

A Minimax Lemma and its Applications

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Dedicated to the memory of Riccardo Damasio.

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We prove an easy version of the minimax theorem with no topological assumption. We deduce from it some domination criteria as well as applications to integral representation and to p -summing operators.

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1. Introduction

The main result of this paper is a minimax lemma which we prove, in various degrees of generality, in Section 3, using of a convexification technique based on finitely additive integration. In the subsequent sections we obtain applications to several, apparently independent problems which are rarely recognized as minimax problems. In Section 4 we examine domination criteria for functions defined on arbitrary sets and obtain a non linear generalisation of a well known Theorem of Ky Fan [14]. In Section 5 we study a weaker form of domination similar to absolute continuity and involving convergence of functions to zero. In Section 6 we establish several integral representation theorems, extending the classical findings of Choquet [9] and of Strassen [31]. Eventually in Section 7 we obtain applications to summable families of functions similar to the results of Grothendieck and Pietsch. In all these applications the general structure of our minimax lemma often permits extensions of the classical versions or just alternative proofs.

As is well known, the problem of finding sufficient conditions for the minimax equality

$$\inf_{f \in \mathcal{F}} \sup_{x \in X} f(x) = \sup_{x \in X} \inf_{f \in \mathcal{F}} f(x) \quad (1)$$

originated in the theory of zero sum games with the classical work of Von Neumann [34] and had immediate applications in several fields, e.g. the theory of sequential statistical decisions of Wald [35]. The abstract mathematical problem received great impulse from the infinite dimensional generalisations obtained by Ky Fan [13, 15] and Maurice Sion [30], later extended or improved by a number of other authors including Ha [18], Kindler [22], König [24], Simons [29] (who also discusses the

different approaches) and Terkelsen [32]. The conditions originally considered by Fan and Sion (and in some more general form also by much of the following literature) involve (a) compactness of the space X , (b) some degree of convexity of $\mathcal{F}$ and concavity in X , (c) some form of semicontinuity of the functions in $\mathcal{F}$. More recent contributions have replaced convexity or concavity with assumptions of a purely topological nature, such as connectedness (see e.g. [19], [24] and [32]).

The approach to the minimax problem we follow in this work is based on the simple observation that even if, in the general case, the left hand side of (1) strictly exceeds the right hand one, we may still find a convenient extension $\widehat{\mathcal{F}}$ of the set $\mathcal{F}$ with the property that

$$\inf_{h \in \widehat{\mathcal{F}}} \sup_{x \in X} h(x) = \sup_{x \in X} \inf_{h \in \mathcal{F}} h(x). \quad (2)$$

In Theorem 3.1 we prove that one such extension is the integral hull, $\text{Int}(\mathcal{F})$, provided that the family $\mathcal{F}$ is (i) pseudo concave on X and (ii) pointwise lower bounded. If we add to (i)–(ii) the further assumption that $\mathcal{F}$ is B -convex, a newly defined property related to the existence of sub barycentres, then we recover the original minimax equality (1). A comparison with the traditional assumptions of this literature, particularly compactness, shows that our result is indeed a generalization of those of Fan and Sion.

1.1. Notation

If X and Y are non empty sets the symbol $\mathfrak{F}(X, Y)$ (resp. $\mathfrak{F}(X)$) denotes the family of all functions which map X into Y (resp. into $\mathbb{R}$). The topology of pointwise convergence assigned to $\mathfrak{F}(X)$ is referred to as the X -topology and the prefix X will be used to mean that a given class or operation is defined relative to such a topology. The set of all evaluations e_x at some $x \in X$ will be denoted by $\mathcal{E}(X)$. The symbol $\mathbb{P}(X)$ (resp. $\mathbb{P}(\mathcal{A})$) will designate the family of all finitely additive probabilities defined on the power set of X (resp. defined on the algebra $\mathcal{A}$ of subsets of X). As usual, $ba(\mathcal{A})$ is the vector space spanned by $\mathbb{P}(\mathcal{A})$. If $m \in \mathbb{P}(X)$ and $f \in L^1(m)$, we shall use the symbols $\int f dm$ or $m(f)$ interchangeably. If $\mathcal{F} \subset \mathfrak{F}(X)$, we write

$$\mathbb{P}(X; \mathcal{F}) = \{m \in \mathbb{P}(X) : \mathcal{F} \subset L^1(m)\}. \quad (3)$$

Most often we shall be concerned with the set $\mathbb{P}(\mathcal{F}; \mathcal{E}(X_0))$ for some $\mathcal{F} \subset \mathfrak{F}(X)$ and $X_0 \subset X$.

2. Barycentrical convexity

In minimax problems two properties are important: a form of boundedness and some extension of the notion of concavity/convexity for functions defined on an abstract set.

Definition 2.1. A family $\mathcal{F} \subset \mathfrak{F}(X)$ is *pointwise lower bounded* if the inequality $\inf_{f \in \mathcal{F}} f(x) > -\infty$ holds for every $x \in X$.

If $\mathcal{F}$ consists of the Y -sections of some function $F \in \mathfrak{F}(X \times Y)$, then pointwise lower boundedness is implicit in the classical assumptions that Y is compact and that the X -sections of F are lower semicontinuous on Y .

Concerning convexity, we define two distinct notions which involve a sequence in X and which, for this reason, we qualify as “*pseudo*”. When the intervening sequence is replaced with a single point – e.g. when X is compact and $\mathcal{F}$ consists of lower semicontinuous functions – this qualification is dropped.

Definition 2.2. A family $\mathcal{F} \subset \mathfrak{F}(X)$ is *pseudo convex* on X if for all $x, x' \in X$ and $0 \leq t \leq 1$ there exists a sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in X such that

$$tf(x) + (1 - t)f(x') + 2^{-n} \geq f(x_n), \quad f \in \mathcal{F}, \quad n \in \mathbb{N}. \quad (4)$$

Pseudo concavity is defined similarly. The definition of convexity for a family of functions on an abstract set X is due to Fan [13] and was used by Sion [30, 2.2] under the name of convexlikeness and by Le Cam [26, Definition 7] who called it subconvexity. Pseudo convexity was introduced by Irle [20, Theorem 3.1]. An intermediate property was examined by König [23, Lemma] and by Terkelsen [32, Theorem 2] who assumed that for each pair $x, x' \in X$ there exists $x_0 \in X$ such that

$$f(x) + f(x') \geq 2f(x_0) \quad f \in \mathcal{F}. \quad (5)$$

Definition 2.3. (a) A *pseudo sub barycentre* of $m \in \mathbb{P}(X; \mathcal{F})$ on $\mathcal{F} \subset \mathfrak{F}(X)$ is a sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ satisfying

$$\int_X f(x)m(dx) + 2^{-n} \geq f(x_n) \quad f \in \mathcal{F}, \quad n \in \mathbb{N}. \quad (6)$$

The set of probabilities admitting a pseudo sub barycentre on $\mathcal{F}$ is denoted by $\mathbb{P}_\beta(X; \mathcal{F})$.

(b) $\mathcal{F}$ is *pseudo B-convex* on X if each $m \in \mathbb{P}(X; \mathcal{F})$ admits a pseudo sub barycentre on $\mathcal{F}$.

To understand the relation between the two definitions above, observe that the point mass at each $x \in X$ is trivially in $\mathbb{P}_\beta(X; \mathcal{F})$ and that $\mathcal{F}$ is pseudo convex on X if and only if every convex combination of point masses is itself an element of $\mathbb{P}_\beta(X; \mathcal{F})$. Thus, if $\mathbb{P}_\beta(X; \mathcal{F})$ is a convex set (e.g. if $\mathcal{F}$ is pseudo B-convex on X) then $\mathcal{F}$ is pseudo convex on X . On the other hand it is easily deduced from Definitions 2.2 and 2.3 that if $\mathcal{F}$ is pseudo convex then $\mathbb{P}_\beta(X; \mathcal{F})$ is a convex set.

Pseudo B-convexity of $\mathcal{F}$ on X may be written as the condition that for each $\varepsilon > 0$ and $m \in \mathbb{P}(X; \mathcal{F})$ the family of sets $\{x \in X : f(x) - m(f) \leq \varepsilon\}$ with $f \in \mathcal{F}$ has non empty intersection. This remark suggests an obvious link with compactness.

Lemma 2.4. A family $\mathcal{F} \subset \mathfrak{F}(X)$ which is pseudo convex on X is also pseudo B-convex on X in either one of the following special cases: (a) $\mathcal{F}$ is finite, (b) X is a compact set and each $f \in \mathcal{F}$ is lower semicontinuous, or (c) $\mathcal{F}$ is totally bounded in the metric of uniform distance on X .

Proof. We start with a useful, general fact (see [4] for details): for any $\mu \in \mathbb{P}(X)$, $\mathcal{H} \subset L^1(\mu)$ finite and $\varepsilon > 0$ there exist $x_1, \dots, x_k \in X$ and convex weights $\alpha_1, \dots, \alpha_k$ such that

$$\sup_{h \in \mathcal{H}} \left| \int h d\mu - \sum_{j=1}^k h(x_j)\alpha_j \right| < \varepsilon. \quad (7)$$

This may be seen by choosing $\eta > 0$ in such a way that, letting $B = \bigcap_{h \in \mathcal{F}} \{|h| < \eta\}$,

$$\sup_{h \in \mathcal{H}} \left| \int h d\mu - \frac{1}{\mu(B)} \int_B h d\mu \right| < \varepsilon/2. \quad (8)$$

Then, since $\mathcal{H}$ is uniformly bounded on B , we construct a finite partition $B_1, \dots, B_k$ of B such that

$$\sup_{h \in \mathcal{H}} \sup_{1 \leq j \leq k} \sup_{x, x' \in B_j} |h(x) - h(x')| < \varepsilon/2. \quad (9)$$

We obtain (7) by selecting $x_j \in B_j$ arbitrarily and setting $\alpha_j = m(B_j)/m(B)$.

Consider now $m \in \mathbb{P}(X; \mathcal{F})$ and $\varepsilon > 0$ as given.

(a) If $\mathcal{F}$ is finite, we obtain from (7)

$$\int f dm \geq -\varepsilon + \sum_{j=1}^k f(x_j) \alpha_j \geq -2\varepsilon + f(x_\varepsilon) \quad f \in \mathcal{F} \quad (10)$$

in which the existence of x_ε follows from pseudo convexity of $\mathcal{F}$.

(b) If each $f \in \mathcal{F}$ is lower semicontinuous and X is compact, the sets of the form $\{f - m(f) \leq \varepsilon\}$, with $f \in \mathcal{F}$, are compact and have the finite intersection property, by (a). Thus, (b) follows. To prove (c), cover $\mathcal{F}$ with a finite number of disks of radius ε with respect to the metric of uniform distance and let $f_1, \dots, f_n \in \mathcal{F}$ be their centres. Then, $\bigcap_{i=1}^n \{f_i - m(f_i) \leq \varepsilon\} \subset \bigcap_{f \in \mathcal{F}} \{f - m(f) \leq 3\varepsilon\}$ and the claim follows again from the finite intersection property. $\square$

Although related to one another, pseudo B -convexity and compactness are independent properties.

Lemma 2.5. *Let $\mathcal{F} \subset \mathfrak{F}(X)$ be pseudo convex on X . $m \in \mathbb{P}_\beta(X; \mathcal{F})$ if and only if $m \in \mathbb{P}(X; \mathcal{F})$ and the set*

$$\mathcal{H}(m) = \overline{\text{co}}^X \left(\bigcup_{f \in \mathcal{F}} \{h \in \mathfrak{F}(X) : h \leq f - m(f)\} \right) \quad (11)$$

does not contain positive constant functions (with $\overline{\text{co}}^X(\cdot)$ indicating the X -closed convex hull).

Proof. If m admits a pseudo sub barycentre on $\mathcal{F}$, then

$$\sup_{h \in \mathcal{H}(m)} \inf_x h(x) \leq \sup_{f \in \mathcal{F}} \inf_x f(x) - m(f) \leq 0$$

so that $\mathcal{H}(m)$ contains no positive constants. Conversely, fix $\varepsilon > 0$ and let ϕ be a X -continuous linear functional that separates $\{\varepsilon \mathbf{1}_X\}$ from $\mathcal{H}(m)$. Obviously ϕ admits the representation as $\phi(h) = \sum_{i=1}^k a_i h(x_i)$ for given $(a_1, x_1), \dots, (a_k, x_k) \in \mathbb{R} \times X$. We conclude that

$$\sup_{f \in \mathcal{F}, b \in \mathfrak{F}(X, \mathbb{R}_+)} \sum_{i=1}^k a_i [f(x_i) - m(f) - b(x_i)] < \varepsilon \sum_{i=1}^k a_i. \quad (12)$$

The fact that $\mathfrak{F}(X, \mathbb{R}_+)$ is a convex cone implies $a_i \geq 0$ for $i = 1, \dots, k$, the strict inequality in (12) requires $\sum_{i=1}^k a_i > 0$. Let $\alpha_i = a_i / \sum_{i=1}^k a_i$. If $\mathcal{F}$ is pseudo convex on Y then there exists $x_\varepsilon^m \in X$ satisfying $\varepsilon > \sum_{i=1}^k \alpha_i f(x_i) - m(f) \geq f(x_\varepsilon^m) - m(f)$ for each $f \in \mathcal{F}$. $\square$

3. Main Theorem

The preceding properties deliver an elementary version of the minimax lemma.

Theorem 3.1. *Let $\mathcal{F} \subset \mathfrak{F}(X)$ be pointwise lower bounded and pseudo concave on X . Then,*

$$\inf_{m \in \mathbb{P}(\mathcal{F}; \mathcal{E}(X))} \sup_{x \in X} \int_{\mathcal{F}} f(x) m(df) = \sup_{x \in X} \inf_{f \in \mathcal{F}} f(x) \quad (13)$$

and the infimum over $\mathbb{P}(\mathcal{F}; \mathcal{E}(X))$ is attained. If, in addition, $\mathcal{E}(X)$ is pseudo B -convex on $\mathcal{F}$, then

$$\inf_{f \in \mathcal{F}} \sup_{x \in X} f(x) = \sup_{x \in X} \inf_{f \in \mathcal{F}} f(x). \quad (14)$$

Proof. Write $\eta = \sup_{x \in X} \inf_{f \in \mathcal{F}} f(x)$, for brevity. Observe that we have for any $m \in \mathbb{P}(\mathcal{F}; \mathcal{E}(X))$,

$$\sup_{x \in X} \int_{\mathcal{F}} f(x) m(df) \geq \eta \quad (15)$$

so that the left hand side is always the largest between the two terms in (13). It is thus enough to show that the converse of (15) holds for some $m \in \mathbb{P}(\mathcal{F}; \mathcal{E}(X))$, a fact which is non trivial only in the case $\eta < +\infty$ to which we shall limit attention. Form the convex cone $\mathcal{K} \subset \mathfrak{F}(\mathcal{F})$ spanned by the set $\{e_x - \eta : x \in X\}$. All elements of $\mathcal{K}$ are lower bounded functions while, by the definition of η and the fact that $\mathcal{F}$ is pseudo concave in X , $\mathcal{K}$ admits no element $k \geq 1$. It follows from [8, Proposition 1] that there exists $m_0 \in \mathbb{P}(\mathcal{F}; \mathcal{E}(X))$ such that $\sup_{k \in \mathcal{K}} \int_{\mathcal{F}} k(f) m_0(df) \leq 0$ i.e., in view of (15), such that

$$\sup_{x \in X} \inf_{f \in \mathcal{F}} f(x) \geq \sup_{x \in X} \int_{\mathcal{F}} f(x) m_0(df) = \inf_{m \in \mathbb{P}(\mathcal{F}; \mathcal{E}(X))} \sup_{x \in X} \int_{\mathcal{F}} f(x) m(df). \quad (16)$$

Assume in addition that $m_0 \in \mathbb{P}_\beta(\mathcal{F}; \mathcal{E}(X))$ and let $\langle f_n \rangle_{n \in \mathbb{N}}$ be its pseudo sub barycentre on $\mathcal{E}(X)$. Then,

$$2^{-n} + \int_{\mathcal{F}} f(x) m_0(df) \geq f_n(x)$$

for each $x \in X$ and $n \in \mathbb{N}$ and consequently

$$2^{-n} + \sup_{x \in X} \inf_{f \in \mathcal{F}} f(x) \geq \sup_{x \in X} f_n(x) \geq \inf_{f \in \mathcal{F}} \sup_{x \in X} f(x)$$

which proves the second claim. $\square$

A special case of Theorem 3.1, treated by Sion [30], is that of a function $H \in \mathfrak{F}(X \times Y)$ whose Y -sections H_y are lower semicontinuous and form a concave family on X and whose X -sections H_x are upper semicontinuous and form a convex family on Y and Y is a compact space.

The equality (13) is established in Theorem 3.1 under minimal assumptions if one accepts to replace $\mathcal{F}$ with its *integral hull* defined as

$$\text{Int}(\mathcal{F}) = \left\{ \int_{\mathcal{F}} f(\cdot) m(df) : m \in \mathbb{P}(\mathcal{F}; \mathcal{E}(X)) \right\}. \quad (17)$$

The importance of the integral hull was clearly understood by Dynkin [12] (his definition is slightly different) who refers to a set $\mathcal{F}$ such that $\mathcal{F} = \text{Int}(\mathcal{F})$ as a convex measurable space.

All properties of $\mathcal{F}$ involving shape – such as positivity, monotonicity and the like – are preserved in passing from $\mathcal{F}$ to $\text{Int}(\mathcal{F})$. Moreover, if X is a Banach space and $\mathcal{F}$ the unit sphere of its dual space then $\text{Int}(\mathcal{F}) = \mathcal{F}$. On the other hand, properties involving limits, such as continuity, do not carry over unless they hold uniformly in $\mathcal{F}$: e.g. if $\mathcal{F}$ is equicontinuous then so is $\text{Int}(\mathcal{F})$. Thus, a solution of a given problem that may be found in $\text{Int}(\mathcal{F})$ rather than $\mathcal{F}$ may still be acceptable in several instances.

In geometric terms, it is clear that $\text{Int}(\mathcal{F})$ is a convex set containing $\mathcal{F}$. On the other hand, it follows from (7) that for given $\varepsilon > 0$ and $X_0 \subset X$ finite there exist points $f_1, \dots, f_k \in \mathcal{F}$ and convex weights $\alpha_1, \dots, \alpha_k$ such that

$$\sup_{x \in X_0} \left| \int_{\mathcal{F}} f(x) m(df) - \sum_{j=1}^k f_j(x) \alpha_j \right| < \varepsilon.$$

Thus $\text{Int}(\mathcal{F}) \subset \overline{\text{co}}^X(\mathcal{F})$ (the converse inclusion requires the additional assumptions of Corollary 4.2 below) from which we conclude:

Lemma 3.2. *If $\mathcal{F} \subset \mathfrak{F}(X)$ is pointwise lower bounded and pseudo concave on X , then*

$$\inf_{h \in \overline{\text{co}}^X(\mathcal{F})} \sup_{x \in X} h(x) = \sup_{x \in X} \inf_{h \in \overline{\text{co}}^X(\mathcal{F})} h(x) \quad (18)$$

We easily recover a local version of Theorem 3.1 similar to a result of Ha [18, Theorem 4].

Corollary 3.3. *Let $\{\mathcal{F}_\alpha : \alpha \in \mathfrak{A}\}$ be a family of subset of $\mathcal{F} \subset \mathfrak{F}(X)$ each of which pointwise lower bounded and pseudo concave on X . Define*

$$\mathcal{M}_\alpha = \{m \in \mathbb{P}(\mathcal{F}; \mathcal{E}(X)) : m(\mathcal{F}_\alpha^c) = 0\} \quad \text{and} \quad \mathcal{M} = \bigcup_{\alpha \in \mathfrak{A}} \mathcal{M}_\alpha.$$

Then,

$$\inf_{\mu \in \mathcal{M}} \sup_{x \in X} \int_{\mathcal{F}} f(x) \mu(df) = \inf_{\alpha \in \mathfrak{A}} \sup_{x \in X} \inf_{f \in \mathcal{F}_\alpha} f(x) \quad (19)$$

Proof. Fix $\alpha \in \mathfrak{A}$. Given that any $m \in \mathbb{P}(\mathcal{F}_\alpha; \mathcal{E}(X))$ extends to some $\mu \in \mathcal{M}_\alpha$, by Theorem 3.1 we have

$$\sup_{x \in X} \inf_{f \in \mathcal{F}_\alpha} f(x) = \inf_{m \in \mathbb{P}(\mathcal{F}_\alpha; \mathcal{E}(X))} \sup_{x \in X} \int_{\mathcal{F}_\alpha} f(x) m(df) = \inf_{\mu \in \mathcal{M}_\alpha} \sup_{x \in X} \int_{\mathcal{F}} f(x) m(df)$$

from which we easily obtain (19). $\square$

For each $\mu \in \mathcal{M}$ the set $\mathcal{F}$ is μ -a.s. pointwise lower bounded and pseudo concave.

A useful generalization of Theorem 3.1 permits to drop concavity upon passing to the free vector space generated by X (see e.g. [25, p. 137]). This may be represented as the space $\mathfrak{F}_0(X)$ of all real valued functions on X with finite support. Associating each $x \in X$ with the function $\delta_x \in \mathfrak{F}_0(X)$ which is 1 at x and 0 elsewhere, is an embedding of X into $\mathfrak{F}_0(X)$. We also notice that $\mathfrak{F}_0(X)$ is (isomorphic to) the dual space of $\mathfrak{F}(X)$ relatively to the X -topology via the identity

$$[f, h] = \sum_{x \in X} f(x) h(x) \quad f \in \mathfrak{F}(X), \quad h \in \mathfrak{F}_0(X). \quad (20)$$

More precisely the X -topology on $\mathfrak{F}(X)$ coincides with the weak topology induced by $\mathfrak{F}_0(X)$ via (20).

Theorem 3.4. *Let $\mathcal{H} \subset \mathfrak{F}_0(X, \mathbb{R}_+)$ be a convex set and let $\mathcal{F} \subset \mathfrak{F}(X)$ be pointwise lower bounded. Then,*

$$\min_{F \in \text{Int}(\mathcal{F})} \sup_{h \in \mathcal{H}} [F, h] = \sup_{h \in \mathcal{H}} \inf_{f \in \mathcal{F}} [f, h]. \quad (21)$$

If $\mathcal{F}$ is pointwise bounded, then (21) remains true upon replacing $\mathcal{H}$ with any convex subset of $\mathfrak{F}_0(X)$.

Proof. Under either assumption, $\mathcal{H} \subset \mathfrak{F}_0(X, \mathbb{R}_+)$ and $\mathcal{F}$ pointwise lower bounded or $\mathcal{F}$ pointwise bounded, the collection of all functionals on $\mathcal{H}$ associated with some $f \in \mathcal{F}$ via (20) is pointwise lower bounded. Moreover, the function $[f, h]$ is concave on the convex set $\mathcal{H}$. We can then apply Theorem 3.1 and obtain (13), of which (21) is clearly an equivalent reformulation. $\square$

4. Strong domination properties

We may rewrite Theorem 3.1 to obtain a useful domination condition. It is convenient to adopt the symbol

$$\Delta(X) = \{\delta \in \mathfrak{F}_0(X, \mathbb{R}_+) : [\mathbb{1}_X, \delta] = 1\}. \quad (22)$$

Theorem 4.1. *Let $\mathcal{F}, \mathcal{G} \subset \mathfrak{F}(X)$, with $\mathcal{F}$ pointwise upper bounded. The inequality*

$$\sum_{i=1}^n [g_i, \delta_i] \leq \sup_{f \in \mathcal{F}} \left[f, \sum_{i=1}^n \delta_i \right] \quad (23)$$

holds for every finite subset of $\mathcal{G} \times \Delta(X)$ if and only if there is $m \in \mathbb{P}(X; \mathcal{E}(X))$ with

$$g(x) \leq \int_{\mathcal{F}} f(x) m(df), \quad g \in \mathcal{G}, \quad x \in X. \quad (24)$$

Proof. Define the maps $G \in \mathfrak{F}(\mathcal{G} \times X)$ and $T \in \mathfrak{F}(\mathfrak{F}(X), \mathfrak{F}(\mathcal{G} \times X))$ implicitly by letting

$$G(g, x) = g(x) \quad \text{and} \quad (Tj)(g, x) = j(x) \quad g \in \mathcal{G}, \quad x \in X, \quad j \in \mathfrak{F}(X). \quad (25)$$

Write $\mathcal{H} = \{h \in \mathfrak{F}_0(\mathcal{G} \times X)_+ : h(g, \cdot) \in \Delta(X) \text{ for all } g \in \mathcal{G}\}$. Then, (23) takes the form

$$\sup_{h \in \mathcal{H}} \inf_{f \in \mathcal{F}} [G - Tf, h] \leq 0. \quad (26)$$

while the family $\{G - Tf : f \in \mathcal{F}\} \subset \mathfrak{F}(\mathcal{G} \times X)$ is pointwise lower bounded. By Theorem 3.1, this implies the inequality

$$0 \geq \sup_{h \in \mathcal{H}} \left[\int_{\mathcal{F}} (G - Tf)(\cdot) m(df), h \right] = \sup_{h \in \mathcal{H}} \left[G - \int_{\mathcal{F}} (Tf)(\cdot) m(df), h \right] \quad (27)$$

for some $m \in \mathbb{P}(\mathcal{F}; \mathcal{E}(X)\mathcal{F})$. From this we deduce

$$g(x) \leq \int_{\mathcal{F}} (Tf)(g, x) m(df) = \int_{\mathcal{F}} f(x) m(df) \quad g \in \mathcal{G}, \quad x \in X. \quad (28)$$

The converse implication is obvious. $\square$

Several useful and known results follow easily from Theorem 4.1, upon assuming linearity. One special case is the domination Theorem of Ky Fan [14, Theorem 12, p. 123] in which X is a Banach space, $g \in \mathfrak{F}(X)$ and $\mathcal{F} = \rho S_{X^*}$ (the ball of radius $\rho > 0$ in the dual space X^*). Then, $\text{Int}(\mathcal{F}) = \mathcal{F}$ and g is dominated by a continuous linear functional with norm $\leq \rho$ if and only if

$$\sum_{i=1}^N p_i g(x_i) \leq \rho \left\| \sum_{i=1}^N p_i x_i \right\| \quad p_1, \dots, p_N \in \mathbb{R}_+, \quad \sum_{i=1}^N p_i \leq 1. \quad (29)$$

As is well known, Fan's Theorem has been widely used in game theory to prove that the value of a game has non empty core, see [10], and the condition corresponding to (23) is known in that literature as *balancedness*.

For each index α in some non empty set $\mathfrak{A}$ let X_α be a set, $g_\alpha \in \mathfrak{F}(X_\alpha)$ and let π_α be the projection of $X = \times_\alpha X_\alpha$ on its α -th coordinate. Letting $\mathcal{G} = \{g_\alpha \circ \pi_\alpha : \alpha \in \mathfrak{A}\}$ Theorem 4.1 provides a simple necessary and sufficient criterion for the existence of common extensions with given marginals. This problem was treated by Strassen [31] in a well known paper for the case of measures while a recent characterization for linear functionals was obtained by Bertl and Rigo [3] (but see also [2, Lemma 2]). The present version does not require linearity.

A last implication is the characterisation of the integral hull of a pointwise bounded family of functions.

Corollary 4.2. *If $\mathcal{F} \subset \mathfrak{F}(X)$ is pointwise bounded then $\text{Int}(\mathcal{F}) = \overline{\text{co}}^X(\mathcal{F})$.*

Proof. We already noticed that $\text{Int}(\mathcal{F}) \subset \overline{\text{co}}^X(\mathcal{F})$. If $g \in \overline{\text{co}}^X(\mathcal{F})$, apply Theorem 4.1 with $\mathcal{G} = \{g, -g\}$ and replacing $\mathcal{F}$ with $\mathcal{F} \cup (-\mathcal{F})$. Then (23) is true and, as a consequence, (24) holds with equality. $\square$

5. Weak domination properties

The pointwise domination criterion (24) is made less restrictive if we focus on the behaviour of functions in approaching (or attaining) zero. The characterization of this weaker criterion involves an appropriate decomposition of X .

If $\mathcal{F} \subset \mathfrak{F}(X)$, define the set function (recall (22))

$$I_{\mathcal{F}}(U) = \inf_{h \in \Delta(U)} \sup_{f \in \mathcal{F}} [|f| \wedge 1, h] \quad U \subset X. \quad (30)$$

This function is instrumental to the following

Definition 5.1. Let κ be a cardinal number and $\mathcal{F} \subset \mathfrak{F}(X)$. We speak of a family $\{X_{\alpha} : \alpha \in \mathfrak{A}\}$ of subsets of X as a κ -*exhaustion* induced by $\mathcal{F}$ if:

- (i) $\mathfrak{A}$ has cardinality $\leq \kappa$,
- (ii) $I_{\mathcal{F}}(X_{\alpha}) > 0$ for every $\alpha \in \mathfrak{A}$, and
- (iii) $\sup_{(x,f) \in X_0 \times \mathcal{F}} |f(x)| = 0$, where $X_0 = X \setminus \bigcup_{\alpha \in \mathfrak{A}} X_{\alpha}$.

A κ -exhaustion is *residual* if condition (iii) is replaced with

- (iii') $\lim_d f(x_d) = 0$ for each net $\langle x_d \rangle_{d \in D}$ which is eventually in $\bigcap_{i=1}^n X_{\alpha_i}^c$ for every $\alpha_1, \dots, \alpha_n \in \mathfrak{A}$.

Theorem 5.2. Let κ be an infinite cardinal number and $\mathcal{F} \subset \mathfrak{F}(X, [0, 1])$. Then,

- (a) $\mathcal{F}$ induces a κ -exhaustion of X if and only if there exists $\mathcal{G} \subset \text{Int}(\mathcal{F})$ of cardinality $\leq \kappa$ and such that

$$\sup_{g \in \mathcal{G}} g(x) = 0 \quad \text{implies} \quad \sup_{f \in \mathcal{F}} f(x) = 0 \quad x \in X; \quad (31)$$

- (b) $\mathcal{F}$ induces a residual, κ -exhaustion of X if and only if there exists $\mathcal{G} \subset \text{Int}(\mathcal{F})$ of cardinality $\leq \kappa$ and such that for every net $\langle x_d \rangle_{d \in D}$ in X

$$\sup_{g \in \mathcal{G}} \lim_d g(x_d) = 0 \quad \text{implies} \quad \sup_{f \in \mathcal{F}} \lim_d f(x_d) = 0. \quad (32)$$

Proof. Assume that $\{X_{\alpha} : \alpha \in \mathfrak{A}\}$ is a κ -exhaustion of X induced by $\mathcal{F}$ and fix $\alpha \in \mathfrak{A}$. It follows from Theorem 3.4 that

$$I_{\mathcal{F}}(X_{\alpha}) = \sup_{F \in \text{Int}(\mathcal{F})} \inf_{\gamma \in \Delta(X_{\alpha})} [F, h] = \sup_{F \in \text{Int}(\mathcal{F})} \inf_{x \in X_{\alpha}} F(x). \quad (33)$$

From the assumption we infer the existence of $g_{\alpha} \in \text{Int}(\mathcal{F})$ such that

$$\inf_{x \in X_{\alpha}} g_{\alpha}(x) \geq I_{\mathcal{F}}(X_{\alpha})/2 > 0.$$

Let $\mathcal{G} = \{g_{\alpha} : \alpha \in \mathfrak{A}\}$. The set $\mathcal{G}$ has cardinality $\leq \kappa$; moreover, if $\sup_{\alpha \in \mathfrak{A}} g_{\alpha}(x) = 0$ then necessarily $x \in X_0$ and so $\sup_{f \in \mathcal{F}} f(x) = 0$, as in (31). If, on the other hand, $\langle x_d \rangle_{d \in D}$ is a net such that $\lim_d g(x_d) = 0$ for all $g \in \mathcal{G}$ then, no matter the choice of $\alpha_1, \dots, \alpha_k \in \mathfrak{A}$, we must have $x_d \in \bigcap_{i=1}^k X_{\alpha_i}^c$ for all $d \in D$ sufficiently large and, if the exhaustion is residual, $\lim_d f(x_d) = 0$ for all $f \in \mathcal{F}$, as in (32). This proves the direct implication of both claims, (a) and (b).

Assume conversely that $\mathcal{G} \subset \text{Int}(\mathcal{F})$ has cardinality $\leq \kappa$ and satisfies (31), e.g. when (32) holds true. Let $\mathfrak{A} = \mathcal{G} \times \mathbb{N}$ and write each element of $\mathfrak{A}$ as $\alpha = (g_{\alpha}, p_{\alpha})$.

Define the sets $X_\alpha = \{x \in X : g_\alpha(x) > 1/p_\alpha\}$, $\alpha \in \mathfrak{A}$, (34)

and $X_0 = X \setminus \bigcup_{\alpha \in \mathfrak{A}} X_\alpha$. Since κ is infinite, the cardinality of $\mathfrak{A}$ does not exceed κ and for each $\alpha \in \mathfrak{A}$ we have

$$I_{\mathcal{F}}(X_\alpha) = \inf_{h \in \Delta(X_\alpha)} \sup_{f \in \mathcal{F}} [f, h] \geq \inf_{h \in \Delta(X_\alpha)} [g_\alpha, h] \geq 1/p_\alpha. \quad (35)$$

Moreover, if $x \in X_0$ we deduce that $\sup_\alpha g_\alpha(x) = 0$ so that $\sup_{f \in \mathcal{F}} f(x) = 0$, by (31). Thus $\{X_\alpha : \alpha \in \mathfrak{A}\}$ is a κ -exhaustion. Eventually, let $\langle x_d \rangle_{d \in D}$ be a net in X and fix $g \in \mathcal{G}$ arbitrarily. Let $\alpha_i(g) = (g, 1/i) \in \mathfrak{A}$ for $i = 1, 2, \dots$. If the net is eventually in $\bigcap_{i=1}^n X_{\alpha_i(g)}^c$ for each $n \in \mathbb{N}$ then we deduce that $\lim_d g(x_d) = 0$. Thus if $\langle x_d \rangle_{d \in D}$ is eventually in any intersection $\bigcap_{i=1}^n X_{\alpha_i}^c$ this implies, by (32), that $\lim_d f(x_d) = 0$ for all $f \in \mathcal{F}$. In other words, the κ -exhaustion is residual. This proves the converse implication for both claims. $\square$

In the special case in which $\kappa = \aleph_0$ Theorem 5.2 simplifies considerably as the collection $\mathcal{G}$ may be replaced, with no loss of generality, with a σ convex combination of its elements, which is still an element of $\text{Int}(\mathcal{F})$. This conclusion may be applied in the context of the following examples.

Example 5.3. If E is a Banach lattice, $\mathcal{F} = S_{E^*} \cap E_+^*$ (so that $\mathcal{F} = \text{Int } \mathcal{F}$) and $X = S_E \cap E_+$ then, by Theorem 5.2, $\mathcal{F}$ induces a countable exhaustion of X if and only if there exists a strictly positive linear functional on E . $\square$

Example 5.4. Let $\mathcal{F}$ be a family of capacities on a Boolean algebra X (i.e. each $f \in \mathcal{F}$ is an increasing function with values in $[0, 1]$ and such that $f(0) = 0$ and $f(1) = 1$). Then $\text{Int}(\mathcal{F})$ consists of capacities as well. If $\mathcal{F}$ induces a countable exhaustion of X , this is equivalent, by Theorem 5.2, to the existence of a capacity ν such that $\nu(x) = 0$ if and only if $f(x) = 0$ for all $f \in \mathcal{F}$. $\square$

The exhaustion technique exploited above was inspired by the approach of Kelley [21, Theorem 4] to the so-called Maharam problem for additive set functions on a Boolean algebra. Kelley's proof, based on the intersection number, has been extended by Galvin and Prikry [16] and, more recently, by Balcar et al. [1].

6. Integral representation theorems

Theorem 4.1 implies a partial extension of the integral representations of Choquet [9, Théorème 1] and of Strassen [31, Theorem 1]. For the former we make use of the concept of sufficient subset, a special case of which is the notion of boundary in the theory of Banach spaces, see e.g. [17, Definition 1.1].

Definition 6.1. A subset $Z \subset X$ is *sufficient* for X relative to the family $\mathcal{H} \subset \mathfrak{F}(X)$, in symbols $Z \geq_{\mathcal{H}} X$, if the following is true:

$$\sum_{h \in \mathcal{H}} \delta(h) h(x) \leq \sup_{z \in Z} \sum_{h \in \mathcal{H}} \delta(h) h(z) \quad x \in X, \delta \in \Delta(\mathcal{H}). \quad (36)$$

We designate with the symbol $\tau(\mathcal{H})$ the initial topology induced by $\mathcal{H}$ on X . If $\mathcal{H}$ consists of bounded functions, it also induces a topology on the space $ba(X)$

(considered as the dual space of the set of bounded functions), necessarily weaker than the corresponding weak* topology. We denote the latter topology by $w^*(\mathcal{H})$.

Theorem 6.2. *Let $\mathcal{H} \subset \mathfrak{F}(X)$ consist of bounded functions. Let $\mathcal{Z}$ be a family of subsets of X , linearly ordered by inclusion. Then $Z \geq_{\mathcal{H}} X$ for every $Z \in \mathcal{Z}$ if and only if for each $x \in X$ there exists $m_x \in \mathbb{P}(X; \mathcal{H})$ such that $m_x(Z^c) = 0$ for all $Z \in \mathcal{Z}$ and*

$$h(x) \leq \int h(y) m_x(dy), \quad h \in \mathcal{H}. \quad (37)$$

Proof. Fix $x \in X$ and $Z \in \mathcal{Z}$ and apply Theorem 4.1 upon replacing X with $\mathcal{H}$, $\mathcal{G}$ with $\{e_x\}$ and $\mathcal{F}$ with $\mathcal{E}(Z)$. We obtain $m'_x \in \mathbb{P}(Z; \mathcal{H})$ satisfying (37). Define $m_x(E) = m'_x(E \cap Z)$ for all $E \subset X$. The set

$$\mathcal{R}_x(\mathcal{H}; Z) = \{m \in \mathbb{P}(X) : m(Z^c) = 0 \text{ and } m \text{ satisfies (37)}\} \quad (38)$$

is thus a non empty, $w^*(\mathcal{H})$ compact set. Moreover, for given $Z, Z' \in \mathcal{Z}$ the inclusion $Z \subset Z'$ implies $\mathcal{R}_x(\mathcal{H}; Z) \subset \mathcal{R}_x(\mathcal{H}; Z')$. But then it is enough to select for each $x \in X$ an element of $\bigcap_{Z \in \mathcal{Z}} \mathcal{R}_x(\mathcal{H}; Z)$. The converse implication is obvious. $\square$

In other words each $x \in X$ is the sub barycentre of some element of $\mathbb{P}(X; \mathcal{H})$ which represents it. If Z is the set of extreme points of a convex, compact subset X of a locally convex space E and $\mathcal{H}$ the dual of E (as in the original formulation of Choquet [9]) it is then obvious that $Z \geq_{\mathcal{H}} X$ and that the measure m_x which represents x relatively to $\mathcal{H}$ may be chosen to be a countably additive, regular Borel measure. The latter property may conflict with the condition $m_x(Z^c) = 0$ if Z is not a Borel set, as noted by Bishop and De Leeuw [5]. Theorem 6.2 requires instead no special assumption on X , Z or $\mathcal{F}$ and raises no measurability issue. At the same time it does not permit a deeper characterization of representing measures through additional properties.

It is obvious from the definition that for a symmetric family $\mathcal{H}$ of bounded functions, a net converges uniformly on X if and only if it converges uniformly on any of its sufficient subsets. This suggests that sufficient subsets may induce a topological characterization of $\mathcal{H}$. Indeed several authors have exploited Choquet integral representation to deduce compactness criteria, including Rainwater [28, p. 25], Godefroy [17, Theorem I.2], Bourgain and Talagrand [6, Théorème 1] and, recently, Pfizner [27]. All of these results assume linearity of X and of $\mathcal{H}$. In our general setting similar conclusions may be reached but under additional assumptions on the set of representing measures $\mathcal{R}(\mathcal{H}; Z) = \bigcup_{x \in X} \mathcal{R}_x(\mathcal{H}; Z)$.

Corollary 6.3. *Let $\mathcal{H} \subset \mathfrak{F}(X)$ be a symmetric set of bounded functions that separate the points of X and let $Z \geq_{\mathcal{H}} X$ be $\tau(\mathcal{H})$ closed. Assume the existence of a $w^*(\mathcal{H})$ closed subset $\mathcal{R} \subset \mathcal{R}(\mathcal{H}; Z)$ such that*

$$\mathcal{R} \cap \mathcal{R}_x(\mathcal{H}; Z) \neq \emptyset \quad x \in X. \quad (39)$$

Then, a sequence in $\mathcal{H}$ converges pointwise on X if and only if it converges pointwise on Z .

Proof. Of course $\mathbb{P}(X)$ is $w^*(\mathcal{H})$ compact and, under the present assumptions, so is $\mathcal{R}$. Let $\langle x_\alpha \rangle_{\alpha \in \mathfrak{A}}$ be a net in X and choose $m_\alpha \in \mathcal{R} \cap \mathcal{R}_{x_\alpha}(\mathcal{H}; Z)$. By assumption, the corresponding net $\langle m_\alpha \rangle_{\alpha \in \mathfrak{A}}$ admits a cluster point $m \in \mathcal{R}$ and an element $x \in X$ such that $m \in \mathcal{R}_x(\mathcal{H}; Z)$. But then, along some subnet $\langle x_\beta \rangle_{\beta \in \mathfrak{B}}$

$$h(x) = \int_Z h(z)m(dz) = \lim_{\beta} \int_Z h(z)m_\beta(dz) = \lim_{\beta} h(x_\beta) \quad h \in \mathcal{H}.$$

This implies that $(X, \tau(\mathcal{H}))$ is compact and Hausdorff, and that so is $(Z, \tau(\mathcal{H}))$. The restriction $\hat{h}$ of each $h \in \mathcal{H}$ to Z is of course continuous and, by (37), the value $h(x)$ of h at x may be viewed as the action $\phi_x(\hat{h})$ on $\hat{h}$ of a continuous linear functional ϕ_x defined on the class of all continuous functions over the topological space $(Z, \tau(\mathcal{H}))$. Consequently we may write $h(x) = \int h(z)\mu_x(dz)$ with μ_x a countably additive, regular probability measure on the Borel subsets of Z . But then pointwise convergence on Z implies pointwise convergence on X as a simple consequence of bounded convergence. $\square$

If X is $\tau(\mathcal{H})$ compact and each $h \in \mathcal{H}$ is linear the condition of Corollary 6.3 holds true.

Specifying a linear structure induces further integral representation results, near to the original findings of Strassen [31] and of Cartier et al. [7].

Lemma 6.4. *Let Z be a countable and symmetric subset of a topological vector space X and $\mathcal{F} \subset \mathfrak{F}(X)$ a pointwise bounded family of sublinear functionals. A linear functional φ on X which satisfies the condition*

$$(\forall z \in Z)(\forall E \subset \mathcal{F}, \text{ finite})(\exists f \in E^c) : \varphi(z) \leq f(z) \quad (40)$$

admits the representation

$$\varphi(y) = \int_{\mathcal{F}} T(f, y)m(df), \quad y \in \text{lin}(Z) \quad (41)$$

in which (a) $m \in \mathbb{P}(\mathcal{F})$ vanishes on finite sets and (b) $T \in \mathfrak{F}(\mathcal{F} \times X)$ is such that $T(f, \cdot)$ is a linear functional $\leq f$ for each $f \in \mathcal{F}$. If X is an F -space and each $f \in \mathcal{F}$ is continuous, then (41) extends to $\overline{\text{lin}}(Z)$.

Proof. We easily obtain from Hahn-Banach a family $\{\chi_f : f \in \mathcal{F}\}$ of linear functionals on X , each of which satisfying the inequality $\chi_f \leq f$. Let $z_1, z_2, \dots$ be an enumeration of Z . Proceeding recursively, for each $n \in \mathbb{N}$ we can, by (40), choose $f_n \in \mathcal{F} \setminus \{f_1, \dots, f_{n-1}\}$ such that $\varphi(z_n) \leq f_n(z_n)$ and, using again Hahn-Banach, obtain a linear functional $t_n \leq f_n$ defined on X and such that $t_n(z_n) = f_n(z_n)$. Define $T(f, z)$ implicitly by letting

$$T(f_n, z) = t_n(z), \quad n = 1, 2, \dots \quad \text{or else} \quad T(f, z) = \chi_f(z), \quad f \notin \{f_1, f_2, \dots\}. \quad (42)$$

By construction,

$$T(f, \cdot) \leq f, \quad f \in \mathcal{F} \quad \text{and} \quad \varphi(y) \leq \inf_{\{E \subset \mathcal{F}: \text{finite}\}} \sup_{f \notin E} T(f, y), \quad y \in \text{lin}(Z).$$

As an immediate consequence of Theorem 4.1 and symmetry of Z we deduce that for each finite $E \subset \mathcal{F}$ there exists $\bar{m}_E \in \mathbb{P}(E^c)$ such that the probability $m_E \in \mathbb{P}(\mathcal{F})$, defined by letting $m_E(A) = \bar{m}_E(A \cap E^c)$, satisfies (41). The family of weak* closed subsets of $\mathbb{P}(\mathcal{F})$ obtained by letting E range over all finite subsets of $\mathcal{F}$ has the finite intersection property so that the claim follows. This establishes the first claim. If X is an F -space, then $\mathcal{F}$ is uniformly bounded and, by the inequality $T(f, \cdot) \leq f$, so is the family $T(f, \cdot)$ for $f \in \mathcal{F}$. The last claim follows from uniform convergence. $\square$

Condition (40) is satisfied e.g. if $\mathbb{F}$ is an infinite, X -separable set and if we have $\varphi(z) < \sup_{f \in \mathcal{F}} f(z)$ for each $z \in Z$.

While Lemma 6.4 does not use any form of measurability, if we introduce some topological assumptions we obtain a representation similar to that of Strassen [31, Theorem 1]. The main difference is that in our formulation the existence of an *a priori* given probability space on $\mathcal{F}$ is not assumed.

We denote by $\overline{\mathcal{H}}^X$ the X -closure of $\mathcal{H} \subset \mathfrak{F}(X)$ and by $\mathcal{B}_X(\mathcal{H})$ the σ algebra generated by the X -open subsets of $\mathcal{H}$.

Theorem 6.5. (Strassen) *Let X be a real vector space and $\mathcal{F} \subset \mathfrak{F}(X)$ a pointwise bounded family of sublinear functionals. A linear functional φ on X satisfies the condition*

$$\varphi(x) \leq \sup_{f \in \mathcal{F}} f(x), \quad x \in X \quad (43)$$

if and only if it may be represented in the form

$$\varphi(x) = \int T(x, f) \lambda(df), \quad x \in X \quad (44)$$

in which

- (a) λ is a Radon probability on $\mathcal{B}_X(\overline{\mathcal{F}}^X)$;
- (b) $T(x, \cdot)$ is $\mathcal{B}_X(\overline{\mathcal{F}}^X)$ measurable for each $x \in X$;
- (c) for each $x, y, z \in X$ and $a, b \in \mathbb{R}$ there exists a λ null set $N(a, b; y, z) \in \mathcal{B}_X(\overline{\mathcal{F}}^X)$ such that

$$T(x, f) \leq f(x) \text{ and } T(ay + bz, f) = aT(y, f) + bT(z, f), \quad f \notin N(a, b; y, z). \quad (45)$$

Moreover, if X is a separable topological vector space, $\mathcal{F}$ is X -closed and φ and each $f \in \mathcal{F}$ are continuous, then one may choose T such that (45) holds for all f outside some fixed λ null set and, if X is an F -space, even for all $f \in \mathcal{F}$.

Proof. Given that (43) remains unchanged if we replace $\mathcal{F}$ with its X -closure, we can assume with no loss of generality that $\mathcal{F}$ is X -closed and thus X -compact as well as Hausdorff. By Theorem 4.1 we can write $\varphi(x) \leq \int_{\mathcal{F}} f(x) m(df)$ for some $m \in \mathbb{P}(\mathcal{F}; \mathcal{E}(X))$ and all $x \in X$. Evaluators are continuous functions of $\mathcal{F}$ if the latter set is given the X -topology and therefore by the Riesz-Markoff representation Theorem, we may replace m with a regular Borel (and thus Radon) probability $\lambda \in \mathbb{P}(\mathcal{B}_X(\mathcal{F}))$. The rest of the proof is very similar to the original proof of Strassen.

If L is the vector subspace of $\mathfrak{F}(\mathcal{F}, X)$ spanned by elements of the form $x\mathbb{1}_E$, with $E \in \mathcal{B}_X(\mathcal{F})$, then φ admits a linear extension $\tilde{\varphi}$ to L satisfying

$$\tilde{\varphi}(h) \leq \int f(h(f))\lambda(df), \quad h \in L. \quad (46)$$

This follows from the Hahn-Banach Theorem once observed that the right hand side of (46) is sublinear on L . Write $\mu_x(E) = \tilde{\varphi}(x\mathbb{1}_E)$. Given that μ_x is additive and that $\mu_x \ll \lambda$ we conclude that μ_x is itself a regular Borel measure on $\mathcal{B}_X(\mathcal{F})$ admitting a Radon-Nikodym derivative denoted by T_x . Write $T(x, f) = T_x(f)$. Properties (a) and (b) are clear; (c) follows from the linearity of $\tilde{\varphi}$. Sufficiency is also clear since, by (c),

$$\varphi(x) = \int T(x, f)\lambda(df) \leq \int f(x)\lambda(df) \leq \sup_{f \in \mathcal{F}} f(x).$$

Assume now that X is a separable topological vector space and denote by X_0 the countable, rational vector subspace of X which is dense in X . For each f outside of the null set $N = \bigcup_{a,b \in \mathbb{Q}, y,z \in X_0} N(a, b; y, z)$ $T(\cdot, f)$ is a linear functional on X_0 and $\leq f$ (and thus continuous). Consider the extension $T'(\cdot, f)$ of $T(\cdot, f)$ to the whole of X obtained by continuity. Let $U(x, f) = T'(x, f)\mathbb{1}_{N^c}(f)$. It is obvious that U satisfies properties (a)–(c) for each $f \notin N$. At the same time, if $\langle x_n \rangle_{n \in \mathbb{N}}$ is a sequence in X_0 converging to x we have

$$\varphi(x) = \lim_n \varphi(x_n) = \lim_n \int_{N^c} T(x_n, f)\lambda(df) = \int U(x, f)\lambda(df) \quad (47)$$

by bounded convergence.

Assume eventually that X is an F space and consider the set Ψ of sublinear functionals satisfying

$$\psi(x) \leq \sup_{f \in \mathcal{F}} f(x), \quad x \in X. \quad (48)$$

Ψ is of course pointwise bounded, X -closed and each $\psi \in \Psi$ is continuous, by uniform boundedness. If X is separable the X -topology on Ψ is metrizable, [33, s. 307, p. 267]. Consider covering the closure of the above set N with finitely many balls of radius 2^{-k} with centres $h_1^k, \dots, h_I^k$ and, for each $i = 1, \dots, I$, let χ_i^k be a linear functional $\leq h_i^k$. Let $E_1^k, \dots, E_I^k$ be the disjoint collection obtained by the cover above and define

$$V^k(x, f) = \sum_{i=1}^I \chi_i^k(x)\mathbb{1}_{E_i^k}(f), \quad f \in \bar{N}, \quad x \in X. \quad (49)$$

Of course $V^k(x, \cdot)$ is $\mathcal{B}_X(\mathcal{F})$ measurable, $V^k(\cdot, f)$ is a linear functional in Ψ for each $f \in \bar{N}$ and $V^k(\cdot, f) \leq h_f^k$ for some $h_f^k \in \mathcal{F}$ such that $\|h_f^k - f\| \leq 2^{-k}$. Because Ψ is metrizable we can extract a subsequence (still indexed by k) which X -converges in Ψ to a linear limit $V(\cdot, f)$. Observe that for fixed $x \in X$ we have $V(x, f) = \lim_k V^k(x, f) \leq f(x) + \lim_n (h_f^k - f)(x) = f(x)$. Eventually, for each $x \in X$ the function $V(x, \cdot)$ is $\mathcal{B}_X(\mathcal{F})$ measurable, since the pointwise limit of measurable functions. The proof is then complete upon replacing T in (44) with $U(x, f) + V(x, f)\mathbb{1}_N(f)$. $\square$

7. Summable functions

We introduce the following family of functions:

Definition 7.1. A function $g \in \mathfrak{F}(X)$ is said to be *summable along* $\mathcal{F} \subset \mathfrak{F}(X)$, in symbols $g \in \mathcal{S}_{\mathcal{F}}(X)$, if the series $\sum_n g(x_n)a(x_n)$ converges for every sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in X and every $a \in \mathfrak{F}(X)$ such that

$$\sup_{f \in \mathcal{F}} \sum_n |f(x_n)a(x_n)| < +\infty. \quad (50)$$

Corollary 7.2. Let $\mathcal{F} \subset \mathfrak{F}(X)$ be pointwise bounded. Then, $g \in \mathcal{S}_{\mathcal{F}}(X)$ if and only if

$$|g(x)| \leq C_g \int_{\mathcal{F}} |f(x)| m_g(df) \quad x \in X \quad (51)$$

for some $C_g \geq 0$ and $m_g \in \mathbb{P}(\mathbb{F}; \mathcal{E}(X))$. If $\mathcal{F}$ is X -closed then m_g may be chosen to be a regular, Borel probability on $\mathcal{B}_X(\mathcal{F})$.

Proof. Assume that $g \in \mathcal{S}_{\mathcal{F}}(X)$ and let $\langle x_n \rangle_{n \in \mathbb{N}}$ and $a \in \mathfrak{F}(X)$ satisfy (50). The series $\sum_n g(x_n)a(x_n)$ converges absolutely. Let $h_1, h_2, \dots \in \mathfrak{F}_0(X, \mathbb{R}_+)$ be such that

$$\sup_{f \in \mathcal{F}} [|f|, h_k] \leq 2^{-k} \quad k \in \mathbb{N}. \quad (52)$$

Then,

$$+\infty > \sup_{f \in \mathcal{F}} \sum_k [|f|, h_k] = \sup_{f \in \mathcal{F}} \sum_x |f(x)| \sum_k h_k(x) = \sup_{f \in \mathcal{F}} \sum_n |f(x_n)a(x_n)| \quad (53)$$

where $x_1, x_2, \dots$ is an enumeration of the countable set $\bigcup_k \{h_k > 0\}$ and $a \in \mathfrak{F}(X, \mathbb{R}_+)$ is defined via

$$a(x) = \sum_k h_k(x) \quad \text{if} \quad \sup_{f \in \mathcal{F}} |f(x)| + |g(x)| > 0 \quad \text{or else} \quad a(x) = 0. \quad (54)$$

By assumption, $+\infty > \sum_n |g(x_n)a(x_n)| = \sum_k [|g|, h_k]$ and therefore $\lim_k [|g|, h_k] = 0$. Since every sequence $\langle h_k \rangle_{k \in \mathbb{N}}$ in $\mathfrak{F}_0(X, \mathbb{R}_+)$ for which $\lim_k \sup_{f \in \mathcal{F}} [|f|, h_k] = 0$ admits a subsequence satisfying (52), we conclude that

$$\lim_k \sup_{f \in \mathcal{F}} [|f|, h_k] = 0 \quad \text{implies} \quad \lim_k [|g|, h_k] = 0. \quad (55)$$

Observing that the function $[\cdot, \cdot]$ is separately homogeneous, we deduce that the inclusion $g \in \mathcal{S}_{\mathcal{F}}(X)$ implies the existence of $C_g > 0$ such that $[|g|, \delta] \leq C_g \sup_{f \in \mathcal{F}} [|f|, \delta]$ for each $\delta \in \Delta(X)$ so that (51) follows from Theorem 4.1.

Conversely, if (51) holds, and if $\langle x_n \rangle_{n \in \mathbb{N}}$ and $a \in \mathfrak{F}(X)$ satisfy (50), then

$$+\infty > C_g \sup_{f \in \mathcal{F}} \sum_n |f(x_n)a(x_n)| \geq C_g \sum_n \int_{\mathcal{F}} |f(x_n)a(x_n)| m(df) \geq \sum_n |g(x_n)a(x_n)|.$$

The last claim is an obvious consequence of well known results once noted that e_x is an X -continuous function on $\mathcal{F}$ and that $\mathcal{F}$ is X -compact by virtue of Tychonoff theorem. $\square$

Corollary 7.2 is a fully non linear extension of a well known result of Grothendieck-Pietsch [11, p.60] which concerns p -summing operators with $p \geq 1$, i.e. bounded linear operators $T \in \mathfrak{F}(X, Y)$ (Y a Banach space) which satisfy the condition

$$\sum_n \|Tx_n\|^p < \infty \quad \text{whenever} \quad \sum_n |x^*x_n|^p < \infty, \quad x^* \in S_{X^*}. \quad (56)$$

This criterion may be equivalently formulated as the condition

$$\lim_k \sum_{x \in X} \|Tx\|^p h_k(x) = 0 \quad \text{whenever} \quad \lim_k \sup_{x^* \in S_{X^*}} \sum_{x \in X} |x^*x|^p h_k(x) = 0 \quad (57)$$

for every sequence $\langle h_k \rangle_{k \in \mathbb{N}}$ in $\mathfrak{F}_0(X, \mathbb{R}_+)$, see [11, p.59], which corresponds to the inclusion $g \in \mathcal{S}_{\mathcal{F}}(X)$ when $\mathcal{F}$ consists of element of the form $f(x) = |x^*x|^p$ for some $x^* \in S_{X^*}$ and $g(x) = \|Tx\|^p$. Condition (51) is then a restatement of the inequality of Grothendieck and Pietsch.

Corollary 7.2 rests on an implicit Banach space structure which is worth making explicit. Assume that $\mathcal{F}$ is pointwise bounded and, with no loss of generality, that $\sup_{f \in \mathcal{F}} |f(x)| > 0$ for al $x \in X$. The space

$$\ell_{\mathcal{F}}(X) = \left\{ h \in \mathfrak{F}(X) : \sup_{f \in \mathcal{F}} \sum_x |f(x)h(x)| < +\infty \right\} \quad (58)$$

contains then $\mathfrak{F}_0(X)$. Endowed with pointwise order and with the norm

$$\|h\| = \sup_{f \in \mathcal{F}} \sum_x |f(x)h(x)|, \quad (59)$$

$\ell_{\mathcal{F}}(X)$ becomes a Banach lattice on which the bilinear form

$$\langle f, h \rangle = \sum_x f(x)h(x) \quad f \in \mathcal{F}, \quad h \in \ell_{\mathcal{F}}(X) \quad (60)$$

permits to associate with each $f \in \mathcal{F}$ an element of $S_{\ell_{\mathcal{F}}(X)^*}$.

Corollary 7.3. *Let $\mathcal{F} \subset \mathfrak{F}(X)$ be a pointwise bounded set satisfying the condition $\sup_{f \in \mathcal{F}} |f(x)| > 0$ for each $x \in X$. If $\varphi \in \ell_{\mathcal{F}}(X)^*$ then the associated function $T\varphi$ defined as $T\varphi(x) = \varphi(\delta_x)$ belongs to $\mathcal{S}_{\mathcal{F}}(X)$. In addition,*

$$\varphi(h) = \sum_{x \in X} T\varphi(x)h(x) \quad h \in \ell_{\mathcal{F}}(X) \quad (61)$$

if and only if φ is order continuous (in symbols $\varphi \in \ell_{\mathcal{F}}(X)^o$).

Proof. Let $\varphi \in \ell_{\mathcal{F}}(X)^*$, fix $h \in \ell_{\mathcal{F}}(X)$ and define $h_\alpha \in \ell_{\mathcal{F}}(X)$ as the restriction of h to some finite subset X_α of X . Then,

$$\sum_{x \in X_\alpha} |T\varphi(x)h(x)| = |\varphi(h_\alpha \text{sign}(T\varphi))| \leq \|\varphi\| \|h_\alpha\| \leq \|\varphi\| \|h\| \quad (62)$$

we conclude that $T\varphi \in \mathcal{S}_{\mathcal{F}}(X)$. If φ satisfies (61) it is clearly order continuous.

If, conversely, φ is order continuous then the net $\langle h_\alpha \rangle_{\alpha \in \mathfrak{A}}$ (with $\mathfrak{A}$ being directed by inclusion of the finite subsets of X) is order convergent to h so that

$$\varphi(h) = \lim_{\alpha} \varphi(h_\alpha) = \lim_{\alpha} \sum_{x \in X_\alpha} T\varphi(x)h(x) = \sum_{x \in X} T\varphi(x)h(x). \quad \square$$

The map T defined in Corollary 7.3 thus establishes a linear isomorphism between $\ell_{\mathcal{F}}(X)^o$ and $\mathcal{S}_{\mathcal{F}}(X)$.

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