $(X, \|\cdot\|)$ is *weak*-closed* (*relatively weak*-closed*) if

$$\overline{\widehat{A}}^{w^*} = \widehat{A} \quad (\overline{\widehat{A}}^{w^*} \subseteq \widehat{X}).$$

The Banach-Dieudonné Theorem, [3, Theorem 4.44] says that a convex subset C of a Banach space $(X, \|\cdot\|)$ is weak*-closed if, and only if, for each $n \in \mathbb{N}$, $nB_X \cap C$ is weak compact. Here, $B_X := \{x \in X : \|x\| \leq 1\}$, i.e., B_X is the closed unit ball in $(X, \|\cdot\|)$.

For a proper function $f : X \to (-\infty, \infty]$ defined on a Banach space $(X, \|\cdot\|)$ the *closed convex envelop* of f, denoted f_c, is defined as follows

$$f_c(x) := \inf\{r \in \mathbb{R} : (x, r) \in \overline{\mathrm{co}}(\mathrm{epi}(f))\} \quad \text{for each } x \in X.$$

Here we use the convention that if $x \in X$ and $\{r \in \mathbb{R} : (x, r) \in \overline{\mathrm{co}}(\mathrm{epi}(f))\} = \varnothing$, then $\inf\{r \in \mathbb{R} : (x, r) \in \overline{\mathrm{co}}(\mathrm{epi}(f))\} = \infty$. f_c is the largest lower semicontinuous convex function that is smaller than f. Therefore, if $x_0 \in X$ and $y^* \in \partial f(x_0)$, then

$f(x_0) + y^*(x - x_0) \leq f_c(x) \leq f(x)$ for all $x \in X$. In particular, $f(x_0) = f_c(x_0)$. Thus, $\partial f(x_0) \subseteq \partial f_c(x_0)$. Furthermore, $\mathrm{epi}(f_c) = \overline{\mathrm{co}}(\mathrm{epi}(f))$ and so

$$f^*(x^*) = \sup_{(x,r)\in \mathrm{epi}(f)} (x^*, -1)(x, r) = \sup_{(x,r)\in\overline{\mathrm{co}}(\mathrm{epi}(f))} (x^*, -1)(x, r) = f_c^*(x^*) \text{ for all } x^* \in X^*. \quad (*)$$

This follows from the general fact that if y^* is a continuous linear functional on a normed linear space $(Y, \|\cdot\|')$, then we have $\sup_{x\in A} y^*(x) = \sup_{x\in\overline{\mathrm{co}}(A)} y^*(x)$. Here, $Y = X \times \mathbb{R}$, $y^* = (x^*, -1)$ and $A = \mathrm{epi}(f)$.

Theorem 2. (Main Theorem) *Let $f : X \to (-\infty, \infty]$ be a proper function defined on a Banach space $(X, \|\cdot\|)$ and let $g : X^* \to (-\infty, \infty]$ be a proper lower semicontinuous convex function such that*

(i) $\mathrm{Dom}(g)$ *is an open subset of* $\mathrm{Dom}(f^*)$ *and*

(ii) $\partial g(x^*) \cap \widehat{X} \neq \varnothing$ *for all* $x^* \in \mathrm{Dom}(g)$.

If we have $\partial f^*(x^*) \cap \widehat{X} \neq \varnothing$ for all $x^* \in \mathrm{Dom}(g)$, then for every $0 < \varepsilon$ and every $x_0^* \in \mathrm{Dom}(f^*)$, $\{x \in X : f^*(x_0^*) - \varepsilon \leq (x_0^* - f)(x)\}$ is relatively weak-closed. In particular, if f is bounded from below, i.e., $0 \in \mathrm{Dom}(f^*)$, then for every $\gamma \in \mathbb{R}$, $\{x \in X : f(x) \leq \gamma\}$ is relatively weak*-closed.*

Proof. Let shall first consider the case when $f : X \to (\infty, \infty]$ is a proper lower semicontinuous convex function. Let $x_0^* \in \mathrm{Dom}(f^*)$ and let ε be an arbitrary positive real number, then set

$$C := \{x \in X : f^*(x_0^*) - \varepsilon \leq (x_0^* - f)(x)\}.$$

Clearly, C is a norm closed and convex subset of $(X, \|\cdot\|)$. So to prove that C is weak*-closed it is sufficient to show that for each $n \in \mathbb{N}$, $nB_X \cap C$ is weak compact. To this end, let $n \in \mathbb{N}$ and let y_0^* be a fixed element of $\mathrm{Dom}(g)$. Let ε' be any positive real number greater than $n\|y_0^* - x_0^*\| + \varepsilon + f^*(y_0^*) - f^*(x_0^*)$ and let $x \in nB_X \cap C$. Then,

$$\begin{aligned}
f^*(y_0^*) &= [f^*(x_0^*) - \varepsilon] + [\varepsilon + f^*(y_0^*) - f^*(x_0^*)] \\
&\leq (x_0^* - f)(x) + [\varepsilon + f^*(y_0^*) - f^*(x_0^*)], \quad \text{since } x \in C \\
&= (y_0^* - f)(x) + (x_0^* - y_0^*)(x) + [\varepsilon + f^*(y_0^*) - f^*(x_0^*)] \\
&\leq (y_0^* - f)(x) + \|x_0^* - y_0^*\|\|x\| + [\varepsilon + f^*(y_0^*) - f^*(x_0^*)] \\
&\leq (y_0^* - f)(x) + n\|x_0^* - y_0^*\| + [\varepsilon + f^*(y_0^*) - f^*(x_0^*)], \quad \text{since } \|x\| \leq n \\
&\leq (y_0^* - f)(x) + \varepsilon'.
\end{aligned}$$

Therefore, $x \in \{y \in X : f^*(y_0^*) - \varepsilon' \leq (y_0^* - f)(y)\}$ and so

$$nB_X \cap C \subseteq \{y \in X : f^*(y_0^*) - \varepsilon' \leq (y_0^* - f)(y)\}.$$

By Theorem 1, $\{y \in X : f^*(y_0^*) - \varepsilon' \leq (y_0^* - f)(y)\}$ is relatively weakly compact and so $nB_X \cap C$ is weak compact. Hence, $\{x \in X : f^*(x_0^*) - \varepsilon \leq (x_0^* - f)(x)\}$ is weak*-closed.

We now consider the general case. It follows from Equation $(*)$ that $\mathrm{Dom}(f^*) = \mathrm{Dom}(f_c^*)$ and so $\mathrm{Dom}(g)$ is an open subset of $\mathrm{Dom}(f_c^*)$. Further, by $(*)$ again, $\partial f^*(x^*) = \partial f_c^*(x^*)$ for all $x^* \in X^*$ and so $\partial f_c^*(x^*) \cap \widehat{X} \neq \varnothing$ for all $x^* \in \mathrm{Dom}(g)$.

Therefore, by the special case that we just considered, we have that for every $0 < \varepsilon$, $\{x \in X : f_c^*(x_0^*) - \varepsilon \leq (x_0^* - f_c)(x)\}$ is relatively weak*-closed. However, as we have $f^*(x_0^*) = f_c^*(x_0^*)$ and $f_c(x) \leq f(x)$ for all $x \in X$ we have that

$$\{x \in X : f^*(x_0^*) - \varepsilon \leq (x_0^* - f)(x)\} \subseteq \{x \in X : f_c^*(x_0^*) - \varepsilon \leq (x_0^* - f_c)(x)\} \text{ for each } 0 < \varepsilon.$$

This completes the proof of the general case. □

It follows from [4, Lemma 2.1] and [4, Lemma 2.6] that if W is a nonempty symmetric weakly compact subset of a Banach space $(X, \|\cdot\|)$ and $x_0^* \in X^*$, then there exists a proper lower semicontinuous function $g : X^* \to (-\infty, \infty]$ such that $\text{Dom}(g) = \{x^* \in X^* : \sup_{w \in W} |(x^* - x_0^*)(w)| < 1\}$ and $\partial g(x^*) \cap \widehat{X} \neq \varnothing$ for all $x^* \in \text{Dom}(g)$. Hence we have the following corollary.

Corollary 3. ([2]) *Let $f : X \to (-\infty, \infty]$ be a proper bounded from below function defined on a Banach space $(X, \|\cdot\|)$. If there exists a nonempty symmetric weakly compact subset W of X and a $x_0^* \in X^*$ such that*

$$S := \{x^* \in X^* : \sup_{w \in W} |(x^* - x_0^*)(w)| < 1\} \subseteq Dom(f^*)$$

and $\partial f^*(x^*) \cap \widehat{X} \neq \varnothing$ for all $x^* \in S$, then for every $\gamma \in \mathbb{R}$, $\{x \in X : f(x) \leq \gamma\}$ is relatively weak-closed.*

For a subset A of a vector space V, over the real numbers, the *core* of A, denoted $\text{Cor}(A)$ is the set of points $a \in A$ where for each $x \in V \setminus \{a\}$ there exists a $0 < \delta$ such that $a + \lambda(x - a) \in A$ for all $0 < \lambda < \delta$. Let us also recall that a subset of a topological space (X, τ) is called an *F_σ-subset* if it is a countable union of closed sets.

Proposition 4. (Folklore) *Suppose that F is a convex F_σ-subset of a Banach space $(X, \|\cdot\|)$. Then $0 \in \text{Cor}(F)$ if, and only if, $0 \in \text{int}(F)$.*

Proof. We need only show that if $0 \in \text{Cor}(F)$, then $0 \in \text{int}(F)$, as the converse is obvious. So let us suppose that $0 \in \text{Cor}(F)$. We will first show that $\varnothing \neq \text{int}(F)$. Since $0 \in \text{Cor}(F)$, for every $x \in X$, there exists an $n \in \mathbb{N}$ such that $(1/n)x \in F$. Thus, $X = \bigcup_{n \in \mathbb{N}} nF$. Since F is an F_σ-subset of $(X, \|\cdot\|)$ there exist closed subsets $\{F_m : m \in \mathbb{N}\}$ of $(X, \|\cdot\|)$ such that $F = \bigcup_{m \in \mathbb{N}} F_m$. Therefore, $X = \bigcup_{(m,n) \in \mathbb{N}^2} nF_m$. Hence, by the Baire Category Theorem, there exists a pair of numbers $(m_0, n_0) \in \mathbb{N}^2$ such that $\varnothing \neq \text{int}(n_0 F_{m_0})$. Thus, $\varnothing \neq \text{int}(F_{m_0})$ and so $\varnothing \neq \text{int}(F_{m_0}) \subseteq \text{int}(F)$. Let $x \in \text{int}(F)$, then there exists a $0 < r$ such that $B[x, r] \subseteq F$. Since $0 \in \text{Cor}(F)$, there exists a $0 < \delta$ such that $\delta(-x) \in F$. Let $\lambda := 1/(1 + \delta)$ and $y := -\delta x$, then $0 < \lambda < 1$ and $0 = \lambda y + (1 - \lambda)x$. Since F is convex, $0 \in \lambda y + (1 - \lambda)B[x, r] \subseteq F$. However,

$$\lambda y + (1-\lambda)B[x,r] = \lambda y + (1-\lambda)(x + rB_X) = \lambda y + (1-\lambda)x + (1-\lambda)rB_X = (1-\lambda)rB_X.$$

Therefore, $0 \in B(0, (1-\lambda)r) \subseteq B[0, (1-\lambda)r] \subseteq F$; which shows that $0 \in \text{int}(F)$. □

For a subset A of a vector space V, over the real numbers, with $0 \in A$, the *directional asymptotic cone* of A, denoted A_∞ is the set $\{x \in V : \lambda x \in A \text{ for all } \lambda \in [0, \infty)\}$.

Let $(X, \|\cdot\|)$ be a normed linear space and let K be a subset of X^* with $0 \in K$. We define $\mathcal{E}(K)$ as the class of all proper functions $\varphi : X \to (-\infty, \infty]$ such that:

(i) $\varphi^* : X^* \to (-\infty, \infty]$ with $\varphi^*(0) = 0$;

(ii) $K_\infty \subseteq \mathrm{Dom}(\varphi^*) \subseteq K$;

(iii) $\sup\{\varphi^*(\eta x^*) : 0 < \eta \text{ and } \eta x^* \in \mathrm{Dom}(\varphi^*)\} = \infty$ for all $x^* \in K \setminus K_\infty$;

(iv) $\partial\varphi^*(x^*) \cap \widehat{X} \neq \varnothing$ for all $x^* \in \mathrm{Dom}(\varphi^*)$.

Lemma 5. *Let $(X, \|\cdot\|)$ be a Banach space. Suppose that K is a subset of X^* such that $0 \in K$ and $\mathbb{R}_+K = X^*$. If $\varphi \in \mathcal{E}(K)$, then $\mathrm{Dom}(\varphi^*)$ is an open subset of $(X^*, \|\cdot\|^*)$ with $0 \in \mathrm{Dom}(\varphi^*)$.*

Proof. First we will show that $0 \in \mathrm{int}(\mathrm{Dom}(\varphi^*))$. Since φ^* is lower semicontinuous, $\mathrm{Dom}(\varphi^*) = \bigcup_{n\in\mathbb{N}}\{x^* \in X^* : \varphi^*(x^*) \leq n\}$ is a F_σ-set. So by Proposition 4 we need only show that $0 \in \mathrm{Cor}(\mathrm{Dom}(\varphi^*))$. To this end, let y^* be an arbitrary element of $X^* \setminus \{0\}$. Since $\mathbb{R}_+K = X^*$ there exists a $0 < \delta < \infty$ and a $k^* \in K$ such that $y^* = \delta k^*$. We now consider two cases.

Case 1: If $k^* \in \mathrm{Dom}(\varphi^*)$, then $[0, \delta^{-1}y^*) = [0, k^*) \subseteq \mathrm{Dom}(\varphi^*)$, as $\mathrm{Dom}(\varphi^*)$ is convex and $0 \in \mathrm{Dom}(\varphi)$.

Case 2: $k^* \in K \setminus \mathrm{Dom}(\varphi^*) \subseteq K \setminus K_\infty$. Then by Property (iii) of the definition of $\mathcal{E}(K)$, $\sup\{\varphi^*(\eta k^*) : 0 < \eta \text{ and } \eta k^* \in \mathrm{Dom}(\varphi^*)\} = \infty$. In particular this means that there exists an $0 < \eta < \infty$ such $\eta k^* \in \mathrm{Dom}(\varphi^*)$. Therefore, $(\eta\delta^{-1})y^* \in \mathrm{Dom}(\varphi^*)$. Again, since $\mathrm{Dom}(\varphi^*)$ is convex and $0 \in \mathrm{Dom}(\varphi)$, $[0, (\eta\delta^{-1})y^*) \subseteq \mathrm{Dom}(\varphi^*)$.

This shows that $0 \in \mathrm{Cor}(\mathrm{Dom}(\varphi^*))$ and hence $0 \in \mathrm{int}(\mathrm{Dom}(\varphi^*))$. Next, we show that $\mathrm{Dom}(\varphi^*)$ is an open subset of $(X^*, \|\cdot\|^*)$. Let x^* be an arbitrary element of $\mathrm{Dom}(\varphi^*)$; we will show that there exists a $1 < \lambda < \infty$ such that $\lambda x^* \in \mathrm{Dom}(\varphi^*)$. Note that if $x^* \in K_\infty$, then clearly such a $1 < \lambda < \infty$ exists. So we can confine our attention to the case when $x^* \in \mathrm{Dom}(\varphi^*) \setminus K_\infty \subseteq K \setminus K_\infty$. Then, by property (iii) of the definition of $\mathcal{E}(K)$, $\sup\{\varphi^*(\eta x^*) : 0 < \eta \text{ and } \eta x^* \in \mathrm{Dom}(\varphi^*)\} = \infty$. However, since

$$\begin{aligned}\varphi^*(\mu x^*) &= \varphi^*(\mu x^* + (1-\mu)0) \leq \mu\varphi^*(x^*) + (1-\mu)\varphi^*(0) \\ &= \mu\varphi^*(x^*) \leq \mu|\varphi^*(x^*)| \leq |\varphi^*(x^*)|\end{aligned}$$

for all $0 < \mu \leq 1$, this means that there must exist a $0 < \eta < \infty$ such that $1 < \eta < \infty$, $\eta x^* \in \mathrm{Dom}(\varphi^*)$ and $|\varphi^*(x^*)| < \varphi^*(\eta x^*)$. Let $\lambda := \eta$, then $1 < \lambda < \infty$ and $\lambda x^* \in \mathrm{Dom}(\varphi^*)$. This completes the proof that there exists a $1 < \lambda < \infty$ such that $\lambda x^* \in \mathrm{Dom}(\varphi^*)$. Now, since $0 \in \mathrm{int}(\mathrm{Dom}(\varphi^*))$, there exists a $0 < r < \infty$ such that $B[0, r] \subseteq \mathrm{Dom}(\varphi^*)$. Let $\mu := \lambda^{-1}$. Then $0 < \mu < 1$ and

$$x^* = \mu(\lambda x^*) + (1-\lambda)0 \in \mu(\lambda x^*) + (1-\mu)B[0, r] \subseteq \mathrm{Dom}(\varphi^*)$$

since, $\mathrm{Dom}(\varphi^*)$ is convex. However, $\mu(\lambda x^*) + (1-\mu)B[0, r] = B[x^*, (1-\mu)r]$ and so

$$x^* \in B(x^*, (1-\mu)r) \subseteq B[x^*, (1-\mu)r] \subseteq \mathrm{Dom}(\varphi^*);$$

which completes the proof. □

Note that Lemma 5 essentially says that if $0 \in K \subseteq X^*$ for some Banach space $(X, \|\cdot\|)$ and $\mathbb{R}_+K = X^*$ and $\mathcal{E}(K) \neq \varnothing$, then K is, in some sense, quite large.

Corollary 6. ([6]) *Let $(X, \|\cdot\|)$ be a Banach space. Let $K \subseteq X^*$ with $0 \in K$, $\mathbb{R}_+K = X^*$ and $\mathcal{E}(K) \neq \varnothing$ and let $f : X \to (-\infty, \infty]$ be a proper function such that $\partial f^*(x^*) \cap \widehat{X} \neq \varnothing$ for all $x^* \in K$. Then for every $\gamma \in \mathbb{R}$, $\{x \in X : f(x) \leq \gamma\}$ is relatively weakly compact.*

Proof. Let $\gamma \in \mathbb{R}$ and let $\varphi \in \mathcal{E}(K)$. By Lemma 5, $\mathrm{Dom}(\varphi^*)$ is an open subset of $(X^*, \|\cdot\|^*)$ with

$$0 \in \mathrm{Dom}(\varphi^*) = \mathrm{int}(\mathrm{Dom}(\varphi^*)) \subseteq K \subseteq \mathrm{Dom}(\partial f^*) \subseteq \mathrm{Dom}(f^*).$$

Now, φ^* is a proper lower semicontinuous convex function such that $\partial\varphi^*(x^*) \cap \widehat{X} \neq \varnothing$ for all $x^* \in \mathrm{Dom}(\varphi^*)$ and $\partial f^*(x^*) \cap \widehat{X} \neq \varnothing$ for all $x^* \in K$ and so $\partial f^*(x^*) \cap \widehat{X} \neq \varnothing$ for all $x^* \in \mathrm{Dom}(\varphi^*)$. Therefore, by Theorem 1, $\{x \in X : f(x) \leq \gamma\}$ is relatively weakly compact. $\square$

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