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On the Roots of Convex Functions

Daniel Solow
Department of Operations, Weatherhead School of Management,
Case Western Reserve University, Cleveland, OH, U.S.A.
daniel.solow@case.edu

Fangqi Fu
Magnolia Drive, Cleveland, OH, U.S.A.
fxf92@case.edu

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The number of real roots of a convex function of one variable is shown to be 0, 1, 2, or infinite and, in the latter case, the infinite number of roots must be an interval on the real line. Necessary and sufficient conditions are provided on the convex function for each of these cases to arise. For example, a necessary and sufficient condition for there to be an infinite interval of roots is for the function to have three distinct roots. An algorithm is also presented for finding all of the roots, subject to computational limitations. At the end of the paper, two generalizations are suggested for future directions of research and an open conjecture is presented of a sufficient condition for a system of n convex equations in n unknowns to have a solution.

Keywords: Solving convex equations, convex functions, finding roots, roots of convex functions.

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1. Introduction, notation, and prior results

Throughout the history of mathematics, there have been many instances where it is necessary to solve an equation, that is, to find the root of a function – linear, quadratic, polynomial, and nonlinear. For such problems, mathematicians have been concerned with determining conditions on the function for which the existence and uniqueness of a root is guaranteed. When closed-form solutions are not available, algorithms are developed for attempting to find a root. For example, Newton's method is an iterative procedure for attempting to find the root of a differentiable nonlinear function.

A related question of interest is *how many* roots a function has. One of the most famous such problems is determining the number of roots of a polynomial of degree n. This problem was solved in 1799 by Gauss [1], resulting in the Fundamental Theorem of Algebra where it is proved that a polynomial of degree n has exactly n complex roots.

In this paper, the *real* roots of a convex function f are studied. As such, from here on, the word "root" refers to a real root. One application of this problem arises in finding a point on the boundary of an n-dimensional convex set C that lies on the line segment connecting a point in the interior of C to a point in the exterior of C.

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In the work here, it is shown that a convex function can have 0, 1, 2, or an infinite number of roots and, in the latter case, those roots must be in the form of a (possibly unbounded) interval on the real line. It is clear that a convex function can have 0 roots (consider $f(x) = x^2 + 1$), 1 root (consider $f(x) = x^2$) and 2 roots (consider $f(x) = x^2 - 1$). In Section 2 it is shown that the only other possibility is for a convex function to have an infinite number of roots that constitute an interval. Necessary and sufficient conditions are also identified on the function for each of the foregoing cases to occur. In Section 3, an algorithm for finding all of the roots of a convex function is developed subject, of course, to computation limitations. The work is summarized in Section 4, where two generalizations are given as directions for future research, including an open conjecture regarding a sufficient condition for a system of n convex equations in n unknowns to have a solution. The remainder of this section is devoted to notation and known facts about convex functions that are used to obtain the results in Section 2.

1.1. Notation and preliminary results on convex functions

Throughout this paper, let R be the set of real numbers and $f : R \longrightarrow R$ be a real-valued function of one variable. The lower case letters x, y, z are real numbers.

The function f is *convex* if for any two numbers $x < y$, the graph of the function over the interval $[x, y]$ lies on or below the line segment connecting $(x, f(x))$ and $(y, f(y))$, as seen in Figure 1.1 and stated more formally in the following definition.

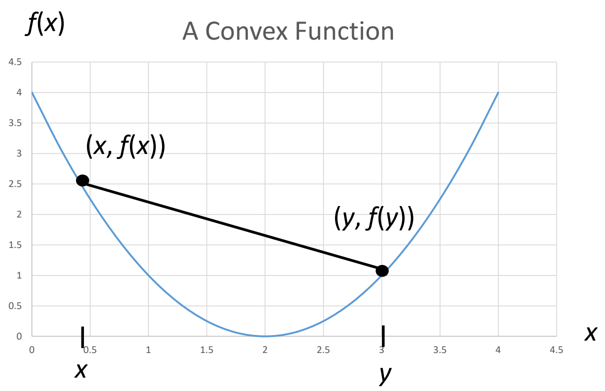

Figure 1.1: A convex function of one variable

Definition 1.1. A real-value function f of one variable is *convex* if and only if for all real numbers x and y with $x < y$ and for all real numbers λ with $0 \leq \lambda \leq 1$,

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y). \tag{1}$$

A related concept of interest is a *convex set*, in which the set contains the line segment connecting any two points in the set, as stated in the following definition.

Definition 1.2. A subset C of R is *convex* if and only if for all $x, y \in C$ and for all real numbers λ with $0 \leq \lambda \leq 1$,

$$(1 - \lambda)x + \lambda y \in C. \tag{2}$$

The concept of *monotonicity* is also useful for the work here, so,

Definition 1.3. The function f is *monotone increasing (decreasing)* if and only if for all $x \leq y$, $f(x) \leq (\geq) f(y)$ and is *monotone* if f is either monotone increasing or monotone decreasing. The function f is *strictly monotone increasing (decreasing)* if and only if for all $x < y$, $f(x) < (>) f(y)$ and is *strictly monotone* if f is either strictly monotone increasing or strictly monotone decreasing.

A great deal is known about convex functions because of their importance in minimization. For example, a convex function f is continuous on R. While f might not be differentiable, for every real number $\bar{x}$ there is a *subgradient*, that is, a real number a that provides the slope of a tangent line to the graph of f drawn through the point $(\bar{x}, f(\bar{x}))$, as seen in Figure 1.2. There can be many different subgradients at a point $\bar{x}$ – and hence many tangent lines to f at $\bar{x}$ – and so let

$$\partial f(\bar{x}) = \{a \in R : a \text{ is a subgradient of } f \text{ at } \bar{x}\}.$$

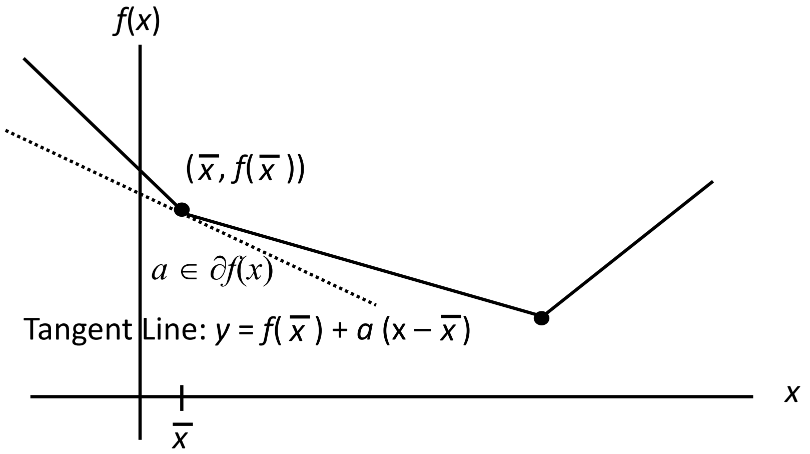

Figure 1.2: Using a subgradient to create a tangent line

Facts regarding convex functions that are used in this paper are summarized in the following proposition and the proofs are available in Rockafeller [2].

Proposition 1.4. *For a convex function f:*

(a) *f is continuous on R.*

(b) *For every real number x, $\partial f(x)$ is nonempty and convex.*

(c) *For every real number x, $0 \in \partial f(x)$ if and only if x is a* global minimizer *of f, that is, for all real numbers y, $f(y) \geq f(x)$.*

(d) *The graph of f lies entirely above any tangent line drawn at any point (see Figure 1.2). That is, for every real number x and for every $a \in \partial f(x)$, the following* subgradient inequality *holds:*

$$f(y) \geq f(x) + a(y - x), \quad \textit{for all real numbers } y. \tag{3}$$

A final result regarding convex functions that is needed in this work is the following.

Proposition 1.5. *For any x and for any subgradient $a \in \partial f(x)$ with $a \neq 0$, the Newton iterate $y = x - f(x)/a$ satisfies $f(y) \geq 0$.*

The foregoing proposition follows from (3) on setting $y = x - f(x)/a$, resulting in

$$f(y) \geq f(x) + a(y - x) = f(x) + a[x - f(x)/a - x)] = 0.$$

2. The number of roots of a convex function

In this section, the main result that a convex function has either 0, 1, 2, or an infinite number of roots is proved. In the latter case, it is shown that the infinite number of roots must be an interval on the real line. Necessary and sufficient conditions on the function f are also provided for each of the foregoing case. It should be noted that all results obtained here apply equally to concave functions.

2.1. Convex functions with 0 roots

A convex function can have no root as seen, for example, by the simple function $f(x) = x^2 + 1$. Necessary and sufficient conditions for this to happen are that the graph of the function lies either entirely below or entirely above the x-axis, as seen in Figure 2.1 and stated in the following proposition.

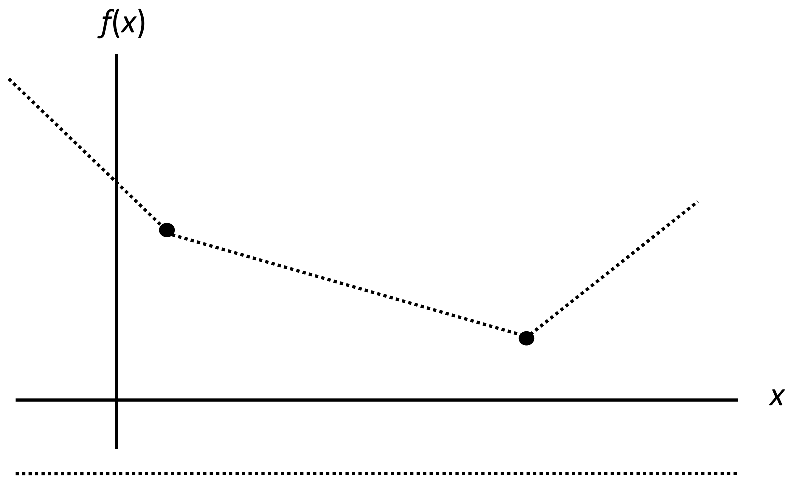

Figure 2.1: Convex functions with 0 roots

Proposition 2.1. *A necessary and sufficient condition for a convex function f to have no root is that either*

(a) *there is a constant $c < 0$ such that for all $x, f(x) = c$ or else*

(b) *for all $x, f(x) > 0$.*

Proof of necessity. By contradiction, assume that neither (a) nor (b) is true. Since (b) is not true, there is an x such that $f(x) \leq 0$. However, because f has no root, it must be that $f(x) < 0$. Now because (a) is not true, f is not identically equal to $f(x) < 0$, so there is a real number y such that $f(y) \neq f(x)$.

Suppose, without loss of generality, that $f(y) < f(x)$ and let $a \in \partial f(x)$. Now $a \neq 0$ for otherwise by Proposition 1.4(c), x is a global minimizer of f which is not the case because $f(y) < f(x)$. Then by Proposition 1.5, the Newton iterate $z = x - f(x)/a$ satisfies $f(z) > 0$ (since $f(z)$ cannot be 0). But then the fact that $f(x) < 0$ and $f(z) > 0$ would mean that f has a root because f is continuous from Proposition 1.4(a).

Proof of sufficiency. This proof is obvious because in case (a), the graph of f lies strictly below the x-axis and in case (b), the graph of f lies strictly above the x-axis.

2.2. Convex functions with exactly 1 root

A convex function can have exactly one root, say x^*, as seen, for example, by the function $f(x) = x^2$. A necessary and sufficient condition for a root x^* to be unique is that either (a) 0 is a subgradient at x^* and 0 is not a subgradient at any other point or else (b) 0 is not a subgradient at x^* and f is monotone, as seen in Figure 2.2 and stated in the following proposition.

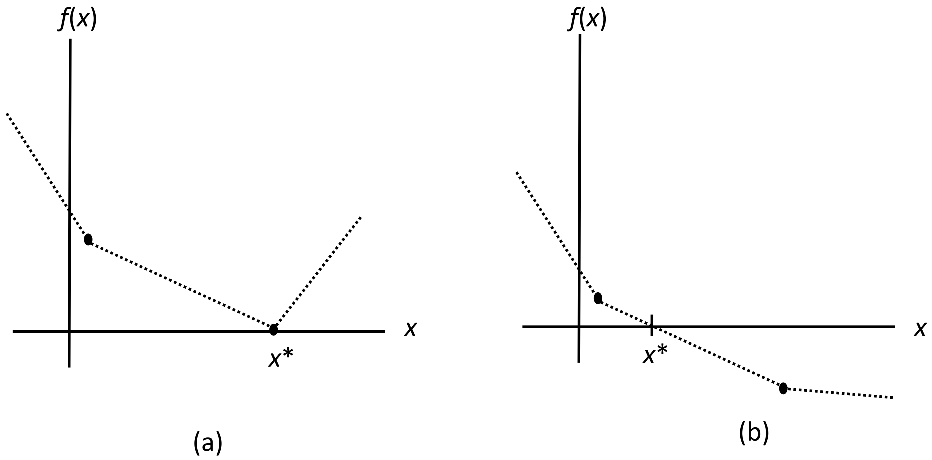

Figure 2.2: Convex functions with exactly 1 root

Proposition 2.2. *A necessary and sufficient condition for a convex function f to have exactly one root is that there is a real number x^* such that $f(x^*) = 0$ and either*

(a) $0 \in \partial f(x^*)$ *and for all* $x \neq x^*, 0 \notin \partial f(x)$ *or else*

(b) $0 \notin \partial f(x^*)$ *and f is monotone.*

Proof of necessity.

Case 1: $0 \in \partial f(x^*)$. In this case, by Proposition 1.4(c), x^* is a global minimizer of f and hence, for all x, $f(x) \geq f(x^*) = 0$. If there is an $x \neq x^*$ such that $0 \in \partial f(x)$, then x would also be a global minimizer of f and so $f(x) = 0$, which cannot happen because there is only one root.

Case 2: $0 \notin \partial f(x^*)$. In this case, it must be shown that f is monotone. To that end, let $0 \neq a \in \partial f(x^*)$ and assume, without loss of generality, that $a < 0$. By the subgradient inequality, it follows that for all $y < x^*$,

$$f(y) \geq f(x^*) + a(y - x^*) = a(y - x^*) > 0.$$

It is first shown that f is monotone decreasing on $(-\infty, x^*]$. If not, then there are numbers $y < z < x^*$ for which $0 \neq f(y) < f(z)$. Let $0 < \lambda < 1$ be such that $z = (1 - \lambda)x^* + \lambda y$. But then

$$f(z) = f((1 - \lambda)x^* + \lambda y) \leq (1 - \lambda)f(x^*) + \lambda f(y) = \lambda f(y) < f(y) < f(z),$$

which cannot happen.

It remains to show that f is monotone decreasing on $[x^*, +\infty)$. If not, then there are numbers $x^* < y < z$ for which $f(y) < f(z)$. If $f(y) > 0$, then since $0 \notin \partial f(x^*)$, x^* is not a global minimizer and so there is a $w > x^*$ such that $f(w) < f(x^*) = 0$, but then there would be another root of f between w and y, which cannot happen.

Thus, $f(y) \leq 0$ and, in fact, $f(y) < 0$. Now if $f(z) > 0$, then again there would be another root so $f(z) \leq 0$ and, in fact, $f(z) < 0$. Now let $b \in \partial f(z)$. It must be that $b > 0$, for otherwise, by the subgradient inequality, $f(y) \geq f(z) + b(y - z) \geq f(z)$, which cannot happen. But then the Newton iterate $v = z - f(z)/b$ satisfies $v > z$. By Proposition 1.5, $f(v) \geq 0$ and, in fact, $f(v) > 0$. As $f(z) < 0$, this would mean that there is another root of f between z and v. This contradiction shows that f is monotone decreasing on $[x^*, +\infty)$.

Proof of sufficiency. The proof proceeds by cases, so assume first that $0 \in \partial f(x^*)$ and for all $x \neq x^*, 0 \notin \partial f(x)$. As $0 \in \partial f(x^*)$, by Proposition 1.4(c), x^* is a global minimizer of f, that is, for all y, $f(y) \geq f(x^*) = 0$. To see that x^* is the only root of f, suppose, to the contrary, that $y^* \neq x^*$ is also a root of f. Then $f(y^*) = 0$ is also a global minimizer of f and so, by Proposition 1.4(c), $0 \in \partial f(y^*)$, which contradicts the assumption that for all $x \neq x^*, 0 \notin \partial f(x)$.

For the second case, assume that $0 \notin \partial f(x^*)$ and f is monotone. Let $a \in \partial f(x^*)$ and assume, without loss of generality, that $a < 0$. To see that x^* is the only root of f, suppose, to the contrary, that $y^* \neq x^*$ is also a root of f. Now by the subgradient inequality, for all x, $f(x) \geq f(x^*) + a(x - x^*)$. In particular,

$$\text{for all } x < x^*, f(x) \geq f(x^*) + a(x - x^*) = a(x - x^*) > 0 = f(x^*). \tag{4}$$

As a result of (4), it must be that $y^* > x^*$ because if $y^* < x^*$, it would follow that $f(y^*) > 0$, which is not the case. Now, the fact that $0 \notin \partial f(x^*)$ means, by Proposition 1.4(c), that x^* is not a global minimizer of f. Thus, there is a y such that $f(y) < f(x^*) = 0$ and from (4), $y > x^*$. A final contradiction that completes this second case – proves the sufficiency, and completes the proof – is obtained in each of the following two subcases:

Subcase (a): $x^* < y^* < y$. In this case, let λ be a real number with $0 < \lambda < 1$ such that $y^* = (1 - \lambda)x^* + \lambda y$. As $f(x^*) = f(y^*) = 0$ and $f(y) < 0$, a contradiction is obtained from the convexity of f because

$$0 = f(y^*) = f((1 - \lambda)x^* + \lambda y) \leq (1 - \lambda)f(x^*) + \lambda f(y) = \lambda f(y) < 0.$$

Subcase (b): $x^* < y < y^*$. The fact that $f(x^*) = 0 > f(y) < 0 = f(y^*)$ contradicts the fact that f is monotone decreasing.

2.3. Convex functions with exactly 2 roots

A convex function can have exactly two roots as seen, for example, by the function $f(x) = x^2 - 1$. A necessary and sufficient condition for this to happen is that there is a point where the function is strictly negative that is surrounded by a point to the left having a strictly negative subgradient and a point to the right having a strictly positive subgradient, as seen in Figure 2.3 and stated in the following proposition.

Proposition 2.3. *A necessary and sufficient condition for a convex function f to have exactly two roots is that there are real numbers x, y, z with $x < y < z$ such that $f(y) < 0$ and for which there are subgradients $a \in \partial f(x)$ and $b \in \partial f(z)$ with $a < 0$ and $b > 0$.*

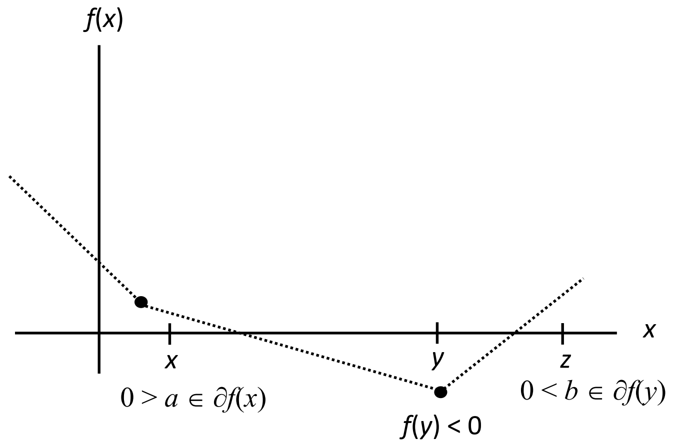

Figure 2.3: Convex functions with exactly 2 roots

Proof of necessity. Let $x < z$ be the two roots of f with $a \in \partial f(x)$ and $b \in \partial f(z)$. Now $a \leq 0$, for otherwise, $f(z) \geq f(x) + a(z - x) > 0$ by the subgradient inequality at x, which cannot happen since $f(x) = f(z) = 0$. If $a = 0$, then by Proposition 1.4(c), x is a global minimizer of f and a contradiction is reached by showing that f has a third root. In particular, for any y with $x < y < z$ and λ with $0 < \lambda < 1$ for which $y = (1-\lambda)x + \lambda z$, it must be that $f(y) = f((1-\lambda)x + \lambda z) \leq (1-\lambda)f(x) + \lambda f(z) = 0$. But since x is a global minimizer, $f(y) \geq f(x) = 0$ and so it must be that $f(y) = 0$, which would mean that f has three roots. This contradiction means that $a < 0$. Similarly, $b > 0$.

It remains to show that there is a y with $x < y < z$ such that $f(y) < 0$. To that end, let y be a minimizer of the continuous function f over the nonempty compact interval $[x, z]$. Now $f(y) \leq f(x) = 0$ and $f(y) \neq 0$, for otherwise there would be three roots. Thus, $f(y) < 0$ and so this portion of the proof is complete.

Proof of sufficiency. Suppose that there are real numbers x, y, z with $x < y < z$ such that $f(y) < 0$ and for which there are subgradients $a \in \partial f(x)$ and $b \in \partial f(z)$ with $a < 0$ and $b > 0$. It is shown first that there is a root x^* with $x^* < y$. In particular, if $f(x) \geq 0$ then there is a root of f between x and y. On the other hand, if $f(x) < 0$, then the Newton iterate $w = x - f(x)/a$ satisfies $w < x < y$ and, by Proposition 1.5, $f(w) \geq 0$. As such, there is a root of f between w and y. In either case, there is a root of f, say x^*, with $x^* < y$. Similarly, there is a root of f, say z^*, with $z^* > y$.

To see that x^* and z^* are the only roots of f, suppose that $v \neq x^*, z^*$ is also a root. Assume, without loss of generality, that $y < v$. Now $v \notin (y, z^*)$ for otherwise, there is a $0 < \lambda < 1$ with $v = (1 - \lambda)y + \lambda z^*$ and

$$0 = f(v) = f((1 - \lambda)y + \lambda z^*) \leq (1 - \lambda)f(y) + \lambda f(z^*) = (1 - \lambda)f(y) < 0.$$

Thus, $v > z^*$. But then there is a $0 < \alpha < 1$ with $z^* = (1 - \alpha)y + \alpha v$ and $0 = f(z^*) = f((1 - \alpha)y + \alpha v) \leq (1 - \alpha)f(y) + \alpha f(v) = (1 - \alpha)f(y) < 0$. This contradiction shows that f cannot have a root other than x^* and z^*.

2.4. Convex functions with an infinite number of roots

The only other possibility is for a convex function to have an infinite number of roots and, in this case, $X = \{x^* : f(x^*) = 0\}$ is an interval on the real line. A necessary and sufficient condition for there to be an infinite number of roots in the form of an interval is for the function to have three distinct roots, as shown in Figure 2.4 and stated in the following two propositions.

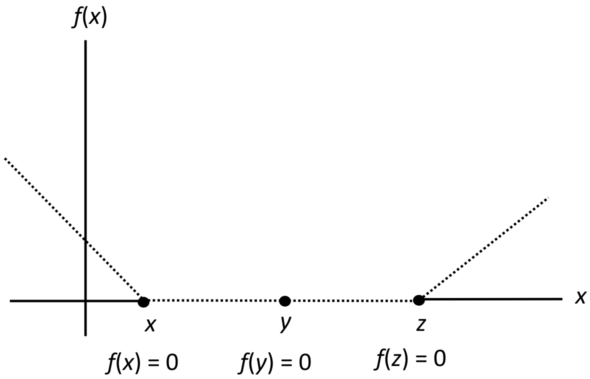

Figure 2.4: Convex functions with an infinite number of roots

Proposition 2.4. *A necessary and sufficient condition for a convex function f to have an infinite number of roots is that the function has three distinct roots.*

Proof of necessity. This is clearly true.

Proof of sufficiency. Let $x^* < y^* < z^*$ be three distinct roots of a convex function f. It is now shown that every $w \in (x^*, z^*)$, $f(w) = 0$. Let $w \in (x^*, z^*)$ and λ with $0 < \lambda < 1$ be such that $w = (1 - \lambda)x^* + \lambda z^*$. Then

$$f(w) = f((1 - \lambda)x^* + \lambda z^*) \leq (1 - \lambda)f(x^*) + \lambda f(z^*) = 0.$$

If $f(w) < 0$ then either $w \in (x^*, y^*)$ or else $w \in (y^*, z^*)$. Assume, without loss of generality, that we have $w \in (x^*, y^*)$ and let α with $0 < \alpha < 1$ be such that $y^* = (1 - \alpha)w + \alpha z^*$. It follows that

$$0 = f(y^*) = f((1 - \alpha)w + \alpha z^*) \leq (1 - \alpha)f(w) + \alpha f(z^*) = (1 - \alpha)f(w) < 0.$$

This contradiction means that $f(w) = 0$ and hence that all points in (x^*, z^*) are roots of f and so there are an infinite number of roots, thus completing the proof.

Proposition 2.5. *If the convex function f has an infinite number of roots, then the set $X = \{x : f(x) = 0\}$ is an interval on the real line.*

Proof. To see that X is an interval, it suffices to show that X is a convex set. To that end, let $x, z \in X$ and $0 < \lambda < 1$ with $y = (1 - \lambda)x + \lambda z$. From the subgradient inequality, it follows that $f(y) = f((1 - \lambda)x + \lambda z) \leq (1 - \lambda)f(x) + \lambda f(z) = 0$. It must be shown that $f(y) = 0$. So suppose that $f(y) < 0$. Now because f has an infinite number of roots, let $w \neq x, z$ be another root of f, for which either $w < y$ or $w > y$. Assume, without loss of generality, that $w < y$. Then either $w < x$ or $w > x$. If $w < x$, then let $0 < \alpha < 1$ such that $x = (1 - \alpha)w + \alpha y$. It would then follow that $0 = f(x) = f((1 - \alpha)x + \alpha y) \leq (1 - \alpha)f(x) + \alpha f(y) = \alpha f(y) < 0$, which cannot happen. On the other hand, if $x < w < y$, then let $0 < \alpha < 1$ such that $w = (1 - \alpha)x + \alpha y$.

It would then follow that

$$0 = f(w) = f((1-\alpha)x + \alpha y) \le (1-\alpha)f(x) + \alpha f(y) = \alpha f(y) < 0,$$

which also cannot happen. This contradiction means that $f(y) = 0$ and so the set X of roots of f is convex and hence is an interval, completing the proof.

3. An algorithm for finding all roots of a convex function

The goal now is to develop an algorithm that finds all of the roots of a convex function or determines that there are no roots subject, of course, to "computational limitations". One computational limitation can arise due to *truncation errors*. For example, consider an algorithm that generates a point x for which $f(x) = 10^{-10}$. It may not be possible to determine whether this is the true value of $f(x)$ or whether, due to truncation errors, the true value is $f(x) = 0$.

For an algorithm that generates an infinite sequence of points, another such limitation arises when the computer is made to terminate with a finite subsequence. When that happens, it is not possible to determine what would happen to the infinite sequence based on the finite subsequence. For example, consider an algorithm that, in theory, generates an infinite sequence of points but terminates finitely with the sequence $(x^k)_{k=1}^n$ for which each $f(x^k) = c$, for some constant c. Unfortunately, it is not possible to conclude that $f(x^k) = c$, for all $k = 1, 2, \ldots$. As another example, consider an algorithm that, in theory, generates an infinite sequence of points but terminates finitely with the sequence $(x^k)_{k=1}^n$ that is increasing monotonically. In this case it is not possible to determine whether the infinite sequence will continue to increase monotonically.

With these limitations in mind, an algorithm is now proposed for finding all of the roots of a convex function f.

3.1. The steps of the algorithm

The starting point for the algorithm is the *Subgradient Newton Method* proposed in 2007 by Solow and Li [3] for finding a root of f or determining that none exists. What is needed is an extension capable of finding two roots or an infinite number of roots, the latter being a significant challenge. Fortunately, however, if the function f has an infinite number of roots, Proposition 2.5 ensures that these roots constitute an interval on the real line, for which it would suffice to find the infimum and supremum of the set of roots (which may or may not be finite). To that end, let

$$X = \{x : f(x) = 0\} \quad \text{with} \quad x_* = \inf X \quad \text{and} \quad x^* = \sup X.$$

Thus, the proposed algorithm works in two phases: one attempts to find x_* and the other attempts to find x^*, each of which is described now.

Phase 1: A procedure for attempting to find inf X

Like the subgradient Newton method, the Phase 1 procedure requires a starting point x^0 and subgradient $0 \neq a^0 \in \partial f(x^0)$. As the only functions for which there is no such point are constant functions, it is assumed that the user provides this initial point x^0 and $a^0 \neq 0$. Without loss of generality, suppose further that $a < 0$ for which the following modified version of the subgradient Newton method is proposed for finding x_*.

Phase 1

Step 0: Compute the Newton iterate $x^1 = x^0 - f(x^0)/a^0$ and set $k = 1$.

Step 1: If $f(x^k) = 0$, stop.

Step 2: Choose any $a^k \in \partial f(x^k)$. If $a^k \geq 0$, stop.

Step 3: Compute the Newton iterate $x^{k+1} = x^k - f(x^k)/a^k$; set $k = k + 1$; go to Step 1.

The foregoing Phase 1 procedure will either terminate finitely or else generate an infinite sequence of points. The consequences of each of these two situations are given in the following two propositions, respectively.

In the subsequent proofs, note that during the Phase 1 procedure, each point x^k for $k \geq 1$ is a Newton iterate and, as such, satisfies $f(x^k) \geq 0$ (see Proposition 1.5). Note also that each time the Phase 1 procedure does not stop in Step 1 with $f(x^k) = 0$, it must be that

$$f(x^k) > 0. \tag{5}$$

Furthermore, each time the Phase 1 procedure does not stop in Step 2, the subgradient $a^k \in \partial f(x^k)$ satisfies $a^k < 0$. This, in turn, means that $(x^1, \ldots, x^k)$ is monotone increasing because, from the fact that x^k is a Newton iterate,

$$x^k - x^{k-1} = -f(x^{k-1})/a^{k-1}. \tag{6}$$

Proposition 3.1. *If the Phase 1 procedure terminates in Step 1, then $x_* = x^k$ while if the Phase 1 procedure stops in Step 2, then f has no root.*

Proof. To see that $x_* = x^k$ when the Phase 1 procedure terminates in Step 1 with $f(x^k) = 0$, it suffices to show that for all $x < x^k$, $f(x) > 0$, which means that f has no root less than the root x^k. So, using the fact that the Newton iterate $x^k = x^{k-1} - f(x^{k-1})/a^{k-1}$, where $0 > a^{k-1} \in \partial f(x^{k-1})$, it follows from the subgradient inequality and (6) that for any $x < x^k$,

$$\begin{aligned}
f(x) &\geq f(x^{k-1}) + a^{k-1}(x - x^{k-1}) \\
&= f(x^{k-1}) + a^{k-1}(x - x^k + x^k - x^{k-1}) \\
&= f(x^{k-1}) + a^{k-1}(x^k - x^{k-1}) + a^{k-1}(x - x^k) \\
&= f(x^{k-1}) - f(x^{k-1}) + a^{k-1}(x - x^k) > 0.
\end{aligned}$$

It remains to show that if the Phase 1 procedure stops in Step 2 with $a^k \geq 0$, then f has no root. As the Phase 1 procedure did not stop in Step 1, from (5), it must be that $f(x^k) > 0$. So consider first the case in which $a^k = 0$. This means that x^k is a global minimizer of f thus, for all x, $f(x) \geq f(x^k) > 0$ and so f has no root.

Finally, consider the case in which $a^k > 0$. From (6) it follows that $(x^1, \ldots, x^k)$ is not monotone increasing. In [3] it is shown that this means f has no root, thus completing the proof.

Proposition 3.2. *If the Phase 1 procedure generates an infinite sequence* $(x^k)_{k=1}^{\infty}$*, then this sequence is strictly monotonically increasing and, if bounded above, converges to* x_* *while if unbounded above, then* f *has no root.*

Proof. The fact that the sequence is monotonically increasing follows from (6) since for each k, $f(x^k) > 0$ by (5) and $a^k < 0$. In [3] it is shown that if the sequence is not bounded above, then f has no root while if the sequence is bounded above, then the sequence converges to a root, say y, of f. It remains to show that $x_* = y$. To that end, it suffice to show that there is no $x < y$ for which $f(x) = 0$. Suppose, to the contrary, that $x < y$ and $f(x) = 0$. Let k be an integer for which $x^k > x$ and $0 > a^k \in \partial f(x^k)$. Noting from (5) that $f(x^k) > 0$, it would then follow by the subgradient inequality that $0 = f(x) \geq f(x^k) + a^k(x - x^k) > 0$, which is a contradiction, completing the proof.

Assuming that the Phase 1 procedure finds x_*, it is now necessary to initiate Phase 2 for attempting to find x^*.

Phase 2: A procedure for attempting to find sup X

The approach proposed here to find x^* starts by searching for a point $y \geq x_*$ together with a subgradient $b \in \partial f(y)$ for which $b > 0$. If no such point is found, then either $x^* = x_*$ or else $x^* = +\infty$. On the other hand, if such a point is found, then a subgradient Newton method similar to that used in Phase 1 results in finding x^*. Here, then, are the steps for the proposed Phase 2 procedure, given x_*.

Phase 2

Step 0: Choose a fixed step size, say $\Delta y > 0$, set $y^1 = x_*$, and $j = 1$.

Step 1: Choose any $b^j \in \partial f(y^j)$. If $b^j > 0$, then go to Step 3.

Step 2: Set $y^{j+1} = y^j + \Delta y$, $j = j + 1$, and go to Step 1.

Step 3: Set $x^1 = y^j - f(y^j)/b^j$ and $k = 1$.

Step 4: If $f(x^k) = 0$, stop.

Step 5: Choose any $a^k \in \partial f(x^k)$, noting that $a^k > 0$.

Step 6: Compute $x^{k+1} = x^k - f(x^k)/a^k$; set $k = k + 1$; go to Step 4.

The foregoing Phase 2 procedure will either generate an infinite sequence, $(y_j)_{j=1}^{\infty}$ in Steps 0, 1, and 2; terminate finitely in Step 4; or else generate an infinite sequence of points $(x^k)_{k=1}^{\infty}$. The consequences of each of these three situations are given in the following three propositions, respectively.

Proposition 3.3. *If Steps 0, 1, and 2 of the Phase 2 Algorithm generate an infinite sequence,* $(y_j)_{j=1}^{\infty}$*, then either* $x^* = x_*$ *or* $x^* = +\infty$*, in which case* $X = [x_*, +\infty)$*.*

Proof. Recall from these steps that each $b^j \in \partial f(y^j)$ satisfies $b^j \leq 0$ and that $(y_j) \to +\infty$ as $j \to \infty$. It is first shown that for all $j \geq 1$, $f(y^j) \leq 0$ for otherwise, there is a $j \geq 1$ such that $f(y^j) > 0$. But then since $b^j \leq 0$, by the subgradient inequality, $0 = f(x_*) \geq f(y^j) + b^j(x_* - y^j) > 0$, which cannot happen. Next, it is shown that for all $x > x_*$, $f(x) \leq 0$ for otherwise, there is an $x > x_*$ such that

$f(x) > 0$. But then, choosing $j > 1$ such that $y_j > x$ and letting λ with $0 \leq \lambda \leq 1$ be such that $x = (1-\lambda)x_* + \lambda y^j$, it follows from the definition of convexity that $0 < f(x) \leq f((1-\lambda)x_* + \lambda y^j) \leq (1-\lambda)f(x_*) + \lambda f(y^j) \leq 0$, which cannot happen.

Given that for all $x > x_*$, $f(x) \leq 0$, it must be that either for all $x > x_*$, $f(x) < 0$ or else there is an $x > x_*$ such $f(x) = 0$. In the former case, f has no root $x > x_*$, which means that $x^* = x_*$. On the other hand, if there is an $x > x_*$ such $f(x) = 0$, then for all $y > x_*$, $f(y) = 0$. To see why, let $a \in \partial f(x)$. It is shown that $a = 0$ and hence x is a global minimizer of f. If $a < 0$, then by the subgradient inequality, $0 = f(x_*) \geq f(x) + a(x_* - x) > 0$, which cannot happen. On the other hand, if $a > 0$, then for $j > 1$ with $y^j > x$, it again follows from the subgradient inequality that $0 \geq f(y^j) \geq f(x) + a(y^j - x) > 0$, which cannot happen. This means that $a = 0$ and so x is a global minimizer of f, that is, for all y, $f(y) \geq f(x) = 0$. In particular, for all $y > x_*$, $f(y) \geq 0$ and also $f(y) \leq 0$ and so it must be that $f(y) = 0$. This means that $x^* = +\infty$ and so $X = [x_*, +\infty)$, completing the proof.

Proposition 3.4. *If the Phase 2 Algorithm stops in Step 4, then $x^* = x^k$.*

Proof. As in Phase 1, each point generated by the algorithm is a Newton iterate and, as such, satisfies $f(x^i) \geq 0$ (see Proposition 1.5). As is now shown, each point x^i also satisfies

$$x^i \geq x_* \tag{7}$$

and

$$a^i > 0 \tag{8}$$

The proof of (7) and (8) is by induction. To that end, let $x^0 = y^j - f(y^j)/b^j$ and $a^0 = b^j > 0$ (from Step 3 of the Phase 2 algorithm) and so $x^1 = x^0 - f(x^0)/a^0$. Then for $i = 1$, we have:

Proof of (7): To see that $x^1 \geq x_*$, note that by the subgradient inequality,

$$\begin{aligned} 0 = f(x_*) &\geq f(x^0) + a^0(x_* - x^0) = f(x^0) + a^0(x_* - x^1) + a^0(x^1 - x^0) \\ &= f(x^0) + a^0(x_* - x^1) - a^0(f(x^0)/a^0) = a^0(x_* - x^1). \end{aligned}$$

Dividing both sides of the foregoing inequality by $a^0 = b^j > 0$ and adding x^1 to both sides provides the desired result that $x^1 \geq x_*$.

Proof of (8): To see that $a^1 > 0$, suppose that $a^1 \leq 0$. Then because the Phase 2 Algorithm did not stop in Step 4, $f(x^1) > 0$ and from (b), $x_* - x^1 \leq 0$, it follows from the subgradient inequality that

$$0 = f(x_*) \geq f(x^1) + a^1(x_* - x^1) \geq f(x^1) > 0.$$

This contradiction establishes the claim.

To complete the induction proof, assume that properties (a) and (b) are true for $i-1$. To see that those properties are true for i, repeat the foregoing proofs of (a) and (b), substituting $i-1$ for the superscript 0 and i for the superscript 1.

Having established the properties in (7) and (8), it is now shown that if the algorithm stops in Step 4 with $f(x^k) = 0$, then $x^* = x^k$. To see that x^k is the largest root of f, it will be shown that for all $y > x^k$, $f(y) > 0$ and so y cannot be a root.

So let $y > x_k$, then, recalling that $x^k = x^{k-1} - f(x^{k-1})/a^{k-1}$ and that $a^{k-1} > 0$ from (8), it follows that

$$\begin{aligned} f(y) &\geq f(x^{k-1}) + a^{k-1}(y - x^{k-1}) \\ &= f(x^{k-1}) + a^{k-1}(y - x^k) + a^{k-1}(x^k - x^{k-1}) \\ &= f(x^{k-1}) + a^{k-1}(y - x^k) - a^{k-1}(f(x^{k-1})/a^{k-1} \\ &= a^{k-1}(y - x^k) > 0. \end{aligned}$$

□

Proposition 3.5. *If Steps 4, 5, and 6 of the Phase 2 Algorithm generate an infinite sequence,* $(x^k)_{k=1}^{\infty}$*, then* $(x^k)_{k=1}^{\infty}$ *is strictly monotonically decreasing and converges to* x^**.*

Proof. It is first shown that the sequence $(x^k)_{k=1}^{\infty}$ is strictly monotonically decreasing. As in the proof of Proposition 3.2, this follows from (6) since for each $k \geq 1$, $f(x^k) > 0$ by (5) and $a^k > 0$. To ensure that the sequence converges, it is now shown that the sequence is bounded below by x_*, for suppose not. Let $k \geq 1$ be such that $x^{k-1} > x_*$ but $x_k < x_*$. From the subgradient inequality and the fact that $a^{k-1} > 0$, it follows that

$$\begin{aligned} 0 = f(x_*) &\geq f(x^{k-1}) + a^{k-1}(x_* - x^{k-1}) \\ &= f(x^{k-1}) + a^{k-1}(x_* - x^k + x^k - x^{k-1}) \\ &= f(x^{k-1}) + a^{k-1}(x^k - x^{k-1}) + a^{k-1}(x_* - x^k) \\ &= f(x^{k-1}) - f(x^{k-1}) + a^{k-1}(x_* - x^k) > 0. \end{aligned}$$

The foregoing contradiction means that the sequence is bounded below by x_*. As such, the sequence converges to a point, x, that is a root of f (see [3]).

To complete the proof, it is then necessary to show that $x^* = x$. To that end, it suffices to show that there is no root of f with $y > x$, so assume, to the contrary, that $y > x$ with $f(y) = 0$. Let k be an integer for which $x^k < y$ with $0 < a^k \in \partial f(x^k)$. Noting from (5) that $f(x^k) > 0$, it would then follow by the subgradient inequality that $0 = f(y) \geq f(x^k) + a^k(y - x^k) > 0$, which is a contradiction, completing the proof.

In the event that $x_* = x^*$, then there is only one root of f. On the other hand, if $-\infty < x_* < x^* < +\infty$, then it remains to determine whether x_* and x^* are the only two roots of f or whether $X = [x_*, x^*]$. To do so, consider any point $x \in (x_*, x^*)$, for which, necessarily, $f(x) \leq 0$. If $f(x) < 0$, then x_* and x^* are the only roots of f while if $f(x) = 0$, then $X = [x_*, x^*]$.

3.2. Computational experiments

Computational experiments were run to validate the foregoing algorithm for the piecewise linear, quadratic, and exponential functions in the following examples. The specific parameter values used, the starting point, the final set X of roots, and the number of iterations (function evaluations) are also reported. Whenever a root is found, a step size of 1 is used to find a starting point for searching for a second root. The stopping criteria are when $|f(x)| < 10^{-10}$ or when 10000 iterations are exceeded.

Example 3.6. Piecewise linear function:

$$f(x) = \begin{cases} -k(x-a)+c & x < a, \\ c & a \le x \le b, \\ k(x-b)+c & x > b. \end{cases}$$

(a) Parameters: $k = 2$, $a = -1$, $b = 4$, $c = 0$; starting point: -2; $X = [-1, 4]$; iterations: 6.

(b) Parameters: $k = 2$, $a = -1$, $b = 4$, $c = -2$; starting point: -3; $X = \{-2, 5\}$; iterations: 8.

Example 3.7. Quadratic function: $f(x) = (x-a)^2 - b$.

(a) Parameters: $a = 1$, $b = -1$; starting point: 2; $X = \emptyset$; iterations: 1.

(b) Parameters: $a = 1$, $b = 0$; starting point: 2; $X = \{1\}$; iterations: 34.

(c) Parameters: $a = 1$, $b = 4$; starting point: 3; $X = \{-1, 3\}$; iterations: 13.

Example 3.8. Exponential function: $f(x) = e^{(x-a)} - b$.

(a) Parameters: $a = 0$, $b = 2$; starting point: 3; $X = \{\ln 2\}$; iterations: 8.

(b) Parameters: $a = 1$, $b = 0$; starting point: 3; $(x^k) \to -\infty$; iterations: 69. After 69 iterations, the algorithm stopped with $|f(x)| < 10^{-10}$ but this is not a root because this function asymptotes the x-axis.

(c) Parameters: $a = 2$, $b = -300$; starting point: 3; $X = \emptyset$; iterations: 2.

4. Summary and future research

In this work it is shown that the number of roots of a convex function f is either 0, 1, 2, or an infinite interval on the real line. Necessary and sufficient conditions are provided on the convex function f for each of these cases to arise and an algorithm is given for finding all of the roots, subject to computational limitations.

One direction for future research is based on the observation that the results here generalize the fact that a linear function has 0, 1, or an infinite number of roots to convex functions, that are shown to have 0, 1, 2, or an infinite interval of roots.

One is naturally led to defining

$$\mathcal{F}^k = \{f : R \to R : f \text{ is continuous and has } 0, 1, ..., k, \text{ or an infinite interval of roots}\}.$$

Just as linear and convex functions are useful subclasses of $\mathcal{F}^1$ and $\mathcal{F}^2$, respectively, it might be possible to identify useful subclasses of $\mathcal{F}^k$ for $k \ge 3$.

Another direction for future research is to generalize the results for a real-valued convex function of one variable to a system of n convex functions of n variables. For example, a system of n linear equations in n unknowns has 0, 1, or an infinite number of solutions. What can be said about the number of solutions of a system of n convex equations in n unknowns?

We end this paper with an open conjecture of a sufficient condition for a system of n convex equations in n unknowns to have a solution, that is, for there to be an n-vector $\mathbf{x}^* \in R^n$ such that for a convex function $f : R^n \longrightarrow R^n$, $f(\mathbf{x}^*) = \mathbf{0}$. By such a convex function is meant that each coordinate function $f_i : R^n \longrightarrow R^1$ is convex. At each $\mathbf{x} \in R^n$, there is a subgradient vector $\mathbf{a}^i \in R^n$ for which $f_i(\mathbf{y}) \geq f_i(\mathbf{x}) + \mathbf{a}^i(\mathbf{y} - \mathbf{x})$ for all $\mathbf{y} \in R^n$. The $(n \times n)$-matrix A, in which row i is $\mathbf{a}^i$, is the *subgradient matrix* of $f : R^n \longrightarrow R^n$ at $\mathbf{x} \in R^n$. With this understanding, the following open conjecture was proposed by B. Curtis Eaves in the Department of Operations Research at Stanford University in the late 1970s:

Conjecture: *Given a convex function $f : R^n \longrightarrow R^n$, if there is a real number $\epsilon > 0$ such that for all $\mathbf{x} \in R^n$ and for all subgradient matrices $A \in \partial f(\mathbf{x})$, $|det(A)| > \epsilon$, then there is an $\mathbf{x}^* \in R^n$ such that $f(\mathbf{x}^*) = \mathbf{0}$.*

The foregoing conjecture is true for $n = 1$ but, to the best of the author's knowledge, remains open for $n > 1$.

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