Journal of Convex Analysis
Volume 29 (2022), No. 4, 1225–1250

Subconvex Functions, Relations and Sets

Jean-Paul Penot
Sorbonne Universités, UPMC Univ Paris 6, CNRS,
Laboratoire Jacques-Louis Lions, Paris, France
penotj@ljll.math.upmc.fr

Received: May 29, 2020
Accepted: May 20, 2022

We consider a class of subsets of a Banach space whose normal cones satisfy a kind of uniformity property. Here the normal cone is associated with a general subdifferential as defined in an appendix for the sake of clarity. We examine to what extent this property depends on the choice of the subdifferential. We relate this property to approximate convexity in the sense of Ngai, Luc and Théra, subsmoothness in the sense of Aussel, Daniilidis, Thibault and super-regularity in the sense of Lewis, Luke and Malick, showing coincidence in appropriate spaces. This class of sets is larger than the classes of amenable subsets and prox-regular subsets. We also explore related properties for functions and multifunctions. We devise some permanence properties under usual operations.

Keywords: Approximate convexity, approximate monotonicity, error bound, normal cone, subconvexity, subdifferential.

2020 Mathematics Subject Classification: 49J52, 49J53, 52A01, 52A30, 58C06, 90C26.

1. Introduction

In analysis, introducing some uniformity is often fruitful. That is well known in the geometry of Banach spaces and for the concept of strict (or circa) differentiability which is the appropriate notion for inverse functions and implicit functions.

In nonsmooth analysis several authors (including [6], [9], [10], [11] and the present author in [27], [30]) explored various ways of introducing for general closed subsets some uniformity or approximate uniformity that are satisfied by convex subsets or smooth subsets. In the present paper we return to one of these proposals dealing with normal cones to sets and subdifferentials of functions and we relate it to the notion of subsmooth sets studied in [2] using essentially the Clarke subdifferential and more recently in [3] using essentially the Fréchet and the limiting Fréchet subdifferentials that inspired the present paper. The striking point of the paper [3] by D. Azé is that it introduces a one-sided notion compared with the symmetric definition of submonotonicity by Spingarn [37] (see also [12], [13], [23], [24], [25]). A comparison was in order. A link between the notion of Fréchet set in [3] (or its present variants) and the more classical notion studied in the just quoted papers is given in the easy Proposition 2.14 below. Subsmooth sets are also studied at length in the survey paper [38] which was not known to the author before the submission of the first version of the present paper.

The author is indebted to D. Azé for many discussions and for providing a preprint of the paper [20] by A. Lewis, D. Luke and J. Malick. The later reference is focussed on applications of wide interest to algorithms involving projections and involves

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

a notion of super-regularity using the limiting subdifferential in finite dimensions. Here we use a general subdifferential ∂ satisfying the axioms of [32] recalled, or rather adapted, in the appendix for the sake of self-containedness. Our study is conceived as a test for these axioms: they avoid specific constructions or arguments and allow some versatility as in [1], [15], [40], but since they intend to gather all classical subdifferentials without discarding some of them, one may wonder whether are they able to prove some interesting relationships among notions which seem to depend on the specific choice of a subdifferential.

The purpose of the present study is to try to answer such a question. In particular, we examine to what extent the properties of the class of sets we study are dependent or independent of the choice of a subdifferential. We also tentatively introduce related notions for functions and relations (or multimaps or multifunctions). The choice of an appropriate definition in such a case is debatable and the author much hesitated. The objective here is not to devise calculus rules but to delineate notions that are coherent enough. Our study completes previous works by D. Aussel, A. Daniilidis and L. Thibault [2], H. V. Ngai and J.-P. Penot [24], [25] and, more recently, D. Azé [3], A. Lewis, D. Luke and J. Malick [20]. Although the spirit of the paper is likely to be new, many of its results are already present in the literature as the author realized when writing the references and later, when receiving from L. Thibault the survey [38]. Thus, thanks to the help of L. Thibault [39], a detailed comparison with [38] is presented at the end of the paper. Let us note that the variety of the terminologies reflects a diversity of approaches and aims. The approach here is focussed on a pointwise property whereas in many cases a local or even global viewpoint prevails. We are not sure that adding another one would serve to some unification because the notion of approximate convexity is clearly fundamental. But the terminology *subconvex* has already been used in [27, Def. 4.3] and is naturally related to the classical notion of *submonotonicity* ([36], [37]).

The classes of sets and functions we consider are rather large as they contain the classes of convex sets, of smooth sets and various classes of sets that have attracted the attention as amenable sets, paraconvex sets, primal lower-nice sets, proximally smooth sets, prox-regular sets... We do not deal with the latter classes that have stronger regularity properties, referring to [4], [10], [11], [19], [35] for the study of such classes. In order to keep the paper short enough we do not recall the different notions of normal cone or subdifferential we use; we refer to the many monographs and papers that expose the topic. The appendix is intended to help the reader in the use of the general properties of subdifferentials that are used. Also, as our notation is similar to the ones in [5], [7], [8], [17], [33], we assume that the reader familiar with one of these books will not be puzzled.

2. Subconvex sets

The following notions are variants of the notions considered in [30, Def. 1] and, more closely, variants of the concept of *Fréchet set at* $\overline{x}$ introduced in [3, Lemma 3.1]. Here N or N_∂ is the *normal cone* associated with a subdifferential ∂ (in the sense of the appendix or other axiomatic approach) by the relation $N(E, x) := \partial \iota_E(x)$, where ι_E is the *indicator* function of E given by $\iota_E(x) = 0$ for $x \in E$, $\iota_E(x) = +\infty$ if $x \in X \backslash E$. The terminology of the next definition is justified by the fact that in

[32] the cone $N^m(E,x) := \mathbb{R}_+\partial d_E(x)$, where $d_E(x) := \inf_{w\in E} \|x-w\|$, is called the *metric normal cone* to E at x. In view of condition (S6) of the appendix, one always has $N^m(E,x) \subset N(E,x)$. When equality holds for any subset E, one says that ∂ is *normally consistent* or *geometrically consistent (*[17, Definition 4.7]). In Euclidean spaces or Hilbert spaces the conditions of the next definition can be interpreted in terms of angles or approximate perpendicularity between normals vectors and secants.

Definition 2.1. (a) Given a subdifferential ∂, a subset E of a normed vector space X is said to be *∂-subconvex* at $\overline{x} \in E$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any x, $x' \in E \cap B(\overline{x}, \delta)$, $x^* \in N_\partial(E,x)$ one has

$$\langle x^*, x'-x\rangle \leq \varepsilon \|x^*\| \, . \, \|x'-x\| \, .$$

(b) It is said to be *∂-m-subconvex* at $\overline{x} \in E$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any x, $x' \in E \cap B(\overline{x}, \delta)$, $x^* \in N^m(E,x) := \mathbb{R}_+\partial d_E(x)$ the preceding inequality holds.

(c) Given a subdifferential ∂, a subset E of a normed vector space X is said to be *∂-metrically subconvex* at $\overline{x} \in E$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any x, $x' \in E \cap B(\overline{x}, \delta)$, $x^* \in \partial d_E(x)$ one has

$$\langle x^*, x'-x\rangle \leq \varepsilon \|x'-x\| \, .$$

Here ∂ is an arbitrary subdifferential and not just the firm (or Fréchet) subdifferential ∂_F as in [3] in which the first and the last notions are considered under the name of "Fréchet subset". On the other hand, a priori this definition is less restrictive than the one in [2] and [42] in which x^* belongs to $\partial_C d_E(x)$ or $N_C(E,x)$ since the Clarke subdifferential ∂_C (resp. normal cone N_C) is the largest of usual subdifferentials. (resp. normal cones). It incorporates the notion of super-regularity of A. Lewis, D. Luke and J. Malick [20, Def. 4.4] in which $\partial = \partial_P$ the proximal subdifferential, or, in finite dimensions the limiting subdifferential ∂_L. When there is no ambiguity about the choice of ∂, one can drop ∂- in "∂-subconvex", but in general these notions depend on the choice of the subdifferential ∂.

In view of the inclusions $N^m(E,x) \subset N(E,x)$ by axiom (S6) in the appendix, and $\partial d_E(\overline{x}) \subset B_{X^*}$, the properties of the preceding definition are ranked in decreasing strength (here we omit "at $\overline{x}$"):

$$E \text{ is } \partial\text{-subconvex} \Rightarrow E \text{ is } \partial\text{-m-subconvex} \Rightarrow E \text{ is } \partial\text{-metrically subconvex.}$$

When $N^m(E,x)$ is dense in $N(E,x)$, then E is ∂-subconvex at $\overline{x}$ if and only if E is ∂-m-subconvex at $\overline{x}$. That is the case for the Fréchet subdifferential ∂_F and the Clarke subdifferential ∂_C.

Example 2.2. There exist subconvex sets that are nonconvex. That is the case for the epigraph E_f of the function f on $\mathbb{R}$ given by $f(x) := (x-x^2)_+ := \max(x-x^2, 0)$ at $\overline{x} := (0,0)$. □

The strength of these properties also depend on the nature of the subdifferential ∂. Obviously, the larger the subdifferential is, the more demanding the condition is.

Example 2.3. ([20], [38, Example 7.10]). Let $f : [-1, 1] \to \mathbb{R}$ be the even function which is affine on $[1/(n+1), 1/n]$ and satisfies $f(0) = 0$, $f(1/n) = 1/n^2$ and let G be its graph. For $a_n := (1/n, 1/n^2)$ the Clarke tangent cone $T_C(G, a_n) = \{0\}$ so that $N_C(G, a_n) = \mathbb{R}^2$ and G is not subconvex at $(0, 0)$.

However, since $(f(u) - f(v))/\|u - v\| \to 0$ as $u, v \to 0$ with $u \neq v$, one sees that $\langle x^*, w - x\rangle/\|x^*\|\|w - x\| \to 0$ as $w, x \to 0$ with $w, x \in G$, $w \neq x$, $x^* \in N_F(G, x)$. Thus G is ∂_F-subconvex at $(0, 0)$. Note that the epigraph E of f is convex. □

Clearly a convex set E is ∂-subconvex at any of its points for any subdifferential ∂, since for a convex subset E one has $\langle x^*, x' - x\rangle \leq 0$ for any $x, x' \in E$, $x^* \in N(E, x)$ and since, in view of Axiom (S2a) in the appendix, all the usual concepts of normal cone coincide for convex subsets. The next proposition shows that this last property still holds for ∂-subconvex subsets when one takes for ∂ a subdifferential containing ∂_F.

Proposition 2.4. (a) *For any subdifferential ∂, if E is ∂-subconvex at $\overline{x} \in E$ then the associated normal cone $N(E, \overline{x}) := N_\partial(E, \overline{x})$ is contained in $N_F(E, \overline{x})$, the Fréchet (or firm) normal cone. Thus, if E is ∂-subconvex at $\overline{x} \in E$ for a subdifferential ∂ containing ∂_F, the Fréchet (or firm) subdifferential, then one has $N(E, \overline{x}) = N_F(E, \overline{x})$.*

(b) *If E is ∂-m-subconvex at $\overline{x} \in E$ then one has $N^m(E, \overline{x}) \subset N_F(E, \overline{x})$, with equality if ∂ is larger than ∂_F.*

(c) *If E is ∂-metrically subconvex at $\overline{x} \in E$ then one has $\partial d_E(\overline{x}) \subset \partial_F d_E(\overline{x})$ and $N^m(E, \overline{x}) \subset N_F(E, \overline{x})$, with equality if ∂ is larger than ∂_F.*

In particular, if E is ∂_C-subconvex at $\overline{x} \in E$, then for all subdifferentials between ∂_F and ∂_C the corresponding normal cones coincide at $\overline{x}$. It is not clear that such a property holds when one takes for ∂ a subdifferential smaller than ∂_C, in particular ∂_F as in [3]. Such a question has been an incentive to consider the case ∂ is an arbitrary subdifferential. The recurrent assumption that ∂ is a subdifferential larger than ∂_F is mild.

Proof. (a) and (b) follow from the choice $x = \overline{x}$ in Definition 2.1. Assertion (c) is an immediate application of the definition of $\partial_F d_E(\overline{x})$. □

Corollary 2.5. *If ∂ is a subdifferential larger than ∂_F and if $E \subset X$ is ∂-subconvex at $\overline{x} \in E$ then E is strongly consistent at $\overline{x}$ in the sense that $N(E, \overline{x}) \cap B_{X^*} = \partial d_E(\overline{x})$ hence E is normally consistent at $\overline{x}$: $N^m(E, \overline{x}) = N(E, \overline{x})$.*

In the sequel we say that E is *strongly consistent* if for all $x \in E$ we have the equality $N(E, x) \cap B_{X^*} = \partial d_E(x)$ and we say that ∂ is *strongly consistent* if all closed subsets are strongly consistent. We also say that E is *quasi-consistent* (resp. *almost consistent*) at $\overline{x}$ if $N(E, x)$ is contained in the weak*closure of $N^m(E, x)$ (resp. the closure of $N^m(E, x)$).

Proof. The inclusion $\partial d_E(\overline{x}) \subset N(E, \overline{x}) \cap B_{X^*}$ is a consequence of the axioms in the appendix. When E is ∂-subconvex at $\overline{x}$ and $\partial_F \subset \partial$ one has

$$N(E, \overline{x}) \cap B_{X^*} \subset N_F(E, \overline{x}) \cap B_{X^*} = \partial_F d_E(\overline{x}) \subset \partial d_E(\overline{x}),$$

hence equalities and $N(E, \overline{x}) \subset \mathbb{R}_+ \partial d_E(\overline{x}) = N^m(E, \overline{x})$ and again equality holds. □

The preceding definitions can be given a quantitative form. For instance, one can define the *index of subconvexity* $\mu(\cdot)$ of E at $\overline{x} \in E$ as

$$\mu(r) := \sup\{\langle x^*, \frac{w-x}{\|w-x\|}\rangle : w,\ x \in E \cap B(\overline{x}, r),\ w \neq x,\ x^* \in N(E,x) \cap B_{X^*}\}.$$

It is a modulus (i.e. $\mu(r) \to 0$ as $r \to 0$) if and only if E is subconvex at $\overline{x}$.

We observe that, by homogeneity, E is ∂-subconvex (resp. ∂-m-subconvex) at $\overline{x} \in E$ if and only if

(a$_1$) (resp. (b$_1$)) for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, x' \in E \cap B(\overline{x}, \delta)$, $x^* \in N(E,x) \cap B_{X^*}$ (resp. $x^* \in N^m(E,x) \cap B_{X^*}$) one has

$$\langle x^*, x' - x\rangle \leq \varepsilon \|x' - x\|.$$

However, it is not clear whether E is ∂-m-subconvex at $\overline{x} \in E$ when in the preceding assertion one takes $x^* \in \partial d_E(x)$ instead of $x^* \in N^m(E,x) \cap B_{X^*}$, i.e. when E is ∂-metrically subconvex at $\overline{x}$.

The next proposition extends the estimates of Definition 2.1.

Proposition 2.6. *Let E be a subset of a normed space X, let $\overline{x} \in E$ and let ∂ be an arbitrary subdifferential. Among the following assertions one has*

$$\text{(a)} \Leftrightarrow \text{(b)} \Rightarrow \text{(c)} \Leftrightarrow \text{(d)} \Leftrightarrow \text{(e)} \Rightarrow \text{(f)} \Leftrightarrow \text{(g)}.$$

If E is almost consistent, that is, for all $x \in E$ one has $N(E,x) \subset \mathrm{cl}(N^m(E,x))$, the closure of $N^m(E,x)$, the assertions (a), (b), (c), (d), (e) *are equivalent. If E is strongly consistent all the assertions* (a),...,(g) *are equivalent:*

(a) *E is ∂-subconvex at $\overline{x} \in E$;*

(b) *for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any $w \in B(\overline{x}, \delta)$, $x \in E \cap B(\overline{x}, \delta)$, $x^* \in N(E,x)$ one has $\langle x^*, w - x\rangle \leq \|x^*\| d_E(w) + \varepsilon \|x^*\| . \|w - x\|$;*

(c) *for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any $w \in B(\overline{x}, \delta)$, $x \in E \cap B(\overline{x}, \delta)$, $x^* \in N^m(E,x) := \mathbb{R}_+ \partial d_E(x)$ one has*
$$\langle x^*, w - x\rangle \leq \|x^*\| d_E(w) + \varepsilon \|x^*\| . \|w - x\|;$$

(d) *for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any $w \in B(\overline{x}, \delta)$, $x \in E \cap B(\overline{x}, \delta)$, $x^* \in \partial d_E(x)$ one has $\langle x^*, w - x\rangle \leq \|x^*\| d_E(w) + \varepsilon \|x^*\| . \|w - x\|$;*

(e) *E is ∂-m-subconvex at $\overline{x} \in E$;*

(f) *for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $w \in B(\overline{x}, \delta)$, $x \in E \cap B(\overline{x}, \delta)$, $x^* \in \partial d_E(x)$ one has $\langle x^*, w - x\rangle \leq d_E(w) + \varepsilon \|w - x\|$;*

(g) *E is ∂-metrically subconvex at $\overline{x} \in E$.*

The terminology in assertion (g) could be changed into "∂-almost approximately convex" since in [25, Thm. 15] this property is characterized and shown to be a consequence of intrinsic approximate convexity. However that would not simplify the terminology. Note that the equivalence (f)$\Leftrightarrow$(g) completes the implications of [25, Thm. 15]; it is also given in [3, Lemma 3.1] in the case $\partial := \partial_F$. After the submission of the present paper we learned that this equivalence as well as the equivalences (a)$\Leftrightarrow$(b), (c)$\Leftrightarrow$(d) are also given in the survey [38] by L. Thibault.

The condition $N(E,x) = \mathrm{w}^*\mathrm{cl}(\mathbb{R}_+\partial d_E(x))$ is fulfilled if $\partial = \partial_F$ or if $\partial = \partial_C$ (see Lemma 2.8 below). It is also satisfied when $\partial = \partial_L$ in view of [3, Prop. 2.1] with $f := d_E$ that even asserts that $N_L(E,x) = \mathbb{R}_+\partial_L d_E(x)$, i.e. that ∂_L is normally consistent.

Example 7.10 in [38] from [20] shows that the implication (g)⇒(a) is not valid, even for a subset of $\mathbb{R}^2$.

Proof. The implications (b)⇒(a), (c)⇒(e), and (f)⇒(g) are obtained by taking $w := x' \in E \cap B(\overline{x}, \delta)$.

(b)⇒(c) stems from the inclusion $N^m(E,x) \subset N(E,x)$. The reverse implication (under the assumption that $N(E,x) = \mathrm{cl}(N^m(E,x)))$ is obtained by a passage to the limit, taking a sequence (x_n^*) in $N^m(E,x))$ which converges to $x^* \in N(E,x)$.

(c)⇔(d) is obvious by homogeneity.

(a)⇒(b) Given $\varepsilon > 0$ we pick $\delta > 0$ as in Definition 2.1. Given $w \in B(\overline{x}, \delta/2)$, $x \in E \cap B(\overline{x}, \delta)$, $x^* \in N(E,x)$, we pick $x_n \in E$ such that $(\|x_n - w\|) \to d_E(w)$, $\|x_n - w\| \le \|\overline{x} - w\| < \delta/2$, so that $x_n \in E \cap B(\overline{x}, \delta)$. Assertion (a) ensures that

$$\langle x^*, x_n - x\rangle \le \varepsilon \|x^*\| \,.\, \|x_n - x\| \le \varepsilon \|x^*\| \,.\, \|x_n - w\| + \varepsilon \|x^*\| \,.\, \|w - x\| \,,$$

hence we have

$$\begin{aligned}\|x^*\| \,.\, \|w - x_n\| \ge \langle x^*, w - x_n\rangle &= \langle x^*, w - x\rangle - \langle x^*, x_n - x\rangle \\ &\ge \langle x^*, w - x\rangle - \varepsilon \|x^*\| \,.\, \|x_n - w\| - \varepsilon \|x^*\| \,.\, \|w - x\| \,,\end{aligned}$$

so that $(1+\varepsilon) \|x^*\| \,.\, \|w - x_n\| \ge \langle x^*, w - x\rangle - \varepsilon \|x^*\| \,.\, \|w - x\|$.

Passing to the limit, one gets

$$(1+\varepsilon) \|x^*\| \, d_E(w) \ge \langle x^*, w - x\rangle - \varepsilon \|x^*\| \,.\, \|w - x\|$$

and, since $d_E(w) \le \|w - x\|$,

$$\|x^*\| \, d_E(w) \ge \langle x^*, w - x\rangle - 2\varepsilon \|x^*\| \,.\, \|w - x\|$$

so that (b) is satisfied with ε, δ changed into 2ε, $\delta/2$.

(e)⇒(c) (resp. (g)⇒(f)) The proof is similar with $x^* \in N^m(E,x)$ (resp. $x^* \in \partial d_E(x)$) instead of $x^* \in N(E,x)$.

(d)⇒(f) is a consequence of the inclusion $\partial d_E(x) \subset B_{X^*}$ valid for any subdifferential (see [32]).

When $N(E,x) \cap B_{X^*}$ the equivalence of assertions (a), (b), (c), (d), (e) follows from the fact that for all $x^* \in N(E,x)$ one has $x^* \in \|x^*\| \, \partial d_E(x)$. In particular when X is reflexive, the next lemma ensures that for $\partial = \partial_C$ these assertions are equivalent. □

Example 2.7. (a) A convex subset is ∂-subconvex around each of its points. More generally, in view of the next proposition, an approximately convex subset is a subconvex subset.

(b) If G is the graph of a map $g : U \to V$ that is circa-differentiable (in other words, strictly differentiable) at $\overline{u} \in U$, then G is a ∂_C-subconvex subset of $X := U \times V$ at $\overline{x} := (\overline{u}, g(\overline{u}))$, as easily seen.

(c) If E is a submanifold of class C^1 of a Banach space X, then E is a ∂_C-subconvex subset of X at each of its points. □

The following lemma bears on an arbitrary subset but for specific subdifferentials.

Lemma 2.8. (a) *For any subset S of a Banach space X, and any $\overline{x} \in S$, the sets $N_F(S, \overline{x})$ and $\partial_F d_S(\overline{x})$ are closed. If X is reflexive, these sets are weakly closed.*

(b) *If X is reflexive, then $N_C(S, \overline{x})$ is the norm closure of $\mathbb{R}_+ \partial_C d_S(\overline{x})$.*

Proof. (a) The closedness of $\partial_F d_S(\overline{x})$ for any $\overline{x} \in S$ (and in fact for any function f finite at $\overline{x}$) is a well known fact (see [6], [33, Prop. 4.10] for example). We give a proof for completeness. Let $\overline{x}^*$ be in the closure of $\partial_F f(\overline{x})$. Given $\varepsilon > 0$, let $x^* \in \partial_F f(\overline{x}) \cap B(\overline{x}^*, \varepsilon/2)$ and let $\delta > 0$ be such that for all $x \in B(\overline{x}, \delta)$ one has $f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x}\rangle - \frac{1}{2}\varepsilon \|x - \overline{x}\|$. Then, for $x \in B(\overline{x}, \delta)$ one has

$$\begin{aligned} f(x) - f(\overline{x}) &\geq \langle \overline{x}^*, x - \overline{x}\rangle - \|\overline{x}^* - x^*\| \, \|x - \overline{x}\| - \frac{1}{2}\varepsilon \|x - \overline{x}\| \\ &\geq \langle \overline{x}^*, x - \overline{x}\rangle - \varepsilon \|x - \overline{x}\| , \end{aligned}$$

so that $\overline{x}^* \in \partial_F f(\overline{x})$. Taking for f the indicator function of S (resp. d_S), we get that $N_F(F, \overline{x})$ (resp. $\partial_F d_S$) is closed.

When X is reflexive, the convex sets $N_F(S, \overline{x})$ and $\partial_F d_S(\overline{x})$ are weakly closed.

(b) Given r, $s \in \mathbb{R}_+$ with $r \leq s$ one has $r\partial_C d_S(\overline{x}) \subset s\partial_C d_S(\overline{x})$ by axiom (S6) in the appendix since $d_S \leq r^{-1} s d_S$ and $r^{-1} s d_S$ is null on S. It easily follows that $\mathbb{R}_+ \partial_C d_S(\overline{x})$ is convex. Thus, in view of [7, 2.4.2], [8, Prop. 5.4], or [33, Prop. 5.25], its weak closure $N_C(S, \overline{x})$ is its norm closure. □

The stronger result that follows is due to a personal communication with D. Azé in the case $\partial = \partial_F$. It shows a distinctive property of ∂-subconvex subsets.

Theorem 2.9. *Let E be a subset of a general Banach space X and let ∂ be a subdifferential larger than ∂_F. If E is ∂-metrically subconvex at $\overline{x}$ then $\partial d_E(\overline{x})$ is weak* closed and coincides with $\partial_F d_E(\overline{x})$.*

If E is ∂-subconvex at $\overline{x}$ and if ∂ is larger than the Fréchet subdifferential ∂_F then $N_\partial(E, \overline{x})$ is weak closed and coincides with $N_F(E, \overline{x})$.*

Proof. The first assertion is an immediate consequence of the definition by a passage to the weak* limit of a converging net of $\partial d_E(\overline{x})$.

If E is ∂-subconvex at $\overline{x}$ and if ∂ is larger than ∂_F Proposition 2.4 ensures that $N_\partial(E, \overline{x}) = N_F(E, \overline{x})$ which is convex. Thus the second assertion stems from the Krein-Šmulian Theorem (see [33, Thm. 1.11] for instance) again from a passage to the weak* limit of a bounded net of $N_\partial(E, \overline{x})$. □

The next result has been given in [2, Lemma 3.8] when X is a reflexive space and E is ∂_C-subconvex at $\overline{x}$. We note that adopting [22, Def. 1.1 (ii)], the relation $N_L(E, \overline{x}) = N_F(E, \overline{x})$ holds in any Banach space when E is ∂_F-subconvex at $\overline{x}$. We also recall that E is ∂_F-metrically subconvex at $\overline{x}$ if and only if E is ∂_F-subconvex at $\overline{x}$ since $\partial_F d_E(x) = N_F(E, x) \cap B_{X^*}$ for all $x \in E$.

Theorem 2.10. *Let E be a closed subset of an Asplund space X and let ∂ be a subdifferential larger than ∂_F. If E is ∂-subconvex at $\overline{x}$, then $N_\partial(E,\overline{x}) = N_F(E,\overline{x})$. Moreover, E is F-soft at $\overline{x}$ in the sense of* [31], [33, Def. 4.99] *or C-F regular at $\overline{x}$ in the sense that $N_C(E,\overline{x}) = N_F(E,\overline{x})$ and one has*

$$N_\partial(E,\overline{x}) = N_F(E,\overline{x}) = N_L(E,\overline{x}) = N_C(E,\overline{x}).$$

If E is just ∂-metrically subconvex at $\overline{x}$, then $\partial_C d_E(\overline{x}) = \partial_L d_E(\overline{x}) = \partial_F d_E(\overline{x})$.

Proof. We know that when E is ∂-subconvex at $\overline{x}$ we have $N_\partial(E,\overline{x}) \subset N_F(E,\overline{x})$, hence $N_\partial(E,\overline{x}) = N_F(E,\overline{x})$ when $\partial_F \subset \partial$. Let $\overline{x}^* \in N_L(E,\overline{x})$. Definition 6.1 of [33] or [22, Thm. 2.34] yields sequences $(x_n) \to \overline{x}$ in E and $(x_n^*) \xrightarrow{*} \overline{x}^*$ (weak* convergence) such that $x_n^* \in N_F(E,x_n)$ for each n. Given $\varepsilon > 0$, Definition 2.1 yields some $\delta > 0$ such that for n so large that $x_n \in B(\overline{x},\delta)$ one has

$$\langle x_n^*, w - x_n\rangle \leq \varepsilon \|x_n^*\| . \|w - x_n\| \text{ for } w \in B(\overline{x},\delta)$$

in view of the inclusion $\partial_F \subset \partial$. Since (x_n^*) is bounded there exists some $c > 0$ such that, passing to the limit, one gets

$$\langle \overline{x}^*, w - \overline{x}\rangle \leq \varepsilon \limsup_n \|x_n^*\| . \|w - \overline{x}\| \leq \varepsilon c \|w - \overline{x}\| \text{ for } w \in B(\overline{x},\delta).$$

Thus $\overline{x}^* \in N_F(E,\overline{x}) \subset N_\partial(E,\overline{x})$ and equality $N_L(E,\overline{x}) = N_F(E,\overline{x})$ holds. Since $N_F(E,\overline{x})$ is convex and weak* closed by Theorem 2.9 and since $N_C(E,\overline{x})$ is the weak* closed convex hull of $N_L(E,\overline{x})$ (see [7], [33, Thm. 6.10]), one gets $N_C(E,\overline{x}) = N_L(E,\overline{x})$.

When E is just ∂-metrically subconvex at $\overline{x}$, and $\overline{x}^* \in \partial_L d_E(\overline{x})$, using [2] (or [33, Lemma 6.7]), we take sequences $(x_n) \to \overline{x}$ and $(x_n^*) \xrightarrow{*} \overline{x}^*$ such that $x_n^* \in \partial_F d_E(x_n)$ and $x_n \in E$ for all n. Since $\partial_F d_E(x_n) \subset \partial d_E(x_n) \subset B_{X^*}$, again, a passage to the limit in the relation $\langle x_n^*, w - x_n\rangle \leq \varepsilon \|w - x_n\|$ for $w \in B(\overline{x},\delta)$ yields $\overline{x}^* \in \partial_F d_E(\overline{x})$ and $\partial_L d_E(\overline{x}) = \partial_F d_E(\overline{x})$. The convexity and weak* closedness of $\partial_F d_E(\overline{x})$ allows to conclude that $\partial_C d_E(\overline{x}) = \partial_L d_E(\overline{x})$. □

Corollary 2.11. *Let E be a closed subset of an Asplund space X. If E is ∂_F-subconvex at $\overline{x} \in E$, then $\partial_C d_E(\overline{x}) = \partial_F d_E(\overline{x})$ and $N_C(E,\overline{x}) = N_F(E,\overline{x})$.*

Remark 2.12. The theorem and its corollary do not assert that $\partial_F d_E$ coincides with $\partial_L d_E$ near $\overline{x}$. However, when E is ∂_F-subconvex at each point of $E \cap V$, where V is a neighborhood of $\overline{x}$, then the assertions (a)–(e) of Proposition 2.6 are equivalent.

Subconvex sets share some continuity properties with convex sets. Here we say that a multimap $Q : X \rightrightarrows X^*$ is *conically outer semicontinuous at $\overline{x} \in X$* if for any $\varepsilon > 0$ one can find some $\delta > 0$ such that for all $x \in B(\overline{x},\delta)$ one has

$$Q(x) \subset V(Q(\overline{x}),\varepsilon) := \{x^* \in X^* : d(x^*, Q(\overline{x})) \leq \varepsilon \|x^*\|\}.$$

Proposition 2.13. *Let E be a subset of a finite dimensional space X. If E is ∂-subconvex subset at $\overline{x} \in E$ for a subdifferential larger than ∂_F, then the normal multimap $N(E,\cdot)$ is conically outer semicontinuous at $\overline{x} \in X$.*

Proof. Assume the contrary, so that there exist $\alpha > 0$ and sequences $(x_n) \to \overline{x}$, (x_n^*) such that $x_n \in E$ and $x_n^* \in N(E,x_n)\backslash V(N(E,\overline{x}),\alpha)$ for all n. Without loss of generality one may assume that $\|x_n^*\| = 1$ for all $n \in \mathbb{N}$ and that (x_n^*) converges to some x^* in the unit sphere. Given $\varepsilon \in]0, \alpha[$, let $\delta > 0$ be as in Definition 2.1 (a). For n large enough one has $x_n \in B(\overline{x}, \delta)$ hence

$$\langle x_n^*, w - x_n \rangle \le \varepsilon \|w - x_n\| \qquad \forall w \in B(\overline{x}, \delta).$$

Passing to the limit we obtain

$$\langle x^*, w - \overline{x} \rangle \le \varepsilon \|w - \overline{x}\| \qquad \forall w \in B(\overline{x}, \delta),$$

so that $x^* \in N_F(E, \overline{x})$, ε being arbitrarily small. Since $N_F(E, \overline{x}) \subset N(E, \overline{x})$, one gets a contradiction with $\|x_n^* - x^*\| > \alpha$ and $(x_n^*) \to x^*$. □

Let us detect some links with another class of sets studied by Aussel, Daniilidis and Thibault [2] in the case ∂ is the Clarke subdifferential ∂_C and Ngai and the author [25]. The next result could be deduced from the preceding corollary. However a direct proof is immediate and valid in any Banach space X.

Theorem 2.14. (a) *A subset E of X is ∂-metrically subconvex at $\overline{x} \in E$, if, and only if, ∂d_E is ∂-approximately monotone at $\overline{x}$ on E, i.e. for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $u, v \in E \cap B(\overline{x}, \delta)$ and any $u^* \in \partial d_E(u)$, $v^* \in \partial d_E(v)$ one has*

$$\langle u^* - v^*, u - v \rangle \ge -\varepsilon \|u - v\| .$$

(b) *A subset E of a normed space X is a ∂-subconvex subset at $\overline{x} \in E$, if, and only if, it is ∂-subsmooth at $\overline{x}$ in the sense that $N_\partial(E, \cdot) \cap B_{X^*}$ is ∂-approximately monotone at $\overline{x}$ on E, i.e. for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any u, $v \in E \cap B(\overline{x}, \delta)$ and any $u^* \in N_\partial(E, u) \cap B_{X^*}$, $v^* \in N_\partial(E, v) \cap B_{X^*}$ one has*

$$\langle u^* - v^*, u - v \rangle \ge -\varepsilon \|u - v\| .$$

Proof. (a) The "if" assertion is obtained in taking $u^* = 0$ in the preceding relations; owing to the fact valid for any subdifferential that $0 \in \partial d_E(u)$ for $u \in E$. The "only if" assertion is obtained by adding sides by sides the inequalities

$$\langle u^*, u - v \rangle \ge -\varepsilon \|u - v\| , \quad \text{and} \quad \langle v^*, v - u \rangle \ge -\varepsilon \|u - v\| .$$

The proof of (b) is similar. □

This equivalence entails that when ∂ is valuable in the sense that a mean value theorem is valid for ∂, a ∂-subconvex set at $\overline{x}$ is approximately convex at $\overline{x}$ in the sense of [23], in view of the results in [13] and [24], [25].

It also allows to use the following slight extension of [2, Prop. 3.9].

Proposition 2.15. *Let E be a closed subset of an Asplund space X and let ∂, ∂' be two subdifferentials such that $\partial_F f(x) \subset \partial f(x)$, $\partial' f(x) \subset \partial_L f(x)$ for all $x \in X$ and all lower semicontinuous functions f. Then E is ∂'-subconvex at $\overline{x} \in E$ if E is ∂-subconvex at $\overline{x}$.*
The same conclusion holds if X is reflexive and if $\partial_F f(x) \subset \partial f(x)$ and $\partial' f(x) \subset \partial_L f(x)$ for all $x \in X$ and all lower semicontinuous functions f.

Let us observe that the assumption on ∂ and ∂' can be substituted with an assertion about normal cones. Moreover the statement can be made symmetric in ∂ and ∂'.

Proof. Let ∂, ∂' be such that for any lower semicontinuous function f and any $x \in X$ one has $\partial_F f(x) \subset \partial f(x)$ and $\partial' f(x) \subset \partial_L f(x)$. Assume E is ∂-subconvex at $\overline{x} \in E$; then E is ∂_F-subconvex at $\overline{x}$. By [2, Prop. 3.9], E is ∂_L-subconvex at $\overline{x}$. Then it is ∂'-subconvex at $\overline{x}$.

The proof of the second assertion is similar. □

3. Preservation under usual operations

For practical uses, it is important to show that subconvexity is preserved under usual operations. We start with products. Again ∂ is an arbitrary subdifferential.

Proposition 3.1. *Let A (resp. B) be a ∂-subconvex subset at $\overline{x} \in A$ (resp. $\overline{y} \in B$) of a Banach space X (resp. Y). Then $C := A \times B$ is ∂-subconvex at $\overline{z} := (\overline{x}, \overline{y})$.*

If A (resp. B) is a ∂-metrically subconvex (resp. ∂-m-subconvex) subset at $\overline{x} \in A$ (resp. $\overline{y} \in B$) then $C := A \times B$ is ∂-metrically subconvex (resp. ∂-m-subconvex) at $\overline{z} := (\overline{x}, \overline{y})$.

Proof. Given $\varepsilon > 0$ let $\delta_A > 0$ (resp. $\delta_B > 0$) be such that for any $x, u \in A \cap B(\overline{x}, \delta_A)$, $y, v \in B \cap B(\overline{y}, \delta_B)$, $x^* \in N(A, x) \cap B_{X^*}$, $y^* \in N(B, y) \cap B_{Y^*}$ one has

$$\langle x^*, u - x \rangle \leq \varepsilon \|u - x\|, \qquad \langle y^*, v - y \rangle \leq \varepsilon \|v - y\|.$$

Endowing $Z := X \times Y$ with the sum norm, setting $\delta := \min(\delta_A, \delta_B)$, for $w := (u, v)$, $z := (x, y)$ in $C \cap B(\overline{z}, \delta)$, for $z^* := (x^*, y^*) \in N(C, z) \cap B_{Z^*}$, in view of condition (S3b) of the appendix one has $x^* \in N(A, x) \cap B_{X^*}$, $y^* \in N(B, y) \cap B_{Y^*}$, hence

$$\langle z^*, w - z \rangle = \langle x^*, u - x \rangle + \langle y^*, v - y \rangle \leq \varepsilon \|u - x\| + \varepsilon \|v - y\| = \varepsilon \|w - z\|.$$

Thus C is ∂-subconvex.

The ∂-metrically subconvex (resp. ∂-m-subconvex) case is obtained by observing that for $(x', y') \in Z$ one has $d_C(x', y') = d_A(x') + d_B(y')$, so that for $z^* := (x^*, y^*) \in \partial d_C(x, y)$ one has $x^* \in \partial d_A(x)$, $y^* \in \partial d_B(y)$, and the conclusion follows as above. □

In the sequel we use the notion of *coderivative* of a map $g : X \to Y$ between two normed spaces, a special case of the notion of coderivative D^*G of a multimap $G : X \rightrightarrows Y$ associated with a subdifferential ∂ defined for $(x, y) \in G$ by

$$D^*G(x, y)(y^*) := \{x^* \in X^* : (x^*, -y^*) \in N(G, (x, y))\} \qquad y^* \in Y^*.$$

In the case of a map g, we simply write $D^*g(x)$ instead of $D^*G(x, y)$.

The following results about inverse images are close to [3, Thm. 3.1] which was devoted to the case $\partial = \partial_F$.

Theorem 3.2. *Let $g : X \to Y$ be Lipschitzian around $\overline{x} \in X$, let $C \subset Y$ be ∂-subconvex at $\overline{y} := g(\overline{x}) \in C$. Then, under the following assumptions, $B := g^{-1}(C)$ is ∂-subconvex at $\overline{x}$ and $N(B, \overline{x}) = D^*g(\overline{x})(N(C, g(\overline{x})))$:*

(a) *the graph G of g is ∂-subconvex at $(\overline{x}, \overline{y})$;*

(b) *for some $\beta > 0$ one has $N(B, x) \cap B_{X^*} \subset D^*g(x)(N(C, g(x)) \cap \beta B_{Y^*})$ for all x near $\overline{x}$.*

Assumption (b) is satisfied if $\partial = \partial_F$ and, for x near $\overline{x}$, g is strictly (or circa)-differentiable at x with $g'(x)$ surjective (see [33, Thm. 2.111]).

Proof. Without loss of generality, we assume that the Lipschitz rate of g near $\overline{x}$ is $\kappa \geq 1$ and $\beta \geq 1$. Given $\varepsilon > 0$, let $\gamma > 0$ be such that

$$\langle y^*, v - y\rangle \leq \varepsilon \|y^*\| . \|v - y\|$$

for any $v, y \in C \cap B(\overline{y}, \gamma)$, $y^* \in N(C, y)$ and let $\alpha > 0$ be such that for any (x, y), $(u, v) \in G \cap B((\overline{x}, \overline{y}), \alpha)$, $(x^*, y^*) \in N(G, (x, y))$ one has

$$\langle (x^*, y^*), (u, v) - (x, y)\rangle \leq \varepsilon \|(x^*, y^*)\| . \|(u - x, v - y)\| .$$

Then, given u, $x \in B \cap B(\overline{x}, \eta)$ with $\eta := \min((\kappa + 1)^{-1}\alpha, \kappa^{-1}\gamma)$, $x^* \in N(B, x)$, setting $v := g(u)$, $y := g(x)$, and taking $y^* \in N(C, y)$ such that $x^* \in D^*g(x)(y^*)$, $\|y^*\| \leq \beta \|x^*\|$, one has

$$\begin{aligned}
\langle x^*, u - x\rangle &\leq \langle y^*, v - y\rangle + \varepsilon \|(x^*, -y^*)\| .(\|u - x\| + \|v - y\|), \\
&\leq \langle y^*, v - y\rangle + \varepsilon\beta(\kappa + 1) \|x^*\| \|u - x\| \\
&\leq \varepsilon\kappa\beta \|x^*\| . \|u - x\| + \varepsilon(\kappa + 1)\beta \|x^*\| . \|u - x\| .
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, that shows that B is ∂-subconvex at $\overline{x}$.

Moreover, in the case $(x, y) = (\overline{x}, \overline{y})$, using the relation $D^*G(\overline{x}, \overline{y}) = D^*_F G(\overline{x}, \overline{y})$, given $x^* \in N(B, \overline{x})$, taking $y^* \in N(C, \overline{y})$ such that $x^* \in D^*G(\overline{x}, \overline{y})(y^*)$, one sees that $x^* \in N_F(B, \overline{x})$ and when $N_F(B, \overline{x}) \subset N(B, \overline{x})$ one gets the announced equality. □

Corollary 3.3. *Let E (resp. F) be a ∂-subconvex subset of X at $\overline{x} \in E \cap F$. Assume there exists some $\beta > 0$ such that for all x near $\overline{x}$ one has*

$$N(E \cap F, x) \cap B_{X^*} \subset N(E, x) \cap \beta B_{X^*} + N(F, x) \cap \beta B_{X^*}.$$

Then $E \cap F$ is ∂-subconvex at $\overline{x}$ and if ∂ is larger than ∂_F one has $N(E \cap F, \overline{x}) = N(E, \overline{x}) + N(F, \overline{x})$.

Proof. The first assertion stems from the theorem by taking for g the diagonal map $x \mapsto (x, x)$ whose transpose map is the map $(x^*, y^*) \mapsto x^* + y^*$, observing that $E \cap F = g^{-1}(E \times F)$, that $N(E \times F, (x, y)) \subset N(E, x) \times N(F, y)$ by axiom (S3b) and that the graph of g is convex and is even a linear subspace. When ∂ is larger than ∂_F one has

$$N(E \cap F, \overline{x}) \subset N(E, \overline{x}) + N(F, \overline{x}) = N_F(E, \overline{x}) + N_F(F, \overline{x}) \subset N_F(E \cap F, \overline{x}).$$

Then, the inclusion $N_F \subset N$ entails equality in the preceding relations. □

In the next statement ∂ is said to be a *sundering* subdifferential if it satisfies the sum rule $\partial(f + g)(x) \subset \partial f(x) + \partial g(x)$ for f Lipschitzian around x and g lower semicontinuous. This condition is satisfied for ∂_C, ∂_G (the Ioffe's subdifferential), and, in the case of Asplund spaces, for $\partial = \partial_L$.

Corollary 3.4. *Let ∂ be a sundering subdifferential. Let E and F be ∂-metrically subconvex subsets of X at $\overline{x} \in E$. Assume there exists some $\beta > 0$ such that for all x near $\overline{x}$ one has $d(E \cap F, x) \leq \beta d(E, x) + \beta d(F, x)$. Then $E \cap F$ is ∂-metrically subconvex at $\overline{x}$. If moreover ∂ is normally consistent, then $E \cap F$ is ∂-subconvex at $\overline{x}$.*

Proof. Since ∂ is sundering, setting $B := E \cap F$ and using axiom (S6) in the appendix, one has $\partial d_B(x) \subset \beta\partial d_E(x) + \beta\partial d_F(x)$ for all x near $\overline{x}$. Thus, for $x^* \in \partial d_B(x)$, one can find $u^* \in \partial d_E(x)$, $v^* \in \partial d_F(x)$ such that $x^* = \beta u^* + \beta v^*$ and, given $\varepsilon > 0$, one can find $\delta > 0$ such that for all x, $x' \in E \cap F \cap B(\overline{x}, \delta)$, one has

$$\langle u^*, x' - x\rangle \leq \frac{1}{2}\varepsilon\beta^{-1} \|x' - x\|, \qquad \langle v^*, x' - x\rangle \leq \frac{1}{2}\varepsilon\beta^{-1} \|x' - x\|,$$

and one gets $\langle x^*, x' - x\rangle \leq \varepsilon \|x' - x\|$. If ∂ is normally consistent, then $E \cap F$ is ∂-subconvex since then ∂-metric subconvexity coincides with ∂-subconvexity. □

Remark 3.5. Under the additional assumption that there exists some $c > 0$ such that for any $x^* \in \partial d_{E\cap F}(x)$, one can find $u^* \in \partial d_E(x)$, $v^* \in \partial d_F(x)$ such that $x^* = \beta u^* + \beta v^*$ and $\|u^*\| \leq c \|x^*\|$, $\|v^*\| \leq c \|x^*\|$, one sees that $E \cap F$ is ∂-m-subconvex at $\overline{x}$ whenever E and F are ∂-m-subconvex at $\overline{x}$.

Theorem 3.6. *Let X and Y be Banach spaces, let $g : X \to Y$ be of class C^1 around $\overline{x} \in X$, with $g'(\overline{x})(X) = Y$, let $C \subset Y$ be ∂-m-subconvex (resp. ∂-metrically subconvex) at $\overline{y} := g(\overline{x})$ and let $B := g^{-1}(C)$. Then, the set B is ∂-m-subconvex (resp. ∂-metrically subconvex) at $\overline{x}$.*

This result is a slight improvement of [3, Thm. 3.1] in which $\partial = \partial_F$, g is of class C^1 over an open subset containing B, and the set C is metrically subconvex at each point v of a neighborhood V of $\overline{y}$, with $N_F(C, v) = N_C(C, v)$ for all $v \in V \cap C$. For other results dealing with inequalities such as $d_B \leq \beta d_C \circ g$ see [34].

Proof. Since $g'(\overline{x})$ is onto, the Lyusternik-Graves Theorem (see [33, Thm. 2.67] for instance) asserts that there exist some $\rho, \sigma, \kappa \in \mathbb{P} :=]0, +\infty[$ such that $B(\overline{x}, \rho) \subset U$, and that for all $w \in B(\overline{x}, \rho)$, $y \in B(\overline{y}, \sigma)$ there exists some $x \in g^{-1}(y)$ such that $\|w - x\| \leq \kappa \|g(w) - y\|$. Taking $w \in B(\overline{x}, \rho)$ such that $\|g(w) - \overline{y}\| < \sigma/2$, hence $d(g(w), C) < \sigma/2$, for all $r \in]d(g(w), C), \sigma/2[$ we pick $y \in C$ such that $\|g(w) - y\| < r$. Then $y \in B(\overline{y}, \sigma)$. Picking $x \in g^{-1}(y)$ such that $\|w - x\| \leq \kappa \|g(w) - y\| < \kappa r$ and passing to the infimum on r, we obtain $d(w, B) \leq \kappa d(g(w), C)$. Using quasi-homotonicity of ∂ (or conditions (S6)) and condition (S3) of the appendix, for $x \in U' := B(\overline{x}, \rho)$ one has the inclusions

$$\partial d_B(x) \subset \kappa\partial(d_C \circ g)(x) \subset \kappa\partial d_C(g(x)) \circ g'(x)$$

valid for any subdifferential ∂. Given $\varepsilon > 0$, $x^* \in \partial d_B(x)$, let $y^* \in \partial d_C(g(x))$ be such that $x^* := \kappa y^* \circ g'(x)$. Since $g'(\overline{x})$ is an open map, using the continuity of g' at $\overline{x}$ and Radstrom's cancellation law, we see that there exists some $\lambda > 0$ such that $B_Y \subset \lambda g'(x)(B_X)$ for all x in a neighborhood V of $\overline{x}$ contained in U and one gets

$$\|y^*\| = \sup\langle y^*, B_Y\rangle \leq \sup\langle y^*, \lambda g'(x)(B_X)\rangle = \lambda\kappa^{-1} \sup\langle x^*, B_X\rangle = \lambda\kappa^{-1} \|x^*\|.$$

Let $k > 0$ be the Lipschitz rate of g on $V \subset U'$, a suitably reduced neighborhood of $\overline{x}$, let $\xi \in]0, \frac{1}{2}\varepsilon\lambda^{-1}]$, and let $\rho' > 0$ be such that $B(\overline{x}, \rho') \subset V$ and

$$\|g(x) - g(x') - g'(\overline{x})(x - x')\| \leq \xi \|x - x'\| \qquad \forall x, x' \in B(\overline{x}, \rho).$$

Given $\varepsilon > 0$, since C is ∂-m subconvex at $\overline{y}$, there exists $\sigma' \in]0, \rho']$ such that $\langle y^*, y' - y\rangle \leq \frac{1}{2}\varepsilon\kappa^{-1}\lambda^{-1} \|y^*\| . \|y' - y\|$ for any $y', y \in C \cap B(g(\overline{x}), \sigma')$, $y^* \in N^m(C, y)$.

Then, for x, $x' \in B \cap B(\overline{x}, \rho'')$, with $\rho'' = \min(\rho', \sigma/k)$, one has

$$\begin{aligned}\langle x^*, x' - x\rangle &= \kappa\langle y^*, g'(x)(x' - x)\rangle \le \kappa\langle y^*, g(x') - g(x)\rangle + \kappa\xi \|y^*\| . \|x' - x\| \\ &\le \frac{\kappa\varepsilon}{2k\lambda} \|y^*\| . \|g(x') - g(x)\| + \xi\lambda \|x^*\| . \|x' - x\| \le \varepsilon \|x^*\| . \|x' - x\|\end{aligned}$$

By homogeneity that shows that B is ∂-m-subconvex at $\overline{x}$.

The proof in the ∂-metrically subconvex case is similar and even simpler. □

4. Subconvex relations

In the sequel, we identify a relation (or correspondence or multimap or multifunction) $G : X \rightrightarrows Y$ between two sets with its graph. By analogy with the convex case it would be natural, but not very convenient, to say that G is ∂-subconvex at $(\overline{x}, \overline{y}) \in G$ if the graph of G is ∂-subconvex at $(\overline{x}, \overline{y})$. In this case we prefer to say that G is *graphically ∂-subconvex* at $(\overline{x}, \overline{y}) \in G$ in order to reserve the term "∂-subconvex" to the notion of the next definition that promotes the viewpoint that multimaps are generalized maps. Still, with the notion of graphical subconvexity, one disposes of the following property. It also uses the notion of pseudo-Lipschitz property (or Aubin property) recalled below which is more realistic than the Lipschitz property when dealing with multimaps. This property reminds an estimate on the derivative of a locally Lipschitz map and a property that is specific of the Fréchet subdifferential.

Proposition 4.1. *If the graph of $G : X \rightrightarrows Y$ is ∂-subconvex at $(\overline{x}, \overline{y})$ and if G is pseudo-Lipschitzian with rate λ around $(\overline{x}, \overline{y}) \in G$, then for any $\kappa > \lambda$ there exists some $\delta > 0$ such that for any $(x, y) \in G \cap B((\overline{x}, \overline{y}), \delta)$ and any $y^* \in Y^*$, $x^* \in D^*G(x, y)(y^*)$ one has $\|x^*\| \le \kappa \|y^*\|$.*

Proof. Saying that G is *pseudo-Lipschitzian with rate* λ around $(\overline{x}, \overline{y})$ means that there exists some $\rho > 0$ such that for any $w, x \in B(\overline{x}, \rho)$, $y \in G(x) \cap B(\overline{y}, \rho)$ one can find some $z \in G(w)$ such that $\|z - y\| \le \lambda \|w - x\|$. Given $\kappa > \lambda$, $\lambda' := \max(\lambda, 1)$, let $\alpha > 0$ be so small that $(1 - \alpha\lambda')^{-1}(\lambda + \alpha\lambda') \le \kappa$. By definition, the set G being ∂-subconvex at $(\overline{x}, \overline{y})$, one can find some $\delta \in]0, \rho/\max(\lambda + 1, 2)[$ such that for any (x, y), $(w, z) \in G \cap B((\overline{x}, \overline{y}), \delta)$, and for any $y^* \in Y^*$, $x^* \in D^*G(x, y)(y^*)$ one has

$$\langle x^*, w - x\rangle - \langle y^*, z - y\rangle \le \alpha(\|x^*\| + \|y^*\|) \max(\|w - x\|, \|z - y\|). \qquad (*)$$

Since G is pseudo-Lipschitzian with rate λ around $(\overline{x}, \overline{y})$, given $(x, y) \in G \cap B((\overline{x}, \overline{y}), \delta)$, $w \in B(x, \delta)$ one can find $z \in G(w)$ such that $\|z - y\| \le \lambda \|w - x\|$. Then one has $(w, z) \in G \cap B((\overline{x}, \overline{y}), \rho)$ since

$$\max(\|\overline{x} - w\|, \|\overline{y} - z\|) \le \max(\|\overline{x} - x\| + \delta, \|\overline{y} - y\| + \lambda\delta) \le \max(2\delta, (\lambda + 1)\delta) \le \rho.$$

Since $w \in B(x, \delta)$ is arbitrary, taking $y^* \in Y^*$ and $x^* \in D^*G(x, y)(y^*)$ in (*), one has

$$\delta \|x^*\| \le \lambda\delta \|y^*\| + \alpha(\|x^*\| + \|y^*\|)\lambda'\delta$$

or

$$\|x^*\| \le (1 - \alpha\lambda')^{-1}(\lambda + \alpha\lambda') \|y^*\| \le \kappa \|y^*\|. \qquad □$$

The definition we choose for subconvexity of a multimap is as follows. In it the roles of the source space X and of the target space Y are not symmetric. Moreover, the definition is more demanding than requiring that the graph of G be subconvex at $(\overline{x}, \overline{y})$. We also present variants that are still more exacting.

Definition 4.2. Given a subdifferential ∂, a multimap $G : X \rightrightarrows Y$ between two normed spaces is said to be *∂-subconvex* at $(\overline{x}, \overline{y}) \in G$ if for any $\varepsilon > 0$ one can find some $\delta > 0$ such that for any $(x, y), (x', y') \in G \cap B((\overline{x}, \overline{y}), \delta)$, any $y^* \in Y^*$, any $x^* \in D^*G(x, y)(y^*)$ one has

$$\langle x^*, x' - x \rangle - \langle y^*, y' - y \rangle \leq \varepsilon \max(\|x^*\|, \|y^*\|) \|x' - x\| .$$

The multimap G is said to be *∂-prosubconvex* if this inequality can be replaced by

$$\langle x^*, x' - x \rangle - \langle y^*, y' - y \rangle \leq \varepsilon \|y^*\| . \|x' - x\|. \qquad \square$$

Among the reasons why we choose this definition for ∂-subconvexity is the fact that a related property has been used in [34] in order to prove an error bound property. It is clearly satisfied by a map that is circa (or strictly) differentiable at $\overline{x}$. Also the following proposition shows a coherence with Definition 2.1.

Let us note immediate observations stemming from Proposition 4.1.

Proposition 4.3. *If $G : X \rightrightarrows Y$ is ∂-subconvex at $(\overline{x}, \overline{y})$ and if G is pseudo-Lipschitzian with rate $\lambda > 0$ around $(\overline{x}, \overline{y}) \in G$, then G is ∂-prosubconvex at $(\overline{x}, \overline{y})$.*

Proposition 4.4. *If $g : X \to Y$ is a λ-Lipschitzian map around $\overline{x}$ and if its graph is ∂-subconvex at $(\overline{x}, g(\overline{x}))$, then g is ∂-subconvex at $(\overline{x}, g(\overline{x}))$ and even ∂-prosubconvex at $(\overline{x}, g(\overline{x}))$.*

Proof. In such a case, for w, $x \in B(\overline{x}, \delta)$, $y := g(x)$, $z := g(w)$ the relation

$$\langle x^*, w - x \rangle - \langle y^*, z - y \rangle \leq \varepsilon \max(\|x^*\|, \|y^*\|)(\|w - x\| + \|z - y\|) \tag{1}$$

yields $\qquad \langle x^*, w - x \rangle - \langle y^*, z - y \rangle \leq \varepsilon(\lambda + 1) \max(\lambda, 1) \|y^*\| \|w - x\|. \qquad \square$

Proposition 4.5. *If a subset S of X is ∂-subconvex at $\overline{x} \in S$ then for any normed space Y and $\overline{y} \in Y$ the relation G with graph $S \times Y$ is ∂-subconvex at $(\overline{x}, \overline{y})$.*
Conversely, if for some normed space Y and $\overline{y} \in Y$ the multimap G with graph $S \times Y$ is ∂-subconvex at $(\overline{x}, \overline{y})$, then S is ∂-subconvex at $\overline{x}$.

Proof. We use the facts that the axioms (S2a), (S3b) imply $N(Y, \overline{y}) = \partial\iota_Y(\overline{y}) = \{0\}$ and $N(S \times Y, (\overline{x}, \overline{y})) \subset N(S, \overline{x}) \times N(Y, \overline{y})$. Thus, if S is ∂-subconvex at $\overline{x} \in S$ then, for any normed space Y and $\overline{y} \in Y$, the relation G with graph $S \times Y$ is ∂-subconvex at $(\overline{x}, \overline{y})$ since for any $(x, y) \in S \times Y$, any $y^* \in Y^*$ and any $x^* \in D^*G(x, y)(y^*)$ one has $y^* = 0$ and the condition of Def 4.2 is satisfied.

Conversely, since $\iota_{S\times Y} = \iota_S \circ p_X$, hence $N(S\times Y, (x, y)) = p_X^{\intercal}(N(S, x)) = N(S, x)\times\{0\}$ by (S3a) if for some normed space Y and some $\overline{x} \in S$, $\overline{y} \in Y$ the multimap $G := S\times Y$ is ∂-subconvex at $(\overline{x}, \overline{y})$, then for any $x^* \in N(S, \overline{x})$ one has $(x^*, 0) \in N(S \times Y, (x, y))$ so that the relation

$$\langle (x^*, 0), (w - x, \overline{y} - \overline{y}) \rangle \leq \varepsilon \|(x^*, 0)\| . \|(w - x, \overline{y} - \overline{y})\| = \varepsilon \|x^*\| . \|w - x\|$$

for w, x close to $\overline{x}$ shows that the condition of Def 2.1 (a) is satisfied. $\square$

While (loosely speaking) the product $F \times G$ of two graphically subconvex multimaps is still graphically subconvex in view of Proposition 3.1, the next result deals with multimaps with values in a product. It could be extended to any finite product.

Proposition 4.6. *Let X, Y, Z be normed spaces, let $F : X \rightrightarrows Y$, $G : X \rightrightarrows Z$ be two multimaps, and let $H := (F, G) : X \rightrightarrows Y \times Z$ be given by $H(x) := F(x) \times G(x)$. Assume that F (resp. G) is ∂-subconvex at the point $(\overline{x}, \overline{y})$ (resp. ∂-subconvex at $(\overline{x}, \overline{z})$), and there exists some $c > 0$ such that for all (x, y, z) near $(\overline{x}, \overline{y}, \overline{z})$ and all $y^* \in Y^*$, $z^* \in Z^*$, $x^* \in D^*H(x, y, z)(y^*, z^*)$ one can find $u^* \in D^*F(x, y)(y^*)$ and $v^* \in D^*G(x, z)(z^*)$ satisfying $x^* = u^* + v^*$, $\|u^*\| + \|v^*\| \leq c(\|x^*\| + \|y^*\| + \|z^*\|)$. Then H is $\partial-$subconvex at $(\overline{x}, (\overline{y}, \overline{z}))$.*

Let us note that the relation $\|u^*\| + \|v^*\| \leq c(\|x^*\| + \|y^*\| + \|z^*\|)$ is automatically satisfied whenever F and G are pseudo-Lipschitzian around $(\overline{x}, \overline{y})$ and $(\overline{x}, \overline{z})$ respectively, $x^* = u^* + v^*$ with $u^* \in D^*F(x, y)(y^*)$ and $v^* \in D^*G(x, z)(z^*)$. Setting $B := F \times Z$, $C := \{(x, y, z) : (x, z) \in G, y \in Y\}$, so that $H = B \cap C$, this result can be seen as similar to Proposition 3.4, but we give a simple direct proof.

Proof. By assumption, for any $\varepsilon > 0$ one can find some $\delta_F > 0$, $\delta_G > 0$ such that for any (x, y), $(x', y') \in F \cap B((\overline{x}, \overline{y}), \delta_F)$, any (x, z), $(x', z') \in G \cap B((\overline{x}, \overline{z}), \delta)$, any $y^* \in Y^*$, $z^* \in Z^*$, any $u^* \in D^*F(x, y)(y^*)$, any $v^* \in D^*G(x, z)(z^*)$ one has

$$\langle u^*, x' - x\rangle - \langle y^*, y' - y\rangle \leq \varepsilon(\|u^*\| + \|y^*\|)\, \|x' - x\|$$

and
$$\langle v^*, x' - x\rangle - \langle z^*, z' - z\rangle \leq \varepsilon(\|v^*\| + \|z^*\|)\, \|x' - x\| \, .$$

Therefore, setting $\delta := \min(\delta_F, \delta_G)$ and taking $(x, y, z) \in H \cap B((\overline{x}, \overline{y}, \overline{z}), \delta)$, for any $x^* \in D^*H(x, y, z)(y^*, z^*)$ the assumptions ensure that one can find $u^* \in D^*F(x, y)(y^*)$, $v^* \in D^*G(x, z)(z^*)$ such that $x^* = u^* + v^*$ and $\|u^*\| + \|v^*\| \leq c(\|x^*\| + \|y^*\| + \|z^*\|)$. Adding sides by sides the preceding inequalities, one gets

$$\langle x^*, x' - x\rangle - \langle (y^*, z^*), (y' - y, z' - z)\rangle \leq \varepsilon(c + 1)(\|x^*\| + \|y^*\| + \|z^*\|)\, \|x' - x\| \, .$$

Since ε is arbitrarily small, the result is established. □

In the next proposition, we observe that the decomposition

$$D^*H(x, y, z)(y^*, z^*) \subset D^*F(x, y)(y^*) + D^*G(x, z)(z^*)$$

can be deduced from a metric estimate under additional assumptions. Here we say that ∂ is *almost consistent* if for any normed space X, any subset E of X and any $\overline{x} \in E$ the metric normal cone $N^m(E, \overline{x})$ is dense in $N(E, \overline{x})$.

Proposition 4.7. *Assume ∂ is almost consistent and sundering as defined in the Appendix. Let X, Y, Z be normed spaces, let $F : X \rightrightarrows Y$, $G : X \rightrightarrows Z$ be two multimaps, and let $H := (F, G) : X \rightrightarrows Y \times Z$ be such that for some $c > 0$, $\rho > 0$ the following estimate holds for $(x, y, z) \in B((\overline{x}, \overline{y}, \overline{z}), \rho)$:*

$$d((x, y, z), H) \leq cd((x, y), F) + cd((x, z), G).$$

If F (resp. G) is ∂-prosubconvex at $(\overline{x}, \overline{y})$ (resp. $(\overline{x}, \overline{z})$), then $H := (F, G)$ is ∂-prosubconvex at $(\overline{x}, (\overline{y}, \overline{z}))$.

Proof. Given $(x,y,z)\in H\cap B((\overline{x},\overline{y},\overline{z}),\rho)$, $y^*\in Y^*$, $z^*\in Z^*$, $x^*\in D^*H(x,y,z)(y^*,z^*)$, since ∂ is almost consistent, there exist some $(r_n)\subset \mathbb{R}_+$, $(x_n^*,-y_n^*,-z_n^*)\in \partial d_H(x,y,z)$ such that $(r_n(x_n^*,y_n^*,z_n^*))\to (x^*,y^*,z^*)$. Using the above estimate, axioms (S3a), (S6) and the fact that ∂ is sundering, one can find $(u_n^*,-\overline{y}_n^*)\in \partial d_F(x,y)$ and $(v_n^*,-\overline{z}_n^*)\in \partial d_G(x,z)$ satisfying

$$(x_n^*,-y_n^*,-z_n^*)=c(u_n^*,-\overline{y}_n^*,0)+c(v_n^*,0,-\overline{z}_n^*).$$

In consequence one has $x_n^*=cu_n^*+cv_n^*$, $y_n^*=c\overline{y}_n^*$, $z_n^*=c\overline{z}_n^*$ and $u_n^*\in D^*F(x,y)(\overline{y}_n^*)$, $v_n^*\in D^*G(x,z)(\overline{z}_n^*)$. Given $\varepsilon>0$, we take $\delta\in]0,\rho[$ be such that for (x,y) and $(x',y')\in F\cap B((\overline{x},\overline{y}),\delta)$, (x,z) and $(x',z')\in G\cap B((\overline{x},\overline{z}),\delta)$ one has

$$\langle u_n^*,x'-x\rangle-\langle \overline{y}_n^*,y'-y\rangle\le \varepsilon\left\|\overline{y}_n^*\right\|.\left\|x'-x\right\|,$$

and
$$\langle v_n^*,x'-x\rangle-\langle \overline{z}_n^*,z'-z\rangle\le \varepsilon\left\|\overline{z}_n^*\right\|.\left\|x'-x\right\|.$$

Adding sides by sides these inequalities after multiplication by c, one gets

$$\langle x_n^*,x'-x\rangle-\langle y_n^*,y'-y\rangle-\langle z_n^*,z'-z\rangle\le\varepsilon(\|y_n^*\|+\|z_n^*\|).\left\|x'-x\right\|.$$

Multiplying both sides by r_n, passing to the limit, since ε is arbitrarily small, we see that H is ∂-prosubconvex at $(\overline{x},\overline{y},\overline{z})$. $\square$

For what concerns graphical ∂-subconvexity, one can use Theorem 3.2 with the map $g:(x,y,z)\mapsto(x,y,x,z)$ and the fact that $H=g^{-1}(F\times G)$.

5. Subconvex functions

There is a difference between sets and functions: whereas the normal cone to a subset E of X at a point $x\in E$ is never empty (since x is always a minimizer of its indicator function ι_E), the subdifferential $\partial f(x)$ of a function f at $x\in \mathrm{dom} f$ may be empty. For this reason, it may seem to be sensible to say that f is ∂-subconvex (resp. ∂-metrically subconvex, resp. ∂-m-subconvex) at $\overline{x}$ if its epigraph E_f is ∂-subconvex (resp. ∂-metrically subconvex, resp. ∂-m-subconvex) at $\overline{x}_f=(\overline{x},f(\overline{x}))$.

Let us examine the implications of such a choice and propose another one which seems to be more attractive.

Hereafter, given $\delta>0$ and $\overline{x}\in \mathrm{dom} f$, we set

$$B_f(\overline{x},\delta):=\{x\in X:\|x-\overline{x}\|<\delta,\ |f(x)-f(\overline{x})|<\delta\}.$$

Proposition 5.1. *Let $f:X\to\mathbb{R}_\infty$ be a function on a normed vector space X finite at some $\overline{x}\in X$ whose epigraph E_f is ∂-subconvex at $\overline{x}_f:=(\overline{x},f(\overline{x}))$. Then, for any $\varepsilon>0$ one can find some $\delta>0$ such that for any w, $x\in B_f(\overline{x},\delta)$ and any $x^*\in\partial f(x)$ one has*

$$f(w)-f(x)-\langle x^*,w-x\rangle\ge-\varepsilon\max(\|x^*\|,1)(\|w-x\|+|f(w)-f(x)|).$$

If moreover f is continuous at $\overline{x}$ then, for any $\varepsilon>0$ one can find some $\eta>0$ such that this inequality holds for any w, $x\in B(\overline{x},\eta)$ and any $x^*\in\partial f(x)$.

In particular, if the set E_f is ∂-subconvex at $\overline{x}_f$ and if f is Lipschitzian around $\overline{x}$, given $\varepsilon>0$, the inequality $f(w)-f(x)-\langle x^*,w-x\rangle\ge-\varepsilon\|w-x\|$ holds for any w, $x\in B(\overline{x},\delta)$ and any $x^*\in\partial f(x)$ with some $\delta>0$ small enough.

Proof. For any ∂ and for any $x \in E$, $x^* \in \partial f(x)$ one has $(x^*, -1) \in N(E_f, x_f)$ for $x_f := (x, f(x))$, as seen in the appendix. Since the set E_f is ∂-subconvex at $\overline{x}_f$, given $\varepsilon > 0$ one can find some $\delta > 0$ such that

$$\langle x^*, w - x\rangle - (r - s) \leq \varepsilon \max(\|x^*\|, 1)(\|w - x\| + |r - s|)$$

for any (w, r), $(x, s) \in E_f \cap B(\overline{x}, \delta) \times]f(\overline{x}) - \delta, f(\overline{x}) + \delta[$ and any $x^* \in \partial f(x)$. Taking w, $x \in B_f(\overline{x}, \delta)$ and $r := f(w)$, $s := f(x)$, we get the inequality of the statement.

If f is continuous at $\overline{x}$ then, given $\delta > 0$ one can find some $\eta > 0$ such that for any w, $x \in B(\overline{x}, \eta)$ one has w, $x \in B_f(\overline{x}, \delta)$, what justifies the second assertion.

When f is Lipschitzian with rate λ around $\overline{x}$, taking $\rho > 0$ small enough, one can ensure that $B(\overline{x}, \rho) \subset B_f(\overline{x}, \delta)$, so that for $w, x \in B(\overline{x}, \rho)$, $x^* \in \partial f(x)$, taking $r := f(w)$ and $s := f(x)$ one has

$$\begin{aligned} \langle x^*, w - x\rangle - (f(w) - f(x)) &\leq \varepsilon \max(\|x^*\|, 1)(\|w - x\| + |f(w) - f(x)| \\ &\leq \varepsilon \max(\|x^*\|, 1)(\lambda + 1) \|w - x\|. \end{aligned}$$

Since we have $\partial f(x) \subset \lambda B_{X^*}$, replacing ε by $\varepsilon := \varepsilon / \max(\lambda, 1)(\lambda + 1)$ one gets the assertion. □

The last property is certainly more attractive than the first one. For this reason we adopt the following definition for a function which is not necessarily locally Lipschitzian.

Definition 5.2. A lower semicontinuous function f on X is said to be ∂-*subconvex* at $\overline{x} \in \mathrm{dom} f$ if for any $\varepsilon > 0$ there exists $\delta := \delta(\varepsilon) > 0$ such that for any w, $x \in B_f(\overline{x}, \delta)$, $x^* \in \partial f(x)$ one has

$$f(w) - f(x) - \langle x^*, w - x\rangle \geq -\varepsilon \max(\|x^*\|, 1) \|w - x\|. \qquad \square$$

A justification for this definition is given by the following result.

Proposition 5.3. *Given $f : X \to \mathbb{R}_\infty$ and $\overline{x} \in \mathrm{dom} f$, if the multimap $E_f := \mathrm{epi} f$ is ∂-subconvex at $\overline{x}_f$ then f is ∂-subconvex at $\overline{x}$.*

Proof. Given $\varepsilon > 0$, let $\delta > 0$ be as in Definition 4.2. Let $w, x \in B_f(\overline{x}, \delta) \subset \mathrm{dom} f$. Setting $r := f(w)$, $s := f(x)$, so that $(w, r), (x, s) \in E_f \cap B(\overline{x}_f, \delta)$, since E_f is ∂-subconvex and since for any $x^* \in \partial f(x)$ one has $x^* \in D^* E_f(x, r)(1)$ one has

$$\langle x^*, w - x\rangle - (f(w) - f(x)) \leq \langle x^*, w - x\rangle - (r - s) \leq \varepsilon \max(\|x^*\|, 1) \|w - x\|$$

and the condition of Definition 5.2 is satisfied. □

Definition 5.2 entails the following property that we call quasi-∂-subconvexity at $\overline{x}$: a function f on X is *quasi-∂-subconvex* at $\overline{x} \in \mathrm{dom} f$ if for any $\varepsilon > 0$, $c > 0$ there exists $\eta := \eta(\varepsilon, c) > 0$ such that for any w, $x \in B_f(\overline{x}, \eta)$, $x^* \in \partial f(x) \cap c B_{X^*}$ one has

$$f(w) - f(x) - \langle x^*, w - x\rangle \geq -\varepsilon \|w - x\|.$$

Of course, a ∂-subconvex function is quasi ∂-subconvex; the converse holds if f is Lipschitzian around $\overline{x}$.

One easily shows that a subset E of X is ∂-subconvex at $\overline{x} \in E$ if, and only if its indicator function ι_E (given by $\iota_E = 0$ on E and $+\infty$ on $X \backslash E$) is ∂-subconvex at $\overline{x}$ if and only if ι_E is quasi ∂-subconvex at $\overline{x}$.

The next propositions also justify Definition 5.2. The first one is obvious since $d_E(x) = 0$ for all $x \in E$ and since $\max(\|x^*\|, 1) = 1$ for $x^* \in \partial d_E(x)$.

Proposition 5.4. *Let E be a closed subset E of a normed space X. If the associated distance function d_E is ∂-subconvex at $\overline{x} \in E$, then E is ∂-metrically subconvex at $\overline{x} \in E$.*

Proposition 5.5. *Let X be an Asplund space and let $f : X \to \mathbb{R}$ be Lipschitzian around $\overline{x} \in X$. If f is ∂-subconvex at $\overline{x}$, or if its epigraph is ∂-subconvex, then f is approximately convex at $\overline{x}$ in the sense that for any $\varepsilon > 0$ there exists some $\delta > 0$ such that for any u, $v \in B(\overline{x}, \delta)$ and any $t \in [0, 1]$, one has*

$$f(tu + (1-t)v) \leq tf(u) + (1-t)f(v) + \varepsilon t(1-t)\|u - v\|.$$

Proof. According to Proposition 5.1, for all $x \in X$ near $\overline{x}$ and $x^* \in \partial f(x)$, for any $\varepsilon > 0$ one can find some $\delta > 0$ such that for any w, $x \in B(\overline{x}, \delta)$ and any $x^* \in \partial f(x)$, $w^* \in \partial f(w)$ one has

$$f(w) - f(x) - \langle x^*, w - x\rangle \geq -\varepsilon \|w - x\|$$

and

$$f(x) - f(w) - \langle w^*, x - w\rangle \geq -\varepsilon \|w - x\|.$$

Adding sides by sides these relations we obtain

$$\langle w^* - x^*, w - x\rangle \geq -2\varepsilon \|w - x\|.$$

That shows that ∂f is approximately monotone at $\overline{x}$. Using [13] or [24, Thm 10] we get that f is ∂-approximately convex at $\overline{x}$. □

Now let us consider sublevel sets. We use a classical error bound property.

Proposition 5.6. *Let U be an open subset of a Banach space X and let $f : U \to \mathbb{R}_+$ be a continuous function that is ∂-subconvex at $\overline{x} \in S := \{x \in U : f(x) \leq 0\}$. Assume there exists some $c > 0$ such that $f \geq cd_S$. Then S is ∂-metrically subconvex at $\overline{x}$.*

Proof. Since $f \geq cd_S$ on U, with equality on S, for $x \in S$ one has $c\partial d_S(x) \subset \partial f(x)$ by axiom (S6) below. Given $\varepsilon > 0$, one can find some $\delta > 0$ such that for any w, $x \in S \cap B(\overline{x}, \delta)$ and $z^* \in \partial f(x)$ one has $\langle z^*, w - x\rangle \leq c\varepsilon \|w - x\|$. Since for any $x^* \in \partial d_S(x)$ one has $z^* := cx^* \in \partial f(x)$, hence

$$\langle x^*, w - x\rangle \leq \varepsilon \|w - x\|,$$

one gets that S is ∂-metrically subconvex at x. □

6. Appendix: subdifferentials

As human beings, the notions of nonsmooth analysis have specific qualities, and also weaknesses. For instance elementary subdifferentials such as ∂_F or ∂_D have poor calculus rules but pleasant compatibility with order or inclusion properties;

the limiting normal cone $N_L(S,\cdot)$ to a subset S has nonclosed values in general; the Clarke or circa subdifferential ∂_C and the associated circa-normal cone $N_C(S,\cdot)$ have fruitful calculus rules but are still larger and $N_C(S,\cdot)$ and the associated tangent cone $T_C(S,\cdot)$ do not reflect the local shape of the set S. Moreover each of these notions requires a specific construction which is not always simple, as in the case of the graded subdifferential ∂_G of Ioffe ([17], [33]). These facts and the observation that they also share some common features led several authors to give an axiomatic, unified approach. In this paper we adopt a simplified list of axioms that are sufficient for our purposes, referring to [32] for a more elaborated study of general properties of subdifferentials. Albeit not all of these properties are used here, we gather them in order to show that nonsmooth analysis has a rich common apparatus.

Here a map $g : U \to Y$ defined on an open subset U of a nvs X with values in another one Y is said to be *circa-differentiable* (or strictly differentiable) at $\overline{x}$ if g is differentiable at $\overline{x}$ and $(\|w-x\|)^{-1}(g(w)-g(x)-g'(\overline{x})(w-x)) \to 0$ as $w, x \to \overline{x}$ with $w \neq x$. This property holds if g is of class C^1 at $\overline{x}$ i.e., if g is differentiable on an open neighborhood of $\overline{x}$ and if its derivative is continuous at $\overline{x}$.

Definition 6.1. We say that ∂ is a *subdifferential* on a class $\mathcal{X}$ of Banach spaces, and on a class $\mathcal{F}$ of functions on the members of $\mathcal{X}$ if for any X, Y of $\mathcal{X}$, any $f \in \mathcal{F}(X)$ and any $\overline{x} \in \mathrm{dom} f := f^{-1}(\mathbb{R})$ a subset $\partial f(\overline{x})$ of X^* exists in such a way that the following properties are satisfied:

(S1) if f attains a local minimum at $\overline{x} \in X$, then $0 \in \partial f(\overline{x})$;

(S2a) if f is convex, then $\partial f(\overline{x}) = \{x^* \in X^* : f \geq x^* + f(\overline{x}) - x^*(\overline{x})\}$;

(S2b) if d is circa-differentiable at $\overline{x}$ then $\partial(f+d)(\overline{x}) = \partial f(\overline{x}) + d'(\overline{x})$;

(S3a) if $f = h \circ g$ with g circa-differentiable at $\overline{x}$, $g'(\overline{x})(X) = Y$, then this implies $\partial f(\overline{x}) = \partial h(g(\overline{x})) \circ g'(\overline{x})$;

(S3b) if $f = h \circ g$ with $X := X_1 \times ... \times X_k$, $\overline{x} := (\overline{x}_1, ..., \overline{x}_k) \in X$, $g := g_1 \times ... \times g_k : X \to \mathbb{R}^k$, $g_i \in \mathcal{F}(X_i)$, $h \in \mathcal{F}(\mathbb{R}^k)$ nondecreasing in each of its k arguments and circa-differentiable at $\overline{r} := g(\overline{x})$ with $D_i h(\overline{r}) \neq 0$ for $i \in \mathbb{N}_k := \{1, ..., k\}$ then $\partial f(\overline{x}) \subset h'(g(\overline{x})) \circ (\partial g_1(\overline{x}_1) \times ... \times \partial g_k(\overline{x}_k))$;

(S4) if $f(x_1, x_2) = \sup(g_1(x_1), g(x_2))$, with $X = X_1 \times X_2$, $g_i \in \mathcal{F}(X_i)$, $g_1(\overline{x}_1) = g_2(\overline{x}_2)$, $g_2 \colon X_2 \to \mathbb{R}$ being circa-differentiable at $\overline{x}_2$ with $g_2'(\overline{x}_2) \neq 0$, and $(\overline{x}_1^*, \overline{x_2}^*) \in \partial f(\overline{x}_1, \overline{x}_2)$ then there exist $\lambda_i \in [0,1]$, $\overline{u}_i^* \in \partial g_i(\overline{x}_i)$ such that $\lambda_1 + \lambda_2 = 1$ and $\overline{x}_i^* = \lambda_i u_i^*$ for $i = 1, 2$;

(S5) if $g : X \to Y$ is circa-differentiable at $\overline{x} \in X$ with $g'(\overline{x})(X) = Y$, if $f \in \mathcal{F}(X)$, $h \in \mathcal{F}(Y)$ are such that $h \circ g \leq f$, with $h(g(\overline{x})) = f(\overline{x}) \in \mathbb{R}$ and such that for any sequences $(\alpha_n) \to 0_+$, $(y_n) \to \overline{y} := h(\overline{x})$ satisfying $(h(y_n)) \to h(\overline{y})$ there exists a sequence $(x_n) \to \overline{x}$ satisfying $f(x_n) \leq h(y_n) + \alpha_n$, $g(x_n) = y_n$ for all n in an infinite subset N of $\mathbb{N}$, then $g'(\overline{x})^{\intercal}(\partial h(\overline{y})) \subset \partial f(\overline{x})$;

(S6) the same relation holds when $f, g, h, \overline{x}, \overline{y}, (\alpha_n)$ are as in the preceding condition, h being the distance function d_S to a closed subset S of Y containing $\overline{y}$ and $(y_n) \to \overline{y}$ being a sequence of S.

A special case of (S6) occurs with *quasi-homotonicity* ([29]): it is the case $X = Y$, $g := I_X$, the identity map and $f = 0$ on S. Then one has $\partial d_S(\overline{x}) \subset \partial f(\overline{x})$ for all $\overline{x} \in E$.

Homotonicity is the stronger property:

$$f, g \in \mathcal{F}(X),\ f \leq g,\ f(\overline{x}) = g(\overline{x}) \Rightarrow \partial f(\overline{x}) \subset \partial g(\overline{x}).$$

It is satisfied by *elementary subdifferentials*, those that satisfy the inclusion

$$\partial f(\overline{x}) + \partial g(\overline{x}) \subset \partial (f+g)(\overline{x}).$$

The following propositions give examples of properties that can be deduced from the above axioms. The first one is a direct consequence of (S3b) when taking for h the sum $h\colon (r_1, r_2) \mapsto r_1 + r_2$. Here p_1, p_2 are the canonical projections of $\mathbb{R}^2$ onto $\mathbb{R}$.

Proposition 6.2. *If $f(x_1, x_2) = g_1(x_1) + g_2(x_2)$ then we have*

$$\partial f(x_1, x_2) \subset \partial g_1(x_1) \circ p_1 + \partial g_2(x_2) \circ p_2.$$

Proposition 6.3. ([32]) *If $f \in \mathcal{F}(X)$ is Lipschitzian with rate κ around $\overline{x} \in X$ then $\partial f(\overline{x}) \subset \kappa B_{X^*}$.*

Several other rules can be satisfied by some subdifferentials. The most important ones are the following sum rules in which $f := g + h$ with $g \in \mathcal{F}(X)$, $h \in \mathcal{L}(X)$ the space of locally Lipschitzian functions on X.

(R) if $x \in X$ is a local minimizer of $f := g + h$, then $0 \in \partial g(x) + \partial h(x)$;

(S) if $x^* \in \partial(g+h)(x)$, then $x^* \in \partial g(x) + \partial h(x)$;

(T) if $x \in X$ is a local minimizer of $f := g + h$, then there exist sequences $(x_n) \to x$, $(y_n) \to x$, (x_n^*), (y_n^*) in X^* such that $(g(x_n)) \to g(x)$, $(x_n^* + y_n^*) \to 0$ and $x_n^* \in \partial g(x_n)$, $y_n^* \in \partial h(y_n)$ for all n.

We say that a subdifferential ∂ is *reliable* (resp. *sundering*, resp. *trustworthy*) if it satisfies condition (R) (resp. condition (S), resp. condition (T)). Obviously one has

$$(S) \Longrightarrow (R) \Longrightarrow (T).$$

Moreover, if ∂ is trustworthy, the limiting subdifferential $\overline{\partial}$ associated with ∂ is reliable. In particular, on the class of Asplund spaces the Fréchet subdifferential ∂_F is trustworthy and the limiting Fréchet subdifferential ∂_L is reliable and even sundering. The graded subdifferential (or approximate G-subdifferential) ∂_G of Ioffe is also sundering on Asplund spaces.

The *normal cone* at $\overline{x}$ to a subset S of X is given by $N(S, \overline{x}) := \partial \iota_S(\overline{x})$, where ι_S is the *indicator function* of S defined by $\iota_S(x) = 0$ if $x \in S$, $+\infty$ else.

Now let us consider an interplay between subdifferentials and normal cones.

For such a purpose, we need to relate the distance function d_E to E to a function e_f associated with f; in [16] where it is introduced, it is denoted by R. What follows completes the study of [16] in which assertion (b) below is proved for a bornological subdifferential.

Lemma 6.4. *Let $f \in \mathcal{F}(X)$, let $e_f(x, r) := (f(x) - r)^+ := \max(f(x) - r, 0)$ and let E be the epigraph of f.*

(a) *Then $d_E \leq e_f$, with equality if f is Lipschitzian with rate at most* 1.

(b) *If $\overline{x} \in \mathrm{dom} f$, $\overline{r} := f(\overline{x})$, then one has $\partial d_E(\overline{x}, \overline{r}) \subset \partial e_f(\overline{x}, \overline{r})$.*

(c) *If $(x^*, r^*) \in \partial e_f(\overline{x}, \overline{r})$ with $\overline{r} := f(\overline{x})$, $r^* \neq 0$ then $-x^*/r^* \in \partial f(\overline{x})$.*

(d) *In particular, $(x^*, -1) \in \partial d_E(\overline{x}, \overline{r})$ with $\overline{r} := f(\overline{x}) \Longrightarrow x^* \in \partial f(\overline{x})$.*

Proof. (a) The first assertion is obvious since for $x \in \mathrm{dom} f$, the sole case of interest, one has $d_E(x,r) \leq \|(x, f(x)) - (x,r)\| = f(x) - r$ when $r \leq f(x)$ and $d_E(x,r) = 0$ when $r \geq f(x)$. When f is Lipschitzian with rate at most $c > 0$, endowing $X \times \mathbb{R}$ with the norm $\|\cdot\|_c$ introduced by A. Ioffe and defined by $\|(x,r)\|_c := c\|x\| + |r|$, for all $(w,s) \in E$ and all $(x,r) \in X \times \mathbb{R}$ one has

$$\begin{aligned}(f(x) - r)^+ &\leq (f(w) - r)^+ + |f(x) - f(w)| \\ &\leq (s-r)^+ + |f(x) - f(w)| \leq |s - r| + c\|w - x\|.\end{aligned}$$

Taking the infimum over $(w,s) \in E$ one obtains $(f(x) - r)^+ \leq d_E(x,r)$ and equality.

(b) stems from (S6).

(c) Noting that $e_f(x,r) + r = \max(f(x), r)$ and applying (S2) and (S4) with h the identity map on $\mathbb{R}$, one gets that for $(x^*, r^*) \in \partial e_f(\overline{x}, f(\overline{x}))$ with $r^* \neq 0$ i.e. $r^* + 1 \neq 1$, there exists $\lambda \in]0,1]$ such that $(1-\lambda)1 = r^* + 1$, i.e. $\lambda = -r^*$ and $u^* := \lambda^{-1}x^* = -x^*/r^* \in \partial f(\overline{x})$.

(d) is an immediate consequence. □

The next proposition completes the assertions of Lemma 6.4 and shows the interest of the function e_f.

Proposition 6.5. *Let $f \in \mathcal{F}(X)$ and let $e_f(x,r) := (f(x) - r)^+$. Then for $\overline{x} \in \mathrm{dom} f$ one has*

$$(x^*, -1) \in \partial e_f(\overline{x}, f(\overline{x})) \Longleftrightarrow x^* \in \partial f(\overline{x}).$$

Proof. Taking $r^* := -1$ in assertion (c) of Lemma 6.4 yields the implication $\Rightarrow$.

For the reverse implication, we note that for all $(x,r) \in X \times \mathbb{R}$ we have

$$f(x) \leq (f(x) - r)^+ + r = e_f(x,r) + r,$$

hence $f(x) = \inf_{r\in\mathbb{R}} m(x,r)$ for $m(x,r) := e_f(x,r) + r$. Moreover, if $(x_n) \to_f \overline{x}$, setting $r_n := f(x_n)$, $\overline{r} := f(\overline{x})$ and $g(x,r) := x$, we have $m(x_n, r_n) = f(x_n)$ and $((x_n, r_n)) \to (\overline{x}, \overline{r})$, so that (S5) implies that for all $x^* \in \partial f(\overline{x})$ we have $(x^*, 0) \in \partial m(\overline{x}, \overline{r})$. Then $(x^*, -1) \in \partial e_f(\overline{x}, \overline{r})$. □

Proposition 6.6. *Let $f \in \mathcal{F}(X)$ and let E be its epigraph. Then for all $\overline{x} \in \mathrm{dom} f$ one has*

$$x^* \in \partial f(\overline{x}) \Longrightarrow (x^*, -1) \in N(E, (\overline{x}, f(\overline{x}))),$$

and

$$(x^*, -1) \in N^m(E, (\overline{x}, f(\overline{x}))) \Longrightarrow x^* \in \partial f(\overline{x}).$$

If f is Lipschitzian with rate at most $c > 0$, then, endowing $X \times \mathbb{R}$ with the norm $\|\cdot\|_c$, the preceding implication is an equivalence, and one has

$$\partial f(\overline{x}) \times \{-1\} = \partial d_E(\overline{x}, f(\overline{x})),$$

Proof. Let $\overline{x} \in \mathrm{dom} f$, $j(x,r) := \iota_E(x,r) + r$ for $(x,r) \in X \times \mathbb{R}$, so that

$$f(x) = \inf\{j(x,r) : r \in \mathbb{R}\}.$$

For any sequence $(x_n) \to_f \overline{x}$ (i.e. $(x_n) \to \overline{x}$ and $(f(x_n)) \to f(\overline{x})$), for $r_n := f(x_n)$, n large enough, one has $r_n \in \mathbb{R}$, $(j(x_n, r_n)) = (f(x_n)) \to f(\overline{x}) = j(\overline{x}, f(\overline{x}))$.

Thus, given $x^* \in \partial f(\overline{x})$, (S5) yields $(x^*, 0) \in \partial j(\overline{x}, f(\overline{x}))$. Applying (S2b) with the linear form $\ell : (x, r) \mapsto r$ one gets that $\partial j(\overline{x}, f(\overline{x})) = \partial \iota_E(\overline{x}, f(\overline{x})) + \ell$, with $\ell = (0, 1) \in X^* \times \mathbb{R}$, hence $(x^*, -1) \in \partial \iota_E(\overline{x}, f(\overline{x})) = N(E, (\overline{x}, f(\overline{x})))$.

Now, for $\overline{x} \in \mathrm{dom} f$, let $x^* \in X^*$, $s \in \mathbb{R}_+$ be such that $(x^*, -1) \in s\partial d_E(\overline{x}, f(\overline{x}))$, so that $s > 0$ and $(x^*/s, -1/s) \in \partial d_E(\overline{x}, f(\overline{x}))$. By Lemma 6.4 we get

$$x^* = (x^*/s)/(1/s) \in \partial f(\overline{x}).$$

If f is Lipschitzian with rate at most c, we have $d_E = e_f$ and Proposition 6.5 ensures that $x^* \in \partial f(x) \Longleftrightarrow (x^*, -1) \in \partial e_f(\overline{x}, \overline{r}) = \partial d_E(\overline{x}, \overline{r})$. □

7. Conclusion

We have focussed our attention at three properties for sets, functions and multifunctions, depending either on the use of distance functions or on the choice of one of the two normal cones associated with a subdifferential. In several usual cases the subdifferential is normally consistent and the last two choices coincide. In order to simplify the reading, the reader may drop two of these properties and retain only one. However a comparison between the techniques associated with each of these choices is of interest.

Another track, which may be of interest, consists in considering a pair (S, S') of subsets, with $S' \subset S$ usually, and requiring that x is restricted to S' in Definition 2.1. The case of greatest interest is $S' := \{\overline{x}\}$, in analogy with the case of star-shaped sets viewed as a generalization of convex sets. As with this situation, one obtains much weaker properties. In order to keep the paper simple enough we have discarded or postponed such a study in our approach, considering that the choice of a subdifferential and the choice of a notion of normality offer a sufficient complexity.

Several questions remain open.

(1) Under what conditions (besides the ones we present) ∂-subconvexity coincides with ∂-metric subconvexity or ∂-m subconvexity?

(2) Can one give counter-examples for the missing equivalences of Proposition 2.6 ?

(3) What can be said about the subgradient flow of the distance function to a subconvex set?

(4) More powerful properties can be described for sets and functions. As mentioned in the introduction, these properties of "metaconvex" sets and functions are discarded here for the sake of simplicity. Can one give interesting and simple passages from subconvexity to such properties?

Acknowledgements. The author is much grateful to D. Azé for several discussions and for providing him with a preprint of his paper [3] which aroused the present study and a preprint of the paper [20] by A. S. Lewis, D. R. Luke, and J. Malick. Shortly after the submission of the present paper, in a private communication from L. Thibault, the author received a detailed comparison of its results with the results of the survey [38] which were not known to the author. The detailed comparison made by L. Thibault in [39] and reported below can be a precious help.

Comparison with the survey [38]

The scope of the survey [38] is much larger than the one of the present paper which in its original form was just devoted to shed light on the notion of Fréchet subset as introduced in [3]. In [38], a full spectrum of variants of subsmoothness (uniform subsmoothness, uniform equi-subsmoothness, hemi-subsmoothness, one-sided subsmoothness...) are given attention while they do not appear here as the author considers that starshapedness is a very weak variant of convexity and that the just observed complexity was already an obstacle. Also the primal viewpoint (tangent cones, directional derivatives) is not considered here whereas the case of multimaps is not dealt with in [38].

Albeit the starting points of these two papers are different (functions in the case of [38], sets here) and although the viewpoint of [38] is a generalization of smoothness emanating from a characterization of strict (or circa-) differentiability by L. Veselý and L. Zajíček [41], whereas here the leading thread is a generalization of convexity, derived from [3], the results of these two papers are close. Amazingly, convexity properties are detected in [38] (for instance [38, Prop. 4.1]) whereas regularity properties such as Proposition 2.13 are given here. They may show that the notions considered in these two papers are natural, even if the terminologies are different and the apparatus is not the same. Nonetheless, we observe that even if the Clarke subdifferential plays a key role in [2] and [38], what makes the notions more restrictive (since ∂_C is the largest of the usual notions of subdifferential), an approach using general properties of subdifferentials is also present in [38] but it requires trustworthiness and it is not in the foreground.

As mentioned above, the present paper has been motivated by an interpretation of the inequality of Definition 2.1 (a) which was a prominent feature of the preprint of [3]. The easy equivalence between subconvexity and submonotonicity of the truncated normal cone (hence approximate convexity in Asplund spaces) of Proposition 2.14 brought us the answer we were looking for. This inequality has also been observed in [38, p. 39] in the case $\partial = \partial_C$. The fact that for any subdifferential, any set S and any $x \in S$ one has $0 \in N(S, w)$, $0 \in \partial d_S(x)$ is the key of this equivalence. This equivalence shows that these two properties are like the two sides of a mountain viewed from two different valleys. However subsmoothness in [38, Def. 3.3] is not related to any subdifferential (and does not involve any limitation on x in X, nor r and s in $\mathbb{P}$), whereas here subconvexity is relative to a specific subdifferential ∂. It is amazing that the very definition of submoothness of a set in [38, Def. 5.1, Def. 7.1] and some results such as [38, Prop. 3.14] evoke submonotonicity and subconvexity, while the definition of subconvexity in the present paper seems to be related to a regularity or approximate permanence property.

The equivalence of the assertions (a) and (b) (resp. (f) and (g)) of Proposition 2.6 has been given in [38, Prop. 7.6] (resp. [38, Prop. 7.5]) in the case $\partial = \partial_C$ and in [3, Lemma 3.1] in the case $\partial = \partial_F$. The notion of metrically subsmooth set has also been considered in [38, Section 7] in the case $\partial = \partial_C$. Its Propositions 7.5 and 7.8 present other characterizations in the cases X is Asplund or Hilbert. The relation $N_F(E, \overline{x}) = N_C(E, \overline{x})$ for E subconvex at $\overline{x}$ has been obtained in [3, Prop. 3.2] and this relation with $\partial_F d_E(\overline{x}) = \partial_C d_E(\overline{x})$ is contained in [38, Lemma 5.15] and

[38, Prop. 7.7] in the case E is ∂_C-metrically regular or even ∂_C-hemi-submooth at $\overline{x}$ without assuming that X is Asplund. The Krein-Šmulian Theorem has been used in [3], [38, p. 45 and p. 108], [42], [43] and various contributions of the author, including [26, Thm. 2.3] among the early ones.

Since any convex set is subconvex, our Theorem 3.2 about inverse images extends [38, Prop. 5.11 (b), Thm. 9.3] inasmuch g is not supposed to be of class C^1; also it avoids the assumption that C is Fréchet regular near $g(\overline{x})$ made in [3, Thm. 3.1] but uses assumption (b) which has some similarity with an assumption of [38, Thm. 9.3]. The coincidence of the assumption of Corollary 3.3 with relation (9.17) of [38] may means that this assumption is natural; but it could be supplemented by more concrete assumptions such as metric estimates. Such metric estimates are considered in [38, relation (9.5), Thm. 9.3 (b)], in Corollary 3.4 and in the proof of Theorem 3.6 above. They are reminiscent of many similar estimates in the literature, in particular [17], [18], [28], [34].

References

[1] D. Aussel, J.-N. Corvelec, M. Lassonde: *Mean value property and subdifferential criteria for lower semicontinuous functions*, Trans. Amer. Math. Soc. 347/10 (1995) 4147–4161.

[2] D. Aussel, A. Daniilidis, L. Thibault: *Subsmooth sets: functional characterizations and related concepts*, Trans. Amer. Math. Soc. 357/4 (2005) 1275–1301.

[3] D. Azé: *Inner conditions for error bounds and metric subregularity of multifunctions*, http://www.optimization-online.org/DBHTML/2017/09/6209.html, corrected in: *Characterizations for existence of multipliers in mathematical programming in Banach spaces*, https://www.researchgate.net/publication/341113349 (2020).

[4] F. Bernard, L. Thibault: *Prox-regularity of functions and sets in Banach spaces*, Set-Valued Analysis 12 (2004) 25–47.

[5] J. M. Borwein, Q. J. Zhu: *Techniques of Variational Analysis*, CMS Books in Mathematics, Canadian Mathematical Society, Springer, New York (2005).

[6] M. Bounkhel, L. Thibault: *On various notions of regularity of sets in non-smooth analysis*, Nonlin. Analysis Theory Meth. Appl. 43 (2002) 223–246.

[7] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Society for Industrial and Applied Mathematics, Philadelphia (1990).

[8] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, P. R. Wolenski: *Nonsmooth Analysis and Control Theory*, Springer, New York (1998).

[9] F. H. Clarke, R. J. Stern, P. R. Wolenski: *Proximal smoothness and the lower C^2 property*, J. Convex Analysis 2 (1995) 117–144.

[10] G. Colombo, V. V. Goncharov: *Variational inequalities and regularity properties of closed sets in Hilbert spaces*, J. Convex Analysis 8/1 (2001) 197–221.

[11] G. Colombo, L. Thibault: *Prox-regular sets and applications*, in: *Handbook of Nonconvex Analysis and Applications*, International Press, Somerville (2010) 99–182.

[12] A. Daniilidis, P. Georgiev: *Approximate convexity and submonotonicity*, J. Math. Analysis Appl. 291 (2004) 292–301.

[13] A. Daniilidis, F. Jules, M. Lassonde: *Subdifferential characterization of approximate convexity: the lower semicontinuous case*, Math. Programming Ser. B 116 (2009) 115–127.

[14] A. Daniilidis, L. Thibault: *Subsmooth sets and metrically subsmooth sets and functions in Banach space*, unpublished manuscript, University of Montpellier 2 (2008).

[15] A. D. Ioffe: *Approximate subdifferentials and applications. I: The finite dimensional theory*, Trans. Amer. Math. Soc. 281 (1984) 389–416.

[16] A. D. Ioffe: *Codirectional compactness, metric regularity and subdifferential calculus*, in: *Constructive, Experimental and Nonlinear Analysis*, M. Théra (ed.), Canadian Math. Soc. Conf. Proc. 27, American Math. Society, Providence (1999) 123–153.

[17] A. D. Ioffe: *Variational Analysis of Regular Mappings. Theory and Applications*, Springer, Berlin (2017).

[18] A. D. Ioffe, J.-P. Penot: *Subdifferentials of performance functions and calculus of set-valued mappings*, Serdica Math. J. 22 (1996) 359–384.

[19] I. Kessis, L. Thibault: *Subdifferential characterization of s-lower regular functions*, Appl. Analysis 94/1 (2015) 85–98.

[20] A. S. Lewis, D. R. Luke, J. Malick: *Local linear convergence for alternating and averaged nonconvex projections*, Found. Comput. Math. 9 (2009) 485–513.

[21] M. Mazade, L. Thibault: *Primal lower-nice functions and their Moreau envelopes*, in: *Computational and Analytical Mathematics*, D. H. Bailey et al. (eds.), Springer Proceedings in Mathematics and Statistics 50, Springer, New York (2013) 521–553.

[22] B. S. Mordukhovich: *Variational Analysis and Generalized Differentiation I*, Springer, New York (2006).

[23] H. V. Ngai, D. T. Luc, M. Théra: *Approximate convex functions*, J. Nonlin. Convex Analysis 1 (2000) 155–176.

[24] H. V. Ngai, J.-P. Penot: *Approximately convex functions and approximately monotone operators*, Nonlin. Analysis 66 (2007) 547–564.

[25] H. V. Ngai, J.-P. Penot: *Approximately convex sets*, J. Nonlin. Convex Analysis 8/3 (2007) 337–371.

[26] J.-P. Penot: *Regularity conditions in mathematical programming*, Math. Programming Study 19 (1982) 167–199.

[27] J.-P. Penot: *Favorable classes of mappings and multimappings in nonlinear analysis and optimization*, J. Convex Analysis 3/1 (1996) 97–116.

[28] J.-P. Penot: *Metric estimates for the calculus of multimappings*, Set-Valued Analysis 5 (1997) 291–308.

[29] J.-P. Penot: *The compatibility with order of some subdifferentials*, Positivity 6/4 (2002) 413–432.

[30] J.-P. Penot: *Delineating nice classes of nonsmooth functions*, Pacific J. Optimization 4/3 (2008) 605–619.

[31] J.-P. Penot: *Softness, sleekness and regularity properties in nonsmooth analysis*, Nonlin. Analysis 68/9 (2008) 2750–2768.

[32] J.-P. Penot: *Towards a new era in subdifferential analysis?*, in: D. H. Bailey et al. (eds.), *Computational and Analytical Mathematics*, Springer Proceedings in Mathematics and Statistics 50, Springer, New York (2013) 635–665.

[33] J.-P. Penot: *Calculus Without Derivatives*, Graduate Texts in Mathematics 266, Springer, New York (2013).

[34] J.-P. Penot: *Error bounds and multipliers in constrained optimization problems with tolerance*, SIAM J. Optimization 29/1 (2019) 522–540.

[35] S. Rolewicz: *Paraconvex analysis*, Control Cybernetics 34/3 (2005) 951–965.

[36] J. E. Spingarn: *Submonotone subdifferentials of Lipschitz functions*, Trans. Amer. Math. Soc. 264/1 (1981) 77–89.

[37] J. E. Spingarn: *Submonotone mappings and the proximal point algorithm*, Numer. Funct. Anal. Opt. 4/2 (1981) 123–150.

[38] L. Thibault: *Subsmooth functions and sets*, Linear Nonlin. Analysis 4 (2018) 157–269.

[39] L. Thibault: personal communication, May 30th (2020).

[40] L. Thibault, D. Zagrodny: *Integration of subdifferentials of lower semicontinuous functions on Banach spaces*, J. Math. Analysis Appl. 189 (1995) 33–58.

[41] L. Veselý, L. Zajíček: *Delta-convex mappings between Banach spaces and applications*, Dissertationes Mathematicae 289, Warszawa (1989) 1–48.

[42] X. Y. Zheng, K. F. Ng: *Linear regularity for a collection of subsmooth sets in Banach spaces*, SIAM J. Optimization 19 (2008) 62–76.

[43] X. Y. Zheng, K. F. Ng: *Calmness for a L-subsmooth multifunctions in Banach spaces*, SIAM J. Optimization 19 (2009) 1648–1673.