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Necessary Optimality Conditions for Convex Continuous-Time Optimization Problems

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We consider convex continuous-time optimization problems with inequality type of constraints. By using subdifferential approximation of the considered problem and new alternative theorem for convex inequalities in functional spaces, necessary optimality conditions are obtained. Moreover, the main result is refined when affine functions are separately considered from other convex non-affine functions.

Keywords: Continuous-time programming, convex programming, optimality conditions, theorems of the alternative.

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1. Introduction

Consider the following maximization problem

$$\text{(P)}\qquad \begin{aligned} &\text{maximize} \quad J(x(\cdot)) = \int_0^T f(t,x(t))dt \\ &\text{subject to} \quad g(t,x(t)) \geq 0 \quad \text{a.e. in} \quad [0,T], \quad x(\cdot) \in L_\infty([0,T],\mathbb{R}^n), \end{aligned}$$

where $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ and $g : [0,T] \times \mathbb{R}^n \to \mathbb{R}^m$ are given functions.

This type of optimization problem is now known as continuous-time linear programming problem and was first considered by Bellman [4] in 1953, related to production-inventory "bottleneck processes". He formulated a corresponding dual problem, provided duality relations and suggested some computational procedures. Since then, Bellman's theory has been extended to wider classes of continuous-time linear problems.

Some early investigations on optimality conditions for Karush-Kuhn-Tucker type for continuous-time nonlinear problems were given by Hanson and Mond [8] and further generalizations of the theory of optimality conditions and duality for continuous-time nonlinear problems were considered in Abraham and Buie [1], Reiland and Hanson [13], Scott and Jefferson [14] and Zalmai [15, 16, 17, 18], where authors assumed some kind of constraint qualifications and used direct methods. Many other authors have also approached this matter.

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Considering necessary conditions for continuous-time programming, Brandão et al. [5] followed the approach of Zalmai [17], but unlike Zalmai, they treated the nonsmooth case. In both articles ([5] and [17]), the authors used Gordan's Transposition Theorem from Zalmai [16]. As shown in [3], this result was not valid, so Arutyunov et al. provided an alternative theorem. By using this alternative theorem, do Monte and de Oliveira have dealt with a smooth case of continuous-time programming problem in their work [7], where they came up with very valuable results not only for problems with inequality constraints but also for problems with both equality and inequality constraints. In [9] continuous-time optimization problems with vector valued objective function are considered. Applying the alternative theorem to the corresponding systems, new necessary optimality conditions are obtained. In the case when some constraint or cost functions are linear, refined version of the main theorem is given. Theorems of the alternative (also known as transposition theorems) play an important role both in linear and nonlinear programming and are crucial in deriving optimality conditions for wide classes of extremal problems. In this manuscript we will use the same direct approach to provide new zero order necessary optimality conditions for convex continuous-time problems involving inequality constraints, without differentiability assumption.

This paper is organized as follows. In Section 2 we give some notation, state the Theorem of alternative and define the regularity condition. In Section 3 we prove our main results, first for the general problem and then for a special case where cost function and some of the constraint functions are affine. There's also an example included.

2. Preliminaries

In the first subsection we state the notation set used in the paper and the second subsection presents some auxiliary results that we will use below.

2.1. Notation

By $\overline{\mathbb{R}}$ we denote the set of extended reals, i.e. $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

The *domain* of an arbitrary function $f : L \to \overline{\mathbb{R}}$, where L is a linear space, is denoted by $dom(f)$, i.e. $dom(f) = \{x \in L : f(x) < \infty\}$.

Let $(E, \|\cdot\|)$ be a normed space. For a closed convex subset X of E we denote by $T_X(x)$ the *tangent cone* to the set X at a point $x \in X$.

All vectors in the text below are column vectors and we use prime for transposition. For vector $w \in \mathbb{R}^p$ we write $w \leq 0$ if $w_i \leq 0$ for each $i = 1, ..., p$ and $w < 0$ if $w_i < 0, i = 1, ..., p$. For a vector $x \in E$ and a linear bounded functional $\varphi \in E^*$, we denote the value of φ at x by $\langle \varphi, x\rangle$. We also use this notation for inner product of vectors $\varphi, x \in \mathbb{R}^n$.

For a function $g : [0, 1] \times E \to \overline{\mathbb{R}}$ and a point $t \in [0, 1]$ such that $g(t, \cdot)$ is convex, we denote by $\partial_x g(t, x)$ the *subdifferential* of $g(t, \cdot)$ at a point $x \in E$.

2.2. Auxiliary results

Let us present the theorem of the alternative by Arutyunov et al. [3], which will be a powerful tool in what follows. We will also state a certain regularity condition that has to be satisfied for applying this theorem.

Let $(E, \|\cdot\|)$ be a Banach space with given functions $f_i : [0,T] \times E \to \overline{\mathbb{R}}, i = 1, ..., k$, let $X \subset E$ be closed and convex and let I_1 and I_2 be sets of indices such that $I_1 \sqcup I_2 = \{1, ..., k\}$ ($\sqcup$ stands for disjoint union). Consider the system

$$\begin{cases} f_i(t,x) \leq 0, & i \in I_1 \\ f_i(t,x) < 0, & i \in I_2 \\ x \in X. \end{cases} \tag{1}$$

A solution of system (1) is a function $x(\cdot) \in L_\infty([0,1], X)$ such that for a.e. $t \in [0,1]$ the following holds:

$$f_i(t, x(t)) \leq 0, \, i \in I_1, \;\; f_i(t, x(t)) < 0, \, i \in I_2, \;\; x(t) \in X.$$

We also assume that the following conditions are valid:

(A1) The functions $f_i(t,\cdot)$ are convex and continuous on X, and for a.e. $t \in [0,1]$, $i = 1, ..., k$, we have $X \subset int(dom(f_i(t,\cdot)))$;

(A2) the functions $f_i(\cdot, x)$ are Lebesgue measurable for all $x \in X$, $i = 1, ..., k$;

(A3) for each $K \geq 0$ there exists $M = M(K) \geq 0$ such that for a.e. $t \in [0,1]$ and for all $i = 1, ..., k$ we have $(x \in X \wedge |x| \leq K) \Rightarrow |f_i(t,x)| \leq M$.

Let $\mathcal{I}(t,x)$ be the set of active indices of the system (1), i.e.

$$\mathcal{I}(t,x) := \left\{ i : f_i(t,x) = \max_{j=1,...,k} f_j(t,x) \right\}, \;\; t \in [0,1], \, x \in X.$$

Definition 2.1. We say that the system (1) is *regular*, if there exist a function $\bar{x}(\cdot) \in L_\infty([0,1], X)$, real numbers $R \geq 0$ and $\alpha > 0$ such that for a.e. $t \in [0,1]$ and for all $x \in X$ with $\|x - \bar{x}(t)\| \geq R$, there exists $e = e(t,x) \in -T_X(x)$, $\|e\| = 1$, which satisfies

$$\langle x^*, e \rangle \geq \alpha \quad \forall x^* \in \partial_x f_i(t,x), \, i \in \mathcal{I}(t,x).$$

Theorem 2.2. (Theorem of the Alternative [3]) *Assume that the Banach space E is separable, the system (1) is regular, and for a.e. $t \in [0,1]$ there exists a vector $x = x(t) \in X$ such that $f_i(t, x(t)) < 0$ for each $i \in I_1$. Then, one and only one of the following assertions is valid.*

(i) *There exists a solution $\chi(\cdot)$ for system (1);*

(ii) *there exists a nonzero function $\varphi(\cdot) = (\varphi_1(\cdot), ..., \varphi_k(\cdot)) \in L_\infty([0,1], \mathbb{R}^k_+)$ such that $\varphi_i(t) \not\equiv 0$ for some $i \in I_2$ and*

$$\sum_{i=1}^{k} f_i(t,x)\varphi_i(t) \geq 0 \quad \text{a.e. in } [0,1], \quad \forall x \in X.$$

This theorem is valid if, instead of $[0,1]$, we consider a set T with a measure that is countably additive, regular finite and positive.

3. Optimality conditions

In order to discuss the necessary optimality conditions for (P), we assume that for a.e. $t \in [0,T]$ the functions $f(t,\cdot)$ and $g(t,\cdot)$ are convex on $\mathbb{R}^n$ and that for all $x \in \mathbb{R}^n$ the functions $f(\cdot,x)$ and $g(\cdot,x)$ are Lebesgue measurable. Also let us assume that for each $M \geq 0$, $N \geq 0$ there exist, respectively, $L = L(M) \geq 0$, $K = K(N) \geq 0$ such that for a.e. $t \in [0,T]$

$$(x \in \mathbb{R}^n \wedge \|x\| \leq M) \Rightarrow |f(t,x)| \leq L,$$

$$(x \in \mathbb{R}^n \wedge \|x\| \leq N) \Rightarrow |g_i(t,x)| \leq K, \quad i \in I = \{1,...,m\}.$$

Let $\mathbb{F}$ denote the set of all feasible solutions of (P) (assumed to be nonempty), i.e.,

$$\mathbb{F} = \{x(\cdot) \in L_\infty([0,T],\mathbb{R}^n) : g(t,x(t)) \geq 0 \quad \text{a.e. in } [0,T]\}.$$

Definition 3.1. We say that $\hat{x}(\cdot) \in \mathbb{F}$ is a *global maximizer* of (P) if

$$J(x(\cdot)) \leqslant J(\hat{x}(\cdot)), \quad \forall x(\cdot) \in \mathbb{F}.$$

Lemma 3.2. *If there exists a global maximizer $\hat{x}(\cdot)$ for* (P)*, then the following system*

$$\text{(S)} \qquad \begin{aligned} &\phi_0(t,x) := -\int_0^T \langle \partial_x f(t,\hat{x}(t)), x - \hat{x}(t)\rangle \, dt < 0, \\ &\phi_i(t,x) := -g_i(t,\hat{x}(t)) - \langle \partial_x g_i(t,\hat{x}(t)), x - \hat{x}(t)\rangle \leq 0, \;\; i \in I, \;\; x \in \mathbb{R}^n, \end{aligned}$$

does not have a solution for a.e. $t \in [0,T]$.

Proof. Suppose that $\overline{x}(\cdot)$ is a solution to the system (S). Then we have

$$\int_0^T \langle \partial_x f(t,\hat{x}(t)), \overline{x}(t) - \hat{x}(t)\rangle \, dt > 0, \tag{2}$$

$$g_i(t,\hat{x}(t)) + \langle \partial_x g_i(t,\hat{x}(t)), \overline{x}(t) - \hat{x}(t)\rangle \geq 0, \;\; i \in I, \;\; \text{a.e. in } [0,T]. \tag{3}$$

From (3) we conclude that $g_i(t,\overline{x}(t)) \geq 0$, because

$$g_i(t,\overline{x}(t)) - g_i(t,\hat{x}(t)) \geq \langle \partial_x g_i(t,\hat{x}(t)), \overline{x}(t) - \hat{x}(t)\rangle \,, \;\; i \in I, \;\; \text{a.e. in } [0,T],$$

which means $\overline{x}(\cdot) \in \mathbb{F}$. By integrating the inequality

$$f(t,\overline{x}(t)) - f(t,\hat{x}(t)) \geq \langle \partial_x f(t,\hat{x}(t)), \overline{x}(t) - \hat{x}(t)\rangle$$

on $[0,T]$, we obtain

$$\int_0^T (f(t,\overline{x}(t)) - f(t,\hat{x}(t)))dt \geq \int_0^T \langle \partial_x f(t,\hat{x}(t)), \overline{x}(t) - \hat{x}(t)\rangle \, dt > 0.$$

This leads us to

$$J(\overline{x}(\cdot)) = \int_0^T f(t,\overline{x}(t))dt > \int_0^T f(t,\hat{x}(t))dt = J(\hat{x}(\cdot)),$$

which contradicts the assumption that $\hat{x}(\cdot)$ is a global maximizer for (P). $\square$

Let us translate the notation $\mathcal{I}(t,x)$ from the preceding Definition 2.1 through the above functions ϕ_l as follows

$$\mathcal{I}(t,x) := \left\{ l : \phi_l(t,x) = \max_{p\in\{0\}\cup I} \phi_p(t,x) \right\}, \ \ t \in [0,T],\ x \in \mathbb{R}^n.$$

Definition 3.3. We say that the system (S) is *regular*, if there exist a function $\overline{x}(\cdot) \in L_\infty([0,T],\mathbb{R}^n)$, reals $R \geq 0$ and $\alpha > 0$ such that for a.e. $t \in [0,T]$ and for all $x \in \mathbb{R}^n$ with $\|x - \overline{x}(t)\| \geq R$, there exist $e = e(t,x) \in \mathbb{R}^n$, $\|e\| = 1$, which satisfy

$$\langle \partial_x \phi_l(t,x), e\rangle \geq \alpha, \quad \forall l \in \mathcal{I}(t,x),$$

with ϕ_l as defined in Lemma 3.2. Notice that for a.e. $t \in [0,T]$

$$\langle \partial_x \phi_0(t,x), e\rangle \geq \alpha \quad \Longleftrightarrow \quad \langle -\partial_x f(t,\hat{x}(t)), e\rangle \geq \alpha$$

and $$\langle \partial_x \phi_i(t,x), e\rangle \geq \alpha \quad \Longleftrightarrow \quad \langle -\partial_x g_i(t,\hat{x}(t)), e\rangle \geq \alpha, \quad i \in I,$$

where $\hat{x}(\cdot)$ is a global maximizer for (P).

Theorem 3.4. *Let $\hat{x}(\cdot)$ be a global maximizer for* (P)*. Assume that the system* (S) *is regular and that there exists $z\colon [0,T] \to \mathbb{R}^n$ such that $g(t,z(t)) > 0$ for almost every $t \in [0,T]$. Then, there exists $\hat{u}(\cdot) \in L_\infty([0,T],\mathbb{R}^m)$ satisfying the following conditions:*

(1) $\hat{u}(t) \geq 0$ a.e. in $[0,T]$,

(2) $\hat{u}'(t)g(t,\hat{x}(t)) = 0$ a.e. in $[0,T]$,

(3) $f(t,x(t)) + \hat{u}'(t)g(t,x(t)) \geq f(t,\hat{x}(t)) + \hat{u}'(t)g(t,\hat{x}(t))$, $\forall\, x(\cdot) \in L_\infty([0,T],\mathbb{R}^n)$, *a.e. in* $[0,T]$.

Proof. Since $\hat{x}(\cdot)$ is a global maximizer for (P), according to Lemma 3.2 the system

$$\begin{aligned} &-\int_0^T \langle \partial_x f(t,\hat{x}(t)), x - \hat{x}(t)\rangle\, dt < 0, \\ &-g_i(t,\hat{x}(t)) - \langle \partial_x g_i(t,\hat{x}(t)), x - \hat{x}(t)\rangle \leq 0, \quad i \in I,\ x \in \mathbb{R}^n, \end{aligned}$$

does not have a solution.

Assuming that there exists $z : [0,T] \to \mathbb{R}^n$ such that $g(t, z(t)) > 0$ for almost every $t \in [0,T]$, and knowing that

$$g(t, \hat{x}(t)) \geq g(t, z(t)) + \langle \partial_x g(t, \hat{x}(t)), \hat{x}(t) - z(t) \rangle ,$$

we conclude

$$g(t, \hat{x}(t)) + \langle \partial_x g(t, \hat{x}(t)), z(t) - \hat{x}(t) \rangle \geq g(t, z(t)) > 0, \quad \text{a.e. in } [0,T].$$

It follows from Theorem 2.2 that there exists a nonzero function

$$(\hat{\lambda}(\cdot), \hat{\varphi}_1(\cdot), ..., \hat{\varphi}_m(\cdot)) \in L_\infty([0,T], \mathbb{R}_+^{m+1})$$

such that $\hat{\lambda}(t) \not\equiv 0$ and

$$\begin{aligned} &\hat{\lambda}(t) \int_0^T \langle \partial_x f(s, \hat{x}(s)), x(s) - \hat{x}(s) \rangle \, ds + \\ &\sum_{i=1}^m \hat{\varphi}_i(t)(g_i(t, \hat{x}(t)) + \langle \partial_x g_i(t, \hat{x}(t)), x(t) - \hat{x}(t) \rangle) \leq 0, \end{aligned} \tag{4}$$

for all $x(\cdot) \in L_\infty([0,T], \mathbb{R}^n)$, a.e. in $[0,T]$. Putting $x(\cdot) = \hat{x}(\cdot)$, we get

$$\sum_{i=1}^m \hat{\varphi}_i(t) g_i(t, \hat{x}(t)) \leq 0,$$

hence

$$\sum_{i=1}^m \hat{\varphi}_i(t) g_i(t, \hat{x}(t)) = 0, \tag{5}$$

since $\hat{\varphi}_i(t) \geq 0$ and $g_i(t, \hat{x}(t)) \geq 0$ a.e. in $[0,T], i \in I$.

From (4) it follows that

$$\hat{\lambda}(t) \int_0^T \langle \partial_x f(s, \hat{x}(s)), x(s) - \hat{x}(s) \rangle \, ds + \sum_{i=1}^m \hat{\varphi}_i(t) \langle \partial_x g_i(t, \hat{x}(t)), x(t) - \hat{x}(t) \rangle \leq 0,$$

for all $x(\cdot) \in L_\infty([0,T], \mathbb{R}^n)$. By integrating this inequality on $[0,T]$, we obtain

$$\int_0^T \left(\hat{\eta} \langle \partial_x f(t, \hat{x}(t)), x(t) - \hat{x}(t) \rangle + \sum_{i=1}^m \hat{\varphi}_i(t) \langle \partial_x g_i(t, \hat{x}(t)), x(t) - \hat{x}(t) \rangle \right) dt \leq 0,$$

for all $x(\cdot) \in L_\infty([0,T], \mathbb{R}^n)$, where

$$\hat{\eta} = \int_0^T \hat{\lambda}(t) dt > 0. \tag{6}$$

Setting

$$\hat{u}_i(t) = \frac{\hat{\varphi}_i(t)}{\hat{\eta}} \quad \text{a.e. in } [0,T], \quad i \in I,$$

we obtain for all $x(\cdot) \in L_\infty([0,T],\mathbb{R}^n)$

$$\int_0^T \left\langle \partial_x f(t,\hat{x}(t)) + \sum_{i=1}^m \hat{u}_i(t)\partial_x g_i(t,\hat{x}(t)), x(t) - \hat{x}(t) \right\rangle dt \leq 0. \tag{7}$$

We show that inequality (7) implies

$$\begin{gathered} \left\langle \partial_x f(t,\hat{x}(t)) + \sum_{i=1}^m \hat{u}_i(t)\partial_x g_i(t,\hat{x}(t)), x(t) - \hat{x}(t) \right\rangle = 0, \\ \forall x(\cdot) \in L_\infty([0,T],\mathbb{R}^n), \text{ a.e. in } [0,T]. \end{gathered} \tag{8}$$

If there was some $\overline{x}(\cdot) \in L_\infty(V,\mathbb{R}^n)$ such that

$$\left\langle \partial_x f(t,\hat{x}(t)) + \sum_{i=1}^m \hat{u}_i(t)\partial_x g_i(t,\hat{x}(t)), \overline{x}(t) - \hat{x}(t) \right\rangle > 0 \quad \text{a.e. in } V,$$

where the set $V \subset [0,T]$ does not have measure zero, we could define a function

$$y(t) = \begin{cases} \overline{x}(t), & t \in V, \\ 0, & t \notin V. \end{cases}$$

Then we would have

$$\left\langle \partial_x f(t,\hat{x}(t)) + \sum_{i=1}^m \hat{u}_i(t)\partial_x g_i(t,\hat{x}(t)), y(t) - \hat{x}(t) \right\rangle > 0 \quad \text{a.e. in } [0,T],$$

and consequently

$$\int_0^T \left\langle \partial_x f(t,\hat{x}(t)) + \sum_{i=1}^m \hat{u}_i(t)\partial_x g_i(t,\hat{x}(t)), y(t) - \hat{x}(t) \right\rangle dt > 0,$$

which is a contradiction to (7).

On the other hand, let us assume that there exists $\overline{x}(\cdot) \in L_\infty(V,\mathbb{R}^n)$ such that

$$\left\langle \partial_x f(t,\hat{x}(t)) + \sum_{i=1}^m \hat{u}_i(t)\partial_x g_i(t,\hat{x}(t)), \overline{x}(t) - \hat{x}(t) \right\rangle < 0 \quad \text{a.e. in } V, \tag{9}$$

where set $V \subset [0,T]$ doesn't have measure zero. Defining a function

$$y(t) = \begin{cases} 2\hat{x}(t) - \overline{x}(t), & t \in V, \\ 0, & t \notin V, \end{cases}$$

we obtain

$$\left\langle \partial_x f(t,\hat{x}(t)) + \sum_{i=1}^{m} \hat{u}_i(t)\partial_x g_i(t,\hat{x}(t)), y(t)-\hat{x}(t) \right\rangle$$
$$= \left\langle \partial_x f(t,\hat{x}(t)) + \sum_{i=1}^{m} \hat{u}_i(t)\partial_x g_i(t,\hat{x}(t)), \hat{x}(t)-\overline{x}(t) \right\rangle > 0 \quad \text{a.e. in } [0,T].$$

This inequality again leads us to a contradiction to (7), so (8) must be true.

It follows that

$$\begin{aligned} 0 &= \langle \partial_x f(t,\hat{x}(t)), x(t)-\hat{x}(t)\rangle + \sum_{i=1}^{m} \hat{u}_i(t)\,\langle \partial_x g_i(t,\hat{x}(t)), x(t)-\hat{x}(t)\rangle \\ &\le f(t,x(t)) - f(t,\hat{x}(t)) + \sum_{i=1}^{m} \hat{u}_i(t)(g_i(t,x(t)) - g_i(t,\hat{x}(t))) \\ &= f(t,x(t)) - f(t,\hat{x}(t)) + \hat{u}'(t)g(t,x(t)) - \hat{u}'(t)g(t,\hat{x}(t)), \end{aligned}$$

for all $x(\cdot) \in L_\infty([0,T],\mathbb{R}^n)$, a.e. in $[0,T]$, which proves condition 3. Condition 1 is satisfied by construction and veracity of condition 2 is obvious from (5) and (6). □

Example 3.5. We consider the following problem:

$$\begin{aligned} &\text{maximize} \quad J(x(\cdot)) = \int_0^1 x_1(t)\,dt, \quad x(\cdot) \in L_\infty([0,1],\mathbb{R}^2), \\ &\text{subject to} \quad |x_2(t)-3| + 2x_2(t) - 6 \ge 0 \quad \text{a.e. in } [0,1], \\ &\qquad\qquad\qquad 9 - 3x_1(t) - 3x_2(t) \ge 0 \quad \text{a.e. in } [0,1]. \end{aligned}$$

A global maximizer for this problem is $\hat{x}(t) = (\hat{x}_1(t), \hat{x}_2(t)) = (0,3)$ a.e. in $[0,1]$. The condition $g(t,z(t)) > 0$ a.e. in $[0,1]$ is satisfied for $z(t) = (-2,4)$.

Given $x = (x_1,x_2) \in \mathbb{R}^2$, for almost everywhere in $[0,1]$, we have

$$\phi_0(t,x) = -x_1,$$
$$\phi_1(t,x) = -\begin{cases} \langle (0,1),(x_1,x_2-3)\rangle, & x_2 < 3, \\ \langle (0\times[1,3]),(x_1,x_2-3)\rangle, & x_2 = 3, \\ \langle (0,3),(x_1,x_2-3)\rangle, & x_2 > 3, \end{cases}$$
$$\phi_2(t,x) = 3x_1 + 3x_2 - 9, \quad x \in \mathbb{R}^2.$$

Define the sets

$$R_0 = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 - \frac{x_1}{\theta} \ge 3,\ 4x_1 + 3x_2 \le 9,\ \theta \in [1,3]\},$$
$$R_1 = \{(x_1,x_2) \in \mathbb{R}^2 : 3x_1 + (3+\theta)x_2 \le 9 + 3\theta,\ x_2 - \frac{x_1}{\theta} \le 3,\ \theta \in [1,3]\},$$
$$R_2 = \{(x_1,x_2) \in \mathbb{R}^2 : 4x_1 + 3x_2 \ge 9,\ 3x_1 + (3+\theta)x_2 \ge 9 + 3\theta,\ \theta \in [1,3]\}.$$

Obviously, we have $\bigcup_{i=0}^{2} R_i = \mathbb{R}^2$.

For $x \in intR_0$ we have $\phi_0(t,x) = \max\{\phi_0(t,x), \phi_1(t,x), \phi_2(t,x)\}$ a.e. in [0,1], that is $\mathcal{I}(t,x) = \{0\}$.
For $x \in intR_1$ we have $\phi_1(t,x) = \max\{\phi_0(t,x), \phi_1(t,x), \phi_2(t,x)\}$, a.e. in [0,1], that is $\mathcal{I}(t,x) = \{1\}$.
For $x \in intR_2$ we have $\mathcal{I}(t,x) = \{2\}$ and $\{0,1\} \notin \mathcal{I}(t,x)$ for $x \in R_2$.
For $x \in int(R_0 \cup R_1)$ we have $\mathcal{I}(t,x) = \{0,1\}$ a.e. in [0,1].
For $x \in int(R_1 \cup R_2)$ we have $\mathcal{I}(t,x) = \{1,2\}$, and finally
for $x \in int(R_0 \cup R_2)$ we have $\mathcal{I}(t,x) = \{0,2\}$ a.e. in [0,1].
It is clear that the system

$$\phi_0(t,x) = -x_1 < 0, \quad \phi_1(t,x) = \theta(3 - x_2) \leq 0,$$
$$\phi_2(t,x) = 3x_1 + 3x_2 - 9 \leq 0, \quad x \in \mathbb{R}^2, \theta \in [1,3],$$

is regular with $\hat{x}(t) = (0,3)$ and a.e. in [0,1] we have $e(t,x) = (-\frac{5}{13}, -\frac{12}{13})$ for $x \in int(R_0 \cup R_1)$ and $e(t,x) = (\frac{5}{13}, \frac{12}{13})$ for $x \in R_2$. Conditions 1–3 from Theorem 3.4 are satisfied for $\hat{u}_1(t) = 1$ and $\hat{u}_2(t) = \frac{1}{3}$ when $x_2 < 3$ and for $\hat{u}_1(t) = \hat{u}_2(t) = \frac{1}{3}$ when $x_2 \geq 3$.

In what follows we will examine a similar maximization problem (P'), which is a special case of problem (P).

$$\text{(P')} \quad \begin{aligned} &\text{Maximize} \quad J(x(\cdot)) = \int_0^T (A_0(t)x(t) + b_0(t))dt, \quad x(\cdot) \in L_\infty([0,T], \mathbb{R}^n), \\ &\text{subject to} \qquad\qquad g(t, x(t)) \geq 0 \quad \text{a.e. in } [0,T], \\ &\qquad\qquad\qquad A_1(t)x(t) + b_1(t) \geq 0 \quad \text{a.e. in } [0,T], \end{aligned}$$

where $A_k(t)$ are $l \times n$ matrices, $b_k(t) \in \mathbb{R}^l, k \in \{0,1\}$ and $g : [0,T] \times \mathbb{R}^n \to \mathbb{R}^m$ are given non-affine functions. Having these affine constraint functions, we can obtain new optimality conditions, where at least one of the multipliers that correspond to non-affine constraint functions is nonzero. In addition to all the previous assumptions for function $g(t,x)$, we will consider the following hypotheses:

(i) $A_0(t) \neq 0$ a.e. in $[0,T]$;

(ii) System (S'):

$$\phi_0(t,x) := -\int_0^T A_0(t)(x - \hat{x}(t)) < 0, \quad x \in \mathbb{R}^n,$$
$$\phi_i(t,x) := -g_i(t, \hat{x}(t)) - \langle \partial_x g_i(t, \hat{x}(t)), x - \hat{x}(t) \rangle \leq 0 \text{ a.e. in } [0,T],$$
$$\phi_j(t,x) := -A_{1j}(t)x(t) - b_{1j}(t) \leq 0 \text{ a.e. in } [0,T],$$
$$i \in I_1 = \{1,...,m\}, \quad j \in I_2 = \{1,...,l\},$$

is regular with $I = I_1 \cup I_2$, where $\hat{x}(\cdot)$ is a global maximizer for (P') ;

(iii) There exists at least one solution $\tilde{x}(\cdot) \in L_\infty([0,T],\mathbb{R}^n)$ to the equation

$$A_1(t)x(t) + b_1(t) = 0, \text{ a.e. in } [0,T],$$

which isn't a feasible solution for problem (P').

Theorem 3.6. *Let $\hat{x}(\cdot)$ be a global maximizer for* (P')*. Assume that the hypotheses* (i)*,* (ii) *and* (iii) *hold and there exists some $z : [0,T] \to \mathbb{R}^n$ such that for a.e. $t \in [0,T]$*

$$g(t,z(t)) > 0 \quad \text{and} \quad A_1(t)x(t) + b_1(t) > 0.$$

Then, there exist $\hat{u}(\cdot) \in L_\infty([0,T],\mathbb{R}^m)$ and $\hat{v}(\cdot) \in L_\infty([0,T],\mathbb{R}^l)$ satisfying the following conditions:

(1) $\hat{u}(t) \geq 0,\ \hat{v}(t) \geq 0$ *a.e. in* $[0,T]$ *and* $\hat{u}(t) \not\equiv 0$ *on* $[0,T]$,

(2) $\hat{u}'(t)g(t,\hat{x}(t)) = 0$ *and* $\hat{v}'(t)(A_1(t)\hat{x}(t) + b_1(t)) = 0$ a.e. in $[0,T]$,

(3) $A_0(t)x(t) + \hat{u}'(t)g(t,x(t)) + \hat{v}'(t)A_1(t)x(t) \geq$
$\geq A_0(t)\hat{x}(t) + \hat{u}'(t)g(t,\hat{x}(t)) + \hat{v}'(t)A_1(t)\hat{x}(t),$
$\forall x(\cdot) \in L_\infty([0,T],\mathbb{R}^n)$, a.e. in $[0,T]$.

Proof. Since (P') is a special case of problem (P), the conditions proved in Theorem 3.4 hold, so that the only thing which we have to show is that $\hat{u}(t) \not\equiv 0$. Assume the opposite, $\hat{u}(t) = 0$ a.e. in $[0,T]$. From Condition 3 we have

$$A_0(t)(x(t) - \hat{x}(t)) + \hat{v}'(t)A_1(t)(x(t) - \hat{x}(t)) \geq 0,$$
$$\forall x(\cdot) \in L_\infty([0,T],\mathbb{R}^n), \text{ a.e. in } [0,T].$$

For each mapping $x : [0,T] \to \mathbb{R}^n$ define the mapping $\mathcal{A}(x(\cdot))$ by

$$\mathcal{A}(x(t)) = (A_0(t) + \hat{v}'(t)A_1(t))(x(t) - \hat{x}(t)).$$

Putting $x(\cdot) = \hat{x}(\cdot)$, $\mathcal{A}(\hat{x}(\cdot))$ is obviously equal to zero a.e. in $[0,T]$. We prove that $\mathcal{A}(x(\cdot))$ vanishes on whole space $L_\infty([0,T],\mathbb{R}^n)$. Suppose to the contrary that there exists an admissible point $\overline{x}(\cdot)$ such that

$$\mathcal{A}(\overline{x}(t)) = (A_0(t) + \hat{v}'(t)A_1(t))(\overline{x}(t) - \hat{x}(t)) > 0 \quad \text{a.e. in } V \subset [0,T], \tag{10}$$

where the set V does not have measure zero. Define a function

$$y(t) = \begin{cases} 2\hat{x}(t) - \overline{x}(t), & t \in V, \\ 0, & t \notin V. \end{cases}$$

Now we have

$$\mathcal{A}(y(t)) = (A_0(t) + \hat{v}'(t)A_1(t))(\hat{x}(t) - \overline{x}(t)) \geq 0 \quad \text{a.e. in } V,$$

which implies

$$(A_0(t) + \hat{v}'(t)A_1(t))(\overline{x}(t) - \hat{x}(t)) \leq 0 \quad \text{a.e. in } V.$$

Since this is in contradiction to (10), $\mathcal{A}(x(\cdot))$ must be equal to zero on whole space $L_\infty([0,T],\mathbb{R}^n)$.

From (iii) there exists $\tilde{x}(\cdot) \in L_\infty([0,T],\mathbb{R}^n)$ such that for a.e. $t \in [0,T]$

$$A_1(t)\tilde{x}(t) + b_1(t) = 0,$$

so we obtain

$$\hat{v}'(t)(A_1(t)\tilde{x}(t) + b_1(t)) - \hat{v}'(t)(A_1(t)\hat{x}(t) + b_1(t)) = \hat{v}'(t)A_1(t)(\tilde{x}(t) - \hat{x}(t)) = 0$$

and hence $\mathcal{A}(\tilde{x}(t)) = A_0(t)(\tilde{x}(t) - \hat{x}(t)) = 0$ a.e. in $[0,T]$. Since $A_0(t) \neq 0$ a.e. in $[0,T]$, this is a contradiction to hypothesis (iii), so $\hat{u}(t)$ must be nonzero. $\square$

Example 3.5 also illustrates this case, so the multiplier $\hat{u}_1(t)$ that corresponds to non-affine function must be nonzero.

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