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Subdifferential and Conjugate Calculus of Integral Functions with and without Qualification Conditions

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We characterize the subdifferential and the Fenchel conjugate of convex integral functions by means of respectively the approximate subdifferential and the conjugate of the associated convex normal integrands. The results are stated in Suslin locally convex spaces, and do not require continuity-type qualification conditions on the functions, nor special topological or algebraic structures on the index set. Consequently, when confined to separable Banach spaces, the characterizations of such a subdifferential are obtained using only the exact subdifferential of the given integrand but at nearby points. We also provide some simplifications of our formulas when additional continuity conditions are in force.

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1. Introduction

Given two proper lower semi-continuous (lsc) convex extended real-valued functions $f_1, f_2 : X \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$, defined on a locally convex space X, the well-known Hiriart-Urruty-Phelps Theorem ([9]) asserts that the approximate subdifferential of the sum $f_1 + f_2$ is given for all $x \in X$ and $\varepsilon \geq 0$ by

$$\partial_\varepsilon(f_1+f_2)(x) = \bigcap_{\delta>0} \mathrm{cl}^{w^*}\left(\bigcup_{\substack{\varepsilon_1+\varepsilon_2\leq\varepsilon+\delta\\ \varepsilon_1,\varepsilon_2\geq 0}} \partial_{\varepsilon_1} f_1(x) + \partial_{\varepsilon_2} f_2(x)\right), \tag{1}$$

where cl^{w^*} stands for the weak*-closure in the topological dual space X^* of X, and ∂_ε denotes the approximate subdifferential (6). Another variant of (1) has been established in [7] under a weaker lsc property. The main feature of formula (1) is that (i) it requires no continuity-like conditions on f_1 or f_2 and (ii) under

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appropriate continuity-type qualification conditions, it implies the exact calculus rules of Moreau and Rockafellar ([20]; see, also, [33, Theorem 2.8.7]). We continue in this spirit in the current work by providing a similar formula for continuous sums. Given infinitely many proper and lsc convex extended real-valued functions $f_t\colon X \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$, indexed in a measure space $(T, \mathcal{E}, \mu)$ with a σ-finite nonnegative ($\mathcal{E}$-)complete measure μ, we consider the convex integral function $I_f\colon X \to \mathbb{R}_\infty$ given by

$$I_f(x) := \int_T f_t(x) d\mu, \; x \in X.$$

We prove that the approximate subdifferential of I_f, $\partial_\varepsilon I_f(x)$, can also be characterized by means of the approximate subdifferential of the data functions f_t as follows,

$$\partial_\varepsilon I_f(x) = \mathrm{cl}^{w^*}\left(\bigcup\nolimits_{\varepsilon(\cdot)\in\mathcal{A}_\varepsilon} \int_T \partial_{\varepsilon(t)} f_t(x) d\mu\right), \text{ for } \varepsilon > 0, \tag{2}$$

and, consequently, the (exact) subdifferential of I_f is written

$$\partial I_f(x) = \bigcap_{\varepsilon>0} \mathrm{cl}^{w^*}\left(\bigcup\nolimits_{\varepsilon(\cdot)\in\mathcal{A}_\varepsilon} \int_T \partial_{\varepsilon(t)} f_t(x) d\mu\right), \tag{3}$$

where $\mathcal{A}_\varepsilon$ is defined in (27). As for finite sum formula (1), (2)–(3) hold without any conditions on the involved functions, except a mere condition ((22)–(21)) needed to make our analysis consistent; for instance, to guarantee the properness of I_f, and to determine the sense of the given (strong or weak) integrals.

The term of the right-hand side in each of formulas (2)–(3) requires an appropriate justification due to the use of vector multi-valued integration, leading us to limit ourselves to the framework of Suslin locally convex spaces. This setting is less general but includes most of the spaces that appear in applications ([3]) and, at the same time, it is suitable for the use of measurable selection theorems ([28]). But that does not prevent (2) from recovering (1) in its general setting of locally convex spaces (see Corollary 4.11).

Formulas (2)–(3) will be simplified considerably, eliminating the closure cl^{w^*} and the intersection on ε, when additional natural qualification conditions are imposed. That is, since our approach is based on submerging X in subspaces of functions defined in T and taking their values in X (namely, $L^\infty(T, X)$), our conditions will intrinsically depend on the continuity properties of the associated functional $\tilde{I}_f : L^\infty(T, X) \to \overline{\mathbb{R}}$, defined by

$$\tilde{I}_f(x(\cdot)) = \int_T f(t, x(t)) d\mu.$$

The specification of the topology in $L^\infty(T, X)$ will be crucial in determining the appropriate qualification conditions and, therefore, the desired formulas for $\partial I_f(x)$. While the Mackey-continuity of $\tilde{I}_f$ at some constant function on X (say for example $x_0(\cdot) \equiv x_0 \in \operatorname{dom} I_f$, the effective domain of I_f) will give

$$\partial_\varepsilon I_f(x) = \bigcup\nolimits_{\varepsilon(\cdot)\in\mathcal{A}_\varepsilon} \int_T \partial_{\varepsilon(t)} f_t(x) d\mu, \text{ for } \varepsilon \geq 0,$$

the norm-continuity of $\tilde{I}_f$ at such a point $x_0(\cdot)$ (when X is, in addition, Banach) will result in

$$\partial_\varepsilon I_f(x) = \bigcup_{\substack{\varepsilon(\cdot)\in\mathcal{A}_{\varepsilon-\varepsilon_0} \\ 0\leq\varepsilon_0\leq\varepsilon}} \int_T \partial_{\varepsilon(t)} f_t(x) d\mu + \mathrm{N}^{\varepsilon_0}_{\operatorname{dom} I_f}(x),$$

where $\mathrm{N}^{\varepsilon_0}_{\operatorname{dom} I_f}(x)$ is the ε_0-normal set to $\operatorname{dom} I_f$ (7).

In terms of the Fenchel conjugation (5), and using the continuous infimal convolution $\oint_T$ (see (4)), the conjugate of I_f is shown to satisfy

$$(I_f)^* = \mathrm{cl}^{w^*}\left(\oint_T f_t^* d\mu\right).$$

Thus, provided that the Mackey continuity condition above is verified, the last formula simplifies to (with an exact continuous infimal convolution)

$$(I_f)^* = \oint_T f_t^* d\mu. \tag{4}$$

If, instead, X is in addition Banach and the used continuity condition is with respect to the norm topology in $L^\infty(T, X)$, then

$$(I_f)^* = \left(\oint_T f_t^* d\mu\right) \square\, \sigma_{\operatorname{dom} I_f},$$

where both $\oint_T$ and the (finite-) infimal convolution $\square$ are exact.

In an additional step, we apply a variant of the Brøndsted-Rockafellar Theorem ([33, Theorem 3.1.1]) to derive other characterizations of $\partial I_f(x)$, always free of qualification conditions, which are written by means of exact subdifferentials of the f_t's but at nearby points. For instance, provided that X is a separable reflexive Banach space, we prove that $x^* \in \partial I_f(x)$ if and only if there are sequences $(x_n(\cdot)) \subset L^\infty(T, X)$ and $(x_n^*(\cdot)) \subset L^1(T, X^*)$ such that $(x_n(\cdot))$ norm-converges to x, $\lim_n \int_T f_t(x_n(t))d\mu = I_f(x)$, $x_n^*(t) \in \partial f_t(x_n(t))$, for a.e. $t \in T$,

$$\|\cdot\|\text{-}\lim_n \int_T x_n^*(t)d\mu = x^*, \text{ and } \lim_n \int_T |\langle x_n^*(t), x_n(t) - x\rangle|\, d\mu = 0,$$

where $\|\cdot\|$ also denotes the norm in X^*. This result slightly extends similar characterizations given in [11], where it is established that $(x_n(\cdot)) \subset L^p(T, X)$ for fixed $p \in [1, +\infty[$. The case $p = +\infty$ has also been considered in [18], where the elements $(x_n^*(\cdot))$ used above are taken in the dual space $(L^\infty(T, X))^*$ rather than in $L^1(T, X^*)$.

There are many other contributions on this subject at different stages of generalities that can be found, for instance, in [2], [11], [13], [19], [21], [22], [23], [24], [25], [26], [27], [29], [32], among many others.

This paper is organized as follows: notation and some preliminary results are gathered in Section 2. Subdifferential and duality properties of convex integral functionals are reviewed and investigated in Section 3. In Section 4, we characterize the ε-subdifferential of convex integral functions, defined on Suslin locally convex spaces, and provide the expressions of the associated conjugates.

2. Notation and preliminary results

Let X be a real locally convex space (lcs, for short), and let X^* be its topological dual space with respect to a given dual pairing $\langle\cdot,\cdot\rangle$, defined on $X^* \times X$ by $\langle x^*, x\rangle := x^*(x)$, and endowed with a compatible topology for the pairing $\langle\cdot,\cdot\rangle$. Examples of such a topology on X^* are the *weak* topology* (denoted by w^* or $\sigma(X^*, X)$), the *Mackey topology* (denoted by $\tau(X^*, X)$), or the topology of the norm when X is reflexive.

The *weak topology* on X (denoted by w or $\sigma(X, X^*)$) is also a compatible topology on X. By $\mathcal{N}_X$ we refer to the family of convex closed and balanced neighborhoods of the origin in X, and by $\mathcal{B}(X)$ the Borel σ-algebra of X. When X is a normed space, we denote its closed unit ball by B_X.

We say that X is *Suslin* if it is the image by a continuous mapping of a separable completely metrizable (Polish) space; thus, a Suslin space X is separable but does not need to be metrizable. Many locally convex spaces that are used in applications are Suslin ([5]); for instance, if X is a separable Banach, then the spaces $(X, \|\cdot\|)$, (X, w) and (X^*, w^*) are Suslin.

We use the notation $\mathbb{R}_+ := [0, +\infty[$, $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ and $\mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$. Given a set $A \subset X$, we denote by $\mathrm{cl}(A)$ (or $\overline{A}$, indistinctly), $\mathrm{co}\, A$, $\overline{\mathrm{co}} A$, $\mathrm{int}(A)$, and $\mathrm{ri}(A)$, the *closure*, the *convex hull*, the *closed convex hull*, the *interior*, and the *topological relative interior* of A, respectively. The *indicator function* $\mathrm{i}_A : X \to \mathbb{R}_\infty$ of A is defined by $\mathrm{i}_A(x) := 0$ for $x \in A$, and $\mathrm{i}_A(x) := +\infty$ otherwise. The *characteristic function* $1_A : X \to [0, 1]$ of A is defined by $1_A(x) := 1$ for $x \in A$, and $1_A(x) := 0$ otherwise. Given a set $A \subset X^*$, by $\mathrm{cl}^{w^*}(A)$ we denote its *weak*-closure*. The *polar set* of A is $A^\circ := \{u \in X : \langle x^*, u\rangle \leq 1, \text{ for all } x^* \in A\}$, and the *support function* of A is the function $\sigma_A : X \to \overline{\mathbb{R}}$ defined by $\sigma_A(x) := \sup_{x^* \in A} \langle x^*, x\rangle$, with the convention $\sigma_\emptyset \equiv -\infty$.

A function $\varphi : X \to \overline{\mathbb{R}}$ is said to be *proper* if $\varphi > -\infty$ and its (effective) domain, $\mathrm{dom}\, \varphi := \{x \in X : \varphi(x) < +\infty\}$, is nonempty. The function φ is *lower semi-continuous* (lsc) (convex, resp.) if its epigraph, $\mathrm{epi}\, \varphi := \{(x, \lambda) \in X \times \mathbb{R} : \varphi(x) \leq \lambda\}$, is closed (convex, resp.). If φ is proper, convex and lsc, then we write $\varphi \in \Gamma_0(X)$. The *closed hull* of φ is the function $\mathrm{cl}\, \varphi : X \to \overline{\mathbb{R}}$ such that $\mathrm{epi}(\mathrm{cl}\, \varphi) = \mathrm{cl}(\mathrm{epi}\, \varphi)$.

The *conjugate function* of φ is the function $\varphi^* : X^* \to \overline{\mathbb{R}}$ defined by

$$\varphi^*(x^*) := \sup_X (x^* - \varphi). \tag{5}$$

Given $\varepsilon \geq 0$, an element $x^* \in X^*$ is called an ε-subgradient of φ at $x \in \varphi^{-1}(\mathbb{R})$ if

$$\langle x^*, y - x\rangle \leq \varphi(y) - \varphi(x) + \varepsilon, \text{ for all } y \in X. \tag{6}$$

The set of such elements is called the *ε-subdifferential* of φ at x, and it is denoted by $\partial_\varepsilon \varphi(x)$. The set $\partial\varphi(x) := \partial_0\varphi(x)$ is the *subdifferential* of φ at x. When $x \notin \mathrm{dom}\, \varphi$ we set $\partial_\varepsilon\varphi(x) := \emptyset$. The ε-normal set and the normal cone to a set $A \subset X$ at $x \in X$ are respectively defined by

$$\mathrm{N}_A^\varepsilon(x) := \partial_\varepsilon \mathrm{i}_A(x) \text{ and } \mathrm{N}_A(x) := \mathrm{N}_A^0(x). \tag{7}$$

Given two functions $\varphi, \psi : X \to \overline{\mathbb{R}}$, the (finite) *infimal convolution* of φ and ψ is the function $\varphi \square \psi : X \to \overline{\mathbb{R}}$ defined by

$$(\varphi \square \psi)(x) := \inf\{\varphi(y) + \psi(x - y) : \ y \in X\}.$$

We say that $\varphi \square \psi$ is *exact* at $x \in X$ if there exists $y \in X$ such that $(\varphi \square \psi)(x) = \varphi(y) + \psi(x - y)$. For instance, $\varphi \square \psi$ is exact at $x \in X$ with $\varphi(x_1) + \psi(x_2) = (\varphi \square \psi)(x)$ whenever $x_1 + x_2 = x$ and

$$\partial\varphi(x_1) \cap \partial\psi(x_2) \neq \emptyset. \tag{8}$$

Next we recall some classical results of convex functions and sets, which are used in the sequel and that can be found, e.g., in [20] and [33].

Given $\varphi \in \Gamma_0(X)$, for all $x \in X$ and $\varepsilon \geq 0$

$$\partial_\varepsilon \varphi(x) = \{x^* \in X^* : \varphi(x) + \varphi^*(x^*) \leq \langle x^*, x\rangle + \varepsilon\}. \tag{9}$$

We also have (because the topology on X^* is compatible)

$$(\varphi^*)^* = \varphi, \tag{10}$$

and so, for all $x^* \in X^*$, we have $\partial\varphi^*(x^*) = (\partial\varphi)^{-1}(x^*) := \{x \in X : x^* \in \partial\varphi(x)\}$.

More generally, when the topology on X^* is not necessarily compatible but contains the $\sigma(X^*, X)$-topology, we have that

$$X \cap \partial\varphi^*(x^*) = (\partial\varphi)^{-1}(x^*), \text{ for all } x^* \in X^*. \tag{11}$$

Also, using the fact that $\partial_\varepsilon\varphi \subset \operatorname{dom}\varphi^*$ and $\operatorname{dom}(\varphi\Box\sigma_{\operatorname{dom}\varphi^*}) = \operatorname{dom}\varphi + \operatorname{dom}(\sigma_{\operatorname{dom}\varphi^*})$, we verify that

$$\varphi\Box\sigma_{\operatorname{dom}\varphi^*} = \varphi. \tag{12}$$

If Y is another lcs and $A : Y \to X$ is a linear continuous mapping, then for all $\varepsilon > 0$, $y \in X$

$$(\varphi \circ A)^* = \operatorname{cl}^{w^*}(A^*\varphi^*) \text{ and } \partial_\varepsilon(\varphi \circ A)(y) = \operatorname{cl}^{w^*}(A^*\partial_\varepsilon\varphi(Ay)), \tag{13}$$

where A^* is the adjoint mapping of A, and $A^*\varphi^* : Y^* \to \overline{\mathbb{R}}$ is the convex function given by

$$(A^*\varphi^*)(y^*) := \inf\{\varphi^*(x^*) : A^*x^* = y^*\}.$$

If φ is finite and continuous somewhere in $A(Y)$, then we have for all $\varepsilon \geq 0$, $y \in X$ and $y^* \in Y^*$

$$\partial_\varepsilon(\varphi \circ A)(y) = A^*(\partial_\varepsilon\varphi(Ay))$$

and
$$(\varphi \circ A)^*(y^*) = (A^*\varphi^*)(y^*) = \min\{\varphi^*(x^*) : A^*x^* := y^*\}. \tag{14}$$

Let $(T, \mathcal{E}, \mu)$ be a measure space with μ being a complete σ-finite nonnegative measure. For simplicity reasons, sometimes we may assume that μ is finite and, in particular, $\mu(T) = 1$ (see, e.g., [10]). Indeed, given a positive function $\eta \in L^1(T)$, we have $\eta_1 := \eta(\cdot)/(1 + \eta(\cdot)) \in L^1(T) \cap L^\infty(T)$ and the function

$$\eta_0 := \frac{\eta_1(\cdot)}{\|\eta_1(\cdot)\|_{L^1(T)}} \in L^1(T) \cap L^\infty(T)$$

satisfies $\|\eta_0(\cdot)\|_{L^1(T)} = 1$. Hence, the measure $\tilde{\mu}$ defined on $\mathcal{E}$ by

$$\tilde{\mu}(A) := \textstyle\int_A \eta_0(t)d\mu, \; A \in \mathcal{E}, \tag{15}$$

is a nonnegative complete measure such that $\tilde{\mu}(T) = 1$ and $d\tilde{\mu} = \eta_0(\cdot)d\mu$.

Suppose that X and X^* are Suslin. A function $\varphi : T \to X$ is said to be *weakly measurable* (written $\varphi \in \mathcal{M}(T, \mathcal{E}, X)$) if the scalar function $\langle x^*, \varphi(\cdot)\rangle : t \in T \mapsto \langle x^*, \varphi(t)\rangle$ is $\mathcal{E}$-measurable, for every $x^* \in X^*$. In the current setting of Suslin spaces, such a measurability property is equivalent to the classical measurability of φ in the sense that $\varphi^{-1}(V) \in \mathcal{E}$ for every open set $V \subset X$ or, equivalently, that φ is the limit of a sequence of simple measurable functions (see, e.g., [31, Lemma 2] and [3, Theorem III.36]).

A measurable function $\varphi : T \to X$ is said to be *weakly integrable*, if the functions $\langle x^*, \varphi(\cdot)\rangle$, $x^* \in X^*$, are integrable, and for each $A \in \mathcal{E}$ there exists $x_A \in X$ such that

$$\langle x^*, x_A\rangle = \int_A \langle x^*, \varphi(t)\rangle \, d\mu, \text{ for all } x^* \in X^*.$$

The weak integral of φ is $\texttt{w-}\int_T \varphi(t)d\mu := x_T$.

Given the equivalence relation that two functions are equivalent if they coincide in the complememt of a negligible set, we denote by $L^1_w(T,X)$ the vector space of equivalence classes of weakly integrable functions.

A measurable function $\varphi : T \mapsto X$ is said to be *(strongly) integrable*, if

$$\textstyle\int_T \sigma_V(\varphi(t)) < +\infty, \tag{16}$$

for some $V \in \mathcal{N}_{X^*}$, that is, $\sigma_V(\varphi(\cdot))$ belongs to the usual space $L^1(T)$. We denote by $L^1(T,X)$ the vector space of equivalence classes of integrable functions. Since X^* is Suslin, the function $\sigma_V(\varphi(\cdot))$ is measurable for being the supremum of a countable family of measurable functions, and the above integral makes sense. Also, if $\varphi \in L^1(T,X)$ and (16) holds for some $V \in \mathcal{N}_{X^*}$, then for each $x^* \in X^*$ the Cauchy-Schwarz inequality implies

$$\langle x^*, \varphi(t)\rangle \leq \sigma_V(\varphi(t))\sigma_{V^\circ}(x^*), \text{ for a.e. } t \in T.$$

Hence, since $\sigma_{V^\circ}(x^*) \in \mathbb{R}$ (because V° is compact in X, by the Alaoglu-Banach-Bourbaki Theorem), for each fixed $A \in \mathcal{E}$ we have

$$\textstyle\int_A |\langle x^*, \varphi(t)\rangle|\, d\mu \leq \sigma_{V^\circ}(x^*)\int_A \sigma_V(\varphi(t))d\mu < +\infty,$$

and the function $\langle x^*, \varphi(\cdot)\rangle$ is integrable on A. Furthermore, the linear mapping $l_A \colon X^* \to \mathbb{R}$ defined by

$$l_A(x^*) := \textstyle\int_A \langle x^*, \varphi(t)\rangle\, d\mu,$$

satisfies, for all $x^* \in V$,

$$|l_A(x^*)| \leq \textstyle\int_A |\langle x^*, \varphi(t)\rangle|\, d\mu \leq \sigma_{V^\circ}(x^*)\int_A \sigma_V(\varphi(t))d\mu \leq \int_A \sigma_V(\varphi(t))d\mu < +\infty,$$

and this shows that l_A is continuous on X^*, that is, $l_A \in X^{**} = X$ (as $V \in \mathcal{N}_{X^*}$ and we are considering a compatible topology in X^*); that is, φ is weakly integrable. In other words, every integrable function is weakly integrable, and so we call strong integral or, simply, integral of φ the element of X given by

$$\textstyle\int_T \varphi(t)d\mu := \mathtt{w}\text{-}\int_T \varphi(t)d\mu.$$

Equivalently, since X is Suslin, the measurabilty of φ entails the existence of some sequence of measurable simple functions $\varphi_k : T \to X$, say $\varphi_k = \sum_{1\leq k\leq n_k} x_k 1_{A_k}$ (for $x_k \in X$, $A_k \in \mathcal{E}$, $n_k \geq 1$), that converges for a.e. on T to φ, and we have that

$$\textstyle\int_T \varphi(t)d\mu = \lim\limits_k \int_T \varphi_k(t)d\mu = \lim\limits_k \displaystyle\sum_{1\leq k\leq n_k} x_k\mu(A_k).$$

We also define the vector space (of equivalence classes) $L^\infty(T,X)$ as follows

$$L^\infty(T,X) := \left\{ \varphi : T \to X : \begin{array}{l} \varphi \text{ is measurable and } \varphi(T\setminus N) \text{ is bounded} \\ \text{in } X \text{ for some negligible set } N \subset T \end{array} \right\}.$$

If X is additionally a reflexive Banach space with a norm $\|\cdot\|$, then $L^\infty(T,X)$ and $L^1(T,X)$ are normed spaces with norms given respectively by (see, e.g., [5, Ch. 2])

$$\|\varphi\|_\infty := \operatorname{ess\,sup} \|\varphi(t)\| \quad \text{and} \quad \|\varphi\|_1 := \textstyle\int_T \|\varphi(t)\|\, d\mu.$$

In other words, when X is Banach reflexive, the usual vector space of Bochner integrable functions (requiring $\int_T \|\varphi(t)\| \, d\mu < +\infty$) coincides with our definition in (16), provided that we endow X^* with its norm topology which is compatible for the pairing (X, X^*). However, the definition in (16) can be restrictive in non-reflexive Banach spaces and, consequently, the family of integrable functions following our definition can be smaller than the usual family of Bochner integrable functions. But with respect to our goals of characterizing the subdifferential of the integral functions $I_f(x) := \int_T f(t,x)d\mu$, this discrepancy will be an advantage and will provide us with more precise characterizations of the required subdifferential ∂I_f (see Theorem 4.2 and subsequent results).

We recall the Lebesgue decomposition theorem for separable Banach spaces (see, e.g., [17, Theorem 4.1]), ensuring that the dual space of $L^\infty(T,X)$ is written as

$$(L^\infty(T,X))^* = L^1(T,X^*) \oplus \Lambda(T,X^*), \tag{17}$$

where $\Lambda(T,X^*)$ denotes the set of singular measures on T, that is, $\lambda \in \Lambda(T,X^*)$ if and only if there exists a nondecreasing sequence of positive measure sets $(T_n)_n \subset T$ such that $T \setminus (\cup_n T_n)$ is negligible and $\langle \lambda, z(\cdot)1_{T_n} \rangle = 0$, for all $z(\cdot) \in L^\infty(T,X)$ and $n \geq 1$.

Given a set-valued map $F : T \rightrightarrows X$, we say that F is *measurable* if, for each open set $U \subset X$, we have $\{t \in T : F(t) \cap U \neq \emptyset\} \in \mathcal{E}$. By S_F, $S_F^{1,w}$ and S_F^1 we denote the sets of measurable, weakly integrable and integrable selectors of F given respectively by

$$\begin{aligned} S_F &:= \{\varphi : T \to X \text{ measurable} : \varphi(t) \in F(t), \text{ a.e. } t \in T\}, \\ S_F^{1,w} &:= \{\varphi \in L_w^1(T,X) : \varphi(t) \in F(t), \text{ a.e. } t \in T\}, \\ \text{and} \qquad S_F^1 &:= \{\varphi \in L^1(T,X) : \varphi(t) \in F(t), \text{ a.e. } t \in T\}. \end{aligned}$$

The weak integral and the (strong) integral of the set-valued map F are respectively given by

$$\texttt{w-}\textstyle\int_T F(t)d\mu := \left\{\texttt{w-}\int_T \varphi(t)d\mu : \varphi \in S_F^{1,w}\right\},$$

$$\textstyle\int_T F(t)d\mu := \left\{\int_T \varphi(t)d\mu : \varphi \in S_F^1\right\}.$$

A linear subspace $L \subset \mathcal{M}(T,\mathcal{E},X)$ is said to be *decomposable* ([21]–[25]) if, for any measurable set $A \subset T$ and any measurable functions $f \in L$ and $g \in \mathcal{M}^k(T,\mathcal{E},X)$ (that is, $\overline{g(T \setminus N)}$ is compact in X with $N \subset T$ negligible), we have that

$$1_A f + 1_{T \setminus A} g \in L.$$

We have the following lemma.

Lemma 2.1. *Assume that X and X^* are both Suslin. Then, for every functions $\varphi \in L^1(T,X^*)$ and $\psi \in L^\infty(T,X)$, the function $t \longmapsto \langle \varphi(t), \psi(t) \rangle$ is integrable, and the mapping*

$$\langle \varphi, \psi \rangle_{L^1(T,X^*),L^\infty(T,X)} := \int_T \langle \varphi(t), \psi(t) \rangle \, d\mu, \tag{18}$$

is a separating duality pairing for the pair $(L^\infty(T,X), L^1(T,X^*))$.

Proof. The measurability of the function $t \to \langle \varphi(t), \psi(t) \rangle$ easily follows. Take some $V \in \mathcal{N}_X$ such that $\int_T \sigma_V(\varphi(t))d\mu < +\infty$. Then, since $\psi \in L^\infty(T, X)$, there exists some $\lambda > 0$ such that $\psi(t) \in \lambda V$, a.e. $t \in T$. Consequently,

$$|\langle \varphi(t), \psi(t) \rangle| \leq \lambda \sigma_V(\varphi(t)) \sigma_{V^\circ}(\varphi(t)) \leq \lambda \sigma_V(\varphi(t))$$

and so

$$\int_T |\langle \varphi(t), \psi(t) \rangle| \, d\mu \leq \lambda \int_T \sigma_V(\varphi(t)) d\mu < +\infty,$$

that is, the function $t \longmapsto \langle \varphi(t), \psi(t) \rangle$ is integrable.

Finally, according to [31], $L^\infty(T, X)$ and $L^1(T, X^*)$ are decomposable subspaces of measurable functions and (18) is a separating pairing. □

A function $\varphi : T \times X \to \mathbb{R}_\infty$ is said to be a *convex normal integrand* if the mapping $(t, x) \mapsto \varphi_t(x) := \varphi(t, x)$ is $\mathcal{E} \otimes \mathcal{B}(X)$ measurable, and $\varphi(t, \cdot) \in \Gamma_0(X)$ for all $t \in T$. Equivalently, φ is a convex normal integrand if and only if the mapping $t \rightrightarrows \operatorname{epi} \varphi(t, \cdot)$ has closed and convex values and its graph belongs to $\mathcal{E} \otimes \mathcal{B}(X)$ (see [3, Lemma VII-1]). If φ is a convex normal integrand, then so is $\varphi^* : T \times X^* \to \mathbb{R}_\infty$ defined by $\varphi^*(t, x^*) := (\varphi(t, \cdot))^*(x^*)$ (see [3, Corollary VII-2]).

Given an extended real-valued measurable function $\varphi : T \to \overline{\mathbb{R}}$, we define as usual its (extended) integral as

$$\int_T \varphi(t) d\mu := \int_T \varphi^+(t) d\mu - \int_T \varphi^-(t) d\mu$$

if $\int_T \varphi^+(t) d\mu < +\infty$, and $\int_T \varphi(t) d\mu := +\infty$ otherwise. Here $\varphi^+ := \max\{\varphi, 0\}$ and $\varphi^- := \max\{-\varphi, 0\}$, that is, φ^+ and φ^- are the (measurable) positive and negative parts of φ, respectively. If, in addition, φ is bounded from below by an integrable function, then $\int_T \varphi^-(t) d\mu \in \mathbb{R}$ and, consequently, $\int_T \varphi(t) d\mu \in \mathbb{R}$ (we say in this case that φ is integrable) if and only if $\int_T \varphi(t) d\mu < +\infty$. Then, for a convex normal integrand $f \colon T \times X \to \mathbb{R}_\infty$, we consider the integral function $I_f \colon X \to \overline{\mathbb{R}}$ defined by

$$I_f(x) := \int_T f(t, x) d\mu, \tag{19}$$

and introduce the associated convex integral functional $\tilde{I}_f \colon L^\infty(T, X) \to \overline{\mathbb{R}}$ as

$$\tilde{I}_f(x(\cdot)) := \int_T f(t, x(t)) d\mu. \tag{20}$$

Both definitions (19) and (20) will require appropriate integral lower bounds of $f(\cdot, x)$ and $f(\cdot, x(\cdot))$. Namely, we shall frequently use either the condition

$$f(t, \cdot) \geq \langle \gamma(t), \cdot \rangle + \beta(t) \text{ (for all } t \in T \setminus N\text{), with } \gamma \in L^1_w(T, X^*),\ \beta \in L^1(T), \tag{21}$$

or the following one

$$f(t, \cdot) \geq \langle \gamma(t), \cdot \rangle + \beta(t) \text{ (for all } t \in T \setminus N\text{), with } \gamma \in L^1(T, X^*),\ \beta \in L^1(T), \tag{22}$$

where $N \subset T$ is a fixed negligible set. Condition (21) implies that $I_f > -\infty$, and for all $x \in X$ we have $\int_T f^-(t, x) d\mu \in \mathbb{R}$; hence, $I_f(x) \in \mathbb{R}$ iff $\int_T f^+(t, x) d\mu \in \mathbb{R}$.

Similarly, (22) ensures the stronger property $\tilde{I}_f > -\infty$, and for all $x(\cdot) \in L^1(T, X)$ we have $\int_T f^-(t, x(t)) d\mu \in \mathbb{R}$, so that $\tilde{I}_f(x) \in \mathbb{R}$ iff $\int_T f^+(t, x(t)) d\mu \in \mathbb{R}$. It is worth recalling that, since every function from $\Gamma_0(X)$ is bounded from below by a continuous affine function, both conditions (22) and (21) automatically hold when T is finite and μ is the counting measure.

The following lemma gives us some insight on the geometry of $\operatorname{dom} I_f$. The relation in (iii) below can also be found, e.g., in [13].

Lemma 2.2. *Assume that X is Suslin, and let $f : T \times X \to \mathbb{R}_\infty$ be a convex normal integrand satisfying* (22)*. Then the following assertions hold true.*

(i) *If $x \in \operatorname{dom} I_f$, then there exists a negligible set $N \subset T$ such that*

$$x \in \bigcap_{t \in T \setminus N} \operatorname{dom} f(t, \cdot).$$

(ii) *There is some negligible set $N \subset T$ such that*

$$\operatorname{int}(\operatorname{dom} I_f) \subset \operatorname{cl}\left(\bigcap_{t \in T \setminus N} \operatorname{dom} f(t, \cdot) \right).$$

(iii) *If X is finite-dimensional, then there is some negligible set $N \subset T$ such that*

$$\operatorname{int}(\operatorname{dom} I_f) \subset \bigcap_{t \in T \setminus N} \operatorname{dom} f(t, \cdot).$$

Proof. (i) Given $x \in \operatorname{dom} I_f$, we have that $\int_T f(t, x) d\mu < +\infty$ and so, due to (22), $f(t, x) \in \mathbb{R}$ for a.e. $t \in T$.

(ii) We may assume that $\operatorname{int}(\operatorname{dom} I_f) \neq \emptyset$. Using the separability of X, there exists a countable set $D := \{(x_n)_{n \geq 1}\} \subset \operatorname{dom} I_f$ such that $\operatorname{cl}(\operatorname{dom} I_f) = \overline{D}$. Next, by assertion (i), for each $n \geq 1$ there exists some negligible set $N_n \subset T$ such that $x_n \in \cap_{t \in T \setminus N_n} \operatorname{dom} f(t, \cdot) \subset \cap_{t \in T \setminus N} \operatorname{dom} f(t, \cdot)$, where $N := \cup_{n \geq 1} N_n$ is also a negligible set. Hence,

$$D \subset \bigcap_{t \in T \setminus N} \operatorname{dom} f(t, \cdot), \tag{23}$$

and we deduce $\operatorname{int}(\operatorname{dom} I_f) \subset \overline{D} \subset \operatorname{cl}\left(\cap_{t \in T \setminus N} \operatorname{dom} f(t, \cdot) \right)$.

(iii) Assume that X is finite-dimensional and $\operatorname{int}(\operatorname{dom} I_f) \neq \emptyset$. As in (ii), there is a countable set $D \subset \operatorname{dom} I_f$ such that $\operatorname{cl}(\operatorname{dom} I_f) = \overline{D}$. We also have that $\operatorname{cl}(\operatorname{dom} I_f) = \overline{\operatorname{co}} D$, due to the convexity of I_f, while (23) entails

$$\operatorname{co} D \subset \bigcap_{t \in T \setminus N} \operatorname{dom} f(t, \cdot). \tag{24}$$

Hence, since we have assumed that $\operatorname{int}(\operatorname{dom} I_f) \neq \emptyset$, we obtain

$$\operatorname{int}(\operatorname{dom} I_f) = \operatorname{int}(\operatorname{cl}(\operatorname{dom} I_f)) = \operatorname{int}(\overline{\operatorname{co}} D) = \operatorname{ri}(\overline{\operatorname{co}} D) = \operatorname{ri}(\operatorname{co} D),$$

because X is finite-dimensional. Therefore $\operatorname{int}(\operatorname{dom} I_f) = \operatorname{int}(\operatorname{co} D)$ and (24) imply

$$\operatorname{int}(\operatorname{dom} I_f) = \operatorname{int}(\operatorname{co} D) \subset \textstyle\bigcap_{t \in T \setminus N} \operatorname{dom} f(t, \cdot). \qquad \square$$

Proposition 2.3 below reviews the main duality result for the functional $\tilde{I}_f$ defined in (20). It was given firstly in finite-dimensional spaces in [21], and secondly for separable reflexive Banach spaces in [25] (see, also, [2]). The present form of Proposition 2.3 can be found in [31, Theorem] and is valid for general locally convex Suslin spaces.

Proposition 2.3. *Assume that X and X^* are Suslin. Let $f : T \times X \to \mathbb{R}_\infty$ be a convex normal integrand satisfying* (22). *Let $V_1 \subset \mathcal{M}(T, \mathcal{E}, X)$ and $V_2 \subset \mathcal{M}(T, \mathcal{E}, X^*)$ be two decomposable subspaces paired with a separating bilinear mapping. Assume that the (effective) domains of the functionals $\tilde{I}_f : V_1 \subset \mathcal{M}(T, \mathcal{E}, X) \to \mathbb{R}_\infty$ and $\tilde{I}_{f^*} : V_2 \subset \mathcal{M}(T, \mathcal{E}, X^*) \to \mathbb{R}_\infty$ are nonempty. Then $\tilde{I}_f \in \Gamma_0(V_1)$, $\tilde{I}_{f^*} \in \Gamma_0(V_2)$, $(\tilde{I}_f)^* = \tilde{I}_{f^*}$, and $(\tilde{I}_{f^*})^* = \tilde{I}_f$.*

3. Convex functional integrals and duality

In this section, X and its dual X^* are two locally convex Suslin spaces endowed with locally convex and compatible topologies with respect to the pairing $\langle x^*, x\rangle := x^*(x)$ defined on the space $X^* \times X$. We consider the measure space $(T, \mathcal{E}, \mu)$ where the measure μ is σ-finite, nonnegative and complete. Given a convex normal integrand $f : T \times X \to \mathbb{R}_\infty$, we review here some duality properties between the convex integral functionals $\tilde{I}_f : L^\infty(T, X) \to \overline{\mathbb{R}}$ and $\tilde{I}_{f^*} : L^1(T, X^*) \to \overline{\mathbb{R}}$, defined by

$$\tilde{I}_f(x(\cdot)) = \textstyle\int_T f(t, x(t))d\mu \tag{25}$$

and

$$\tilde{I}_{f^*}(x^*(\cdot)) = \textstyle\int_T f^*(t, x^*(t))d\mu. \tag{26}$$

Taking into account Lemma 2.1, we consider the duality pair $(L^\infty(T, X), L^1(T, X^*))$ associated to the separating pairing

$$\langle x^*(\cdot), x(\cdot)\rangle := \textstyle\int_T \langle x^*(t), x(t)\rangle\, d\mu,\ \ x(\cdot) \in L^\infty(T, X),\ \ x^*(\cdot) \in L^1(T, X^*).$$

Hence, the ε-subdifferential of the functional $\tilde{I}_f$ at $x(\cdot) \in (\tilde{I}_f)^{-1}(\mathbb{R})$ $(\subset L^\infty(T, X))$ is the set $\partial_\varepsilon \tilde{I}_f(x(\cdot))$ of subgradients $x^*(\cdot) \in L^1(T, X^*)$ such that

$$\langle x^*(\cdot), y(\cdot) - x(\cdot)\rangle \le \tilde{I}_f(y(\cdot)) - \tilde{I}_f(x(\cdot)) + \varepsilon$$

for all $y(\cdot) \in L^\infty(T, X)$, that is,

$$\textstyle\int_T \langle x^*(t), y(t) - x(t)\rangle\, d\mu \le \int_T f(t, y(t))d\mu - \int_T f(t, x(t))d\mu + \varepsilon.$$

Similarly, the ε-subdifferential of $\tilde{I}_{f^*}$ at $x^*(\cdot) \in (\tilde{I}_{f^*})^{-1}(\mathbb{R})$ $(\subset L^1(T, X^*))$ is the set $\partial_\varepsilon \tilde{I}_{f^*}(x^*(\cdot))$ of subgradients $x(\cdot) \in L^\infty(T, X)$ such that, for all $y^*(\cdot) \in L^1(T, X^*)$,

$$\langle y^*(\cdot) - x^*(\cdot), x(\cdot)\rangle \le \tilde{I}_{f^*}(y^*(\cdot)) - \tilde{I}_{f^*}(x^*(\cdot)) + \varepsilon,$$

that is, $\int_T \langle y^*(t) - x^*(t), x(t)\rangle\, d\mu \le \int_T f^*(t, y^*(t))d\mu - \int_T f^*(t, x^*(t))d\mu + \varepsilon.$

We start by giving different consequences (and characterizations) of condition (22), stating that for some functions $\gamma \in L^1(T, X^*)$ and $\beta \in L^1(T)$ we have, for a.e. $t \in T$,

$$f_t(x) \ge \langle \gamma(t), x\rangle + \beta(t), \quad \text{for all } x \in X,$$

where we denote $f_t := f(t, \cdot)$.

Also, if there exists $x_0(\cdot)\in L^\infty(T,X)$ with $\tilde{I}_f(x_0(\cdot))\in\mathbb{R}$, we have for a.e. $t\in T$

$$f_t^*(x^*(t)) \geq \langle x_0(t), x^*(t)\rangle - f(t,x_0(t)), \quad \text{for all } x^*(\cdot) \in L^1(T,X^*),$$

that is, the function $t \longmapsto f_t^*(x^*(t))$ is bounded from below by the integral function $\langle x_0(\cdot), x^*(\cdot)\rangle - f(\cdot, x_0(\cdot))$. Hence, the function $t \longmapsto f_t^*(x^*(t))$ is integrable iff $\int_T (f_t^*)^+(x^*)d\mu < +\infty$.

We shall use the following notation,

$$\mathcal{A}_\varepsilon := \left\{\varepsilon(\cdot) \in L^1(T,\mathbb{R}_+) : \int_T \varepsilon(t)d\mu \leq \varepsilon\right\},\ \varepsilon \geq 0, \tag{27}$$

and $$\mathcal{A}_\varepsilon^+ := \{\varepsilon(\cdot) \in \mathcal{A}_\varepsilon : \varepsilon(t) > 0,\ \text{a.e. } t \in T\},\ \varepsilon > 0.$$

Proposition 3.1. *Let f be a convex normal integrand such that $\tilde{I}_f(x_0(\cdot)) \in \mathbb{R}$ for some $x_0(\cdot) \in L^\infty(T,X)$. Then condition (22) is equivalent to each one of the following assertions.*

(i) *For all $x(\cdot) \in \operatorname{dom} \tilde{I}_f$ and $\varepsilon \geq 0$, we have*

$$\partial_\varepsilon \tilde{I}_f(x) = \bigcup\nolimits_{\varepsilon(\cdot)\in\mathcal{A}_\varepsilon} \{x^*(\cdot) \in L^1(T,X^*) : x^*(t) \in \partial_{\varepsilon(t)} f_t(x(t)),\ \text{a.e. } t \in T\},$$

and $\partial_\varepsilon \tilde{I}_f(x(\cdot)) \neq \emptyset$ *whenever* $\varepsilon > 0$.

(ii) *The convex functionals $\tilde{I}_f$ and $\tilde{I}_{f^*}$ are proper, $\tilde{I}_f$ is $\sigma(L^\infty(T,X), L^1(T,X^*))$-lsc, $\tilde{I}_{f^*}$ is $\sigma(L^1(T,X^*), L^\infty(T,X))$-lsc, and we have*

$$\tilde{I}_{f^*} = (\tilde{I}_f)^* \text{ and } \tilde{I}_f = (\tilde{I}_{f^*})^*.$$

(iii) *For all $\varepsilon > 0$ and $x(\cdot) \in \operatorname{dom} \tilde{I}_f$ (equivalently, for some $x \in \operatorname{dom} \tilde{I}_f$), we have*

$$\bigcup\nolimits_{\varepsilon(\cdot)\in\mathcal{A}_\varepsilon^+} \int_T \partial_{\varepsilon(t)} f_t(x(t))d\mu \neq \emptyset.$$

(iv) *There exists a function $\zeta \in L^1(T,X^*)$ such that $\int_T f_t^*(\zeta(t))d\mu < +\infty$.*

(v) *For some $U \in \mathcal{N}_X$, there exist $\alpha \in L^1(T)$ and nonnegative function $r \in L^1(T)$ such that, for a.e. $t \in T$,*

$$f_t(x) + r(t)\sigma_{U^\circ}(x) \geq \alpha(t), \text{ for all } x \in X.$$

Proof. We establish the necessary implications in the following order:
(ii) ⇒ (iii) ⇒ (22) ⇒ (iv) ⇒ (ii) ⇒ (i) ⇒ (ii) and the equivalence (22) ⇔ (v).

(ii) ⇒ (iii). Assume that (ii) holds, and fix $\varepsilon > 0$ and $x(\cdot) \in \operatorname{dom} \tilde{I}_f$, so that we have $\partial_{\frac{\varepsilon}{2}} \tilde{I}_f(x(\cdot)) \neq \emptyset$ (e.g., [33, Theorem 2.4.4(iii)]). Take

$$x^*(\cdot) \in \partial_{\frac{\varepsilon}{2}} \tilde{I}_f(x(\cdot)) \quad (\subset \operatorname{dom} \left(\tilde{I}_f\right)^* = \operatorname{dom} \tilde{I}_{f^*} \subset L^1(T,X^*), \text{ by (ii)}).$$

Then the functions $f(\cdot,x(\cdot))$, $f^*(\cdot,x^*(\cdot))$, and $\langle x^*(\cdot), x(\cdot)\rangle$ are integrable, and (ii) yields

$$\begin{aligned}
&\int_T [f(t,x(t)) + f^*(t,x^*(t)) - \langle x^*(t), x(t)\rangle]d\mu \\
&\qquad = \int_T f(t,x(t))d\mu + \int_T f^*(t,x^*(t))d\mu - \int_T \langle x^*(t), x(t)\rangle\, d\mu \\
&\qquad = \tilde{I}_f(x(\cdot)) + \tilde{I}_{f^*}(x^*(\cdot)) - \langle x^*(\cdot), x(\cdot)\rangle \\
&\qquad = \tilde{I}_f(x(\cdot)) + \left(\tilde{I}_f\right)^*(x^*(\cdot)) - \langle x^*(\cdot), x(\cdot)\rangle \leq \frac{\varepsilon}{2}.
\end{aligned}$$

Let us denote

$$\varepsilon(t) := f(t, x(t)) + f^*(t, x^*(t)) - \langle x^*(t), x(t)\rangle + \frac{\varepsilon\eta_0(t)}{2},$$

where $\eta_0 \in L^1(T) \cap L^\infty(T)$ is a positive function such that $\|\eta_0(\cdot)\|_{L^1(T)} = 1$ (see (15)). Then $\varepsilon(\cdot) \in \mathcal{A}_\varepsilon^+$, as $\varepsilon(t) \geq \frac{\varepsilon\eta_0(t)}{2} > 0$ for a.e. $t \in T$, and $x^*(t) \in \partial_{\varepsilon(t)} f_t(x(t))$ for a.e. $t \in T$. In other words, $x^*(\cdot)$ is an integrable selection of the measurable multifunction $t \rightrightarrows \partial_{\varepsilon(t)} f_t(x(t))$, and assertion (iii) holds true.

(iii) $\Rightarrow$ (22). Assume that, for each $\varepsilon > 0$ and $x(\cdot) \in \operatorname{dom} \tilde{I}_f$, there are $\varepsilon(\cdot) \in \mathcal{A}_\varepsilon^+$ and $x^*(\cdot) \in L^1(T, X^*)$ such that $x^*(t) \in \partial_{\varepsilon(t)} f_t(x(t))$ for a.e. $t \in T$. Then, for all such $t \in T$, we have

$$f(t, y) \geq \langle x^*(t), y - x(t)\rangle + f(t, x(t)) - \varepsilon(t), \text{ for all } y \in X,$$

that is, (22) follows with $\gamma := x^*(\cdot) \in L^1(T, X^*)$ and

$$\beta := f(\cdot, x(\cdot)) - \langle x^*(\cdot), x(\cdot)\rangle - \varepsilon(\cdot) \in L^1(T).$$

(iv) $\Rightarrow$ (ii). Assume that $\tilde{I}_{f^*}(\zeta) < +\infty$ for some $\zeta \in L^1(T, X^*)$. Due to the current assumption $\tilde{I}_f(x_0(\cdot)) \in \mathbb{R}$, Proposition 2.3 implies that $\tilde{I}_f = (\tilde{I}_{f^*})^*$, and so $\tilde{I}_f$ is $\sigma(L^\infty(T, X), L^1(T, X^*))$-lsc and proper. The same argument shows that $\tilde{I}_{f^*}$ is proper, $\sigma(L^1(T, X^*), L^\infty(T, X))$-lsc and satisfies $\tilde{I}_{f^*} = (\tilde{I}_f)^*$.

(ii) $\Rightarrow$ (i). Fix $x(\cdot) \in \operatorname{dom} \tilde{I}_f$ and $\varepsilon \geq 0$. Since $\tilde{I}_f$ is assumed to be proper, convex and $\sigma(L^\infty(T, X), L^1(T, X^*))$-lsc, we have that $\partial_\varepsilon \tilde{I}_f(x(\cdot)) \neq \emptyset$ when $\varepsilon > 0$. Take $x^*(\cdot) \in \partial_\varepsilon \tilde{I}_f(x(\cdot)) \subset L^1(T, X^*)$. Then $\tilde{I}_f(x(\cdot)) \in \mathbb{R}$ and (ii) implies that

$$\begin{aligned}\tilde{I}_f(x(\cdot)) + (\tilde{I}_f)^*(x^*(\cdot)) - \langle x^*(\cdot), x(\cdot)\rangle &= \tilde{I}_f(x(\cdot)) + \tilde{I}_{f^*}(x^*(\cdot)) - \langle x^*(\cdot), x(\cdot)\rangle \\ &= \textstyle\int_T (f_t(x(t)) + f_t^*(x^*(t)) - \langle x^*(t), x(t)\rangle) d\mu \leq \varepsilon.\end{aligned}$$

Thus, taking $\varepsilon(t) := f_t(x) + f_t^*(x^*(t)) - \langle x^*(t), x(t)\rangle$, we have that $\varepsilon(\cdot) \in \mathcal{A}_\varepsilon$ and $x^*(t) \in \partial_{\varepsilon(t)} f_t(x(t))$ for a.e. $t \in T$. Conversely, let $x^*(\cdot) \in L^1(T, X^*)$ and $\varepsilon(\cdot) \in \mathcal{A}_\varepsilon$ be such that $x^*(t) \in \partial_{\varepsilon(t)} f_t(x(t))$ for a.e. $t \in T$, that is,

$$f_t(x(t)) + f_t^*(x^*(t)) \leq \langle x^*(t), x(t)\rangle + \varepsilon(t).$$

Then, since $x(\cdot) \in \operatorname{dom} \tilde{I}_f$, we have that $x^*(\cdot) \in \operatorname{dom} \tilde{I}_{f^*}$ and

$$\textstyle\int_T f_t(x(t)) d\mu + \int_T f_t^*(x^*(t)) d\mu \leq \int_T \langle x^*(t), x(t)\rangle\, d\mu + \varepsilon. \tag{28}$$

Consequently, because we have, for all $z(\cdot) \in L^\infty(T, X)$,

$$\langle x^*(\cdot), z(\cdot)\rangle - \tilde{I}_f(z(\cdot)) = \textstyle\int_T (\langle x^*(t), z(t)\rangle - f(t, z(t)))\, d\mu \leq \int_T f_t^*(x^*(t)) d\mu,$$

inequality (28) gives rise to

$$\begin{aligned}\tilde{I}_f(x(\cdot)) + (\tilde{I}_f)^*(x^*(\cdot)) \leq \tilde{I}_f(x(\cdot)) + \tilde{I}_{f^*}(x^*(\cdot)) &= \textstyle\int_T f_t(x(t)) d\mu + \int_T f_t^*(x^*(t)) d\mu \\ &\leq \textstyle\int_T \langle x^*(t), x(t)\rangle\, d\mu + \varepsilon,\end{aligned}$$

which shows that $x^*(\cdot) \in \partial_\varepsilon \tilde{I}_f(x(\cdot))$.

(i) $\Rightarrow$ (ii). Assumption (i) implies that the convex functional $\tilde{I}_f$ is proper and $\sigma(L^\infty(T,X),L^1(T,X^*))$-lsc. Moreover, given any $x(\cdot)\in \operatorname{dom}\tilde{I}_f$, for each $\varepsilon>0$ assertion (i) yields some $x^*(\cdot)\in\partial_\varepsilon\tilde{I}_f(x(\cdot))$ and $\varepsilon(\cdot)\in\mathcal{A}_\varepsilon$ such that $x^*(t)\in\partial_{\varepsilon(t)}f_t(x(t))$ for a.e. $t\in T$. This implies that

$$\langle x^*(t),x(t)\rangle\le f_t(x(t))+f_t^*(x^*(t))\le\langle x^*(t),x(t)\rangle+\varepsilon(t),\quad\text{for a.e. } t\in T,$$

and the functions $f(\cdot,x(\cdot))$ and $f^*(\cdot,x^*(\cdot))$ are integrable. Hence, by integrating,

$$\textstyle\int_T f_t^*(x^*(t))d\mu\le-\tilde{I}_f(x(t))+\int_T\langle x^*(t),x(t)\rangle\,d\mu+\int_T\varepsilon(t)d\mu<+\infty,$$

and $x^*(\cdot)\in\operatorname{dom}\tilde{I}_{f^*}$. Therefore, according to Proposition 2.3, we have $(\tilde{I}_{f^*})^*=\tilde{I}_f$ and $\tilde{I}_{f^*}=(\tilde{I}_f)^*$, that is, $\tilde{I}_{f^*}$ is proper.

(22) $\Rightarrow$ (v). Assume that (22) holds with $\gamma\in L^1(T,X^*)$ and $\beta\in L^1(T)$. Then there exists some $U\in\mathcal{N}_X$ such that $\int_T\sigma_U(\gamma(t))d\mu<+\infty$. Moreover, using the Cauchy-Schwartz inequality, for a.e. $t\in T$ we have

$$f_t(x)\ge\langle\gamma(t),x\rangle+\beta(t)\ge-\sigma_{U^\circ}(x)\sigma_U(\gamma(t))+\beta(t),\quad\text{for all } x\in X,$$

and (v) follows by taking $r(\cdot):=\sigma_U(\gamma(\cdot))$ $(\in L^1(T,\mathbb{R}_+))$ and $\alpha(\cdot):=\beta(\cdot)$.

(v) $\Rightarrow$ (22). By (v) there exists $U\in\mathcal{N}_X$ such that $f_t(x)+r(t)\sigma_{U^\circ}(x)\ge\alpha(t)$ for a.e. $t\in T$ and for all $x\in X$. Observe that the convex function σ_{U° is continuous on X, as U° is w^*-compact by the Alaoglu-Banach-Bourabaki theorem. Hence, since $x_0(t)\in\operatorname{dom}f_t$ for a.e. $t\in T$, by Lemma 2.2(i), for a.e. $t\in T$ the Moreau-Rockafellar Theorem implies that

$$(f_t+r(t)\sigma_{U^\circ})^*(0)=\left(f_t^*\square\mathrm{i}_{r(t)U^\circ}\right)(0)\le-\alpha(t),$$

where the infimal convolution is exact. Hence, there is some $u^*\in-r(t)U^\circ=r(t)U^\circ$ such that $f_t^*(u^*)\le-\alpha(t)$, and the measurable multifunction $F:T\rightrightarrows X^*$ defined by

$$F(t):=\{u^*\in r(t)U^\circ: f_t^*(u^*)\le-\alpha(t)\},$$

has nonempty closed convex values (a.e. $t\in T$). Consequently, the measurable selection theorem ([31]) gives rise to a measurable selection $\gamma(\cdot)$ of F, that is, for a.e. $t\in T$ we have $f_t^*(\gamma(t))\le-\alpha(t)$ and therefore

$$f_t(x)\ge\langle\gamma(t),x\rangle+\alpha(t)\quad\text{for all } x\in X.$$

Moreover, because $\gamma(t)\in r(t)U^\circ$ we infer that $\int_T\sigma_U(\gamma(t))d\mu\le\int_T r(t)d\mu<+\infty$, and $\gamma\in L^1(T,X^*)$.

Finally, the implication (22) $\Rightarrow$ (iv) is straightforward. □

The following result is the counterpart of Proposition 3.1 for weak integrals. Note that the duality arguments used before are not directly applicable here, because the pair $(L^\infty(T,X),\ L_w^1(T,X^*))$ does not need to form a duality pair. It is worth observing that, generally, we cannot take strong integrals instead of weak integrals in the following result (see Example 4.6).

Corollary 3.2. *Let f be a convex normal integrand such that $I_f(x_0) < +\infty$, for some $x_0 \in X$. Then the following assertions are equivalent:*

(i) *Condition* (21) *holds with some $\gamma \in L^1_w(T, X^*)$ and $\beta \in L^1(T)$.*

(ii) *For any $\varepsilon > 0$ and $x \in \operatorname{dom} I_f$, we have*

$$\bigcup_{\varepsilon(\cdot)\in\mathcal{A}^+_\varepsilon} \mathtt{w}\text{-}\int_T \partial_{\varepsilon(t)} f_t(x) d\mu \neq \emptyset.$$

Proof. It is clear that (ii) implies (i). For the opposite implication, we assume (i) and introduce the convex normal integrand $\tilde{f} : T \times X \to \mathbb{R}_\infty$ given by

$$\tilde{f}(t, x) := f(t, x) - \langle \gamma(t), x\rangle - \beta(t); \tag{29}$$

hence, for a.e. $t \in T$, we have $\tilde{f}(t, x) \geq 0$ for all $x \in X$.

Obviously, the convex normal integrand $\tilde{f}$ satisfies (22), and we obtain

$$\tilde{I}_{\tilde{f}}(x_0 1_T(\cdot)) = \int_T f(t, x_0) d\mu - \int_T (\langle \gamma(t), x_0\rangle + \beta(t)) d\mu < +\infty.$$

Then, for every $x \in \operatorname{dom} I_f$ and $\varepsilon > 0$, we have $x(\cdot) \equiv x1_T(\cdot) \in \operatorname{dom} \tilde{I}_{\tilde{f}}$, and Proposition 3.1(iii) yields some $\varepsilon(\cdot) \in \mathcal{A}^+_\varepsilon$ such that

$$\emptyset \neq \int_T \partial_{\varepsilon(t)} \tilde{f}_t(x) d\mu.$$

Take $x^*(\cdot) \in L^1(T, X^*)$ such that $x^*(t) \in \partial_{\varepsilon(t)} \tilde{f}(t, x)$, for a.e. $t \in T$. Then

$$x^*(t) \in \partial_{\varepsilon(t)} \tilde{f}_t(x) = \partial_{\varepsilon(t)} f_t(x) - \gamma(t), \text{ a.e. } t \in T,$$

and $\tilde{x}^*(\cdot) := x^*(\cdot) + \gamma(\cdot) \in L^1_w(T, X^*)$ (because $L^1(T, X^*) \subset L^1_w(T, X^*)$) is a selection of the multifunction $t \rightrightarrows \partial_{\varepsilon(t)} f_t(x)$. □

4. Characterization of the ε-subdifferential of I_f

As in the previous section, X and its dual X^* are two locally convex Suslin spaces endowed with compatible topologies with respect to a given pairing $\langle \cdot, \cdot\rangle$ defined on $X^* \times X$. We consider the measure space $(T, \mathcal{E}, \mu)$ where the measure μ is σ-finite, nonnegative and complete. In this section, we characterize the ε-subdifferential of the convex integral function $I_f : X \to \mathbb{R}_\infty$, defined by

$$I_f(x) = \int_T f(t, x) d\mu,$$

where $f : X \times T \to \mathbb{R}_\infty$ is a convex normal integrand.

We begin by the following example to justify the need of using approximate subdifferentials $\partial_{\varepsilon(t)} f_t$ with t-varying approximations $\varepsilon(\cdot)$.

Example 4.1. Given $T := [0, 1]$ with the Lebesgue measure, we consider the integral function $I_f : \mathbb{R} \to \mathbb{R}_\infty$ defined by $I_f(x) := \int_0^1 \frac{x^2}{t} dt$, that is, I_f is the indicator function of the singleton $\{0\}$, so that $\partial I_f(0) = \partial \mathrm{i}_{\{0\}}(0) = \mathbb{R}$.

We verify, on the one hand, that for every $\varepsilon \geq 0$

$$\bigcap_{\delta>\varepsilon} \operatorname{cl} \left(\int_0^1 \partial_\delta f_t(0) dt \right) = [-4\sqrt{\varepsilon}, 4\sqrt{\varepsilon}]. \tag{30}$$

Indeed, given $t \in (0,1)$ and $\delta > \varepsilon$, we have that $\alpha \in \partial_\delta f_t(0)$ if and only if

$$\frac{-2\sqrt{\delta}}{\sqrt{t}} = \sup_{x<0} \frac{x^2+t\delta}{tx} \leq \alpha \leq \inf_{x>0} \frac{x^2+t\delta}{tx} = \frac{2\sqrt{\delta}}{\sqrt{t}}.$$

So,
$$\int_0^1 \partial_\delta f_t(0)dt = \left\{ \int_0^1 \alpha(t)dt : \frac{-2\sqrt{\delta}}{\sqrt{t}} \leq \alpha(t) \leq \frac{2\sqrt{\delta}}{\sqrt{t}},\ a.e.\ t \in [0,1] \right\}$$
$$= [-4\sqrt{\delta}, 4\sqrt{\delta}], \tag{31}$$

and we get $\bigcap_{\varepsilon>0} \mathrm{cl}\left(\int_0^1 \partial_\varepsilon f_t(0)dt\right) = \{0\} \subsetneq \partial I_f(0)$.

On the other hand, given $0 < \delta < \varepsilon < 1$, we consider the function $\delta(\cdot)$ defined on $[0,1]$ by

$$\delta(t) := \frac{\delta^2}{4t},\ \text{if}\ t \in [t_\delta, 1],\ \text{and}\ \delta(t) := \frac{\delta^2 t}{4},\ \text{otherwise},$$

where $t_\delta := e^{1-\frac{1}{\delta}}$ $(\in]0,1[)$. Then we get

$$\int_0^1 \delta(t)dt = \int_0^{t_\delta} (\delta^2 t/4)dt + \int_{t_\delta}^1 (\delta^2/(4t))dt$$
$$= \delta^2 t_\delta^2/8 + (\delta^2/4)(-1+1/\delta) \leq \delta^2/8 + \delta/4 \leq \delta \leq \varepsilon,$$

so that $\delta(\cdot) \in \mathcal{A}_\varepsilon$, and (31) entails

$$\int_0^1 \partial_{\delta(t)} f_t(0)dt = \left\{ \int_0^1 \alpha(t)dt : \begin{array}{r} -\delta \leq \alpha(t) \leq \delta,\ \text{a.e.}\ t \in [0,t_\delta] \\ \frac{-\delta}{t} \leq \alpha(t) \leq \frac{\delta}{t},\ \text{a.e.}\ t \in [t_\delta, 1] \end{array} \right\}$$
$$= [-\delta t_\delta - 1 - 1/\delta, \delta t_\delta + 1 + 1/\delta].$$

Consequently, since $\delta t_\delta + 1 + 1/\delta \to +\infty$ as $\delta \to 0$, we deduce

$$\bigcup_{0<\delta<\varepsilon} \int_0^1 \partial_{\delta(t)} f_t(0)dt = \bigcup_{0<\delta<\varepsilon} [-\delta t_\delta - 1 - 1/\delta, \delta t_\delta + 1 + 1/\delta] = \mathbb{R},$$

which gives us
$$\bigcap_{\varepsilon>0} \mathrm{cl}\left(\bigcup_{\varepsilon(\cdot)\in\mathcal{A}_\varepsilon} \int_0^1 \partial_{\varepsilon(t)} f_t(0)dt \right) = \mathbb{R} = \partial I_f(0).$$

We give now the characterization of the subdifferential set $\partial_\varepsilon I_f(x)$ $(\varepsilon \geq 0)$. The first two expressions below do not require any qualification condition on the data functions $f_t := f(t,\cdot)$, $t \in T$. These expressions are simplified in (i)–(ii) under additional continuity conditions. The formula in statement (ii) is a known result in separable and reflexive Banach spaces (see [13] and references therein). The counterpart of formula (35) for the ε-subdifferential is given in Corollary 4.8.

Theorem 4.2. *Let f be a convex normal integrand satisfying condition* (22)*. Then, for every $x \in X$, we have*

$$\partial_\varepsilon I_f(x) = \mathrm{cl}\left(\bigcup_{\varepsilon(\cdot)\in\mathcal{A}_\varepsilon} \int_T \partial_{\varepsilon(t)} f_t(x)d\mu \right),\ \textit{for all}\ \varepsilon > 0, \tag{32}$$

and consequently,
$$\partial I_f(x) = \bigcap_{\varepsilon>0} \mathrm{cl}\left(\bigcup_{\varepsilon(\cdot)\in\mathcal{A}_\varepsilon} \int_T \partial_{\varepsilon(t)} f_t(x)d\mu \right). \tag{33}$$

Moreover, the following assertions hold:

(i) *If $\tilde{I}_f$ is continuous with respect to the Mackey topology $\tau(L^\infty(T,X),L^1(T,X^*))$ at some $x_0(\cdot)\equiv x_0\in \operatorname{dom} I_f$, then*

$$\partial_\varepsilon I_f(x)=\textstyle\bigcup_{\varepsilon(\cdot)\in\mathcal{A}_\varepsilon}\int_T\partial_{\varepsilon(t)}f_t(x)d\mu,\ \text{for all } \varepsilon\geq 0, \tag{34}$$

(ii) *If X is a (separable) Banach space and $\tilde{I}_f$ is continuous with respect to the norm topology on $L^\infty(T,X)$ at some $x_0(\cdot)\equiv x_0\in\operatorname{dom} I_f$, then*

$$\partial I_f(x)=\textstyle\int_T\partial f_t(x)d\mu+\mathrm{N}_{\operatorname{dom} I_f}(x). \tag{35}$$

Proof. To prove formula (32), it is enough to establish the inclusion "$\subset$" when $\partial_\varepsilon I_f(x)\neq\emptyset$; that is, $\tilde{I}_f(x)=I_f(x)\in\mathbb{R}$. Thus, the mapping $\tilde{I}_f:L^\infty(T,X)\to\mathbb{R}_\infty$ is proper, convex and $\sigma(L^\infty(T,X),L^1(T,X^*))$-lsc, due to Proposition 3.1. We also have that $I_f=\tilde{I}_f\circ A$, where $A:X\to L^\infty(T,X)$ is the linear mapping given by

$$Au:=u. \tag{36}$$

Notice that A is τ_X-$\sigma(L^\infty(T,X),L^1(T,X^*))$-continuous, where τ_X is the initial topology on X, with the adjoint being the mapping $A^*:L^1(T,X^*)\to X^*$ defined by

$$A^*(u^*)=\textstyle\int_T u^*(t)d\mu. \tag{37}$$

Next, taking into account (13), Proposition 3.1 gives rise to

$$\begin{aligned}\partial_\varepsilon I_f(x)&=\operatorname{cl}\left(A^*\partial_\varepsilon\tilde{I}_f(Ax)\right) &(38)\\ &=\operatorname{cl}\left(\textstyle\bigcup_{\varepsilon(\cdot)\in\mathcal{A}_\varepsilon}\int_T x^*(t)d\mu : x^*\in L^1(T,X^*),\ x^*(t)\in\partial_{\varepsilon(t)}f_t(x)\text{ a.e. } t\in T\right)\\ &=\operatorname{cl}\left(\textstyle\bigcup_{\varepsilon(\cdot)\in\mathcal{A}_\varepsilon}\int_T\partial_{\varepsilon(t)}f_t(x)d\mu\right), &(39)\end{aligned}$$

and the desired inclusion "$\subset$" in (32) follows. Formula (33) comes easily from (32) thanks to the relation $\partial I_f(x)=\cap_{\varepsilon>0}\partial_\varepsilon I_f(x)$.

(i) To prove this statement, we first check that the operator $A:X\to L^\infty(T,X)$ defined in (36) is τ_X-$\tau(L^\infty(T,X),L^1(T,X^*))$-continuous. Consider a net $(z_i)_i\subset X$ which τ_X-converges to $z\in X$ and a (symmetric) $\sigma(L^1(T,X^*),L^\infty(T,X))$-compact set $K\subset L^1(T,X^*)$, then there exists a weak symmetric convex neighborhood U of 0 in $L^\infty(T,X)$ such that $K\subset U^\circ$. Since A is τ_X-$\sigma(L^\infty(T,X),L^1(T,X^*))$-continuous, as we have commented before, we find some i_0 such that for all i posterior to i_0 we have $z_i 1_T - z1_T = A(z_i-z)\in U$ and, so,

$$0\leq\sup_{z^*(\cdot)\in K}\langle z^*(\cdot),z_i1_T(\cdot)-z1_T(\cdot)\rangle\leq\sup_{z^*(\cdot)\in U^\circ,u(\cdot)\in U}\langle z^*(\cdot),u(\cdot)\rangle\leq 1.$$

Hence, $z_i1_T(\cdot)-z1_T(\cdot)\in K^\circ$ eventually for i, and A is τ_X-$\tau(L^\infty(T,X),L^1(T,X^*))$-continuous. Therefore $\partial_\varepsilon I_f(x)=A^*\partial_\varepsilon\tilde{I}_f(Ax)$ (by (14)) and, like in (39) above, assertion (i) follows again by Proposition 3.1.

(ii) We consider here the pair $(L^\infty(T,X),(L^\infty(T,X))^*)$, where $L^\infty(T,X)$ is endowed with its norm topology and $(L^\infty(T,X))^*$ is the corresponding topological dual space;

that is, $(L^\infty(T,X))^* = L^1(T,X^*) \oplus \Lambda(T,X^*)$ due to the Lebesgue decomposition theorem. We also consider the $\|\cdot\|_X$-$\tau(L^\infty(T,X), L^1(T,X^*))$-continuous linear mapping $A_1 : X \to L^\infty(T,X)$ given by

$$A_1 u := u, \tag{40}$$

whose adjoint mapping $A_1^* : (L^\infty(T,X))^* \to X^*$ satisfies

$$A_1^*(u^*) = \textstyle\int_T u^*(t) d\mu, \text{ for all } u^* \in L^1(T,X^*). \tag{41}$$

Then, using the relation $I_f = \tilde{I}_f \circ A$ together with (14), we get

$$\partial I_f(x) = A_1^* \partial \tilde{I}_f(A_1 x). \tag{42}$$

Moreover, on account of (17), for each given $x^*(\cdot) \in \partial \tilde{I}_f(A_1 x)$ $(\subset (L^\infty(T,X))^*)$ there are some elements $x_1^*(\cdot) \in L^1(T,X^*)$ and $y^* \in \Lambda(T,X^*)$ such that

$$x^*(\cdot) = x_1^*(\cdot) + y^*, \tag{43}$$

and so, for all $y(\cdot) \in L^\infty(T,X)$,

$$\langle x_1^*(\cdot) + y^*, y(\cdot) - A_1 x \rangle \le \tilde{I}_f(y(\cdot)) - \tilde{I}_f(A_1 x). \tag{44}$$

We choose a nondecreasing sequence of positive measurable sets $(T_n)_n \subset T$ such that the set $T \setminus (\cup_n T_n)$ is negligible and

$$\langle y^*, z(\cdot) 1_{T_n} \rangle = 0, \quad \text{for all} \quad z(\cdot) \in L^\infty(T,X) \text{ and all } n \ge 1.$$

Then, on the one hand, for any given $y(\cdot) \in \operatorname{dom} \tilde{I}_f \subset L^\infty(T,X)$ and $n \ge 1$, relation (44) yields

$$\begin{aligned}\textstyle\int_{T_n} \langle x_1^*(t), y(t) - x \rangle \, d\mu &= \langle x_1^*(\cdot), (y(\cdot) - A_1 x) 1_{T_n}(\cdot) \rangle \\ &= \langle x_1^*(\cdot) + y^*, (y(\cdot) - A_1 x) 1_{T_n}(\cdot) \rangle \\ &\le \tilde{I}_f((y(\cdot) - A_1 x) 1_{T_n}(\cdot) + A_1 x) - \tilde{I}_f(A_1 x) \\ &= \textstyle\int_{T_n} f_t(y(t)) d\mu - \int_{T_n} f_t(x) d\mu.\end{aligned}$$

But we have that $\tilde{I}_f(y(\cdot)) \in \mathbb{R}$, $\tilde{I}_f(A_1 x) = I_f(x) \in \mathbb{R}$ and $\langle x_1^*(\cdot), y(\cdot) - A_1 x \rangle \in L^1(T)$, and the last inequality yields, as $n \to +\infty$,

$$\langle x_1^*(\cdot), y(\cdot) - A_1 x \rangle = \textstyle\int_T \langle x_1^*(t), y(t) - x \rangle \, d\mu \le \tilde{I}_f(y(\cdot)) - \tilde{I}_f(A_1 x), \tag{45}$$

showing that $x_1^*(\cdot) \in \partial \tilde{I}_f(A_1 x) \cap L^1(T,X^*)$. Thus, according to Proposition 3.1(i), we have that $x_1^*(t) \in \partial f_t(x)$ for a.e. $t \in T$, and we deduce

$$A_1^* x_1^*(\cdot) \in \textstyle\int_T \partial f_t(x) d\mu. \tag{46}$$

On the other hand, for any y in $\operatorname{dom} I_f$, we have that $A_1 y \in \operatorname{dom} \tilde{I}_f \subset L^\infty(T,X)$ and (44) implies

$$\begin{aligned}&\langle y^*, A_1(y - x) \rangle = \left\langle y^*, (y - x) 1_{T \setminus T_n} \right\rangle \\ &\le -\left\langle x_1^*(\cdot), (y - x) 1_{T \setminus T_n} \right\rangle + \tilde{I}_f((y - x) 1_{T \setminus T_n} + A_1 x) - \tilde{I}_f(A_1 x).\end{aligned} \tag{47}$$

Observe that, for a.e. $t \in \cup_n T_n$,

$$\left\langle x_1^*(t), (y-x)1_{T\setminus T_n}(t)\right\rangle \to 0 \text{ and } f_t((y-x)1_{T\setminus T_n}(t) + A_1 x) \to f_t(x).$$

and so, since $\left\langle x_1^*(\cdot), (y-x)1_{T\setminus T_n}(\cdot)\right\rangle \le |\langle x_1^*(\cdot), A_1(y-x)\rangle|$, by the dominance convergence theorem

$$\lim_n \left\langle x_1^*(\cdot), (y-x)1_{T\setminus T_n}\right\rangle = 0.$$

Also, taking into account condition (22), the Fatou lemma assures that

$$\limsup_{n\to\infty} \tilde{I}_f((y-x)1_{T\setminus T_n} + A_1 x) \le \tilde{I}_f(x) = I_f(x).$$

Therefore, taking limits in (47), for all $y \in \operatorname{dom} I_f$

$$\langle A_1^* y^*, y-x\rangle = \langle y^*, A_1 y - A_1 x\rangle \le 0,$$

and we infer that $A_1^* y^* \in \mathrm{N}_{\operatorname{dom} I_f}(x)$. Finally, combining this with (42), (43), and (46), we derive that

$$\partial I_f(x) = A_1^* \partial \tilde{I}_f(A_1 x) \subset \textstyle\int_T \partial f_t(x) d\mu + \mathrm{N}_{\operatorname{dom} I_f}(x),$$

and the proof is finished as the opposite of this inclusion holds straightforwardly. □

Remark 4.3. When $X = \mathbb{R}^n$, the continuity assumption used in Theorem 4.2(ii) is equivalent to the nonemptiness of $\operatorname{int}(\operatorname{dom} I_f)$; formula (35) is known in this case (see [14, Theorem 4] and [18, Corollary 5.1]). Other consequences of the continuity of I_f are discussed in Lemma 2.2. Formula (35) was also obtained in [16] for continuous convex integral functions defined on separable Banach spaces.

We give next the counterpart of Theorem 4.2 for weak integrals.

Corollary 4.4. *Let f be a convex normal integrand satisfying* (21)*. Then, for every $x \in X$, we have*

$$\partial_\varepsilon I_f(x) = \operatorname{cl}\left(\textstyle\bigcup_{\varepsilon(\cdot)\in\mathcal{A}_\varepsilon} \mathtt{w}\text{-}\int_T \partial_{\varepsilon(t)} f_t(x) d\mu\right), \textit{ for all } \varepsilon > 0,$$

and, consequently, $\quad \partial I_f(x) = \bigcap_{\delta>0} \operatorname{cl}\left(\bigcup_{\delta(\cdot)\in\mathcal{A}_\delta} \mathtt{w}\text{-}\int_T \partial_{\delta(t)} f_t(x) d\mu\right).$

Moreover, the assertions (i) *and* (ii) *in Theorem* 4.2 *are also valid when the weak integrals are used instead of (strong) ones.*

Proof. Again we use the convex normal integrand $\tilde{f} : T \times X \to \mathbb{R}_\infty$ defined in (29), that is,

$$\tilde{f}(t,u) := f(t,u) - \langle \gamma(t), u\rangle - \beta(t), \tag{48}$$

where $\gamma(\cdot)$ in $L_w^1(T,X)$ and $\beta(\cdot) \in L^1(T,\mathbb{R}_+)$ are as (21), together with the associated functional $\tilde{I}_{\tilde{f}} : L^\infty(T,X) \to \mathbb{R}_\infty$ defined by

$$\tilde{I}_{\tilde{f}}(u) := \textstyle\int_T \tilde{f}(t,u) d\mu.$$

Hence, $\tilde{f}(t,u) \geq 0$ for a.e. $t \in T$ and all $u \in X$, and Theorem 4.2 applies and yields, for all $x \in \operatorname{dom} I_f$ and $\varepsilon > 0$,

$$\begin{aligned}\partial_\varepsilon I_{\tilde{f}}(x) &= \operatorname{cl}\left(\bigcup_{\varepsilon(\cdot)\in\mathcal{A}_\varepsilon} \int_T \partial_{\varepsilon(t)} \tilde{f}_t(x) d\mu\right) \\ &= \operatorname{cl}\left(\bigcup_{\varepsilon(\cdot)\in\mathcal{A}_\varepsilon} \int_T \left(\partial_{\varepsilon(t)} f_t(x) - \gamma(t)\right) d\mu\right) \\ &= \mathtt{w}\text{-}\int_T \gamma(t) d\mu + \operatorname{cl}\left(\bigcup_{\varepsilon(\cdot)\in\mathcal{A}_\varepsilon} \mathtt{w}\text{-}\int_T \partial_{\varepsilon(t)} f_t(x) d\mu\right).\end{aligned}$$

Consequently, the desired formula of $\partial_\varepsilon I_f(x)$ follows as

$$\partial_\varepsilon I_f(x) = \partial_\varepsilon I_{\tilde{f}}(x) + \mathtt{w}\text{-}\int_T \gamma(t) d\mu = \operatorname{cl}\left(\bigcup_{\varepsilon(\cdot)\in\mathcal{A}_\varepsilon} \mathtt{w}\text{-}\int_T \partial_{\varepsilon(t)} f_t(x) d\mu\right).$$

Next, the formula of $\partial I_f(x)$ comes by intersecting this last set over $\varepsilon > 0$. Finally, the proof of assertions (i) and (ii) in Theorem 4.2 for weak integrals follows the same pattern. □

Remark 4.5. Since the sets $\cup_{\varepsilon(\cdot)\in\mathcal{A}_\varepsilon} \int_T \partial_{\varepsilon(t)} f_t(x) d\mu$ and $\cup_{\varepsilon(\cdot)\in\mathcal{A}_\varepsilon} \mathtt{w} - \int_T \partial_{\varepsilon(t)} f_t(x) d\mu$ are convex, in the case of a reflexive Banach space X the expressions involving closures in Theorem 4.2 and Corollary 4.4 are still valid if such closures are taken with respect to the norm topology of X^*.

The following example draws the attention to the choice of the functions γ involved in (22)–(21), and the influence they have in the determination of the subdifferential of I_f.

Example 4.6. Take $X = l^2$, the space of real sequences $(x_n)_{n\geq 1}$ with the property $\sum_{n\geq 1} x_n^2 < +\infty$, and let $(e_n)_{n\geq 1}$ be its canonical basis. Let the integral function $I_f : l^2 \to \mathbb{R}$ be defined as

$$I_f(x) = \sum_{n\geq 1} \frac{1}{n} \langle e_n, x\rangle = \langle a_0, x\rangle_{l^2}, \text{ where } a_0 := (1/n)_{n\geq 1} \in l^2,$$

that is, I_f is a linear form; hence,

$$\partial_\varepsilon I_f(x) = \{a_0\}, \text{ for all } x \in l^2 \text{ and } \varepsilon \geq 0.$$

The function I_f above corresponds to our model integral function with $T = \mathbb{N}$, $\mathcal{E}$ is the discrete σ-algebra of $\mathbb{N}$, μ is the counting measure, and $f(n,x) := (1/n)\langle e_n, x\rangle$ is the underlying convex normal integrand.

Notice that $\sum_{n\geq 1} |(1/n)\langle e_n, x\rangle| \in \mathbb{R}$, for all $x \in l^2$, and

$$\sum_{n\geq 1} \|(1/n)e_n\| \mu(n) = \sum_{n\geq 1} 1/n = +\infty.$$

Hence, since l^2 is a Hilbert space, the function $\gamma : T \to l^2$ defined as $\gamma(n) := (1/n)e_n$ satisfies

$$\gamma \in L_w^1(T, l^2) \setminus L^1(T, l^2),$$

that is, γ satisfies condition (21) but not (22). Therefore we can compute the ε-subdifferential of I_f at 0 by applying Corollary 4.4: for each $\varepsilon \geq 0$, we have

$$\partial_\varepsilon I_f(0) = \operatorname{cl}\left(\bigcup_{\varepsilon(\cdot)\in\mathcal{A}_\varepsilon} \mathtt{w}\text{-}\textstyle\int_T \partial_{\varepsilon(t)} f_t(x) d\mu\right) = \left\{(1/n)_{n\geq 1}\right\},$$

as expected. At the same time, the (strong) integrals $\int_T \partial_{\varepsilon(t)} f_t(x) d\mu$, $\varepsilon(\cdot) \in \mathcal{A}_\varepsilon$, are all empty, as

$$\textstyle\int_T \partial_{\varepsilon(t)} f_t(x) d\mu = \displaystyle\sum_{n\geq 1} (1/n) e_n, \quad \text{and} \quad \sum_{n\geq 1} \|(1/n) e_n\| = \sum_{n\geq 1} 1/n = +\infty.$$

In other words, $\emptyset = \operatorname{cl}\left(\bigcup_{\varepsilon(\cdot)\in\mathcal{A}_\varepsilon} \int_T \partial_{\varepsilon(t)} f_t(x) d\mu\right) \subsetneqq \partial_\varepsilon I_f(0)$, for all $\varepsilon \geq 0$.

Theorem 4.2 can be rewritten in terms of the continuous infimal convolution of the conjugate functions f_t^*, $t \in T$, which is the function $\oint_T f_t^* d\mu : X^* \to \overline{\mathbb{R}}$ defined by

$$\begin{aligned}\textstyle\oint_T f_t^* d\mu(x^*) &:= \textstyle\oint_T f_t^*(x^*) d\mu \\ &:= \inf\left\{\textstyle\int_T f_t^*(x^*(t)) d\mu : \int_T x^*(t) d\mu = x^*,\ x^*(\cdot) \in L^1(T, X^*)\right\}.\end{aligned}$$

Recall that $\operatorname{cl}^{w^*}\left(\oint_T f_t^* d\mu\right)$ denotes the weak* closure of the function $\oint_T f_t^* d\mu$.

Theorem 4.7. *Let f be a convex normal integrand satisfying (22) and the condition $I_f(x_0) < +\infty$ for some $x_0 \in X$. Then we have*

$$(I_f)^* = \operatorname{cl}^{w^*}\left(\textstyle\oint_T f_t^* d\mu\right). \tag{49}$$

Moreover, the following assertions hold:

(i) *With the assumptions of Theorem 4.2(i), for each $x^* \in X^*$ there exists some $x^*(\cdot) \in L^1(T, X^*)$ such that $\int_T x^*(t) d\mu = x^*$ and*

$$(I_f)^*(x^*) = \textstyle\oint_T f_t^*(x^*) d\mu = \int_T f_t^*(x^*(t)) d\mu. \tag{50}$$

(ii) *With the assumptions of Theorem 4.2(ii), for each $z^* \in X^*$ there exist some $x^*(\cdot) \in L^1(T, X^*)$ and $y^* \in X^*$ such that $\int_T x^*(t) d\mu + y^* = z^*$ and*

$$(I_f)^*(z^*) = \left(\left(\textstyle\oint_T f_t^* d\mu\right) \square \sigma_{\operatorname{dom} I_f}\right)(z^*) = \textstyle\int_T f_t^*(x^*(t)) d\mu + \sigma_{\operatorname{dom} I_f}(y^*). \tag{51}$$

Proof. We proceed as in the proof of Theorem 4.2 and write $I_f = \tilde{I}_f \circ A$, where $\tilde{I}_f \colon L^\infty(T, X) \to \mathbb{R}_\infty$, and $A \colon X \to L^\infty(T, X)$ is the τ_X-$\sigma(L^\infty(T, X), L^1(T, X^*))$-continuous linear mapping defined in (36), and whose adjoint is computed in (37). Then, since $\tilde{I}_f$ is proper, convex and $\sigma(L^\infty(T, X), L^1(T, X^*))$-lsc, due to Proposition 3.1, (49) and (50) follow by applying (13) and (14), respectively.

We are going to prove (51). Assume first that $z^* \notin \operatorname{dom}(I_f)^*$. Since $I_f \in \Gamma_0(X)$, and $(I_f)^* = (I_f)^* \square \sigma_{\operatorname{dom} I_f}$ by (12), (49) yields

$$\begin{aligned} +\infty = (I_f)^*(z^*) &= \left((I_f)^* \square \sigma_{\operatorname{dom} I_f}\right)(z^*) \\ &= \left(\operatorname{cl}\left(\textstyle\oint_T f_t^* d\mu\right) \square \sigma_{\operatorname{dom} I_f}\right)(z^*) \\ &\leq \left(\left(\textstyle\oint_T f_t^* d\mu\right) \square \sigma_{\operatorname{dom} I_f}\right)(z^*) \leq \left(\textstyle\oint_T f_t^* d\mu\right)(z^*). \end{aligned}$$

Hence, (51) obviously holds with $x^*(\cdot) \equiv \eta_0(\cdot)z^* \in L^1(T, X^*)$ and $y^* = 0$, where $\eta_0(\cdot) \in L^1(T)$ is such that $\int_T \eta_0(t)d\mu = 1$ (see (15)).

Now we take $z^* \in \mathrm{dom}(I_f)^*$; hence, $(I_f)^*(z^*) \in \mathbb{R}$ because we have $I_f \in \Gamma_0(X)$ and $(I_f)^* \in \Gamma_0(X^*)$. Given $\varepsilon > 0$, by [33, Theorem 3.1.4] we choose elements $z_\varepsilon \in X$ and $z_\varepsilon^* \in \partial I_f(z_\varepsilon)$ such that

$$|(I_f)^*(z_\varepsilon^*) - (I_f)^*(z^*)| \leq \varepsilon \quad \text{and} \quad \|z_\varepsilon^* - z^*\| \leq \varepsilon. \tag{52}$$

Thus, according to Theorems 3.1 and 4.2(ii), there are

$$x_\varepsilon^*(\cdot) \in \partial \tilde{I}_f(z_\varepsilon 1_T) \cap L^1(T, X^*) \quad \text{and} \quad y_\varepsilon^* \in \mathrm{N}_{\mathrm{dom}\, I_f}(z_\varepsilon) \tag{53}$$

such that
$$x_\varepsilon^*(t) \in \partial f_t(z_\varepsilon), \text{ for a.e. } t \in T, \tag{54}$$

and
$$z_\varepsilon^* = \int_T x_\varepsilon^*(t)d\mu + y_\varepsilon^*. \tag{55}$$

At the same time, Theorem 4.2 ensures that

$$x_\varepsilon^* := \int_T x_\varepsilon^*(t)d\mu = A_1^* x_\varepsilon^*(\cdot) \in \partial I_f(z_\varepsilon),$$

where $A_1 : X \to L^\infty(T, X)$ is the $\|\cdot\|_X$-$\tau(L^\infty(T, X), L^1(T, X^*))$-continuous linear mapping defined in (40) and (41). Moreover, since we have

$$(\partial I_f)^{-1}(x_\varepsilon^*) = \partial (I_f)^*(x_\varepsilon^*) \quad \text{and} \quad \partial \sigma_{\mathrm{dom}\, I_f}(y_\varepsilon^*) = (\partial \mathrm{I}_{\mathrm{cl}(\mathrm{dom}\, I_f)})^{-1}(y_\varepsilon^*)$$

by (11), we obtain that $z_\varepsilon \in \partial \sigma_{\mathrm{dom}\, I_f}(y_\varepsilon^*) \cap \partial (I_f)^*(x_\varepsilon^*)$. So, (55), (8) and (12) imply

$$(I_f)^*(z_\varepsilon^*) = ((I_f)^* \square \sigma_{\mathrm{dom}\, I_f})(z_\varepsilon^*) = (I_f)^*(x_\varepsilon^*) + \sigma_{\mathrm{dom}\, I_f}(y_\varepsilon^*). \tag{56}$$

The rest of the proof is divided into four steps.

Step 1. We prove in this step that, for each given $\varepsilon > 0$,

$$(I_f)^*(z_\varepsilon^*) = \int_T f_t^* \left(x_\varepsilon^*(t)\right) d\mu + \sigma_{\mathrm{dom}\, I_f}(y_\varepsilon^*). \tag{57}$$

The inequality "$\leq$" in (57) comes from (56) and the following relation, which is a consequence of statement (i) of the current theorem,

$$(I_f)^*(x_\varepsilon^*) = \mathrm{cl}\left(\oint_T f_t^* d\mu\right)(x_\varepsilon^*) \leq \int_T f_t^* \left(x_\varepsilon^*(t)\right) d\mu.$$

Also, taking into account that $z_\varepsilon \in \mathrm{dom}\, I_f$, by using (9) and (54) we obtain that

$$\int_T f_t^* \left(x_\varepsilon^*(t)\right) d\mu = \int_T (\langle x_\varepsilon^*(t), z_\varepsilon \rangle - f_t\left(z_\varepsilon\right))d\mu = \langle x_\varepsilon^*, z_\varepsilon \rangle - I_f\left(z_\varepsilon\right) \leq (I_f)^*(x_\varepsilon^*),$$

which leads us, again by (56), to the inequality "$\geq$" in (57).

Step 2. We show here that the net $(x_\varepsilon^*(\cdot))_\varepsilon \subset (L^\infty(T, X))^*$ has a cluster point $u^*(\cdot) + \lambda^*$, with $u^*(\cdot) \in L^1(T, X^*)$ and $\lambda^* \in \Lambda(T, X^*)$, satisfying

$$\langle \lambda, y 1_{T_n} \rangle = 0, \text{ for all } y \in L^\infty(T, X) \text{ and } n \geq 1, \tag{58}$$

for some nondecreasing sequence of positive measure sets $(T_n)_n \subset T$ such that $T \setminus (\cup_n T_n)$ is negligible.

Consequently, the continuity properties of the operator A_1 will imply that the corresponding net $(x_\varepsilon^*)_\varepsilon$ is $\sigma(X^*,X)$-convergent to $A_1^*u^*(\cdot)+A_1^*\lambda^*$, while (52) and (55) will ensure that the corresponding net $(y_\varepsilon^*)_\varepsilon$ is $\sigma(X^*,X)$-convergent to some $y^*\in X^*$ such that

$$z^* = A_1^*u^*(\cdot)+A_1^*\lambda^*+y^*. \tag{59}$$

To show the existence of such $u^*(\cdot)$ and λ^*, we use the current continuity assumption to find some $m,\rho>0$ such that for all $y(\cdot)\in\rho B_{L^\infty(T,X)}$ we have $\tilde{I}_f(x_0+y(\cdot))\le m$, that is,

$$\begin{aligned}
\langle x_\varepsilon^*(\cdot),y(\cdot)\rangle &\le \langle x_\varepsilon^*(\cdot),(z_\varepsilon-x_0)1_T\rangle+\tilde{I}_f(x_01_T+y(\cdot))-I_f(z_\varepsilon)+\varepsilon && \text{(by (53))}\\
&\le \langle z_\varepsilon^*-y_\varepsilon^*,z_\varepsilon-x_0\rangle+m-I_f(z_\varepsilon)+\varepsilon && \text{(by (55))}\\
&\le \langle z_\varepsilon^*,z_\varepsilon-x_0\rangle-I_f(z_\varepsilon)+m+\varepsilon && \text{(by (53))}\\
&\le (I_f)^*(z_\varepsilon^*)-\langle z_\varepsilon^*,x_0\rangle+m+\varepsilon,
\end{aligned}$$

and (52) yields some $r\ge 0$ such that

$$\langle x_\varepsilon^*(\cdot),y(\cdot)\rangle\le (I_f)^*(z^*)-\langle z^*,x_0\rangle+m+2\varepsilon+\varepsilon\,\|x_0\|\le r. \tag{60}$$

Therefore, using (17) and the weak*-compactness of $B_{(L^\infty(T,X))^*}$, we may suppose that the net $(x_\varepsilon^*(\cdot))_\varepsilon$ is $\sigma((L^\infty(T,X))^*,L^\infty(T,X^*))$-convergent when $\varepsilon\downarrow 0$ to some $u^*(\cdot)+\lambda^*$, with $u^*(\cdot)\in L^1(T,X^*)$ and $\lambda^*\in\Lambda(T,X^*)$. The existence of the sequence $(T_n)_n$ as required in (58) comes from the definition of the space $\Lambda(T,X^*)$.

Step 3. We prove in this step statement (ii) when all the f_t's are nonnegative. Given $n\ge 1$, we denote by $I_f^n, I_f^{-n}:X\to\mathbb{R}_\infty$ the convex integral functions given by

$$I_f^n(z):=\int_{T_n} f_t(z)\,d\mu \text{ and } I_f^{-n}(z):=\int_{T\setminus T_n} f_t(z)\,d\mu,$$

so that the sequences $(I_f^n)_n$ and $(I_f^{-n})_n$ are nondecreasing and nonincreasing, respectively. Moreover, invoking the main statement of the current theorem and the fact that $I_f\ge I_f^n$, for all $n\ge 1$ and $\varepsilon>0$ we have that

$$\begin{aligned}
(I_f)^*\left(\int_{T_n} x_\varepsilon^*(t)d\mu\right)) &\le (I_f^n)^*\left(\int_{T_n} x_\varepsilon^*(t)d\mu\right)\\
&= \operatorname{cl}\left(\oint_{T_n} f_t^*d\mu\right)\left(\int_{T_n} x_\varepsilon^*(t)d\mu\right)\le\int_{T_n} f_t^*(x_\varepsilon^*(t))d\mu,
\end{aligned} \tag{61}$$

and, similarly,

$$(I_f^{-n})^*\left(\int_{T\setminus T_n} x_\varepsilon^*(t)d\mu\right)\le\int_{T\setminus T_n} f_t^*(x_\varepsilon^*(t))d\mu. \tag{62}$$

Notice that

$$w^*\text{-}\lim_{\varepsilon\to 0}\int_{T_n} x_\varepsilon^*(t)d\mu=\int_{T_n} u^*(t)d\mu, \tag{63}$$

because of the convergence results of Step 2 we have, for all $z\in X$,

$$\begin{aligned}
\left\langle \int_{T_n} x_\varepsilon^*(t)d\mu, z\right\rangle &= \langle x_\varepsilon^*(\cdot),z1_{T_n}(\cdot)\rangle\\
&\longrightarrow_{\varepsilon\to 0}\langle u^*(\cdot)+\lambda^*,z1_{T_n}(\cdot)\rangle\\
&= \langle u^*(\cdot),z1_{T_n}(\cdot)\rangle=\left\langle\int_{T_n} u^*(t)d\mu,z\right\rangle.
\end{aligned}$$

The same arguments show that

$$w^*\text{-}\lim_{\varepsilon\to 0} \int_{T\setminus T_n} x_\varepsilon^*(t)d\mu = \int_{T\setminus T_n} u^*(t)d\mu + A_1^*\lambda^*, \tag{64}$$

$$w^*\text{-}\lim_{n\to+\infty} \int_{T_n} u^*(t)d\mu = \int_T u^*(t)d\mu =: u^*, \text{ and } w^*\text{-}\lim_{n\to+\infty} \int_{T\setminus T_n} u^*(t)d\mu = 0. \tag{65}$$

Therefore, by making $\varepsilon \downarrow 0$ in (61), and using (63) and the weak*-lower semicontinuity of $(I_f)^*$, we obtain for all $n \geq 1$

$$\liminf_{\varepsilon\downarrow 0} \int_{T_n} f_t^*\left(x_\varepsilon^*(t)\right)d\mu) \geq \liminf_{\varepsilon\downarrow 0} (I_f)^* \left(\int_{T_n} x_\varepsilon^*(t)d\mu\right) \geq (I_f)^* \left(\int_{T_n} u^*(t)d\mu\right). \tag{66}$$

Similarly, for all $n \geq m \geq 1$ we have that $I_f^{-n} \leq I_f^{-m}$, so that $(I_f^{-n})^* \geq (I_f^{-m})^*$ and (62) together with (64) entail that

$$\begin{aligned} &\liminf_{\varepsilon\downarrow 0} \int_{T\setminus T_n} f_t^*\left(x_\varepsilon^*(t)\right)d\mu \geq \liminf_{\varepsilon\downarrow 0} (I_f^{-n})^* \left(\int_{T\setminus T_n} x_\varepsilon^*(t)d\mu\right) \\ &\geq (I_f^{-n})^* \left(\int_{T\setminus T_n} u^*(t)d\mu + A_1^*\lambda^*\right) \geq (I_f^{-m})^* \left(\int_{T\setminus T_n} u^*(t)d\mu + A_1^*\lambda^*\right). \end{aligned} \tag{67}$$

Now, taking into account that the f_t's are supposed nonnegative, using successively (52), (57), (66)-(67) and (65) we get for any $m \geq 1$ (remember that $u^* := \int_T u^*(t)d\mu$)

$$\begin{aligned} &(I_f)^*(z^*) = \liminf_{\varepsilon\downarrow 0} (I_f)^*(z_\varepsilon^*) \\ &= \liminf_{n\to+\infty} \liminf_{\varepsilon\downarrow 0} \left(\int_{T\setminus T_n} f_t^*\left(x_\varepsilon^*(t)\right)d\mu + \int_{T_n} f_t^*\left(x_\varepsilon^*(t)\right)d\mu + \sigma_{\operatorname{dom} I_f}(y_\varepsilon^*)\right) \\ &\geq \liminf_{n\to+\infty} \left((I_f^{-m})^* \left(\int_{T\setminus T_n} u^*(t)d\mu + A_1^*\lambda^*\right) + (I_f)^* \left(\int_{T_n} u^*(t)d\mu\right)\right) + \sigma_{\operatorname{dom} I_f}(y^*) \\ &\geq (I_f^{-m})^*\left(A_1^*\lambda^*\right) + (I_f)^*\left(u^*\right) + \sigma_{\operatorname{dom} I_f}(y^*), \end{aligned}$$

and so, for all $y \in \operatorname{dom} I_f$ ($\subset \operatorname{dom} I_f^{-m}$),

$$(I_f)^*(z^*) \geq \langle A_1^*\lambda^*, y\rangle - \int_{T\setminus T_m} f_t(y)d\mu + (I_f)^*\left(u^*\right) + \sigma_{\operatorname{dom} I_f}(y^*).$$

Thus, using the dominated convergence theorem, as $m \to +\infty$ we get

$$(I_f)^*(z^*) \geq \langle A_1^*\lambda^*, y\rangle + (I_f)^*\left(u^*\right) + \sigma_{\operatorname{dom} I_f}(y^*).$$

In other words, for all $y \in X$ we have

$$(I_f)^*(z^*) \geq \langle A_1^*\lambda^*, y\rangle - \mathrm{i}_{\operatorname{dom} I_f}(y) + (I_f)^*\left(u^*\right) + \sigma_{\operatorname{dom} I_f}(y^*),$$

and by taking the supremum over $y \in X$ we deduce, as $(\mathrm{i}_{\operatorname{dom} I_f})^* = \sigma_{\operatorname{dom} I_f}$,

$$\begin{aligned} (I_f)^*(z^*) &\geq \sigma_{\operatorname{dom} I_f}(A_1^*\lambda^*) + (I_f)^*\left(u^*\right) + \sigma_{\operatorname{dom} I_f}(y^*) \\ &\geq (I_f)^*\left(u^*\right) + \sigma_{\operatorname{dom} I_f}(y^* + A_1^*\lambda^*). \end{aligned}$$

Therefore, taking into account (10) and (12), (59) yields

$$(I_f)^*(z^*) \geq \left((I_f)^* \square \sigma_{\operatorname{dom} I_f}\right)\left(u^* + y^* + A_1^*\lambda^*\right) = \left((I_f)^* \square \sigma_{\operatorname{dom} I_f}\right)(z^*) = (I_f)^*(z^*),$$

and the desired relation follows.

Step 4. We prove statement (ii) in the general case, when the f_t's are not necessarily nonnegative. To this aim, we consider the convex normal integrand $\hat{f} : T \times X \to \mathbb{R}_\infty$ defined by

$$\hat{f}_t(t,z) := f(t,z) - \langle \gamma(t), z\rangle - \beta(t),$$

where $\gamma \in L^1(T,X)$ and $\beta \in L(T)$ come from assumption (22). Then $\hat{f}$ satisfies the same continuity properties as f, $\operatorname{dom} I_{\hat{f}} = \operatorname{dom} I_f$, and we have that

$$(I_{\hat{f}})^*(\cdot) = (I_f)^*(\cdot + \gamma_0) + \beta_0,$$

where $\gamma_0 := \int_T \gamma(t) d\mu$ and $\beta_0 := \int_T \beta(t) d\mu$. Consequently, using Step 3, for all $z^* \in \operatorname{dom}(I_f)^*$ we have that $z^* - \gamma_0 \in \operatorname{dom}(I_{\hat{f}})^*$ and so there exist $x_1^*(\cdot) \in L^1(T,X^*)$ and $y^* \in X^*$ such that $\int_T x_1^*(t) d\mu + y^* = z^* - \gamma_0$ and

$$\begin{aligned}(I_f)^*(z^*) = (I_{\hat{f}})^*(z^* - \gamma_0) - \beta_0 &= \left(\left(\fint_T (\hat{f}_t)^* d\mu\right) \square \sigma_{\operatorname{dom} I_f}\right)(z^* - \gamma_0) - \beta_0 \\ &= \int_T (\hat{f}_t)^*(x_1^*(t)) d\mu + \sigma_{\operatorname{dom} I_f}(y^*) - \beta_0 \\ &= \int_T f_t^*(x_1^*(t) + \gamma(t)) d\mu + \sigma_{\operatorname{dom} I_f}(y^*).\end{aligned}$$

Then, since the following relation is easily verified,

$$\left(\left(\fint_T \hat{f}_t^* d\mu\right) \square \sigma_{\operatorname{dom} I_f}\right)(z^* - \gamma_0) = \left(\left(\fint_T f_t^* d\mu\right) \square \sigma_{\operatorname{dom} I_f}\right)(z^*) + \beta_0,$$

the element $x^*(\cdot) := x_1^*(\cdot) + \gamma(\cdot) \in L^1(T,X^*)$ satsifies $\int_T x^*(t) d\mu + y^* = z^*$ and

$$(I_f)^*(z^*) = \left(\left(\fint_T f_t^* d\mu\right) \square \sigma_{\operatorname{dom} I_f}\right)(z^*) = \int_T f_t^*(x^*(t)) d\mu + \sigma_{\operatorname{dom} I_f}(y^*),$$

as we wanted to prove. □

Corollary 4.8. (Theorem 4.2(ii), continued) *Assume that X is a (separable) Banach space, and let f be a convex normal integrand satisfying condition* (22). *If $\tilde{I}_f$ is continuous with respect to the norm topology on $L^\infty(T,X)$ at some point $x_0(\cdot) \equiv x_0 \in \operatorname{dom} I_f$, then for every $x \in \operatorname{dom} I_f$ and $\varepsilon \geq 0$ we have that*

$$\partial_\varepsilon I_f(x) = \bigcup_{\substack{\varepsilon(\cdot) \in \mathcal{A}_{\varepsilon-\varepsilon_0} \\ 0 \leq \varepsilon_0 \leq \varepsilon}} \int_T \partial_{\varepsilon(t)} f_t(x) d\mu + \mathrm{N}^{\varepsilon_0}_{\operatorname{dom} I_f}(x). \tag{68}$$

Proof. Fix $x \in \operatorname{dom} I_f$ and $\varepsilon \geq 0$. On account of Theorem 4.7(ii), an element $x^* \in X^*$ belongs to $\partial_\varepsilon I_f(x)$ if and only if there are $x^*(\cdot) \in L^1(T,X^*)$ and $y^* \in X^*$ such that $\int_T x^*(t) d\mu + y^* = x^*$, $\int_T f_t^*(x^*(t)) d\mu = \tilde{I}_{f^*}(x^*(\cdot)) \in \mathbb{R}$ (due to Proposition 3.1), and

$$\begin{aligned}I_f(x) + (I_f)^*(x^*) &= I_f(x) + \int_T f_t^*(x^*(t)) d\mu + \sigma_{\operatorname{dom} I_f}(y^*) \\ &\leq \langle x^*, x\rangle + \varepsilon = \left\langle \int_T x^*(t) d\mu, x\right\rangle + \langle y^*, x\rangle + \varepsilon.\end{aligned}$$

Denote $\varepsilon_1(t) := f_t(x) + f_t^*(x^*(t)) - \langle x^*(t), x\rangle$ (≥ 0), so that $x^*(t) \in \partial_{\varepsilon_1(t)} f_t(x)$, and the inequality above yields

$$\int_T \varepsilon_1(t) d\mu + \sigma_{\operatorname{dom} I_f}(y^*) - \langle y^*, x\rangle \leq \varepsilon,$$

that is, $\varepsilon_1(\cdot) \in L^1(T)$ and $\varepsilon_1 := \int_T \varepsilon_1(t)d\mu$ satisfies

$$0 \le \sigma_{\operatorname{dom} I_f}(y^*) - \langle y^*, x\rangle \le \varepsilon - \varepsilon_1.$$

In other words, $y^* \in \mathrm{N}^{\varepsilon-\varepsilon_1}_{\operatorname{dom} I_f}(x)$, and we deduce

$$x^* = \int_T x^*(t)d\mu + y^* \in \int_T \partial_{\varepsilon_1(t)} f_t(x)d\mu + \mathrm{N}^{\varepsilon-\varepsilon_1}_{\operatorname{dom} I_f}(x),$$

that is, the inclusion "$\subset$" in (68) follows. The proof is finished as the opposite inclusion is straightforward. □

We now characterize the subdifferential of I_f in a sequential form that involves exact subgradients of the f_t's at appropriately chosen nearby points, namely at points converging in the $L^\infty(T,X)$-norm to the reference point. The use of $L^p(T,X)$-norms with $1 \le p < +\infty$, instead of $L^\infty(T,X)$, has been considered in [11]. The case $p = +\infty$ has also been investigated in [18] where the dual estimates are taken in the dual space $(L^\infty(T,X))^*$, while the following result considers $x_n^*(\cdot)$ in the smaller space $L^1(T,X^*)$. It is worth recalling that the finite sum version of Corollary 4.9 has been first established in [30] (see, also, [4] and [15]).

Corollary 4.9. *Assume that X is a (separable) reflexive Banach space, and let f be a convex normal integrand satisfying condition* (21)*. Given a point $x \in X$, we have that $x^* \in \partial I_f(x)$ if and only if there are sequences $(x_n(\cdot)) \subset L^\infty(T,X)$ and $(x_n^*(\cdot)) \subset L^1(T,X^*)$ such that*

(i) *$(x_n(\cdot))$ converges to $x1_T(\cdot)$ in the $L^\infty(T,X)$-norm,*

(ii) *$x_n^*(t) \in \partial f_t(x_n(t))$, a.e. $t \in T$,*

(iii) *$x_n^* := \int_T x_n^*(t)d\mu$ converges to x^* in the norm of X^*,*

(iv) *$\int_T f_t(x_n(t))d\mu \to I_f(x)$,*

(v) *$\int_T \langle x_n^*(t), x_n(t) - x\rangle\, d\mu \to 0$.*

Proof. Fix $n \ge 1$. If $x^* \in \partial I_f(x)$, then by Theorem 4.2 and Remark 4.5 we find $T_0 \subset T$, $(\varepsilon_n) \in \mathcal{A}_{1/n^2}$ and $(\tilde{x}_n^*) \subset L^1(T,X^*)$ such that $\mu(T \setminus T_0) = 0$,

$$\tilde{x}_n^* = \int_T \tilde{x}_n^*(t)d\mu, \quad \|\tilde{x}_n^* - x^*\| \le 1/n, \tag{69}$$

and $$\tilde{x}_n^*(t) \in \partial_{\varepsilon_n(t)} f_t(x), \quad \text{for all } t \in T_0;$$

hence, according to Proposition 3.1(i), we also have that $\tilde{x}_n^*(\cdot) \in \partial_{\frac{1}{n^2}} \tilde{I}_f(x1_T(\cdot))$. Next, by the Brøndsted-Rockafellar Theorem [33, Theorem 1.4.1], there are sequences $\lambda_n \in [-1,1]$, $z_n^*(\cdot) \in B_{L^1(T,X^*)}$, $x_n(\cdot) \in L^\infty(T,X)$ and

$$x_n^*(\cdot) \in \partial \tilde{I}_f(x_n(\cdot)) \tag{70}$$

such that $$\|x_n(\cdot) - x1_T(\cdot)\|_{L^\infty(T,X)} + |\langle \tilde{x}_n^*(\cdot), x_n(\cdot) - x1_T(\cdot)\rangle| \le \frac{1}{n}, \tag{71}$$

$$\tilde{I}_f(x_n(\cdot)) + \frac{1}{n}\|x_n(\cdot) - x1_T(\cdot)\|_{L^\infty(T,X)} + \frac{1}{n}|\langle \tilde{x}_n^*(\cdot), x_n(\cdot) - x1_T(\cdot)\rangle| \\ \leq I_f(x) + \langle \tilde{x}_n^*(\cdot), x_n(\cdot) - x1_T(\cdot)\rangle, \tag{72}$$

$$x_n^*(\cdot) - \tilde{x}_n^*(\cdot) = \frac{1}{n} z_n^*(\cdot) + \frac{\lambda_n}{n}\tilde{x}_n^*(\cdot). \tag{73}$$

From (70) together with Proposition 3.1(i), we infer that $x_n^*(t) \in \partial f_t(x_n(t))$ and assertion (ii) follows. From (71) and (73) we obtain that $\|x_n(\cdot) - x1_T(\cdot)\|_{L^\infty(T,X)} \to 0$ and

$$\begin{aligned} &\textstyle\int_T \langle x_n^*(t), x_n(t) - x\rangle \, d\mu \\ &= \textstyle\int_T \langle \tilde{x}_n^*(t), x_n(t) - x\rangle \, d\mu + \frac{1}{n}\int_T \langle z_n^*(t) + \lambda_n \tilde{x}_n^*(t), x_n(t) - x\rangle \, d\mu \\ &\leq \frac{1}{n} + \frac{2}{n^2}, \end{aligned}$$

and assertions (i) and (v) follow. Moreover, from (72), (71) and the lower semicontinuity of $\tilde{I}_f$ (again by Proposition 3.1) we obtain

$$\begin{aligned} I_f(x) &= \tilde{I}_f(x1_T(\cdot)) \leq \liminf_n \tilde{I}_f(x_n(\cdot)) \\ &\leq \limsup_n \tilde{I}_f(x_n(\cdot)) \leq I_f(x) + \limsup_n \langle \tilde{x}_n^*(\cdot), x_n(\cdot) - x1_T(\cdot)\rangle = I_f(x), \end{aligned}$$

and assertion (iv) holds true. Now, using (69) and (73), we get

$$\begin{aligned} \left\|\textstyle\int_T x_n^*(t) d\mu - x^*\right\| &\leq \left\|\textstyle\int_T x_n^*(t) d\mu - \int_T \tilde{x}_n^*(t) d\mu\right\| \\ &\leq \frac{1}{n}\textstyle\int_T \|z_n^*(t)\| \, d\mu + \frac{1}{n}\left\|\int_T \tilde{x}_n^*(t) d\mu\right\| + \frac{1}{n} \\ &\leq \frac{2}{n} + \frac{1}{n^2} + \frac{1}{n}\|x^*\|, \end{aligned}$$

and assertion (iii) holds.

The proof of the corollary is complete because the opposite implication is straightforward, that an element x^* satisfying assertions (i)–(v) is necessary a subgradient of I_f at x. □

Remark 4.10. The limitation to Suslin locally convex spaces in the current work is mainly due to the need to apply measurable selection theorems ([28]). Notice that general nonconvex integral functions over arbitrary Banach spaces are studied in [19], but assuming some regular and uniform Lipschitz conditions on the integrands. This condition implies in particular that the Clarke generalized subdifferential multifunction $t \rightrightarrows \partial^\circ \varphi_t(x)$ is w^*-compact and non-empty valued, allowing the use of a measurable selection theorem from [1]. In our case, the given integrands are assumed to be convex and only lsc, so that the multifunction $t \rightrightarrows \partial f_t(x)$ can be unbounded or even empty-valued, hence the requirement of the separability of the space. However, we can avoid such a restriction when T is finite as we show in Corollary 4.11 below.

We close the paper by recovering formula (1) in general locally convex spaces, based on Theorem 4.2.

Corollary 4.11. *Assume that X is a locally convex space. Given $f_1, f_2 \in \Gamma_0(X)$, for every $x \in \operatorname{dom} f_1 \cap \operatorname{dom} f_1$ we have*

$$\partial_\varepsilon(f_1+f_2)(x) = \operatorname{cl}^{w^*}\left(\bigcup_{\substack{\varepsilon_1+\varepsilon_2\le\varepsilon\\ \varepsilon_1,\varepsilon_2\ge 0}} \partial_{\varepsilon_1} f_1(x) + \partial_{\varepsilon_2} f_2(x)\right), \text{ for all } \varepsilon > 0,$$

and, consequently, $\partial(f_1+f_2)(x) = \bigcap_{\varepsilon>0} \operatorname{cl}^{w^*}(\partial_\varepsilon f_1(x) + \partial_\varepsilon f_2(x))$.

Proof. First observe that condition (22) holds in the current case.

Fix $x \in \operatorname{dom} f_1 \cap \operatorname{dom} f_1$ and $\varepsilon > 0$, and pick any finite-dimensional subspace $L \subset X$ such that $x \in L$. Let us denote by $\tilde{f}_1$ and $\tilde{f}_2$ the restrictions of the functions $f_1 + \mathrm{i}_L$ and $f_2 + \mathrm{i}_L$, respectively, to the subspace L. Hence, $\tilde{f}_1, \tilde{f}_2 \in \Gamma_0(L)$ and formula (32) applied in the finite-dimensional space L yields

$$\partial_\varepsilon(\tilde{f}_1+\tilde{f}_2)(x) = \operatorname{cl}\left(\bigcup_{\substack{\varepsilon_1+\varepsilon_2\le\varepsilon\\ \varepsilon_1,\varepsilon_2\ge 0}} \partial_{\varepsilon_1} \tilde{f}_1(x) + \partial_{\varepsilon_2} \tilde{f}_2(x)\right), \tag{74}$$

where the closure is in the dual space L^* of L. Observe that L^* is isomorphic to the quotient space $X^*/L^\perp$, where $L^\perp$ is the orthogonal space of L. Also, we can easily verify that

$$\partial_\varepsilon(\tilde{f}_1+\tilde{f}_2)(x) = \{x^*_{|L} : x^* \in \partial(f_1+f_2+\mathrm{i}_L)(x)\},$$

and
$$\partial\tilde{f}_i(x) = \{x^*_{|L} : x^* \in \partial(f_i+\mathrm{i}_L)(x)\}, \quad i=1,2,$$

where $x^*_{|L}$ denotes the restriction of $x^* \in X^*$ to L. Then (74) simplifies to

$$\partial_\varepsilon(f_1+f_2)(x) \subset \operatorname{cl}^{w^*}\left(\bigcup_{\substack{\varepsilon_1+\varepsilon\le\varepsilon\\ \varepsilon_1,\varepsilon_2\ge 0}} \partial_{\varepsilon_1} f_1(x) + \partial_{\varepsilon_2} f_2(x) + L^\perp\right),$$

and by taking the intersection over the L's we obtain that

$$\partial_\varepsilon(f_1+f_2)(x) \subset \operatorname{cl}^{w^*}\left(\bigcup_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon\\ \varepsilon_1,\varepsilon_2\ge 0}} \partial_{\varepsilon_1} f_1(x) + \partial_{\varepsilon_2} f_2(x)\right).$$

Thus, we are done since the opposite inclusion is straightforward.

The last statement of the corollary easily follows from the main statement. □

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