Journal of Convex Analysis
Volume 30 (2023), No. 1, 111–130

The Norming Set of a Bilinear Form on $\mathbb{R}^2$ with the Octagonal Norm

Sung Guen Kim*, **Chang Yeol Lee, Ukje Jeong**
Department of Mathematics, Kyungpook National University, Daegu, Republic of Korea
sgk317@knu.ac.kr

Received: April 4, 2022
Accepted: June 5, 2022

An element $(x_1, \dots, x_n) \in E^n$ is called a *norming point* of $T \in \mathcal{L}(^nE)$ if $\|x_1\| = \cdots = \|x_n\| = 1$ and $|T(x_1, \dots, x_n)| = \|T\|$, where $\mathcal{L}(^nE)$ denotes the space of all continuous n-linear forms on E. For $T \in \mathcal{L}(^nE)$, we define

$$\operatorname{Norm}(T) = \{(x_1, \dots, x_n) \in E^n : (x_1, \dots, x_n) \text{ is a norming point of } T\}.$$

Let $\mathbb{R}^2_{o(w)}$ denote $\mathbb{R}^2$ with the octagonal norm with weight $0 < w \neq 1$

$$\|(x, y)\|_{o(w)} = \max\Big\{|x| + w|y|, |y| + w|x|\Big\}.$$

In this paper we classify $\operatorname{Norm}(T)$ for every $T \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})$ with weight $0 < w \neq 1$.

Keywords: Norming points, bilinear forms.

2010 Mathematics Subject Classification: 46A22.

1. Introduction

Let us introduce a brief history of norm attaining multilinear forms and polynomials on Banach spaces. In 1961 Bishop and Phelps [2] initiated and showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, specially bounded linear operators between Banach spaces. The problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [1], where they showed that the Radon-Nikodym property is sufficient for the denseness of norm attaining multilinear forms. Choi and Kim [3] showed that the Radon-Nikodym property is also sufficient for the denseness of norm attaining polynomials. Jiménez-Sevilla and Payá [5] studied the denseness of norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces.

Let $n \in \mathbb{N}$, $n \geq 2$. We write S_E for the unit sphere of a Banach space E. We denote by $\mathcal{L}(^nE)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{(x_1,\cdots,x_n) \in S_E \times \cdots \times S_E} |T(x_1, \cdots, x_n)|$. $\mathcal{L}_s(^nE)$ denote the closed subspace of all continuous symmetric n-linear forms on E. A point $(x_1, \dots, x_n) \in E^n$ is called a *norming point* of T if $\|x_1\| = \cdots = \|x_n\| = 1$ and $|T(x_1, \dots, x_n)| = \|T\|$.

*Corresponding author.

ISSN 0944-6532 / $ 2.50

For $T \in \mathcal{L}(^n E)$, we define

$$\mathrm{Norm}(T) = \{(x_1, \ldots, x_n) \in E^n : (x_1, \ldots, x_n) \text{ is a norming point of } T\}.$$

$\mathrm{Norm}(T)$ is called the *norming set* of T.

Kim ([7], [8]) classified $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}(^2 l_\infty^2)$ or $T \in \mathcal{L}_s(^3 l_1^2)$.

Notice that $(x_1, \ldots, x_n) \in \mathrm{Norm}(T)$ if and only if $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in \mathrm{Norm}(T)$ for some $\epsilon_k = \pm 1$ $(k = 1, \ldots, n)$. Indeed, if $(x_1, \ldots, x_n) \in \mathrm{Norm}(T)$, then

$$|T(\epsilon_1 x_1, \ldots, \epsilon_n x_n)| = |\epsilon_1 \cdots \epsilon_n T(x_1, \ldots, x_n)| = |T(x_1, \ldots, x_n)| = \|T\|,$$

which shows that $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in \mathrm{Norm}(T)$. If $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in \mathrm{Norm}(T)$ for some $\epsilon_k = \pm 1$ $(k = 1, \ldots, n)$, then

$$(x_1, \ldots, x_n) = (\epsilon_1(\epsilon_1 x_1), \ldots, \epsilon_n(\epsilon_n x_n)) \in \mathrm{Norm}(T).$$

The following examples show that $\mathrm{Norm}(T) = \emptyset$ or an infinite set.

Examples 1.1. (a) Let $T((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}) = \displaystyle\sum_{i=1}^{\infty} \frac{1}{2^i} x_i y_i \in \mathcal{L}_s(^2 c_0)$.

We claim that $\mathrm{Norm}(T) = \emptyset$. Obviously, $\|T\| = 1$. Assume that $\mathrm{Norm}(T) \neq \emptyset$. Let $((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}) \in \mathrm{Norm}(T)$. Then,

$$1 = |T((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}})| \leq \sum_{i=1}^{\infty} \frac{1}{2^i} |x_i|\ |y_i| \leq \sum_{i=1}^{\infty} \frac{1}{2^i} = 1,$$

which shows that $|x_i| = |y_i| = 1$ for all $i \in \mathbb{N}$. Hence, $(x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}} \notin c_0$. This is a contradiction. Therefore, $\mathrm{Norm}(T) = \emptyset$.

(b) Let $T((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}) = x_1 y_1 \in \mathcal{L}_s(^2 c_0)$. Then

$\mathrm{Norm}(T) = \{((\pm 1, x_2, x_3, \ldots), (\pm 1, y_2, y_3, \ldots)) \in c_0 \times c_0 : |x_j| \leq 1,\ |y_j| \leq 1 \text{ for } j \geq 2\}$.

(c) Let $T \in \mathcal{L}(^n E)$ and $(x_1, \ldots, x_n) \in \mathrm{Norm}(T)$. By Hahn-Banach theorem there are $f_j \in E^*$ such that $f_j(x_j) = \|f_j\| = 1$ for every $j = 1, \ldots, n$. Let ϕ be a permutation on $\{1, \ldots, n\}$. We define $T_{\phi, f_1, \ldots, f_n} \in \mathcal{L}(^{2n} E)$ by

$$T_{\phi, f_1, \ldots, f_n}(y_1, \ldots, y_{2n}) := \prod_{j=1}^{n} f_{\phi(j)}(y_{\phi(j)})\ T(y_{n+1}, \ldots, y_{2n}).$$

Then $$(x_{\phi(1)}, \ldots, x_{\phi(n)}, x_1, \ldots, x_n) \in \mathrm{Norm}(T_{\phi, f_1, \ldots, f_n}).$$ □

A mapping $P : E \to \mathbb{R}$ is a *continuous n-homogeneous polynomial* if there exists a continuous n-linear form L on the product $E \times \cdots \times E$ such that we have $P(x) = L(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^n E)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$.

An element $x \in E$ is called a *norming point* of $P \in \mathcal{P}(^n E)$ if $\|x\| = 1$ and $|P(x)| = \|P\|$. For $P \in \mathcal{P}(^n E)$, we define

$$\mathrm{Norm}(P) = \{x \in E : x \text{ is a norming point of } P\}.$$

Norm(P) is called the *norming set* of P. Notice that Norm$(P) = \emptyset$ or a finite set or an infinite set.

Kim [6] classified Norm(P) for every $P \in \mathcal{P}(^2 l_\infty^2)$, where $l_\infty^2 = \mathbb{R}^2$ with the supremum norm.

If Norm$(T) \neq \emptyset$, $T \in \mathcal{L}(^n E)$ is called a *norm attaining* n-linear form and if Norm$(P) \neq \emptyset$, $P \in \mathcal{P}(^n E)$ is called a *norm attaining* n-homogeneoue polynomial (see [3]).

For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [4].

It seems to be natural and interesting to classify Norm(T) for every $T \in \mathcal{L}(^n E)$.

Let $\mathbb{R}^2_{o(w)}$ denote $\mathbb{R}^2$with the octagonal norm with weight $0 < w \neq 1$

$$\|(x,y)\|_{o(w)} = \max\Big\{|x| + w|y|, |y| + w|x|\Big\}.$$

Notice that $\|(x,y)\|_{o(w)} = \|(y,x)\|_{o(w)} = \|(x,-y)\|_{o(w)}$ for $(x,y) \in \mathbb{R}^2_{o(w)}$ and that for every $T \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})$, Norm$(T) \neq \emptyset$ since $S_{\mathbb{R}^2_{o(w)}}$ is compact.

In this paper, we classify Norm(T) for every $T \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})$ with weight $0 < w \neq 1$.

2. Main results

Let $0 < w \neq 1$ and $T \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})$ be such that

$$T((x_1,y_1),(x_2,y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1.$$

For simplicity, we will denote T by (a,b,c,d).

Theorem 2.1. *Let $0 < w \neq 1$ and $T \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})$ be such that*

$$T((x_1,y_1),(x_2,y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 = (a,b,c,d)$$

for some $a,b,c,d \in \mathbb{R}$. Then:

(a) *If $0 < w < 1$, then*

$$\begin{aligned} \|T\| &= max\big\{|a|,|b|,|c|,|d|,(1+w)^{-1}(|a|+|c|),(1+w)^{-1}(|a|+|d|),\\ &\quad (1+w)^{-1}(|b|+|c|),(1+w)^{-1}(|b|+|d|),(1+w)^{-2}(|a-b|+|c-d|),\\ &\quad (1+w)^{-2}(|a+b|+|c+d|)\big\}. \end{aligned}$$

(b) *If $1 < w$, then*

$$\begin{aligned} \|T\| &= \max\big\{w^{-2}|a|,w^{-2}|b|,w^{-2}|c|,w^{-2}|d|,(w(1+w))^{-1}(|a|+|c|),\\ &\quad (w(1+w))^{-1}(|a|+|d|),(w(1+w))^{-1}(|b|+|c|),\\ &\quad (w(1+w))^{-1}(|b|+|d|),(1+w)^{-2}(|a-b|+|c-d|),\\ &\quad (1+w)^{-2}(|a+b|+|c+d|)\big\}. \end{aligned}$$

Proof. (a) Let $0 < w < 1$. Notice that

$$\text{ext}\, B_{\mathbb{R}^2_{o(w)}} = \Big\{\pm(1,0), \pm((1+w)^{-1}, \pm(1+w)^{-1}), \pm(0,1)\Big\}.$$

By the bilinearity of T, we have

$$\|T\| = \sup\Big\{|T((x_1,y_1),(x_2,y_2))| : (x_j,y_j) \in \text{ext}\, B_{\mathbb{R}^2_{o(w)}} \text{ for } j = 1,2\Big\}$$

$$
\begin{aligned}
= \ & \max\big\{|T((1,0),(1,0))|, |T((0,1),(0,1))|, |T((1,0),(0,1))|, |T((0,1),(1,0))|, \\
& |T((1,0),\pm((1+w)^{-1},\pm(1+w)^{-1}))|, |T(\pm((1+w)^{-1},\pm(1+w)^{-1}),(1,0))|, \\
& |T((0,1),\pm((1+w)^{-1},\pm(1+w)^{-1}))|, |T(\pm((1+w)^{-1},\pm(1+w)^{-1}),(0,1))|, \\
& |T(\pm((1+w)^{-1},\pm(1+w)^{-1}),((1+w)^{-1},-(1+w)^{-1})|, \\
& |T(((1+w)^{-1},-(1+w)^{-1}),((1+w)^{-1},(1+w)^{-1}))|, \\
& |T(((1+w)^{-1},(1+w)^{-1}),((1+w)^{-1},(1+w)^{-1}))|, \\
& |T(((1+w)^{-1},-(1+w)^{-1}),((1+w)^{-1},-(1+w)^{-1}))|\big\} \\
= \ & \max\big\{|a|,|b|,|c|,|d|,(1+w)^{-1}(|a|+|c|),(1+w)^{-1}(|a|+|d|),(1+w)^{-1}(|b|+|c|), \\
& (1+w)^{-1}(|b|+|d|),(1+w)^{-2}(|a-b|+|c-d|),(1+w)^{-2}(|a+b|+|c+d|)\big\}.
\end{aligned}
$$

(b) Let $w > 1$.

Claim: $\|T\|_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})} = \left\|w^{-2}T\right\|_{\mathcal{L}(^2\mathbb{R}^2_{o(\frac{1}{w})})}$.

Notice that $\|(w^{-1}x, w^{-1}y)\|_{o(w)} = \|(x,y)\|_{o(\frac{1}{w})}$ for $(x,y) \in \mathbb{R}^2$. It follows that

$$
\begin{aligned}
& \left\|w^{-2}T\right\|_{\mathcal{L}(^2\mathbb{R}^2_{o(\frac{1}{w})})} \\
= & \sup_{\|(x_j,y_j)\|_{o(\frac{1}{w})}=1,\ j=1,2} \left|w^{-2}ax_1x_2 + w^{-2}by_1y_2 + w^{-2}cx_1y_2 + w^{-2}dx_2y_1\right| \\
= & \sup_{\|(w^{-1}x_j,w^{-1}y_j)\|_{o(w)}=1,\ j=1,2} \Big|a(w^{-1}x_1)(w^{-1}x_2) + b(w^{-1}y_1)(w^{-1}y_2) \\
& \qquad + c(w^{-1}x_1)(w^{-1}y_2) + d(w^{-1}x_2)(w^{-1}y_1)\Big| \\
= & \|T\|_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})}.
\end{aligned}
$$

By (a), we have

$$
\begin{aligned}
\|T\|_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})} = \ & \left\|(w^{-2}a, w^{-2}b, w^{-2}c, w^{-2}d)\right\|_{\mathcal{L}(^2\mathbb{R}^2_{o(\frac{1}{w})})} \\
= \ & \max\big\{w^{-2}|a|, w^{-2}|b|, w^{-2}|c|, w^{-2}|d|, (1+w^{-1})^{-1}(w^{-2}|a|+w^{-2}|c|), \\
& (1+w^{-1})^{-1}(w^{-2}|a|+w^{-2}|d|), (1+w^{-1})^{-1}(w^{-2}|b|+w^{-2}|c|), \\
& (1+w^{-1})^{-1}(w^{-2}|b|+w^{-2}|d|), (1+w^{-1})^{-2}(w^{-2}|a-b|+w^{-2}|c-d|), \\
& (1+w^{-1})^{-2}(w^{-2}|a+b|+w^{-2}|c+d|)\big\} \\
= \ & \big\{w^{-2}|a|, w^{-2}|b|, w^{-2}|c|, w^{-2}|d|, (w(1+w))^{-1}(|a|+|c|), \\
& (w(1+w))^{-1}(|a|+|d|), (w(1+w))^{-1}(|b|+|c|), \\
& (w(1+w))^{-1}(|b|+|d|), (1+w)^{-2}(|a-b|+|c-d|), \\
& (1+w)^{-2}(|a+b|+|c+d|)\big\}. \quad \square
\end{aligned}
$$

Lemma 2.2. *Let* $0 < w \neq 1$ *and* $T = (a,b,c,d) \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})$ *for some* $a,b,c,d \in \mathbb{R}$. *Then, there exists* $T' = (a^*, b^*, c^*, d^*) \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})$ *with* $a^*, b^*, c^*, d^* \in \{\pm a, \pm b, \pm c, \pm d\}$ *where* $a^* \geq |b^*|$ *and* $a^* \geq c^* \geq d^* \geq 0$ *and* $\|T\| = \|T'\|$.

Proof. If $a < 0$, taking $-T$, we assume that $a \geq 0$. If $|b| > a$, we simply define $T_1 = (|b|, \mathrm{sign}(b)a, \mathrm{sign}(b)d, \mathrm{sign}(b)c)$. Then, $\|T_1\| = \|T\|$.

Hence, we may assume that $a \geq |b|$. If $c < 0$, we define $T_2 = (a, -b, -c, d)$. Then, $\|T_2\| = \|T\|$. Hence, we may assume that $a \geq |b|, c \geq 0$. If $d < 0$, we define $T_3 = (a, -b, c, -d)$. Then, $\|T_3\| = \|T\|$. Hence, we may assume that $a \geq |b|$, $c \geq 0$, $d \geq 0$. If $c < d$, we define $T_4 = (a, b, d, c)$. Then, $\|T_4\| = \|T\|$. Hence, we may assume that $a \geq |b|, c \geq d \geq 0$. If $a < c, b \geq 0$, we define $T_5 = (c, d, a, b)$. Then, $\|T_5\| = \|T\|$. Hence, we may assume that $a \geq |b|, c \geq d$. If $a < c, b < 0$, we define $T_6 = (c, -d, a, -b)$. Then, $\|T_6\| = \|T\|$. Hence, we may assume that $a \geq |b|$ and $a \geq c \geq d \geq 0$. Therefore, we can find a bilinear form T' which satisfies the conditions of the lemma. □

Let $0 < w < 1$ and

$$\begin{aligned}
B_1 &= \Big\{t(0,1) + (1-t)\Big(\frac{1}{1+w},\ \frac{1}{1+w}\Big) : 0 \leq t \leq 1\Big\},\\
B_2 &= \Big\{t(1,0) + (1-t)\Big(\frac{1}{1+w},\ \frac{1}{1+w}\Big) : 0 \leq t \leq 1\Big\},\\
B_3 &= \Big\{t(1,0) + (1-t)\Big(\frac{1}{1+w},\ \frac{-1}{1+w}\Big) : 0 \leq t \leq 1\Big\},\\
B_4 &= \Big\{t(0,-1) + (1-t)\Big(\frac{1}{1+w},\ \frac{-1}{1+w}\Big) : 0 \leq t \leq 1\Big\},\\
A_{ij} &= \Big\{\Big((x_1,y_1),\ (x_2,y_2)\Big) : (x_1,y_1) \in B_i,\ (x_2,y_2) \in B_j\Big\} \text{ for } i,j = 1,\ldots,4.
\end{aligned}$$

Notice that $S_{\mathbb{R}^2_{o(w)}} = \bigcup_{1 \leq k \leq 4} (\pm B_k)$.

Lemma 2.3. *Let $0 < w < 1$ and $T = (a,b,c,d) \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})$ for some $a,b,c,d \in \mathbb{R}$. Then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm(x_1,y_1),\ \pm(x_2,y_2)\Big) : \Big((x_1,y_1),\ (x_2,y_2)\Big) \in \bigcup_{1 \leq i,j \leq 4} (\mathrm{Norm}(T) \bigcap A_{ij})\Big\}.$$

Proof. This is obvious. □

Lemma 2.4. *Suppose that $0 < w < 1$ and $T = (a,b,c,d) \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})$ be such that $\|T\| = 1$. Let $\Big((x_1,y_1),(x_2,y_2)\Big) \in \mathrm{Norm}(T)$, $x_j \geq 0$ for $j = 1,2$ and $0 \leq t,s \leq 1$. Then the following statements hold:*

(1) *If $\big((x_1,y_1),(x_2,y_2)\big) \in A_{11}$, then*

$$\begin{aligned}
&|T((x_1,y_1),(x_2,y_2))|\\
=\ & \Big|T\Big(t(0,1) + (1-t)\Big(\frac{1}{1+w},\frac{1}{1+w}\Big), s(0,1) + (1-s)\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\Big|\\
=\ & ts|b| + t(1-s)\Big|\frac{b+d}{1+w}\Big| + (1-t)s\Big|\frac{b+c}{1+w}\Big| + (1-t)(1-s)\Big|\frac{a+b+c+d}{(1+w)^2}\Big| = 1.
\end{aligned}$$

(2) *If $\big((x_1,y_1),(x_2,y_2)\big) \in A_{12}$, then*

$$\begin{aligned}
&|T((x_1,y_1),(x_2,y_2))|\\
=\ & \Big|T\Big(t(0,1) + (1-t)\Big(\frac{1}{1+w},\frac{1}{1+w}\Big), s(1,0) + (1-s)\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\Big|\\
=\ & ts|d| + t(1-s)\Big|\frac{b+d}{1+w}\Big| + (1-t)s\Big|\frac{a+d}{1+w}\Big| + (1-t)(1-s)\Big|\frac{a+b+c+d}{(1+w)^2}\Big| = 1.
\end{aligned}$$

(3) *If* $\big((x_1,y_1),(x_2,y_2)\big) \in A_{13}$, *then*

$$\begin{aligned}
&|T((x_1,y_1),(x_2,y_2))| \\
= \ & \Big|T\Big(t(0,1)+(1-t)\Big(\frac{1}{1+w},\frac{1}{1+w}\Big), s(1,0)+(1-s)\Big(\frac{1}{1+w},\frac{-1}{1+w}\Big)\Big)\Big| \\
= \ & ts|d|+t(1-s)\Big|\frac{-b+d}{1+w}\Big|+(1-t)s\Big|\frac{a+d}{1+w}\Big|+(1-t)(1-s)\Big|\frac{a-b-c+d}{(1+w)^2}\Big|=1.
\end{aligned}$$

(4) *If* $\big((x_1,y_1),(x_2,y_2)\big) \in A_{14}$, *then*

$$\begin{aligned}
&|T((x_1,y_1),(x_2,y_2))| \\
= \ & \Big|T\Big(t(0,1)+(1-t)\Big(\frac{1}{1+w},\frac{1}{1+w}\Big), s(0,-1)+(1-s)\Big(\frac{1}{1+w},\frac{-1}{1+w}\Big)\Big)\Big| \\
= \ & ts|-b|+t(1-s)\Big|\frac{-b+d}{1+w}\Big|+(1-t)s\Big|\frac{-b-c}{1+w}\Big|+(1-t)(1-s)\Big|\frac{a-b-c+d}{(1+w)^2}\Big|=1.
\end{aligned}$$

(5) *If* $\big((x_1,y_1),(x_2,y_2)\big) \in A_{21}$, *then*

$$\begin{aligned}
&|T((x_1,y_1),(x_2,y_2))| \\
= \ & \Big|T\Big(t(1,0)+(1-t)\Big(\frac{1}{1+w},\frac{1}{1+w}\Big), s(0,1)+(1-s)\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\Big| \\
= \ & ts|c|+t(1-s)\Big|\frac{a+c}{1+w}\Big|+(1-t)s\Big|\frac{b+c}{1+w}\Big|+(1-t)(1-s)\Big|\frac{a+b+c+d}{(1+w)^2}\Big|=1.
\end{aligned}$$

(6) *If* $\big((x_1,y_1),(x_2,y_2)\big) \in A_{22}$, *then*

$$\begin{aligned}
&|T((x_1,y_1),(x_2,y_2))| \\
= \ & \Big|T\Big(t(1,0)+(1-t)\Big(\frac{1}{1+w},\frac{1}{1+w}\Big), s(1,0)+(1-s)\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\Big| \\
= \ & ts|a|+t(1-s)\Big|\frac{a+c}{1+w}\Big|+(1-t)s\Big|\frac{a+d}{1+w}\Big|+(1-t)(1-s)\Big|\frac{a+b+c+d}{(1+w)^2}\Big|=1.
\end{aligned}$$

(7) *If* $\big((x_1,y_1),(x_2,y_2)\big) \in A_{23}$, *then*

$$\begin{aligned}
&|T((x_1,y_1),(x_2,y_2))| \\
= \ & \Big|T\Big(t(1,0)+(1-t)\Big(\frac{1}{1+w},\frac{1}{1+w}\Big), s(1,0)+(1-s)\Big(\frac{1}{1+w},\frac{-1}{1+w}\Big)\Big)\Big| \\
= \ & ts|a|+t(1-s)\Big|\frac{a-c}{1+w}\Big|+(1-t)s\Big|\frac{a+d}{1+w}\Big|+(1-t)(1-s)\Big|\frac{a-b-c+d}{(1+w)^2}\Big|=1.
\end{aligned}$$

(8) *If* $\big((x_1,y_1),(x_2,y_2)\big) \in A_{24}$, *then*

$$\begin{aligned}
&|T((x_1,y_1),(x_2,y_2))| \\
= \ & \Big|T\Big(t(1,0)+(1-t)\Big(\frac{1}{1+w},\frac{1}{1+w}\Big), s(0,-1)+(1-s)\Big(\frac{1}{1+w},\frac{-1}{1+w}\Big)\Big)\Big| \\
= \ & ts|-c|+t(1-s)\Big|\frac{a-c}{1+w}\Big|+(1-t)s\Big|\frac{-b-c}{1+w}\Big|+(1-t)(1-s)\Big|\frac{a-b-c+d}{(1+w)^2}\Big|=1.
\end{aligned}$$

(9) *If* $\big((x_1,y_1),(x_2,y_2)\big)\in A_{31}$, *then*

$$\begin{aligned}
&|T((x_1,y_1),(x_2,y_2))|\\
=\ &\Big|T\Big(t(1,0)+(1-t)\Big(\frac{1}{1+w},\frac{-1}{1+w}\Big),s(0,1)+(1-s)\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\Big|\\
=\ &ts|c|+t(1-s)\Big|\frac{a+c}{1+w}\Big|+(1-t)s\Big|\frac{-b+c}{1+w}\Big|+(1-t)(1-s)\Big|\frac{a-b+c-d}{(1+w)^2}\Big|=1.
\end{aligned}$$

(10) *If* $\big((x_1,y_1),(x_2,y_2)\big)\in A_{32}$, *then*

$$\begin{aligned}
&|T((x_1,y_1),(x_2,y_2))|\\
=\ &\Big|T\Big(t(1,0)+(1-t)\Big(\frac{1}{1+w},\frac{-1}{1+w}\Big),s(1,0)+(1-s)\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\Big|\\
=\ &ts|a|+t(1-s)\Big|\frac{a+c}{1+w}\Big|+(1-t)s\Big|\frac{a-d}{1+w}\Big|+(1-t)(1-s)\Big|\frac{a-b+c-d}{(1+w)^2}\Big|=1.
\end{aligned}$$

(11) *If* $\big((x_1,y_1),(x_2,y_2)\big)\in A_{33}$, *then*

$$\begin{aligned}
&|T((x_1,y_1),(x_2,y_2))|\\
=\ &\Big|T\Big(t(1,0)+(1-t)\Big(\frac{1}{1+w},\frac{-1}{1+w}\Big),s(1,0)+(1-s)\Big(\frac{1}{1+w},\frac{-1}{1+w}\Big)\Big)\Big|\\
=\ &ts|a|+t(1-s)\Big|\frac{a-c}{1+w}\Big|+(1-t)s\Big|\frac{a-d}{1+w}\Big|+(1-t)(1-s)\Big|\frac{a+b-c-d}{(1+w)^2}\Big|=1.
\end{aligned}$$

(12) *If* $\big((x_1,y_1),(x_2,y_2)\big)\in A_{34}$, *then*

$$\begin{aligned}
&|T((x_1,y_1),(x_2,y_2))|\\
=\ &\Big|T\Big(t(1,0)+(1-t)\Big(\frac{1}{1+w},\frac{-1}{1+w}\Big),s(0,-1)+(1-s)\Big(\frac{1}{1+w},\frac{-1}{1+w}\Big)\Big)\Big|\\
=\ &ts|-c|+t(1-s)\Big|\frac{a-c}{1+w}\Big|+(1-t)s\Big|\frac{b-c}{1+w}\Big|+(1-t)(1-s)\Big|\frac{a+b-c-d}{(1+w)^2}\Big|=1.
\end{aligned}$$

(13) *If* $\big((x_1,y_1),(x_2,y_2)\big)\in A_{41}$, *then*

$$\begin{aligned}
&|T((x_1,y_1),(x_2,y_2))|\\
=\ &\Big|T\Big(t(0,-1)+(1-t)\Big(\frac{1}{1+w},\frac{-1}{1+w}\Big),s(0,1)+(1-s)\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\Big|\\
=\ &ts|-b|+t(1-s)\Big|\frac{-b-d}{1+w}\Big|+(1-t)s\Big|\frac{-b+c}{1+w}\Big|+(1-t)(1-s)\Big|\frac{a-b+c-d}{(1+w)^2}\Big|=1.
\end{aligned}$$

(14) *If* $\big((x_1,y_1),(x_2,y_2)\big)\in A_{42}$, *then*

$$\begin{aligned}
&|T((x_1,y_1),(x_2,y_2))|\\
=\ &\Big|T\Big(t(0,-1)+(1-t)\Big(\frac{1}{1+w},\frac{-1}{1+w}\Big),s(1,0)+(1-s)\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\Big|\\
=\ &ts|-d|+t(1-s)\Big|\frac{-b-d}{1+w}\Big|+(1-t)s\Big|\frac{a-d}{1+w}\Big|+(1-t)(1-s)\Big|\frac{a-b+c-d}{(1+w)^2}\Big|=1.
\end{aligned}$$

(15) *If* $\big((x_1,y_1),(x_2,y_2)\big) \in A_{43}$, *then*

$$\begin{aligned}
&|T((x_1,y_1),(x_2,y_2))| \\
= \;& \Big|T\Big(t(0,-1)+(1-t)\Big(\frac{1}{1+w},\frac{-1}{1+w}\Big), s(1,0)+(1-s)\Big(\frac{1}{1+w},\frac{-1}{1+w}\Big)\Big)\Big| \\
= \;& ts|-d| + t(1-s)\Big|\frac{b-d}{1+w}\Big| + (1-t)s\Big|\frac{a-d}{1+w}\Big| + (1-t)(1-s)\Big|\frac{a+b-c-d}{(1+w)^2}\Big| = 1.
\end{aligned}$$

(16) *If* $\big((x_1,y_1),(x_2,y_2)\big) \in A_{44}$, *then*

$$\begin{aligned}
&|T((x_1,y_1),(x_2,y_2))| \\
= \;& \Big|T\Big(t(0,-1)+(1-t)\Big(\frac{1}{1+w},\frac{-1}{1+w}\Big), s(0,-1)+(1-s)\Big(\frac{1}{1+w},\frac{-1}{1+w}\Big)\Big)\Big| \\
= \;& ts|b| + t(1-s)\Big|\frac{b-d}{1+w}\Big| + (1-t)s\Big|\frac{b-c}{1+w}\Big| + (1-t)(1-s)\Big|\frac{a+b-c-d}{(1+w)^2}\Big| = 1.
\end{aligned}$$

Proof. This follows from Theorem 2.1 and the bilinearity of T. $\square$

We let $\Omega = \Big\{(0,1),(0,-1),(1,0),\Big(\frac{1}{1+w},\frac{1}{1+w}\Big),\Big(\frac{1}{1+w},\frac{-1}{1+w}\Big)\Big\}$.

Theorem 2.5. *Let* $0<w<1$ *and* $T \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})$. *Then*

(1) $\mathrm{Norm}(T) \subseteq \displaystyle\bigcup_{(X,Y)\in \mathrm{Norm}(T)\cap(\Omega\times\Omega),\; X\in B_i, Y\in B_j} \Big(A_{ij}\cup A_{ji}\Big)$.

(2) *Let* $(X,Y)\in \mathrm{Norm}(T)$ *be such that* $X = tW_1+(1-t)W_2$, $Y = sW_1'+(1-s)W_2'$ *for some* $W_i, W_j' \in \Omega$ $(i,j=1,2,\ 0\le t,\ s\le 1)$.

If there are W_{i_0}, W_{j_0}' *such that* $(W_{i_0}, W_{j_0}') \notin \mathrm{Norm}(T)$, *then we have* $(t=0$ *or* $1)$ *or* $(s=0$ *or* $1)$.

Proof. (1) Let $S := \{(x,y)\in S_{\mathbb{R}^2_{o(w)}} : x\ge 0\}$ and

$$\mathcal{F} := \bigcup_{(X,Y)\in \mathrm{Norm}(T)\cap(\Omega\times\Omega),\; X\in B_i, Y\in B_j} \Big(A_{ij}\cup A_{ji}\Big).$$

It suffices to show that if $(U,V)\in (S\times S)\backslash\mathcal{F}$, then $(U,V)\notin \mathrm{Norm}(T)$.

Let $(U,V)\in (S\times S)\backslash\mathcal{F}$. Then there are $W_1, W_2, W_1', W_2' \in \Omega$ and $0\le t,s\le 1$ with

$$U = tW_1+(1-t)W_2, V = sW_1'+(1-s)W_2'.$$

Case 1. $(t=0$ or $1)$ and $(s=0$ or $1)$.

Thus $(U,V) = (W_{i_1}, W_{j_1}')$ for some $1\le i_1, j_1\le 2$. Thus $(U,V)\notin \mathrm{Norm}(T)$.

Indeed, assume that we have $(U,V)\in \mathrm{Norm}(T)$. Let $W_{i_1}\in B_{i_0}, W_{j_1}'\in B_{j_0}$ for some $1\le i_0, j_0\le 4$. Thus $(U,V) = (W_{i_1}, W_{j_1}') \in \mathrm{Norm}(T)\cap(\Omega\times\Omega)$ and hence $(U,V)\in \big(A_{i_0j_0}\cup A_{j_0i_0}\big)\subseteq \mathcal{F}$. This is a contradiction.

Case 2. $(0 < t < 1)$ and ($s = 0$ or 1).

For each $j = 1, 2$, $W_j \in B_{i_0}$ for some $1 \leq i_0 \leq 4$ because $\|U\|_{o(w)} = 1$ and $V = W_1'$ or W_2'. Without loss of generality we may assume that $V = W_1'$. Let $W_1' \in B_{j_0}$ for some $1 \leq j_0 \leq 4$. We claim that $(W_j, W_1') \notin \mathrm{Norm}(T)$ for $j = 1, 2$. Suppose not. We may assume that $(W_1, W_1') \in \mathrm{Norm}(T)$. Thus $(W_1, W_1') \in \mathrm{Norm}(T) \cap (\Omega \times \Omega)$. Hence, $(U, V) \in \left(A_{i_0 j_0} \cup A_{j_0 i_0}\right) \subseteq \mathcal{F}$. This is a contradiction.

Claim. $(U, V) \notin \mathrm{Norm}(T)$.

Suppose not. It follows that

$$\begin{aligned} \|T\| &= |T(U,V)| \leq t|T(W_1, W_1')| + (1-t)|T(W_2, W_1')| \\ &< t\|T\| + (1-t)\|T\| = \|T\|, \end{aligned}$$

which is a contradiction. Hence, the claim holds.

Case 3. $(0 < s < 1)$ and ($t = 0$ or 1).

For each $j = 1, 2$, $W_j' \in B_{i_0}$ for some $1 \leq i_0 \leq 4$ because $\|V\|_{o(w)} = 1$ and $U = W_1$ or W_2. Without loss of generality we may assume that $U = W_1$. Let $W_1 \in B_{j_0}$ for some $1 \leq j_0 \leq 4$. We claim that $(W_1, W_j') \notin \mathrm{Norm}(T)$ for $j = 1, 2$. Suppose not. We may assume that $(W_1, W_2') \in \mathrm{Norm}(T)$. Thus $(W_1, W_2') \in \mathrm{Norm}(T) \cap (\Omega \times \Omega)$. Hence, $(U, V) \in \left(A_{j_0 i_0} \cup A_{i_0 j_0}\right) \subseteq \mathcal{F}$. This is a contradiction.

By an analogous argument as in Case 2, $(U, V) \notin \mathrm{Norm}(T)$.

Case 4. $(0 < t < 1)$ and $(0 < s < 1)$.

For each $j = 1, 2$, $W_j \in B_{i_0}$ and $W_j' \in B_{j_0}$ for some $1 \leq i_0, j_0 \leq 4$ because $\|U\|_{o(w)} = \|V\|_{o(w)} = 1$. We claim that $(W_i, W_j') \notin \mathrm{Norm}(T)$ for $i, j = 1, 2$. Suppose not. We may assume that $(W_2, W_2') \in \mathrm{Norm}(T)$. Thus $(W_2, W_2') \in \mathrm{Norm}(T) \cap (\Omega \times \Omega)$. Hence, $(U, V) \in \left(A_{i_0 j_0} \cup A_{j_0 i_0}\right) \subseteq \mathcal{F}$. This is a contradiction.

We will show that $(U, V) \notin \mathrm{Norm}(T)$. Suppose not.

$$\begin{aligned} \|T\| &= |T(U,V)| \\ &\leq ts|T(W_1, W_1')| + t(1-s)|T(W_1, W_2')| + (1-t)s|T(W_2, W_1')| \\ &\quad + (1-t)(1-s)|T(W_2, W_2')| \\ &< ts|T(W_1, W_1')| + t(1-s)\|T\| + (1-t)s|T(W_2, W_1')| \\ &\quad + (1-t)(1-s)|T(W_2, W_2')| \\ &\leq (ts + t(1-s) + (1-t)s + (1-t)(1-s))\|T\| = \|T\|, \end{aligned}$$

which is a contradiction. This completes the proof of (1).

(2) Assume the contrary. Then $0 < t < 1$ and $0 < s < 1$. Without loss of generality we may assume that $i_0 = 1$ and $j_0 = 2$.

Claim. $t(1-s)=0$.

Assume that $t(1-s)>0$. Then

$$\begin{aligned}\|T\| &= |T(X,Y)| \\ &\le ts|T(W_1,W_1')| + t(1-s)|T(W_1,W_2')| + (1-t)s|T(W_2,W_1')| \\ &\quad + (1-t)(1-s)|T(W_2,W_2')| \\ &< ts|T(W_1,W_1')| + t(1-s)\|T\| + (1-t)s|T(W_2,W_1')| \\ &\quad + (1-t)(1-s)|T(W_2,W_2')| \\ &\le (ts+t(1-s)+(1-t)s+(1-t)(1-s))\|T\| = \|T\|,\end{aligned}$$

which is a contradiction. Hence, the claim holds.

Since $0<t<1$ and $0<s<1$, $0<t(1-s)=0$, which is impossible. Therefore, we complete the proof of (2). □

Lemma 2.6. *Let $0<w<1$ and $T=(a,b,c,d)\in\mathcal{L}(^2\mathbb{R}^2_{o(w)})$ for some $a,b,c,d\in\mathbb{R}$ with $\|T\|=1, a\ge|b|=-b$ and $a\ge c\ge d\ge 0$. If $\frac{1}{(1+w)^2}(a+b+c+d)=1$, then $a\le 1-w^2$.*

Proof. By Theorem 2.1, $a+c\le 1+w$. Notice that

$$(1+w)^2 = a+b+c+d = (a+c)-|b|+d \le 1+w-|b|+d.$$

Hence,
$$c\ge d\ge d-|b|\ge w+w^2,$$

which shows that $-c\le -w-w^2$. Therefore,

$$a\le -c+(1+w)\le -w-w^2+(1+w)=1-w^2.$$ □

Lemma 2.7. *Let $0<w<1$ and $T=(a,b,c,d)\in\mathcal{L}(^2\mathbb{R}^2_{o(w)})$ for some $a,b,c,d\in\mathbb{R}$ with $\|T\|=1, a\ge|b|$ and $a\ge c\ge d\ge 0$.*

Then
$$\frac{1}{(1+w)^2}(a-b+c-d) < \frac{1}{(1+w)^2}(a+b+c+d)\le 1$$
or
$$\frac{1}{(1+w)^2}(a+b+c+d) < \frac{1}{(1+w)^2}(a-b+c-d)\le 1.$$

Proof. Assume that
$$\frac{1}{(1+w)^2}(a-b+c-d) = \frac{1}{(1+w)^2}(a+b+c+d)=1.$$

Then $b+d=0$. By Theorem 2.1, $a+c\le 1+w$. It follows that

$$(1+w)^2 = a+c\le 1+w<(1+w)^2,$$

which is a contradiction. Therefore the proof is complete. □

Let $\mathcal{W}\subseteq S_{\mathbb{R}^2_{o(w)}}\times S_{\mathbb{R}^2_{o(w)}}$. We denote

$$\mathrm{Sym}(\mathcal{W}) \ := \ \Big\{(X,Y),(Y,X):(X,Y)\in\mathcal{W}\Big\}.$$

We are now in a position to prove the main result of this paper: In fact, for weight $0 < w < 1$, we classify $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}({}^2\mathbb{R}^2_{o(w)})$.

Theorem 2.8. *Let* $0 < w < 1$ *and* $T = (a, b, c, d) \in \mathcal{L}({}^2\mathbb{R}^2_{o(w)})$ *be such that* $\|T\| = 1$, $a \geq |b|$ *and* $a \geq c \geq d \geq 0$. *Then the following statements hold:*

Case 1. $b > 0$

Subcase 1. $\frac{1}{(1+w)^2}(a+b+c+d) = 1$

Suppose that $a = \frac{1}{1+w}(a+c) = 1$.

If $b = 1$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big), \pm\Big(\frac{1-t}{1+w}, \frac{tw+1}{1+w}\Big)\Big),$$
$$\Big(\pm\Big(\frac{1-t}{1+w}, \frac{tw+1}{1+w}\Big), \pm(0,1)\Big), \Big(\pm(1,0), \pm\Big(\frac{tw+1}{1+w}, \frac{1-t}{1+w}\Big)\Big),$$
$$\Big(\pm\Big(\frac{tw+1}{1+w}, \frac{1-t}{1+w}\Big), \pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big) : 0 \leq t \leq 1\Big\}.$$

If $w^2 < b < 1$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm(1,0), \pm\Big(\frac{tw+1}{1+w}, \frac{1-t}{1+w}\Big)\Big),$$
$$\Big(\pm\Big(\frac{tw+1}{1+w}, \frac{1-t}{1+w}\Big), \pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big) : 0 \leq t \leq 1\Big\}.$$

If $b = w^2$, *then* $\mathrm{Norm}(T) = A_{22}$.

Suppose that $\frac{1}{1+w}(a+c) < a = 1$.

If $b = 1$ *and* $c \neq 0$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm(1,0), \pm(1,0)\Big), \Big(\pm(0,1), \pm(0,1)\Big),$$
$$\Big(\pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big), \pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big)\Big\}.$$

If $b = 1$ *and* $c = 0$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm(1,0), \pm(1,0)\Big), \Big(\pm(0,1), \pm(0,1)\Big),$$
$$\Big(\pm\Big(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\Big), \pm\Big(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\Big)\Big), \pm\Big(\Big(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\Big), \pm\Big(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\Big)\Big)\Big\}.$$

If $b < 1$ *and* $c \neq 0$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm(1,0), \pm(1,0)\Big), \Big(\pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big), \pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big)\Big\}.$$

If $b<1$ and $c=0$, then

$$\mathrm{Norm}(T)=\Big\{\Big(\pm(1,0),\pm(1,0)\Big),\Big(\pm\Big(\frac{1}{1+w},\frac{1}{1+w}\Big),\pm\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big),$$
$$\pm\Big(\Big(\frac{1}{1+w},\frac{-1}{1+w}\Big),\pm\Big(\frac{1}{1+w},\frac{-1}{1+w}\Big)\Big)\Big\}.$$

Suppose that $a<\frac{1}{1+w}(a+c)=1$.

If $T\notin\mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$, *then*

$$\mathrm{Norm}(T)\;=\;\Big\{\Big(\pm\Big(\frac{tw+1}{1+w},\frac{1-t}{1+w}\Big),\pm\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big):0\le t\le 1\Big\}.$$

If $T\in\mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$, *then*

$$\mathrm{Norm}(T)\;=\;\mathrm{Sym}\Big(\Big\{\Big(\pm\Big(\frac{tw+1}{1+w},\frac{1-t}{1+w}\Big),\pm\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big):0\le t\le 1\Big\}\Big).$$

Suppose that $\max\{a,\frac{1}{1+w}(a+c)\}<1$.

If $c\neq 0$, then

$$\mathrm{Norm}(T)\;=\;\Big\{\Big(\pm\Big(\frac{1}{1+w},\frac{1}{1+w}\Big),\pm\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\Big\}.$$

If $c=0$, then

$$\mathrm{Norm}(T)=\Big\{\Big(\pm\Big(\frac{1}{1+w},\frac{1}{1+w}\Big),\pm\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big),$$
$$\pm\Big(\Big(\frac{1}{1+w},\frac{-1}{1+w}\Big),\pm\Big(\frac{1}{1+w},\frac{-1}{1+w}\Big)\Big)\Big\}.$$

Subcase 2. $\frac{1}{(1+w)^2}(a+b+c+d)<1$

Suppose that $a=\frac{1}{1+w}(a+c)=1$.

If $b=1$, then $$\mathrm{Norm}(T)=\Big\{\Big(\pm\Big(\frac{1-t}{1+w},\frac{tw+1}{1+w}\Big),\pm(0,1)\Big),$$
$$\Big(\pm(1,0),\pm\Big(\frac{tw+1}{1+w},\frac{1-t}{1+w}\Big)\Big):0\le t\le 1\Big\}.$$

If $b<1$ and $T\notin\mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$, *then*

$$\mathrm{Norm}(T)\;=\;\Big\{\Big(\pm(1,0),\pm\Big(\frac{tw+1}{1+w},\frac{1-t}{1+w}\Big)\Big):0\le t\le 1\Big\}.$$

If $b<1$ and $T\in\mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$, then

$$\mathrm{Norm}(T)\;=\;\mathrm{Sym}\Big(\Big\{\Big(\pm(1,0),\pm\Big(\frac{tw+1}{1+w},\frac{1-t}{1+w}\Big)\Big):0\le t\le 1\Big\}\Big).$$

Suppose that $\frac{1}{1+w}(a+c)<a=1$.

If $b=1$, then $\mathrm{Norm}(T)=\Big\{\Big(\pm(1,0),\pm(1,0)\Big),\Big(\pm(0,1),\pm(0,1)\Big)\Big\}$.

If $b < 1$, *then* $\quad \mathrm{Norm}(T) = \Big\{\Big(\pm (1,0), \pm(1,0)\Big)\Big\}.$

Suppose that $a < \frac{1}{1+w}(a+c) = 1.$

If $T \notin \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$, *then*

$$\mathrm{Norm}(T) \;=\; \Big\{\Big(\pm (1,0), \pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big)\Big\}.$$

If $T \in \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$, *then*

$$\mathrm{Norm}(T) \;=\; \mathrm{Sym}\,\Big(\Big\{\Big(\pm (1,0), \pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big)\Big\}\Big).$$

Case 2. $b \leq 0$

Subcase 1. $\frac{1}{(1+w)^2}(a+b+c+d) < \frac{1}{(1+w)^2}(a-b+c-d) = 1.$

Suppose that $a = \frac{1}{1+w}(a+c) = 1.$

If $b = -1$ *and* $\sqrt{2}-1 < w \leq \frac{\sqrt{5}-1}{2}$, *then*

$$\begin{aligned}\mathrm{Norm}(T) = \Big\{&\Big(\pm \Big(\frac{1}{1+w}, \frac{-1}{1+w}\Big), \pm\Big(\frac{1-t}{1+w}, \frac{tw+1}{1+w}\Big)\Big),\\ &\Big(\pm \Big(\frac{1-t}{1+w}, -\frac{tw+1}{1+w}\Big), \pm(0,1)\Big), \Big(\pm (1,0), \pm\Big(\frac{tw+1}{1+w}, \frac{1-t}{1+w}\Big)\Big),\\ &\Big(\pm \Big(\frac{tw+1}{1+w}, \frac{t-1}{1+w}\Big), \pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big) : 0 \leq t \leq 1\Big\}.\end{aligned}$$

If $b = -1$ *and* $w = \sqrt{2}-1$, *then*

$$\begin{aligned}\mathrm{Norm}(T) \;=\; \mathrm{Sym}\,\Big(\Big\{&\Big(\pm \Big(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\Big), \pm\Big(\frac{1-t}{\sqrt{2}}, \frac{(\sqrt{2}-1)t+1}{\sqrt{2}}\Big)\Big),\\ &\Big(\pm \Big(\frac{1-t}{\sqrt{2}}, -\frac{(\sqrt{2}-1)t+1}{\sqrt{2}}\Big), \pm(0,1)\Big),\\ &\Big(\pm (1,0), \pm\Big(\frac{(\sqrt{2}-1)t+1}{\sqrt{2}}, \frac{1-t}{\sqrt{2}}\Big)\Big),\\ &\Big(\pm \Big(\frac{(\sqrt{2}-1)t+1}{\sqrt{2}}, \frac{t-1}{\sqrt{2}}\Big), \pm\Big(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\Big)\Big) : 0 \leq t \leq 1\Big\}\Big).\end{aligned}$$

If $w^2 + w \leq |b| < w^2 + 2w < 1$, *then*

$$\begin{aligned}\mathrm{Norm}(T) \;=\; \Big\{&\Big(\pm (1,0), \pm\Big(\frac{tw+1}{1+w}, \frac{1-t}{1+w}\Big)\Big),\\ &\pm\Big(\pm \Big(\frac{tw+1}{1+w}, \frac{t-1}{1+w}\Big), \pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big) : 0 \leq t \leq 1\Big\}.\end{aligned}$$

If $|b| = w^2 + 2w < 1$, *then*

$$\begin{aligned}\mathrm{Norm}(T) &= \mathrm{Sym}\Big(\Big\{\Big(\pm(1,0), \pm\Big(\frac{tw+1}{1+w}, \frac{1-t}{1+w}\Big)\Big),\\ &\pm\Big(\pm\Big(\frac{tw+1}{1+w}, \frac{t-1}{1+w}\Big), \pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big) : 0 \le t \le 1\Big\}\Big).\end{aligned}$$

Suppose that $\frac{1}{1+w}(a+c) < a = 1$.

If $b = -1$, $c \neq 0, T \notin \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm(1,0), \pm(1,0)\Big), \Big(\pm(0,1), \pm(0,1)\Big),$$
$$\Big(\pm\Big(\frac{1}{1+w}, \frac{-1}{1+w}\Big), \pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big)\Big\}.$$

If $b = -1$, $c \neq 0$ *and* $T \in \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$,

$$\mathrm{Norm}(T) = \mathrm{Sym}\Big(\Big\{\Big(\pm(1,0), \pm(1,0)\Big), \Big(\pm(0,1), \pm(0,1)\Big),$$
$$\Big(\pm\Big(\frac{1}{1+w}, \frac{-1}{1+w}\Big), \pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big)\Big\}\Big).$$

If $b = -1$, $c = 0$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm(1,0), \pm(1,0)\Big), \Big(\pm(0,1), \pm(0,1)\Big),$$
$$\Big(\pm\Big(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\Big), \pm\Big(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\Big)\Big), \Big(\pm\Big(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\Big), \pm\Big(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\Big)\Big)\Big\}.$$

If $|b| < 1$, $c \neq 0$ *and* $T \notin \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm(1,0), \pm(1,0)\Big), \Big(\pm\Big(\frac{1}{1+w}, \frac{-1}{1+w}\Big), \pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big)\Big\}.$$

If $(|b| < 1,\ c \neq 0$ *and* $T \in \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)}))$ *or* $(|b| < 1, c = 0)$, *then*

$$\mathrm{Norm}(T) = \mathrm{Sym}\Big(\Big\{\Big(\pm(1,0), \pm(1,0)\Big), \Big(\pm\Big(\frac{1}{1+w}, \frac{-1}{1+w}\Big), \pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big)\Big\}\Big).$$

Suppose that $a < \frac{1}{1+w}(a+c) = 1$.

If $T \notin \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm\Big(\frac{tw+1}{1+w}, \frac{t-1}{1+w}\Big), \pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big) : 0 \le t \le 1\Big\}.$$

If $T \in \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$, *then*

$$\mathrm{Norm}(T) = \mathrm{Sym}\Big(\Big\{\Big(\pm\Big(\frac{tw+1}{1+w}, \frac{t-1}{1+w}\Big), \pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big) : 0 \le t \le 1\Big\}\Big).$$

Suppose that $\max\{a, \frac{1}{1+w}(a+c)\} < 1$.

If $T \notin \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm\Big(\frac{1}{1+w}, \frac{-1}{1+w}\Big), \pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big)\Big\}.$$

If $T \in \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$, *then*

$$\mathrm{Norm}(T) = \mathrm{Sym}\Big(\Big\{\Big(\pm\Big(\frac{1}{1+w}, \frac{-1}{1+w}\Big), \pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big)\Big\}\Big).$$

Subcase 2. $\frac{1}{(1+w)^2}(a-b+c-d) < \frac{1}{(1+w)^2}(a+b+c+d) = 1.$

Suppose that $a < \frac{1}{1+w}(a+c) = 1.$

If $T \notin \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm\Big(\frac{tw+1}{1+w}, \frac{1-t}{1+w}\Big), \pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big) : 0 \le t \le 1\Big\}.$$

If $T \in \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$, *then*

$$\mathrm{Norm}(T) = \mathrm{Sym}\Big(\Big\{\Big(\pm\Big(\frac{tw+1}{1+w}, \frac{1-t}{1+w}\Big), \pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big) : 0 \le t \le 1\Big\}\Big).$$

If $\max\{a, \frac{1}{1+w}(a+c)\} < 1$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big), \pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big)\Big\}.$$

Subcase 3. $\max\Big\{\frac{1}{(1+w)^2}(a+b+c+d), \frac{1}{(1+w)^2}(a-b+c-d)\Big\} < 1.$

Suppose that $a = \frac{1}{1+w}(a+c) = 1.$

If $b = -1$ *and* $T \notin \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm(0,1), \pm(0,1)\Big), \Big(\pm(1,0), \pm\Big(\frac{tw+1}{1+w}, \frac{1-t}{1+w}\Big)\Big) : 0 \le t \le 1\Big\}.$$

If $b = -1$ *and* $T \in \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$, *then*

$$\mathrm{Norm}(T) = \mathrm{Sym}\Big(\Big\{\Big(\pm(0,1), \pm(0,1)\Big), \Big(\pm(1,0), \pm\Big(\frac{tw+1}{1+w}, \frac{1-t}{1+w}\Big)\Big) : 0 \le t \le 1\Big\}\Big).$$

If $|b| < 1$ *and* $T \notin \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm(1,0), \pm\Big(\frac{tw+1}{1+w}, \frac{1-t}{1+w}\Big)\Big) : 0 \le t \le 1\Big\}.$$

If $|b| < 1$ *and* $T \in \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$, *then*

$$\mathrm{Norm}(T) = \mathrm{Sym}\Big(\Big\{\Big(\pm(1,0), \pm\Big(\frac{tw+1}{1+w}, \frac{1-t}{1+w}\Big)\Big) : 0 \le t \le 1\Big\}\Big).$$

Suppose that $\frac{1}{1+w}(a+c) < a = 1.$

If $b = -1$, *then* $\mathrm{Norm}(T) = \Big\{\Big(\pm(1,0), \pm(1,0)\Big), \Big(\pm(0,1), \pm(0,1)\Big)\Big\}.$

If $|b| < 1$, *then* $\mathrm{Norm}(T) = \Big\{\Big(\pm(1,0), \pm(1,0)\Big)\Big\}.$

Suppose that $a < \frac{1}{1+w}(a+c) = 1$.

If $T \notin \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$, *then* $\mathrm{Norm}(T) = \Big\{\Big(\pm(1,0), \pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big)\Big\}$.

If $T \in \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$, *then* $\mathrm{Norm}(T) = \mathrm{Sym}\Big(\Big\{\Big(\pm(1,0), \pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big)\Big\}\Big)$.

Proof. We only present the proofs for each Subcase 1 of Case 1 and Case 2 because the proofs for the other subcases are similar.

By Theorem 2.1, $c \leq \frac{1+w}{2}$ because $\|T\| = 1$. Let us take $(X, Y) \in \mathrm{Norm}(T)$, where $X = (x_1, y_1)$ and $Y = (x_2, y_2)$ for $x_j \geq 0$ $(j = 1, 2)$.

Case 1. $b > 0$

By Lemma 2.7, $\frac{1}{(1+w)^2}(a-b+c-d) < \frac{1}{(1+w)^2}(a+b+c+d) \leq 1$.

Subcase 1. $\frac{1}{(1+w)^2}(a+b+c+d) = 1$.

By Theorem 2.1, $a = \frac{1}{1+w}(a+c) = 1, \frac{1}{1+w}(a+c) < a = 1, a < \frac{1}{1+w}(a+c) = 1$ or $\max\{a, \frac{1}{1+w}(a+c)\} < 1$.

Suppose that $a = \frac{1}{1+w}(a+c) = 1$. Obviously, $c = w \geq d$. Let $b = 1$. Then

$$T = (1, 1, w, w^2 + w - 1) \notin \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)}).$$

By Theorem 2.5(1), $(X, Y) \in A_{11} \cup A_{22}$ because $c \neq 0$.

Let $(X, Y) \in A_{11}$. By Lemma 2.4,

$$\begin{aligned} & |T((x_1, y_1), (x_2, y_2))| \\ = & \Big|T\Big(t(0,1) + (1-t)\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big), s(0,1) + (1-s)\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big)\Big| \\ = & ts|b| + t(1-s)\Big|\frac{b+d}{1+w}\Big| + (1-t)s\Big|\frac{b+c}{1+w}\Big| + (1-t)(1-s)\Big|\frac{a+b+c+d}{(1+w)^2}\Big| = 1. \end{aligned}$$

By Theorem 2.5(2), $t(1-s) = 0$. If $t = 0$, then, for every $0 \leq t \leq 1$, we have

$$\Big(\pm\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big), \pm\Big(t(0,1) + (1-t)\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big)\Big) \in \mathrm{Norm}(T)$$

If $s = 1$, then

$$\Big(\pm\Big(t(0,1) + (1-t)\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big), \pm(0,1)\Big) \in \mathrm{Norm}(T) \text{ for every } 0 \leq t \leq 1.$$

Let $(X, Y) \in A_{22}$. By Lemma 2.4,

$$\begin{aligned} & |T((x_1, y_1), (x_2, y_2))| \\ = & \Big|T\Big(t(1,0) + (1-t)\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big), s(1,0) + (1-s)\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big)\Big| \\ = & ts|a| + t(1-s)\Big|\frac{a+c}{1+w}\Big| + (1-t)s\Big|\frac{a+d}{1+w}\Big| + (1-t)(1-s)\Big|\frac{a+b+c+d}{(1+w)^2}\Big| = 1. \end{aligned}$$

By Theorem 2.5(2), $(1-t)s=0$. If $t=1$, then, for every $0\le t\le 1$,

$$\Big(\pm\Big((1,0),\pm\Big(t(1,0)+(1-t)\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\Big)\Big)\in \mathrm{Norm}(T).$$

If $s=0$, then, for every $0\le t\le 1$,

$$\Big(\pm\Big(t(1,0)+(1-t)\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big),\pm\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\in \mathrm{Norm}(T).$$

Let $b<1$. Then $w^2\le b$. Notice that if $w^2<b<1$, then $T\notin \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$ and that if $b=w^2$, then $T=(1,w^2,w,w)\in \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$. By the above arguments as in the case $b=1$, we get the results asserted in Theorem 2.8.

Suppose that $\frac{1}{1+w}(a+c)<a=1$.

By Theorem 2.5(1), $(X,Y)\in A_{11}\cup A_{22}\cup A_{33}$ because c may be zero. Notice that if $b=1$ and $c\neq 0$, then $\sqrt{2}-1<w<1$ and that if $b=1$ and $c=0$, then $w=\sqrt{2}-1$. Note that if $b<1$ and $c=0$, then $w<\sqrt{2}-1$ and $T=(1,w^2+2w,0,0)$. By the above arguments as in the case $b=1$, we get the results asserted in Theorem 2.8.

Suppose that $a<\frac{1}{1+w}(a+c)=1$.

By Theorem 2.5(1), $(X,Y)\in A_{11}\cup A_{22}$ because $c\neq 0$. By the above arguments as in the case $b=1$, we get the results asserted in Theorem 2.8.

Suppose that $\max\{a,\frac{1}{1+w}(a+c)\}<1$.

By Theorem 2.5(1), $(X,Y)\in A_{11}\cup A_{22}\cup A_{33}$ because c may be zero. Notice that if $c=0$, then $0<w<\sqrt{2}-1$. By the above arguments as in the case $b=1$, we get the results asserted in Theorem 2.8.

Subcase 2. $\frac{1}{(1+w)^2}(a+b+c+d)<1$

By Theorem 2.5(1), $(X,Y)\in A_{11}\cup A_{22}$. By Theorem 2.1,

$$a=\frac{1}{1+w}(a+c)=1,\ \frac{1}{1+w}(a+c)<a=1\ \text{ or }\ a<\frac{1}{1+w}(a+c)=1.$$

Suppose that $a=\frac{1}{1+w}(a+c)=1$.

If $b=1$, then $\sqrt{2}-1<w<1$, $T=(1,1,w,d)\notin \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$ for some $d<w^2+w-1$.

Suppose that $\frac{1}{1+w}(a+c)<a=1$. If $b=1$, then $\sqrt{2}-1<w<1$.

Note that Norm(T) in each step is contained in the norming sets which occurs in each step in the Subcase 1. Since the results in the subcase 2 follows from those in the Subcase 1, we omit the proof of the Subcase 2.

Case 2. $b\le 0$

By Lemma 2.7 we have

$$\frac{1}{(1+w)^2}(a+b+c+d)<\frac{1}{(1+w)^2}(a-b+c-d)=1,$$

$$\frac{1}{(1+w)^2}(a-b+c-d) < \frac{1}{(1+w)^2}(a+b+c+d) = 1,$$

or $$\max\Big\{\frac{1}{(1+w)^2}(a-b+c-d), \frac{1}{(1+w)^2}(a+b+c+d)\Big\} < 1.$$

Subcase 1. $\frac{1}{(1+w)^2}(a+b+c+d) < \frac{1}{(1+w)^2}(a-b+c-d) = 1.$

By Theorem 2.1, $a = \frac{1}{1+w}(a+c) = 1,\ \frac{1}{1+w}(a+c) < a = 1,\ a < \frac{1}{1+w}(a+c) = 1$ or $\max\{a, \frac{1}{1+w}(a+c)\} < 1$.

Suppose that $a = \frac{1}{1+w}(a+c) = 1$.

By Theorem 2.5(1), $(X,Y) \in A_{41} \cup A_{32}$. Obviously, $c = w \geq d$.

Let $b = -1$.

Notice that $\sqrt{2}-1 \leq w \leq \frac{\sqrt{5}-1}{2}$. If $\sqrt{2}-1 < w \leq \frac{\sqrt{5}-1}{2}$, then

$$T = (1, -1, w, -w^2 - w + 1) \notin \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)}).$$

If $w = \sqrt{2}-1$, then $T = (1, -1, \sqrt{2}-1, \sqrt{2}-1) \in \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$. If $b = -1,\ c \neq 0$, $T \notin \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$, then $\sqrt{2}-1 < w < 1$. If $(b = -1,\ c \neq 0,\ T \in \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$ or $(b = -1,\ c = 0)$, then $w = \sqrt{2}-1$.

Let $(X,Y) \in A_{41}$. By Lemma 2.4,

$$\begin{aligned}
&|T((x_1,y_1),(x_2,y_2))| \\
&= \Big|T\Big(t(0,-1) + (1-t)\Big(\frac{1}{1+w}, \frac{-1}{1+w}\Big), s(0,1) + (1-s)\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big)\Big| \\
&= ts|b| + t(1-s)\Big|\frac{b+d}{1+w}\Big| + (1-t)s\Big|\frac{b-c}{1+w}\Big| + (1-t)(1-s)\Big|\frac{a-b+c-d}{(1+w)^2}\Big| = 1.
\end{aligned}$$

By Theorem 2.5(2), $t(1-s) = 0$. If $t = 0$, then, for every $0 \leq t \leq 1$,

$$\Big(\pm\Big(\frac{1}{1+w}, \frac{-1}{1+w}\Big), \pm\Big(t(0,1) + (1-t)\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big)\Big) \in \mathrm{Norm}(T).$$

If $s = 1$, then, for every $0 \leq t \leq 1$,

$$\Big(\pm\Big(t(0,-1) + (1-t)\Big(\frac{1}{1+w}, -\frac{1}{1+w}\Big)\Big), \pm(0,1)\Big) \in \mathrm{Norm}(T).$$

Let $(X,Y) \in A_{32}$. By Lemma 2.4,

$$\begin{aligned}
&|T((x_1,y_1),(x_2,y_2))| \\
&= \Big|T\Big(t(1,0) + (1-t)\Big(\frac{1}{1+w}, \frac{-1}{1+w}\Big), s(1,0) + (1-s)\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big)\Big| \\
&= ts|a| + t(1-s)\Big|\frac{a+c}{1+w}\Big| + (1-t)s\Big|\frac{a-d}{1+w}\Big| + (1-t)(1-s)\Big|\frac{a-b+c-d}{(1+w)^2}\Big| = 1.
\end{aligned}$$

By Theorem 2.5(2), $(1-t)s = 0$. If $t = 1$, then, for every $0 \leq t \leq 1$,

$$\Big(\pm(1,0), \pm\Big(t(1,0) + (1-t)\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big)\Big)\Big) \in \mathrm{Norm}(T).$$

If $s = 0$, then, for every $0 \le t \le 1$,

$$\Big(\pm\Big(t(1,0)+(1-t)\Big(\frac{1}{1+w},\frac{-1}{1+w}\Big)\Big),\Big(\frac{1}{1+w},\frac{1}{1+w}\Big)\Big)\in \mathrm{Norm}(T).$$

Let $|b| < 1$.

Notice that $w^2 + w \le |b| \le w^2 + 2w$. Note that if $w^2 + w \le |b| < w^2 + 2w$, then $T \notin \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$ and that if $|b| = w^2 + 2w$, then $T \in \mathcal{L}_s({}^2\mathbb{R}^2_{o(w)})$. By the above arguments as in the case $b = -1$, we get the results which asserted in Theorem 2.8.

Suppose that $\frac{1}{1+w}(a+c) < a = 1, a < \frac{1}{1+w}(a+c) = 1$ or $\max\{a, \frac{1}{1+w}(a+c)\} < 1$. By Theorem 2.5(1), $(X,Y) \in A_{41} \cup A_{32}$. By the above arguments as in the case $b = -1$, we get the results which asserted in Theorem 2.8.

Subcase 2. $\frac{1}{(1+w)^2}(a-b+c-d) < \frac{1}{(1+w)^2}(a+b+c+d) = 1.$

Notice that $c \neq 0$. Thus by Theorem 2.5(1), $(X,Y) \in A_{11} \cup A_{22}$. By Lemma 2.6, $a \le 1 - w^2$. Thus, $a < \frac{1}{1+w}(a+c) = 1$ or $\max\{a, \frac{1}{1+w}(a+c)\} < 1$. By the analogous arguments as in the subcase 1 of the case 1, we get the results which asserted in Theorem 2.8.

Subcase 3. $\max\big\{\frac{1}{(1+w)^2}(a-b+c-d), \frac{1}{(1+w)^2}(a+b+c+d)\big\} < 1.$

By Theorem 2.5(1), $(X,Y) \in A_{11} \cup A_{22}$. By Theorem 2.1, $a = \frac{1}{1+w}(a+c) = 1$, $\frac{1}{1+w}(a+c) < a = 1$ or $a < \frac{1}{1+w}(a+c) = 1$. By analogous arguments as in Subcase 1 of Case 1, we get the results asserted in Theorem 2.8. This completes the proof. $\Box$

Finally, we classify $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}({}^2\mathbb{R}^2_{o(w)})$ with weight $w > 1$ in terms of $\mathrm{Norm}(w^{-2}T)$ which was classified in Theorem 2.8.

Theorem 2.9. *Let $w > 1$ and $T = (a,b,c,d) \in \mathcal{L}({}^2\mathbb{R}^2_{o(w)})$. Then*

$$\mathrm{Norm}(T) = \big\{\big((w^{-1}x_1, w^{-1}y_1),\ (w^{-1}x_2, w^{-1}y_2)\big) : \big((x_1,y_1),\ (x_2,y_2)\big) \in \mathrm{Norm}(w^{-2}T)\big\},$$

where $w^{-2}T \in \mathcal{L}({}^2\mathbb{R}^2_{o(w^{-1})})$.

Proof. Let $\big((x_1,y_1),\ (x_2,y_2)\big) \in \mathrm{Norm}(w^{-2}T)$. Notice that

$$\|(w^{-1}x_i, w^{-1}y_i)\|_{o(w)} = \|(x_i,y_i)\|_{o(w^{-1})} = 1 \text{ for } i = 1,2.$$

It follows that

$$\begin{aligned}\|T\|_{\mathcal{L}({}^2\mathbb{R}^2_{o(w)})} &= \big\|w^{-2}T\big\|_{\mathcal{L}({}^2\mathbb{R}^2_{o(w^{-1})})} = \big|w^{-2}T\big((x_1,y_1),\ (x_2,y_2)\big)\big| \\ &= \big|T\big((w^{-1}x_1, w^{-1}y_1),\ (w^{-1}x_2, w^{-1}y_2)\big)\big|,\end{aligned}$$

which shows that $\big((w^{-1}x_1, w^{-1}y_1),\ (w^{-1}x_2, w^{-1}y_2)\big) \in \mathrm{Norm}(T)$. Hence,

$$\big\{\big((w^{-1}x_1, w^{-1}y_1), (w^{-1}x_2, w^{-1}y_2)\big) : \big((x_1,y_1), (x_2,y_2)\big) \in \mathrm{Norm}(w^{-2}T)\big\} \subseteq \mathrm{Norm}(T).$$

For the reverse inclusion, let $\big((u_1,v_1),\ (u_2,v_2)\big) \in \mathrm{Norm}(T)$. Then

$$\|(wu_i, wv_i)\|_{o(w^{-1})} = \|(u_i,v_i)\|_{o(w)} = 1 \text{ for } i = 1,2.$$

It follows that

$$\begin{aligned}&\Big|w^{-2}T\Big((wu_1, wv_1),\ (wu_2, wv_2)\Big)\Big| = \Big|T\Big((u_1, v_1),\ (u_2, v_2)\Big)\Big| \\ &= \|T\|_{\mathcal{L}(^2\mathbb{R}^2_{o(w)})} = \Big\|w^{-2}T\Big\|_{\mathcal{L}(^2\mathbb{R}^2_{o(w^{-1})})},\end{aligned}$$

which shows that $\big((wu_1, wv_1),\ (wu_2, wv_2)\big) \in \mathrm{Norm}(w^{-2}T)$. Hence,

$$\Big\{\Big((w^{-1}x_1, w^{-1}y_1), (w^{-1}x_2, w^{-1}y_2)\Big) : \big((x_1, y_1), (x_2, y_2)\big) \in \mathrm{Norm}(w^{-2}T)\Big\} \supseteq \mathrm{Norm}(T).$$

The proof is complete. $\square$

Remark 2.10. By means of Theorem 2.8 and 2.9, we have classified $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})$ with weight $0 < w \neq 1$. $\square$

In [9], Kim posed the following question:
Let E be a real Banach space. Let $T, S \in \mathcal{L}(^2E)$ be such that $\|T\| = \|S\|$ and $\mathrm{Norm}(T) = \mathrm{Norm}(S)$. Is it true that $T = \pm S$?

Kim [9] showed that the question is true for $E = l_1^2$, where $l_1^2 = \mathbb{R}^2$ with the l_1-norm. Note that the question is not true in general: Indeed, let $T = (1, |b|, 0, 0)$ or $(1, -|b|, 0, 0) \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})$ for $0 < |b| < \min\{1, w^2 + 2w\}$ and $0 < w < 1$. Then by Subcase 2 of Case 1 and Subcase 3 of Case 2 of Theorem 2.8 we conclude that $\mathrm{Norm}(T) = \big\{\big(\pm(1, 0), \pm(1, 0)\big)\big\}$.

References

[1] R. M. Aron, C. Finet, E. Werner: *Some remarks on norm-attaining n-linear forms*, in: *Function Spaces*, Edwardsville 1994, Lecture Notes in Pure and Applied Mathematics 172, Dekker, New York (1995) 19–28.

[2] E. Bishop, R. Phelps: *A proof that every Banach space is subreflexive*, Bull. Amer. Math. Soc. 67 (1961) 97–98.

[3] Y. S. Choi, S. G. Kim: *Norm or numerical radius attaining multilinear mappings and polynomials*, J. London Math. Soc. (2) 54 (1996) 135–147.

[4] S. Dineen: *Complex Analysis on Infinite Dimensional Spaces*, Springer, London (1999).

[5] M. Jiménez Sevilla, R. Payá: *Norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces*, Studia Math. 127 (1998) 99–112.

[6] S. G. Kim: *The norming set of a polynomial in $\mathcal{P}(^2l^2_\infty)$*, Honam Math. J. 42/3 (2020) 569–576.

[7] S. G. Kim: *The norming set of a bilinear form on l^2_∞*, Comment. Math. 60/1-2 (2020) 37–63.

[8] S. G. Kim: *The norming set of a symmetric 3-linear form on the plane with the l_1-norm*, New Zealand J. Math. 51 (2021) 95–108.

[9] S. G. Kim: *The norming sets of $\mathcal{L}(^2l^2_1)$ and $\mathcal{L}_s(^2l^3_1)$*, to appear in Bull. Trans. Univ. Brasov, Ser. III Math. Comput. Sci. 2(64) No. 2 (2022) 125–150.