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Competitive Equilibrium in Affine Economies

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Our approach is committed to suitably enriching the duality commodity-price at the level of a single structure instead of the classical usage of dual pairs. To this aim, we study economies having as commodity space the vector lattice of real affine continuous functions on a Bauer simplex playing the role of the price space. Furthermore, the initial endowment is a partition of the unity and the preferences are represented by valuation functions on the extreme points boundary of the price space. The main goal in this paper is the obtainment of a competitive equilibrium existence theorem.

Keywords: Bauer simplex, affine economy, competitive equilibrium.

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1. Introduction

In the economic activity, what is observed is a finite number of agents meeting in the market to exchange their respective products. This fact suggests to study commercial transactions and the behavior of the economy from the analysis of the commodities demanded by the agents.

In an uncertainty world, economies spreading on an infinite horizon are considered, that is, commodities are contingent to possible states of the nature and the time, that will be appraised as continuous variables and therefore the commodity space will have an infinite-dimensional structure. The first seminal work on economies with infinite-dimensional commodity spaces was due to Bewley [8].

Debreu [10] expressed the dupla *commodity-price* in terms of a dual pair, by assuming that agents respond to linear prices: $\langle E, P\rangle$ where E is a real vector space called *commodity space* and P called *price space*, a vector subspace of its algebraic dual E^* (linear functionals on E). Elements $f \in E$ are the *commodity bundles* and $x \in P$ are the *price systems*. The *value* of the bundle $f \in E$ for the price $x \in P$ is $x(f) \in \mathbb{R}$.

The *weak topology* $\sigma(E^*, E)$ is the weakest topology on the algebraic dual E^* making continuous the functions

$$f\colon E^* \to \mathbb{R},\ x \mapsto f(x) = x(f), \quad f \in E.$$

A basic consideration associated to any economy without production is that each consumer $i = 1, \dots, n$ is defined by a consumption set $E_i \subseteq E$, and contributes to the market with a bundle $\omega_i \in E_i$ called *initial endowment*. The bundle $\omega = \omega_1 + \cdots + \omega_n \in E$ consisting of the sum of all the initial endowments is called *total*

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endowment. A *feasible allocation* is any family of bundles $(f_1, \dots, f_n) \in E_1 \times \dots \times E_n$ clearing the market, i.e. $f_1 + \dots + f_n = \omega$.

Economic agents make rational choices by means of some rule allowing them to choose the most preferred bundle according to their possibilities and tastes. To this aim, a *preference* on E_i is a binary relation $\preceq_i$ satisfying:

(a) $f \preceq_i f$;

(b) $f \preceq_i g$ and $g \preceq_i h$, then $f \preceq_i h$;

(c) $f \preceq_i g$ or $g \preceq_i f$.

We shall denote $f \prec_i g$ if $f \preceq_i g$ and $g \not\preceq_i f$.

A situation of equilibrium basically looks for maximizing preferences within the budgets. More specifically, a feasible allocation $(f_1, \dots, f_n)$ is said to be a *competitive equilibrium* if there exists a price system $x \in P$, $x \neq 0$ such that $x(f_i) \leq x(\omega_i)$ for every i, and $f_i \prec_i g_i$ for some $g_i \in E_i$ implies $x(\omega_i) < x(g_i)$.

Along last two decades, order structure on vector spaces has become prominent in the economics equilibrium analysis, mainly in the study of properties related to efficiency. Aliprantis-Brown [4] suggested the most appropriate framework to develop an infinite-dimensional equilibrium theory is a dual pair of vector lattices (relevancy of lattice structures to ensure equilibrium existence is discussed in [6]).

Recall that an *ordered vector space* is a real vector space E endowed with an order relationship " $\leq$ " compatible with its vector structure; that is, if $f, g, h \in E$ and $r \geq 0$, then $f \leq g$ implies $f + h \leq g + h$ and $rf \leq rg$.

If $E_+ = \{f \in E : f \geq 0\}$ denotes the positive cone of an ordered vector space E, then an order relationship on E^* is pointwise defined: $x \leq y$ iff $x(f) \leq y(f)$ for all $f \in E_+$. Hence, the set of positive functionals $E^*_+ = \{x \in E^* : x \geq 0\}$ is well defined.

Definition 1.1. The *order dual* of an ordered vector space E is the ordered vector subspace $E^\sim = E^*_+ - E^*_+$.

A *vector lattice* (also called *Riesz space*) is an ordered vector space satisfying that every finite subset has a greatest lower bound (infimum) and a least upper bound (supremum).

As usual, infimum and supremum of a finite subset $F \subseteq E$ will be denoted by $\bigwedge F$ and $\bigvee F$ respectively. The absolute value of $f \in E$ is $|f| = f \vee -f$.

Definition 1.2. The *Riesz dual* of a vector lattice E is defined as the vector lattice E^+ consisting of Riesz homomorphisms from E to $\mathbb{R}$ (linear functionals on E preserving infima and suprema of finite subsets).

It is clear that E^+ becomes a vector lattice, and that $E^+ \subseteq E^\sim \subseteq E^*$ as topological subspaces for their respective weak topologies.

Economic agents have some idea on the closeness among commodities. By introducing topological spaces, such a situation can be modeled by means of the notion of neighbourhood. As we will see later, the model will acquire a bigger generality, and it will be an adequate framework to the economic theory.

It is possible to consider some compatibility form between topology and order, to this aim a mapping $u\colon E \to \mathbb{R}_+$ is said to be a *lattice seminorm* provided it satisfies:

(a) $u(f+g) \leq u(f) + u(g)$;

(b) $u(rf) = ru(f)$, $r > 0$;

(c) $|f| \leq |g|$ implies $u(f) \leq u(g)$.

Definition 1.3. The *order-bound topology* on a vector lattice is the topology defined by the family of all the lattice seminorms.

In our previous paper [17] we widely studied the order-bound topology. The next lemma can be derived from [19, 6.1 p. 237] and [18, Proposition 4.1.4 p. 88]:

Lemma 1.4. *Let E be a vector lattice endowed with its order-bound topology. Its topological dual E' (continuous linear functionals on E) coincides with the order dual $E^\sim$.*

2. Bauer simplices

Denote by $C(P)$ the vector lattice of real continuous functions on a topological space P, and

$$C^*(P) = \{f \in C(P) : |f| \leq n \text{ for some } n \in \mathbb{N}\}$$

its vector sublattice of bounded elements.

Lemma 2.1. *Every compact Hausdorff space P is homeomorphic to the set*

$$\beta P = \{x \in C(P)^+ : x(1) = 1\}$$

equipped with the weak topology $\sigma(C(P)^+, C(P))$.

Proof. Is well-known that P compact implies that $C(P)$ is Riesz isomorphic to $C^*(P)$, and by the classical Banach-Stone theorem, what is enough to prove is that βP is homeomorphic to the *Stone-Čech compactification* of P; that is, βP is a compact space having P as a dense subspace and for which P is C^*-embedded (i.e. for every $f \in C^*(P)$ there exists a $f^\beta \in C(\beta P)$ such that $f^\beta_{|P} = f$).

On the one hand, βP inherits the product topology of $\mathbb{R}^{C^*(P)}$, and therefore every mapping in the closure $\overline{\beta P}$ is a Riesz homomorphism carrying 1 to 1. Taking into account the continuity of the projections $\pi\colon \mathbb{R}^{C^*(P)} \to \mathbb{R}$, $\pi(x) = x(1)$, it follows that P is C^*-embedded in βP and that

$$\pi\big(\overline{\beta P}\big) \subseteq \overline{\pi(\beta P)} = \{1\}.$$

Consequently, βP is a closed subset of $\mathbb{R}^{C^*(P)}$.

On the other hand, given $f \in C^*(P)$, there exists $n \in \mathbb{N}$ such that $|f| \leq n$, so $|x(f)| \leq x(|f|) \leq n$ for each $x \in \beta P$. Thus, the image of the elements by the projections are bounded, and therefore βP is compact.

As for density, let $x_0 \in \beta P$, $f_1, \ldots, f_n \in C^*(P)$ and $\varepsilon > 0$. If we consider

$$f = |f_1 - x_0(f_1)| \vee \cdots \vee |f_n - x_0(f_n)| \in C^*(P)$$

then $x_0(f) = 0$ and consequently there exists $p_0 \in P$ such that $f(p_0) < \varepsilon$ (otherwise $\varepsilon \leq f(p)$ for every $p \in P$, implying $\varepsilon \leq f$ and therefore $0 < \varepsilon \leq x_0(f)$).

From inequality $f(p_0) < \varepsilon$ it follows that $|f_i(p_0) - x_0(f_i)| < \varepsilon$, $i = 1, \ldots, n$ and then P is actually dense in βP. $\square$

A subset P of a vector space is called *convex* whenever it contains the segment joining every its two points; that is,

$$\lambda p + (1-\lambda)q \in P, \quad p, q \in P,\ 0 \leq \lambda \leq 1.$$

The class of *topological convex spaces* (i.e. convex subspaces of a Hausdorff locally convex vector space) is provided with at least two structures: topology and convex combinations. Thus, two topological convex spaces are regarded as identical if they have the same elements, the same open subsets, and the same convex combinations.

Definition 2.2. Given a topological convex space P, a mapping $f\colon P \to \mathbb{R}$ is said to be *affine* provided it preserves convex combinations; that is,

$$f(\lambda p + (1-\lambda)q) = \lambda f(p) + (1-\lambda) f(q), \quad p, q \in P,\ 0 \leq \lambda \leq 1.$$

We denote by $A(P)$ the ordered vector space of all affine continuous functions from P to $\mathbb{R}$.

The following well-known result was proved by Kadison [14, p. 328]:

Lemma 2.3. *Every compact convex space P is affine homeomorphic to the set*

$$\gamma P = \{x \in A(P)^{\sim} : x(1) = 1\}$$

equipped with the weak topology $\sigma(A(P)^{\sim}, A(P))$.

The *extreme points boundary* of a convex set P is defined as

$$\partial P = \{p \in P : P - \{p\} \text{ remains convex}\}.$$

The next definition was announced by Alfsen in [1, Theorem II.4.1 p. 103], and characterizes a class of compact convex spaces that will be crucial to our purposes.

Definition 2.4. A compact convex space P is said to be a *Bauer simplex* provided it satisfies one of the following three equivalent conditions:

(i) ∂P is closed in P;

(ii) $A(P)$ and $C(\partial P)$ are isomorphic as ordered vector spaces;

(iii) $A(P)$ is a vector lattice.

Feldman-Porter proved in [11, Theorem 1, p. 292] that the order-bound topology in spaces of continuous functions on realcompact spaces is the compact convergence topology. Taking into account that $A(P) = C(\partial P)$ as vector lattices, we may characterize its order-bound topology:

Lemma 2.5. *Let P be a Bauer simplex. The order-bound topology in $A(P)$ is the topology defined by the lattice norm:*

$$\|f\|_\partial = \sup\{f(q) : q \in \partial P\}$$

Denote by $A'(P)$ the topological dual of $A(P)$ (i.e. $\|\ \|_\partial$-continuous linear functionals on $A(P)$). As a consequence of Lemma 1.4 and Lemma 2.1 we obtain:

Corollary 2.6. *If P is a Bauer simplex, then*

$$P = \beta P = \gamma P = \{x \in A(P)' : x(1) = 1\}.$$

3. Affine economies

In the sequel we are assuming that commodity spaces are of the form $A(P)$ for some Bauer simplex P playing the role of price space.

There is a vast literature dealing with the representation of preferences by utility functions (see for instance Bridges-Mehta [9]). Since $A(P) = C(\partial P)$ as vector lattices, we may consider only preferences represented by valuation functions on the extreme points of P, that is, those preferences $\preceq$ for which there exists $q \in \partial P$ such that:

$$f \preceq g \iff f(q) \leq g(q).$$

Under this assumption all information related to preferences would be inherent to the inner structure of the commodity space.

We will use boldface letters referring vector terms:

$$\boldsymbol{f} = (f_1, \ldots, f_n) \in \boldsymbol{A(P)}_+ = A(P)_+ \times \overset{n}{\cdots} \times A(P)_+,$$

$$\boldsymbol{p} = (p_1, \ldots, p_n) \in \boldsymbol{P} = P \times \overset{n}{\cdots} \times P,\ \boldsymbol{q} = (q_1, \ldots, q_n) \in \boldsymbol{\partial P} = \partial P \times \overset{n}{\cdots} \times \partial P$$

$$\boldsymbol{f}(\boldsymbol{p}) = (f_1(p_1), \ldots, f_n(p_n)) \in \mathbb{R}^n, \text{ and } \boldsymbol{f} \cdot \boldsymbol{p} = f_1(p_1) + \cdots + f_n(p_n) \in \mathbb{R}.$$

Definition 3.1. An *affine economy* is a triplet $\langle P, \boldsymbol{\omega}, \boldsymbol{q}\rangle$ where P is a Bauer simplex, $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n) \in \boldsymbol{A(P)}_+$ is a partition of unity (i.e. $\omega_1 + \cdots + \omega_n = 1$) and $\boldsymbol{q} = (q_1, \ldots, q_n) \in \boldsymbol{\partial P}$ representing the preferences.

Given an affine economy $\langle P, \boldsymbol{\omega}, \boldsymbol{q}\rangle$, the *budget set* for $\boldsymbol{p} \in \boldsymbol{P}$ is

$$\boldsymbol{B}(\boldsymbol{p}) = \{\boldsymbol{f} \in \boldsymbol{A(P)}_+ : \boldsymbol{f}(\boldsymbol{p}) \leq \boldsymbol{\omega}(\boldsymbol{p})\}.$$

We may consider the set of the $\boldsymbol{q}$*-preferred bundles* for $\boldsymbol{f} \in \boldsymbol{A(P)}_+$

$$\boldsymbol{S_q(f)} = \{\boldsymbol{g} \in \boldsymbol{A(P)}_+ : \boldsymbol{f}(\boldsymbol{q}) \leq \boldsymbol{g}(\boldsymbol{q})\},$$

and its subset of *strictly* $\boldsymbol{q}$*-preferred bundles*

$$\boldsymbol{S^*_q(f)} = \{\boldsymbol{g} \in \boldsymbol{S_q(f)} : f_i(q_i) \neq g_i(q_i),\ i = 1, \ldots, n\}.$$

Feasible allocations are denoted by

$$\boldsymbol{F} = \Big\{\boldsymbol{f} \in \boldsymbol{A(P)}_+ : f_1 + \cdots + f_n = 1\Big\},$$

and the *weakly Pareto optimal* allocations are

$$\boldsymbol{W} = \{\boldsymbol{f} \in \boldsymbol{F} : \boldsymbol{S^*_q(f)} \cap \boldsymbol{F} = \varnothing\}.$$

Definition 3.2. A feasible allocation $\boldsymbol{f} \in \boldsymbol{F}$ is a *competitive equilibrium* if there exists $p \in P$, $p \neq 0$ such that $\boldsymbol{f} \in \boldsymbol{B}(\boldsymbol{p})$ and $\boldsymbol{S^*_q(f)} \cap \boldsymbol{B}(\boldsymbol{p}) = \varnothing$ for $\boldsymbol{p} = (p, \ldots, p)$.

3.1. An example

The 1-simplex $P = \{(x, y) \in [0, 1] \times [0, 1] : x + y = 1\}$

endowed with the subspace topology induced from the canonical topology in $\mathbb{R}^2$ is a compact convex space. Moreover, its extreme points boundary

$$\partial P = \{e_1, e_2\}, \text{ with } e_1 = (0, 1), e_2 = (1, 0)$$

is closed, and therefore P becomes a Bauer simplex.

Indeed, for any $f^* \in C(\partial P)$, we may define the real continuous function:

$$(x, y) \in P \mapsto f(x, y) = f^*(e_2)x + f^*(e_1)y.$$

Given $0 \leq \lambda \leq 1$, we have:

$$\begin{aligned} f(\lambda(x, y) + (1-\lambda)(x', y')) =& f^*(e_2)(\lambda x + (1-\lambda)x') + f^*(e_1)(\lambda y + (1-\lambda)y') \\ =& \lambda(f^*(e_2)x + f^*(e_1)y) + (1-\lambda)(f^*(e_2)x' + f^*(e_1)y') \\ =& \lambda f(x, y) + (1-\lambda)f(x', y'). \end{aligned}$$

Thus, $f \in A(P)$.

Consider the 2-consumers affine economy

$$\langle P, \boldsymbol{\omega}, \boldsymbol{q} \rangle$$

The initial endowment $\boldsymbol{\omega} = (\omega_1, \omega_2) \in \boldsymbol{A(P)}_+$ is defined by:

$$\omega_1(x, y) = ax + by, \;\; \omega_2(x, y) = (1-a)x + (1-b)y, \quad a, b \in [0, 1],$$

and accordingly with the extreme points $\boldsymbol{q} = (q_1, q_2) \in \boldsymbol{\partial P} = \{e_1, e_2\} \times \{e_1, e_2\}$ representing the preferences, next situations may occur:

(a) if $q_1 = e_1$ and $q_2 = e_2$, then let:

$$f_1(x, y) = y, \;\; f_2(x, y) = x \text{ and } p = \left(\frac{1-b}{1-b+a}, \frac{a}{1-b+a}\right)$$

(b) if $q_1 = e_2$ and $q_2 = e_1$, then let:

$$f_1(x, y) = x, \;\; f_2(x, y) = y \text{ and } p = \left(\frac{b}{1-a+b}, \frac{1-a}{1-a+b}\right)$$

(c) if $q_1 = q_2 = e_1$, then let:

$$f_1(x, y) = (1-a)x + by, \;\; f_2(x, y) = ax + (1-b)y \text{ and } p = e_1$$

(d) if $q_1 = q_2 = e_2$, then let:

$$f_1(x, y) = ax + (1-b)y, \;\; f_2(x, y) = (1-a)x + by \text{ and } p = e_2$$

In any of the above cases (a)–(d), $\boldsymbol{f} = (f_1, f_2) \in \boldsymbol{F} \subseteq \boldsymbol{A(P)}_+$ and $p \in P$, $p \neq 0$ satisfy $\boldsymbol{S^*_q(f)} \cap \boldsymbol{B}(\boldsymbol{p}) = \varnothing$, for $\boldsymbol{p} = (p, p)$ and therefore $\boldsymbol{f}$ becomes a competitive equilibrium supported by p (see Fig. 3.1).

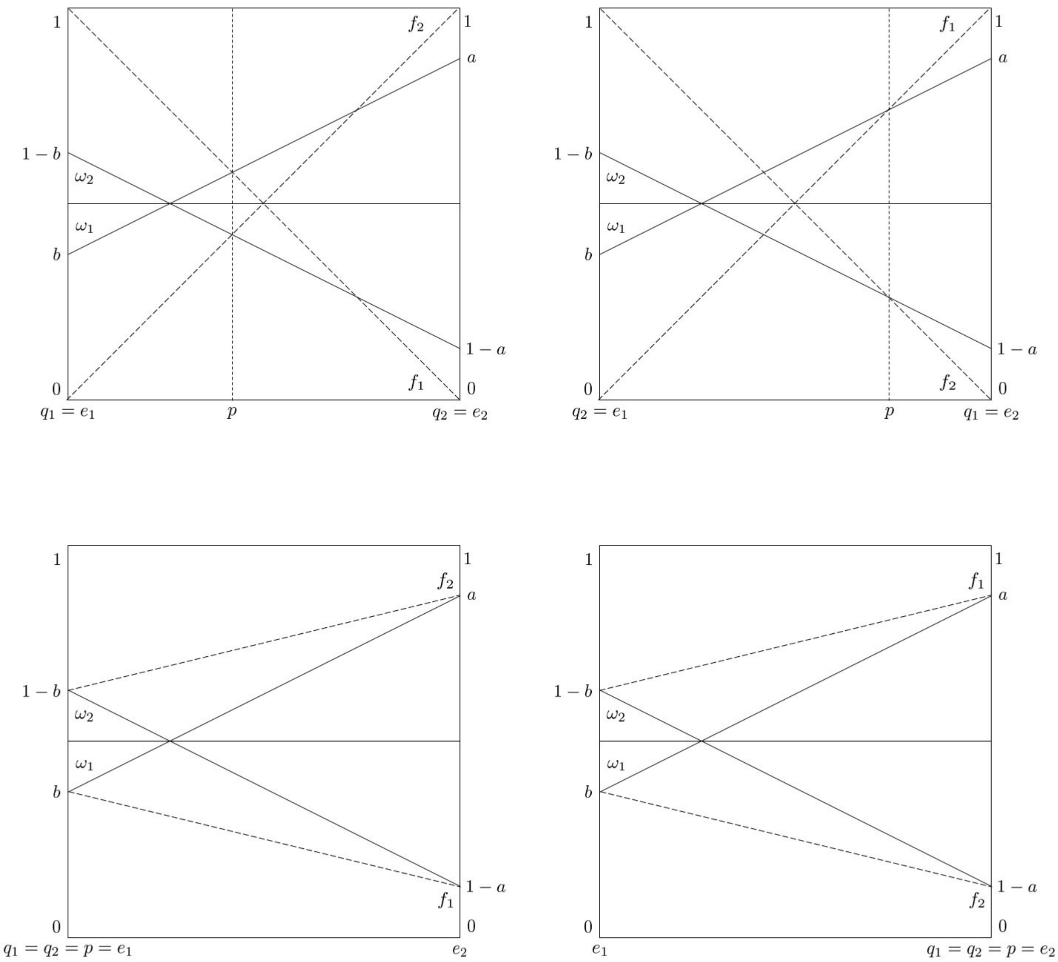

Figure 3.1: Competitive equilibrium in the 1-simplex affine economy

4. Equilibrium existence theorem

Denote by $\sigma(\boldsymbol{A(P)}, \boldsymbol{P})$ the weakest topology in $\boldsymbol{A(P)}$ making continuous the mappings

$$\boldsymbol{p}\colon \boldsymbol{A(P)} \to \mathbb{R}^n, \ \boldsymbol{f} \mapsto \boldsymbol{p}(\boldsymbol{f}) = \boldsymbol{f}(\boldsymbol{p}), \quad \boldsymbol{p} \in \boldsymbol{P}.$$

The next lemma is inspired in a well-known result due to Florenzano [12, Proposition 4.2.2 p. 91]:

Lemma 4.1. $\boldsymbol{W}$ *is a nonempty* $\sigma(\boldsymbol{A(P)}, \boldsymbol{P})$*-compact subset.*

Proof. Since $\boldsymbol{F}$ is a $\sigma(\boldsymbol{A(P)}, \boldsymbol{P})$-closed subset of

$$[\mathbf{0}, \mathbf{1}]_{\boldsymbol{A(P)}} = [0,1]_{A(P)} \times \overset{n}{\cdots} \times [0,1]_{A(P)}, \ \ [0,1]_{A(P)} = \{f \in A(P) : 0 \le f \le 1\}$$

it would be enough to see that $[0,1]_{A(P)}$ is $\sigma(A(P), P)$-compact to prove that $\boldsymbol{F}$ is $\sigma(\boldsymbol{A(P)}, \boldsymbol{P})$-compact.

Let U be a open covering of $[0,1]_{A(P)}$ and denote by τ the product topology on $\mathbb{R}^P$. On the one hand, $\sigma \subseteq \tau$ because $A(P) \subseteq \mathbb{R}^P$ and therefore U becomes a τ-open cover as well. If we prove that $[0,1]_{A(P)}$ is τ-compact, we conclude by extracting a finite subcovering.

Taking into account that

$$[0,1]_{A(P)} = \bigcap_{p \in P} p^{-1}[0,1]$$

is $\sigma(A(P),P)$-closed, then $[0,1]_{A(P)}$ is a τ-closed subset of $[0,1]_{\mathbb{R}^P}$. By virtue of the classical Tychonoff theorem we have that $[0,1]_{\mathbb{R}^P}$ is τ-compact, and therefore $[0,1]_{A(P)}$ is τ-compact as well. Then $\boldsymbol{W}$ is $\sigma(\boldsymbol{A(P)},\boldsymbol{P})$-compact because it is a $\sigma(\boldsymbol{A(P)},\boldsymbol{P})$-closed subset of $\boldsymbol{F}$.

As for the nonemptiness of $\boldsymbol{W}$, define the correspondence

$$\boldsymbol{S}_{\boldsymbol{q}}^{*}\colon \boldsymbol{F} \to \boldsymbol{F},\ \boldsymbol{f} \mapsto \boldsymbol{S}_{\boldsymbol{q}}^{*}(\boldsymbol{f}) \cap \boldsymbol{F}$$

and suppose that $\boldsymbol{S}_{\boldsymbol{q}}^{*}(\boldsymbol{f}) \cap \boldsymbol{F} \neq \varnothing$ for every $\boldsymbol{f} \in \boldsymbol{F}$. Then $\left\{\boldsymbol{S}_{\boldsymbol{q}}^{*-1}(\boldsymbol{f})\right\}_{\boldsymbol{f} \in \boldsymbol{F}}$ forms an open covering of $\boldsymbol{F}$. Since $\boldsymbol{F}$ is $\sigma(\boldsymbol{A(P)},\boldsymbol{P})$-compact, let $\left\{\boldsymbol{S}_{\boldsymbol{q}}^{*-1}(\boldsymbol{f}_i)\right\}_{i=1}^{m}$ be a finite subcover, where $\boldsymbol{f}_i = (f_{i1},\dots,f_{in})$. If

$$\boldsymbol{f} = (f_{11} \vee \cdots \vee f_{m1}, \dots, f_{1n} \vee \cdots \vee f_{mn})$$

then $\boldsymbol{S}_{\boldsymbol{q}}^{*}(\boldsymbol{f}) \cap (\boldsymbol{f}_1,\dots,\boldsymbol{f}_m) = \varnothing$, but in view of the above finite covering there exists some $1 \leq i \leq m$ such that $\boldsymbol{f}_i \in \boldsymbol{S}_{\boldsymbol{q}}^{*}(\boldsymbol{f}) \cap \boldsymbol{F}$, contrary to $\boldsymbol{S}_{\boldsymbol{q}}^{*}(\boldsymbol{f}) \cap \boldsymbol{F} = \varnothing$. □

The next lemma is an adaptation of the well-known separation theorem that can be found in the book of Aliprantis-Burkinshaw [5, Theorem 3.10 p. 142].

Lemma 4.2. *Let F and G be disjoint, convex nonempty subsets of $A(P)$ for P a Bauer simplex. If F or G has nonempty interior for the order-bound topology, then there exists $p \in P$ such that $f(p) \leq g(p)$ for all $f \in F$, $g \in G$.*

Proof. Classical separation theorem states that there exists $x \in A(P)'$ such that $x(f) \leq x(g)$ for all $f \in F$, $g \in G$. Lemma 1.4 ensures that $x \in A(P)' = A(P)^{\sim}$, and by defining $p = \frac{x}{x(1)} \in P$ we conclude that $f(p) \leq g(p)$ for all $f \in F$, $g \in G$. □

We will need making use of an immediate consequence of the Halpern-Bergman fixed-point theorem [13, Theorem 4.1 p. 356] (more comprehensive discussions about this topic can be found in the book of Aliprantis-Border [3, Theorem 17.16 p. 563 and Theorem 17.54 p. 582]).

Lemma 4.3. *Let the simplex $\Delta = \left\{\boldsymbol{s} \in \mathbb{R}^n_+ : s_1 + \cdots + s_n = 1\right\}$. If a correspondence $\Phi : \Delta \twoheadrightarrow \Delta$ satisfies both:*

(i) *if $\{(s^\alpha, t^\alpha)\}_\alpha \to (s,t)$ is a converging net in Δ^2 such that $t^\alpha \in \Phi(s^\alpha)$, then $t \in \Phi(s)$;*

(ii) *for any $s \in \Delta$, $\Phi(s)$ is a convex compact nonempty subset of Δ,*

then, there exists $s \in \Delta$ such that $s \in \Phi(s)$.

Advances in general equilibrium theory have been developed jointly with the existence of at least one equilibrium point. The first result in this direction was obtained by Arrow-Debreu [7, Theorem 1.5.1.I p. 272 and Theorem 4.5.II p. 281] for finite dimensional spaces, by applying some variation of the Brouwer's fixed point theorem.

Under more restrictive hypothesis, although in a more general framework Mas-Colell-Richard [16, Theorem p. 3] obtained a seminal theorem on the existence of a competitive equilibrium (see also Aliprantis [2, Theorem 1.1 p. 416]) for infinite-dimensional dual pairs of locally convex vector lattices. We will greatly replicate their procedure adapted to our model by assuming no additional condition.

Theorem 4.4. *Every affine economy $\langle P, \boldsymbol{\omega}, \boldsymbol{q}\rangle$ has a competitive equilibrium.*

Proof. Lemma 4.1 ensures that weakly Pareto optimal allocations

$$\boldsymbol{W} = \{\boldsymbol{f} \in \boldsymbol{F} : \boldsymbol{S}^*_{\boldsymbol{q}}(\boldsymbol{f}) \cap \boldsymbol{F} = \varnothing\}$$

is a nonempty $\sigma(\boldsymbol{A(P)}, \boldsymbol{P})$-compact subset of $\boldsymbol{A(P)}_+$.

Let $\boldsymbol{f} \in \boldsymbol{W}$ and consider the open convex subset of $\boldsymbol{A(P)}$

$$\boldsymbol{W_f} = \left\{\boldsymbol{g} + \frac{\boldsymbol{\varepsilon}}{\boldsymbol{n}} : \boldsymbol{g} \in \boldsymbol{S_q(f)},\ \varepsilon > 0\right\},\ \frac{\boldsymbol{\varepsilon}}{\boldsymbol{n}} = \left(\frac{\varepsilon}{n}, \dots, \frac{\varepsilon}{n}\right)$$

then, $\boldsymbol{f} \notin \boldsymbol{W_f}$. Moreover, $\boldsymbol{W_f} \cap \boldsymbol{F} \subseteq \boldsymbol{S}^*_{\boldsymbol{q}}(\boldsymbol{f}) \cap \boldsymbol{F} = \varnothing$. By virtue of Lemma 4.2, there exists $\boldsymbol{p} \in \boldsymbol{P}$ such that

$$\boldsymbol{h} \cdot \boldsymbol{p} \leq \left(\boldsymbol{g} + \frac{\boldsymbol{\varepsilon}}{\boldsymbol{n}}\right) \cdot \boldsymbol{p} \tag{1}$$

for all $\boldsymbol{h} \in \boldsymbol{F}$, $\boldsymbol{g} \in \boldsymbol{S_q(f)}$ and $\varepsilon > 0$.

On the one hand, supposing $\boldsymbol{g} \in \boldsymbol{S_q(f)}$, fix i and let $\boldsymbol{h}^i = (h^i_1, \dots, h^i_n)$ such that

$$h^i_j = \begin{cases} g_i + \frac{\varepsilon}{n} \text{ if } j = i \\ f_j + \frac{\varepsilon}{n} \text{ if } j \neq i \end{cases} .$$

Then $\boldsymbol{h}^i \in \boldsymbol{W_f}$ and taking into account that $\boldsymbol{f} \in \boldsymbol{F}$, from equation (1) it follows

$$\sum_{j=1}^{n} f_j(p_j) = \boldsymbol{f} \cdot \boldsymbol{p} \leq \boldsymbol{h}^i \cdot \boldsymbol{p} = \sum_{j \neq i}^{n} f_j(p_j) + g_i(p_i) + \varepsilon \left(\frac{\boldsymbol{1}}{\boldsymbol{n}} \cdot \boldsymbol{p}\right),$$

and therefore $f_i(p_i) \leq g_i(p_i)$. Concluding that

$$\boldsymbol{f}(\boldsymbol{p}) \leq \boldsymbol{g}(\boldsymbol{p}) \text{ if } \boldsymbol{g} \in \boldsymbol{S_q(f)}. \tag{2}$$

On the other hand, since $\boldsymbol{f} \in \boldsymbol{S_q(f)}$, we have

$$\boldsymbol{h} \cdot \boldsymbol{p} \leq \left(\boldsymbol{f} + \frac{\boldsymbol{\varepsilon}}{\boldsymbol{n}}\right) \cdot \boldsymbol{p} = \boldsymbol{f} \cdot \boldsymbol{p} + \varepsilon \left(\frac{\boldsymbol{1}}{\boldsymbol{n}} \cdot \boldsymbol{p}\right) \text{ for all } \varepsilon > 0.$$

This implies

$$\boldsymbol{h}\cdot\boldsymbol{p}\le\boldsymbol{f}\cdot\boldsymbol{p} \text{ if } \boldsymbol{h}\in\boldsymbol{F}. \tag{3}$$

From (2) and (3) we conclude that $\boldsymbol{p}$ belongs to the compact convex set

$$\boldsymbol{K}(\boldsymbol{f})=\{\boldsymbol{p}\in\boldsymbol{P}:\boldsymbol{f}(\boldsymbol{p})\le\boldsymbol{g}(\boldsymbol{p}) \text{ if } \boldsymbol{g}\in\boldsymbol{S_q}(\boldsymbol{f})\}\cap\{\boldsymbol{p}\in\boldsymbol{P}:\boldsymbol{h}\cdot\boldsymbol{p}\le\boldsymbol{f}\cdot\boldsymbol{p} \text{ if } \boldsymbol{h}\in\boldsymbol{F}\},$$

and therefore it is nonempty.

Consider on $\Delta=\Big\{\boldsymbol{s}\in\mathbb{R}^n_+ : s_1+\cdots+s_n=1\Big\}\subseteq\boldsymbol{F}$ the real valued function

$$\varphi(\boldsymbol{s})=\sup_{\boldsymbol{f}\in\boldsymbol{W}}\Big\{\lambda\ge 0:\boldsymbol{f}(\boldsymbol{q})=\lambda\boldsymbol{s}\Big\}$$

which is continuous by virtue of [15, Proposition 2 p. 1045].

If $\boldsymbol{s}\in\Delta\subseteq\boldsymbol{F}$ and $\boldsymbol{f}(\boldsymbol{q})>\boldsymbol{s}$, then $\boldsymbol{f}\notin\boldsymbol{F}$ and consequently $\boldsymbol{s}\in\boldsymbol{W}$. Then, $\varphi(\boldsymbol{s})\ge 1$, and since $\boldsymbol{W}$ is compact (Lemma 4.1), the supremum defining φ is actually a maximum; that is, there exists $\boldsymbol{f^s}\in\boldsymbol{W}$ such that $\boldsymbol{f^s}(\boldsymbol{q})=\varphi(\boldsymbol{s})\boldsymbol{s}$. Therefore, the set

$$\boldsymbol{W}(\boldsymbol{s})=\Big\{\boldsymbol{f^s}\in\boldsymbol{W}:\boldsymbol{f^s}(\boldsymbol{q})=\varphi(\boldsymbol{s})\boldsymbol{s}\Big\},$$

is nonempty.

Let $\Phi:\Delta\twoheadrightarrow\Delta$ the a correspondence defined by

$$\Phi(\boldsymbol{s})=\Big\{\boldsymbol{s}+\boldsymbol{\omega}(\boldsymbol{p})-\boldsymbol{f^s}(\boldsymbol{p}):\boldsymbol{p}\in\boldsymbol{K}(\boldsymbol{f^s}) \text{ for some } \boldsymbol{f^s}\in\boldsymbol{W}(\boldsymbol{s})\Big\}$$

(notice that $\Phi(\Delta)\subseteq\Delta$ because $\boldsymbol{\omega}\cdot\boldsymbol{p}=\boldsymbol{f^s}\cdot\boldsymbol{p}$).

Let $\{(\boldsymbol{s}^\alpha,\boldsymbol{t}^\alpha)\}_\alpha\to(\boldsymbol{s},\boldsymbol{t})$ be a converging net in Δ^2 such that $\boldsymbol{t}^\alpha\in\Phi(\boldsymbol{s}^\alpha)$ (i.e. $\boldsymbol{t}^\alpha=\boldsymbol{s}^\alpha+\boldsymbol{\omega}(\boldsymbol{p}^\alpha)-\boldsymbol{f}^{\boldsymbol{s}^\alpha}(\boldsymbol{p}^\alpha)$ for some $\boldsymbol{p}^\alpha\in\boldsymbol{K}(\boldsymbol{f}^{\boldsymbol{s}^\alpha})$ and some $\boldsymbol{f}^{\boldsymbol{s}^\alpha}\in\boldsymbol{W}(\boldsymbol{s}^\alpha)$). By virtue of Lemma 4.1 and through a subnet, we may assume

$$\boldsymbol{f}^{\boldsymbol{s}^\alpha}\xrightarrow{\sigma(\boldsymbol{A}(\boldsymbol{P}),\boldsymbol{P})}\boldsymbol{f^s}\in\boldsymbol{W}, \text{ and } \boldsymbol{p}^\alpha\to\boldsymbol{p}\in\boldsymbol{P}.$$

We have

$$\boldsymbol{f}^{\boldsymbol{s}^\alpha}(\boldsymbol{q})=\varphi(\boldsymbol{s}^\alpha)\boldsymbol{s}^\alpha\to\boldsymbol{f^s}(\boldsymbol{q})=\varphi(\boldsymbol{s})\boldsymbol{s},$$

(by continuity of φ). So that, $\boldsymbol{f^s}\in\boldsymbol{W}(\boldsymbol{s})$.

Given $\varepsilon>0$, there exists α_0 such that for every $\alpha\ge\alpha_0$

$$(\boldsymbol{f^s}+\varepsilon\boldsymbol{\omega})(\boldsymbol{q})>\boldsymbol{f}^{\boldsymbol{s}^\alpha}(\boldsymbol{q})>(\boldsymbol{f^s}-\varepsilon\boldsymbol{\omega})(\boldsymbol{q}).$$

Then,

$$\boldsymbol{f^s}(\boldsymbol{p}^\alpha)+\varepsilon\boldsymbol{\omega}(\boldsymbol{p}^\alpha)\ge\boldsymbol{f}^{\boldsymbol{s}^\alpha}(\boldsymbol{p}^\alpha)=\boldsymbol{s}^\alpha+\boldsymbol{\omega}(\boldsymbol{p}^\alpha)-\boldsymbol{t}^\alpha\ge\boldsymbol{f^s}(\boldsymbol{p}^\alpha)-\varepsilon\boldsymbol{\omega}(\boldsymbol{p}^\alpha).$$

By taking limits with respect to α and since ε is arbitrary we derive

$$\boldsymbol{f^s}(\boldsymbol{p})\ge\boldsymbol{s}+\boldsymbol{\omega}(\boldsymbol{p})-\boldsymbol{t}\ge\boldsymbol{f^s}(\boldsymbol{p}).$$

By analogous argument, it is immediate to see that $\boldsymbol{f^s}(\boldsymbol{p})\le\boldsymbol{g}(\boldsymbol{p})$ if $\boldsymbol{g}\in\boldsymbol{S_q}(\boldsymbol{f^s})$.

Finally, and since $\boldsymbol{h} \in \boldsymbol{F}$ is arbitrary we have $\boldsymbol{h} \cdot \boldsymbol{p}^{\alpha} \leq \boldsymbol{f}^{\boldsymbol{s}^{\alpha}} \cdot \boldsymbol{p}^{\alpha}$, thus by taking limits we obtain that $\boldsymbol{p} \in \boldsymbol{K}(\boldsymbol{f}^{\boldsymbol{s}})$. Then $\boldsymbol{t} \in \Phi(\boldsymbol{s})$, and condition (i) of Lemma 4.3 holds.

Let us see that condition (ii) holds as well. Let $\boldsymbol{t}, \boldsymbol{t}' \in \Phi(\boldsymbol{s})$ and $0 \leq \lambda \leq 1$. Choose $\boldsymbol{p}, \boldsymbol{p}' \in \boldsymbol{K}(\boldsymbol{f}^{\boldsymbol{s}})$ such that

$$\boldsymbol{t} = \boldsymbol{s} + \boldsymbol{\omega}(\boldsymbol{p}) - \boldsymbol{f}^{\boldsymbol{s}}(\boldsymbol{p}) \text{ and } \boldsymbol{t}' = \boldsymbol{s} + \boldsymbol{\omega}(\boldsymbol{p}') - \boldsymbol{f}^{\boldsymbol{s}}(\boldsymbol{p}').$$

Let $\boldsymbol{p}^{\lambda} = \lambda \boldsymbol{p} + (1-\lambda)\boldsymbol{p}' \in \boldsymbol{K}(\boldsymbol{f}^{\boldsymbol{s}})$, and it is clear that

$$\lambda \boldsymbol{t} + (1-\lambda)\boldsymbol{t}' = \boldsymbol{s} + \boldsymbol{\omega}(\boldsymbol{p}^{\lambda}) - \boldsymbol{f}^{\boldsymbol{s}}(\boldsymbol{p}^{\lambda})$$

obtaining the convexity of $\Phi(\boldsymbol{s})$.

Lemma 4.3 ensures the existence of $\boldsymbol{s} \in \Phi(\boldsymbol{s})$, i.e. there exists $\boldsymbol{f}^{\boldsymbol{s}} \in \boldsymbol{W}(\boldsymbol{s})$ and $\boldsymbol{p} \in \boldsymbol{K}(\boldsymbol{f}^{\boldsymbol{s}})$ such that $\boldsymbol{\omega}(\boldsymbol{p}) = \boldsymbol{f}^{\boldsymbol{s}}(\boldsymbol{p})$. Hence, if $\boldsymbol{g} \in \boldsymbol{S_q}(\boldsymbol{f}^{\boldsymbol{s}}) \cap \boldsymbol{B}(\boldsymbol{p})$, it follows $\boldsymbol{g}(\boldsymbol{p}) \leq \boldsymbol{\omega}(\boldsymbol{p}) = \boldsymbol{f}^{\boldsymbol{s}}(\boldsymbol{p}) \leq \boldsymbol{g}(\boldsymbol{p})$ because $\boldsymbol{p} \in \boldsymbol{K}(\boldsymbol{f}^{\boldsymbol{s}})$, i.e. $\boldsymbol{g}(\boldsymbol{p}) = \boldsymbol{\omega}(\boldsymbol{p})$.

Supposing $\boldsymbol{g} \in \boldsymbol{S}^*_{\boldsymbol{q}}(\boldsymbol{f}^{\boldsymbol{s}}) \cap \boldsymbol{B}(\boldsymbol{p})$, there exists $0 < \lambda < 1$ such that $\lambda \boldsymbol{g} \in \boldsymbol{S}^*_{\boldsymbol{q}}(\boldsymbol{f}^{\boldsymbol{s}})$, and therefore $\lambda \boldsymbol{g}(\boldsymbol{p}) = \lambda \boldsymbol{\omega}(\boldsymbol{p}) < \boldsymbol{\omega}(\boldsymbol{p})$, in contradiction with the equality $\lambda \boldsymbol{g}(\boldsymbol{p}) = \boldsymbol{\omega}(\boldsymbol{p})$ because $\lambda \boldsymbol{g} \in \boldsymbol{S}^*_{\boldsymbol{q}}(\boldsymbol{f}^{\boldsymbol{s}}) \cap \boldsymbol{B}(\boldsymbol{p})$.

By denoting $\check{p} = p_1 \vee \cdots \vee p_n \in C(P)^+$, then $\check{p}(1) = 1$ and therefore $\check{\boldsymbol{p}} = (\check{p}, \ldots, \check{p}) \in \boldsymbol{P}$ (see Corollary 2.6).

On the one hand, and since $\boldsymbol{f}^{\boldsymbol{s}} \in \boldsymbol{B}(\boldsymbol{p})$, we have $\boldsymbol{f}^{\boldsymbol{s}}(\boldsymbol{p}) \leq \boldsymbol{\omega}(\boldsymbol{p})$ which implies $\boldsymbol{f}^{\boldsymbol{s}}(\check{\boldsymbol{p}}) \leq \boldsymbol{\omega}(\check{\boldsymbol{p}})$, and therefore $\boldsymbol{f}^{\boldsymbol{s}} \in \boldsymbol{B}(\check{\boldsymbol{p}})$. If $\boldsymbol{\omega}(\boldsymbol{p}) < \boldsymbol{\omega}(\check{\boldsymbol{p}})$, then

$$1 = \boldsymbol{\omega} \cdot \boldsymbol{p} < \boldsymbol{\omega} \cdot \check{\boldsymbol{p}} = \check{p}(1) = 1,$$

which is a contradiction. Hence, $\boldsymbol{\omega}(\boldsymbol{p}) = \boldsymbol{\omega}(\check{\boldsymbol{p}})$.

On the other hand, if $\boldsymbol{g} \in \boldsymbol{S}^*_{\boldsymbol{q}}(\boldsymbol{f}^{\boldsymbol{s}})$, then $\boldsymbol{g} \notin \boldsymbol{B}(\boldsymbol{p})$ and since $\boldsymbol{\omega}(\check{\boldsymbol{p}}) = \boldsymbol{\omega}(\boldsymbol{p})$, there exists $\omega_i(\check{p}) = \omega_i(p_i) < g_i(p_i) \leq g_i(\check{p})$. Thus, $\boldsymbol{g} \notin \boldsymbol{B}(\check{\boldsymbol{p}})$ and we have proven that $\boldsymbol{f}^{\boldsymbol{s}}$ becomes a competitive equilibrium supported by the price system $\check{p} \in P$. $\square$

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