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Max-Solar Properties of Sets in Normed and Asymmetrically Normed Spaces

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The paper is concerned with new concepts of the max-approximation theory related to solar properties of sets: a max-sun, a local max-sun, max-Chebyshev set, and a max-attractor. Some results on max-solarity are established for uniquely remotal sets (such sets are called absolute max-Chebyshev sets). An example of a nonsingleton (nonclosed) uniquely remotal set on an asymmetrically normed plane is constructed, which gives a negative answer in the problem of whether a uniquely remotal set is a singleton (the unique farthest point problem). Results are obtained both in symmetrically and asymmetrically normed spaces.

Keywords: Max-approximation, farthest point, absolutely max-Chebyshev set, uniquely remotal set, max-sun, local max-sun, point of max-luminosity, Chebyshev center.

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1. Introduction

In the present paper, we consider some max-approximation problems, in which, as in many classical problems of geometric approximation theory, a set is approximated by a singleton (a farthest point, in the max-approximation setting). The max-projection operator F_M (defined in (3) below) associates with each point the set of all its farthest (remotal) points from the set M (unlike the best approximation operator, which associates with a point the set of all its nearest points from M). In the case of strictly convex (asymmetric) spaces, this problem is equivalent to the problem of max-approximation of the convex hull of the set (which is not assumed to be closed). One of the impetuses to the development of the theory of max-approximation was the well-known Efimov-Stechkin-Klee problem on convexity of Chebyshev sets in Hilbert spaces. In this regard, E. Asplund (see, for example, [3], [5], [11]) showed that a Hilbert space contains a nonconvex Chebyshev set if and only if it contains a nonsingleton uniquely remotal set (a so-called absolute max-Chebyshev set; see Definition 3.6(3) below). This result led to the following well-known problem formulated by Asplund for Hilbert spaces:

$$\textit{is each absolute max-Chebyshev set (uniquely remotal set) a singleton?} \qquad (1)$$

At present, both these problems in infinite-dimensional Hilbert spaces remain unanswered (see also Remark 4.4 below). For non-necessarily Hilbert spaces, problem

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(1) is known under the name UFPP – Unique farthest point problem. A number of positive answers to the UFPP in non-Hilbert spaces was given Asplund, Blatter, Bosznay, Klee, Narang, Balashov, Ivanov and Alimov (this list is far from being complete).

Recently, some results on stability of the max-projection (also called antiprojection) with respect to a point and a set were obtained by Balashov, Ivanov, Goncharov, and others (see, for example, [1], [2], [9], [23]).

Problems related to max-approximation are classical in approximation theory. First results on max-approximation were obtained in the middle of the 20th century by Jessen, Motzkin, Straus, Valentine, Blatter, Klee, and others.

For some recent studies, see [1], [2], [9], [18], [23].

Problems of max-approximation are closely related to the Chebyshev center problem – in this problem, one searches for a point that best approximates (represents) the entire given set. Such points are called Chebyshev centers of the set M; see, for example, [4] and Definition 3.3 below. The set of all Chebyshev centers of a set M is denoted by $\mathrm{Z}(M)$. It is well known that if a nonsingleton set M is uniquely remotal (is an absolute max-Chebyshev set), then the max-projection operator F_M onto M is discontinuous at a (unique) Chebyshev center of the set (see, for example, [10]). Thanks to this observation, the definitions of approximatively defined sets M will be given depending on whether the points from the set of Chebyshev centers $\mathrm{Z}(M)$ are excluded from consideration or not. For example, in the definition of an absolute max-Chebyshev set (see Definition 3.6 below), it is required that the operator $\mathrm{F}_M x$ should have singleton values for any $x \in X$, and in the definition of a max-Chebyshev set this condition ($|\mathrm{F}_M x| = 1$) should be satisfied not for all $x \in X$, but for $x \in X \setminus \mathrm{Z}(M)$ (see Definition 4.5).

The paper is organized as follows. In §2, we define an asymmetric norm and an asymmetrically normed space. In §3, we introduce the main concepts related to solar properties in max-approximation: a max-sun, a local max-sun, and a max-attractor. It is worth mentioning that (local) max-solar properties are closely related to algorithms for finding Chebyshev or near-Chebyshev centers for such sets (see [4], [8], [15]). In §4, we establish some theorems for max-attractors, max-suns, and max-Chebyshev sets. In particular, we examine which max-solar conditions imply the singletoness of the set (not necessarily closed). We also study max-Chebyshev sets – such sets have a unique farthest point for any point which is not a Chebyshev center for this set. It is established (Theorem 4.2) that in a complete symmetrizable asymmetric space a max-attractor is a singleton. As a corollary, we show that any local max-sun which is a max-Chebyshev set is necessarily a max-sun. In Theorem 4.7, we establish the following new property of max-suns with a unique Chebyshev center: such a set M always contains the sphere $S(x_0, r(M))$, where $r(M)$ is the Chebyshev radius of the set M, x_0 is a (unique) Chebyshev center of M (note that the inclusion $M \subset B(x_0, r(M))$ always holds). In Theorem 4.10, a new result on local max-solarity is established for sets for which the max-projection operator F is upper semicontinuous and has nonempty compact acyclic values. Theorem 4.14 shows that if the max-projection operator F_M onto a max-approximatively compact set M is single-valued on some set T, then it is continuous on T. Using this result, we show that in a finite-dimensional strictly convex asymmetric space any absolute max-Chebyshev set is a singleton (here it is not assumed that the set is closed). In the concluding § 5, we give an analogue of the classical concept of a δ-sun for max-approximation problems and prove Theorem 5.3, which gives a sufficient condition for a point to be a max-δ-solar point of a set.

It is worth pointing out that, unlike most of the studies on max-approximation, in what follows we do not assume, as a rule, that the approximatively defined sets under consideration are closed or convex.

The main results will be formulated for asymmetrically normed spaces, even though they are also new in the classical setting of normed linear spaces.

2. Asymmetrically normed space

An *asymmetric norm* on a linear real space X is a nonnegative sublinear functional $\|\cdot\,|$ such that, for all $x, y \in X$,

(1) $\|x\,| = 0 \Leftrightarrow x = 0$;

(2) $\|\alpha x\,| = \alpha\|x\,|$ for all $\alpha \geq 0$;

(3) $\|x + y\,| \leq \|x\,| + \|y\,|$.

In general, $\|x\,| \neq \|-x\,|$. The name "asymmetric norm" was given by Krein in 1938.

The function $\|x\|_{\mathrm{sym}} = \max\{\|x\,|, \|-x\,|\}$, $x \in X$, is called the *symmetrization norm*. By an *asymmetric* space we will mean a real linear space equipped with an asymmetric norm. The topology τ of any asymmetric space is generated by open balls $\mathring{B}(x, r) = \{y \in X \mid \|y - x\,| < r\}$. In general, such topology satisfies only the T_1-separation axiom and may fail to be Hausdorff, but even in the Hausdorff case the "closed" ball $B(x, r) = \{y \in X \mid \|y - x\,| \leq r\}$ my fail to be closed (see, for example, [13]). We will also consider the reflected (left) balls $B^-(x, r) = \{y \mid \|x - y\,| \leq r\}$, i.e., the balls with respect to the (mirror) asymmetric norm $\|-\cdot\,|$. Similarly, one defines the left (reflected) open balls $\mathring{B}^-(x, r)$, and the spheres $S^+(x, r)$ and $S^-(x, r)$. For right ("+") objects, the superscripts "+" will be dropped as a rule.

Asymmetric spaces with natural T_1-topology will be simply called *asymmetric spaces* (or T_1-asymmetric spaces). Below, we will also consider the more narrow class of spaces X for which the asymmetric norm $\|\cdot\,|$ is equivalent to the symmetrization norm $\|\cdot\|_{\mathrm{sym}}$ (i.e., there exists a number $K \geq 1$ such that $\frac{1}{K}\|x\|_{\mathrm{sym}} \leq \|x\,| \leq \|x\|_{\mathrm{sym}}$ for all $x \in X$). Such spaces will be called *symmetrizable* asymmetric spaces. The topology τ_{sym} of a symmetrizable space is generated by the open balls of the form $\{y \mid \|y - x\|_{\mathrm{sym}} < r\}$, where $x, \in X$, $r > 0$. Note also that a space in which the balls $B(x, r)$ are closed may fail to be symmetrizable (see [13]).

The theory of asymmetric normed spaces and applications thereof (in particular, in max- and min-approximation theory) is in active development at present (see, for example, [6], [7], [12], [17], [24]).

Below, $X = (X, \|\cdot\,|)$ is a real normed or asymmetrically normed linear space, $\|\cdot\,|$ is a norm or an asymmetric norm on X.

3. Farthest points, basic concepts of max-approximation

In this section, we will introduce or recall basic definitions of max-approximation, which are given in terms of the max-distance function $r(\,\cdot\,, M)$ or the max-projection F_M: a farthest point, a max-attractor, a max-sun, a local max-sun, a max-Chebyshev set, a max-luminosity points, etc.

All definitions and theorems will be given in the case of right approximation, i.e., in the case where the max-distance is measured from a point to the set. A corresponding reformulation for the case of left approximation is clear.

Definition 3.1. Let $x \in X$, $\emptyset \neq M \subset X$. A point $y_0 \in M$ is called a (right) *farthest* point from the set M for a point x if

$$\|y_0 - x\,| = \sup\bigl\{\|y - x\,| \mid y \in M\bigr\} =: r(x, M). \tag{2}$$

In other words, $r(x, M)$ is the radius of the smallest ball with center at x that contains the set M. The function $r(\,\cdot\,, M)$ is known as the max-*distance function*. □

Similarly, one defines the left farthest point in the set M for a point x, i.e., a point $y_0 \in M$ such that $\|x - y_0\,| = \sup\bigl\{\|x - y\,| \mid y \in M\bigr\} =: r^-(x, M)$. Below, we will consider only "right" properties (as in Definition 3.1); a reformulation of the corresponding definitions and results to the case of "left" properties is straightforward.

Definition 3.2. The mapping

$$x \longmapsto \mathrm{F}_M(x) := \bigl\{y \in M \mid \|y - x\,| = r(x, M)\bigr\} \tag{3}$$

is called the *metric* max-*projection* [4], [1] (the max-*projection* or the *metric anti-projection* [9]). So, the set $\mathrm{F}_M(x)$ consists of all farthest points from M for x. □

In the normed case, the max-distance function $r(x, M)$ is convex (*qua* the supremum of convex functions $x \mapsto \|y - x\,|$) and is Lipschitz continuous:

$$|r(x, M) - r(y, M)| \leq \|x - y\| \leq r(x, M) + r(y, M) \qquad \text{for all } x, y \in X. \tag{4}$$

In the asymmetric case, one guarantees only the inequality

$$r(x, M) \leq r(y, M) + \|y - x\,|,$$

from which it follows that the max-distance function $r(x, M)$ is lower semicontinuous, i.e., $r(x, M) \leq \lim r(x_n, M)$ as $\|x_n - x\,| \to 0$.

Definition 3.3. Let $M \subset X$ be a nonempty bounded set. The quantity

$$r(M) := \inf\bigl\{a \geq 0 \mid x \in X,\ M \subset B(x, a)\bigr\} = \inf_{x \in X} r(x, M), \tag{5}$$

is called the *Chebyshev radius* of the set M. A point $x_0 \in X$ at which we have $M \subset B(x_0, r(M))$ is known as a *Chebyshev center* of the set M. The set (possibly empty) of all Chebyshev centers of a nonempty bounded set M is denoted by $\mathrm{Z}(M)$. For more details and a survey on Chebyshev centers, see [4], [5, Chap. 15]; for some results in the asymmetric case, see [17]. □

Let us mention some properties distinguishing the theory of max-approximation from problems of min-approximation. For example, in problems max-approximation an absolute max-existence set is necessarily bounded (of course, this is not so in min-approximation) and an absolute max-existence set need not be closed (in the classical min-approximation theory, an existence set is always closed).

The following clear result is well known:

$$\text{if } y \in \mathrm{F}_M x, \text{ then } y \in \mathrm{F}_M\big(x + t(y-x)\big) \quad \text{for all } t < 0, \tag{6}$$

i.e. if y is a farthest point from M for x, then y is a farthest point also for all points of the ray emanating from x in the direction $\overrightarrow{yx}$. In the case of min-approximation, if $y \in M$ is a luminosity point form M for $x \in X \setminus M$, then the point y remains a nearest point for any point located on the ray $[\overrightarrow{y,x})$ away from the point x to infinity (see, for example, [3], [5, § 1.1]). In the case of max-approximation, this naturally corresponds to the motion along the same ray from the point x in the (opposite) direction $\overrightarrow{xy}$, where y is a farthest point from M until the condition $r(u, M) > r(M)$ holds, where $u \in [x, y)$.

Moreover, if the set M has a Chebyshev center, then, in accordance with what has been just said, the motion takes place until reaching a Chebyshev center (not inclusively), and so the Chebyshev centers can be looked upon as analogues of the point at infinity in the case of min-approximation.

Here are the corresponding definitions in the max-approximation setting.

Definition 3.4. (1) Let $x \in X$, $M \subset X$. A point $y \in \mathrm{F}_M x$ for which there exists a number $\delta > 0$ such that

$$y \in \mathrm{F}_M(z) \qquad \text{for any point } \ z \in \big[x, x + \delta(y-x)\big), \tag{7}$$

will be called an *attraction point* (or a max-attraction point) from the set M for the point x; the point x is called a *local* max-*solar point* from M for x.

(2) A set M is called a max-*attractor* if any point $x \in X$ is a max-solar point for the set M.

(3) A set M is called a *local* max-*sun* if any point $x \notin \mathrm{Z}(M)$ is a max-solar point for M.

(4) A set M is called a max-*sun* if, for any point $x \in X \setminus \mathrm{Z}(M)$, there exists a farthest point $y \in \mathrm{F}_M(x)$ such that

$$y \in \mathrm{F}_M(z) \text{ for any } z \in [x, y) \text{ under the condition that } \|y - z\,| > r(M). \quad \square \tag{8}$$

(5) A point x for which there exists a farthest point $y \in \mathrm{F}_M(x)$ satisfying (8) will be called a max-*solar point* from M for x; the corresponding point y is called a max-*luminosity point* from M for x.

Remark 3.5. Similar (but more restrictive) properties were also considered in [10], [16], [20].

Some properties of max-attractors (under different names) were established in [1], [2], [16] in the normed space case.

Definition 3.6. (1) A set $M \subset X$ is called a max-*existence set* if $\mathrm{F}_M x \neq \emptyset$ for any $x \in X$ such that $r(x, M) > r(M)$.

(2) A set $M \subset X$ is called an *absolute* max-*existence set* (or a remotal set) if $\mathrm{F}_M x \neq \emptyset$ for any $x \in X$ such that $r(x, M) \geq r(M)$.

(3) A set $M \subset X$ is called an *absolute* max-*Chebyshev set* (or a *uniquely remotal set*) if $|\mathrm{F}_M x| = 1$ for any $x \in X$. □

4. Theorems on max-attractors, max-suns and max-Chebyshev sets

We first note the following simple result, which holds in arbitrary T_1-asymmetric spaces.

Lemma 4.1. *If* $y \in \mathrm{F}_M x$ *and* $\|y - x\,| = \|y - z\,| + \|z - x\,|$, *then* $y \in \mathrm{F}_M z$.

Indeed, let $y' \in \mathrm{F}_M z$. Since $y \in \mathrm{F}_M x$, we have

$$B(z, \|y' - z\,|) \subset B(x, \|y - x\,|). \tag{9}$$

In view of the well-known equivalence

$$B(u, \alpha) \subset B(u', \alpha') \iff \|u - u'\,| \leq \alpha' - \alpha \tag{10}$$

it follows from inclusion (9) that

$$\|z - x\,| \leq \|y - x\,| - \|y' - z\,| = \|y - z\,| + \|z - x\,| + \|y' - z\,|,$$

which leads to the required result: $\|y' - z\,| \leq \|y - z\,|$.

The next theorem gives a positive answer to the UFPP (see (1)) for the class of max-attractors.

Theorem 4.2. *In a complete symmetrizable asymmetric space, any* max-*attractor is a singleton.*

Corollary 4.3. *Any* max-*attractor in a Banach space is a singleton.*

A particular case of Corollary 4.3 under a restrictive constraint is due to Baronti and Papini [10] (see also Narang [20]). For the normed space setting, see [1].

Remark 4.4. (1) In relation to Theorem 4.2, we also note that in the spaces $L^1[0,1]$, $L^\infty[0,1]$, c, $c_0(\Gamma)$, ℓ^1, $C(Q)$ (Q is a compact topological space) any absolute max-Chebyshev set is a singleton (see, for example, Panda [21]).

(2) The condition that a space in Theorem 4.2 (and in Theorem 4.14 below) should be symmetrizable is important, because in such spaces the max-distance function $r(\,\cdot\,, M)$ is continuous (in essentially asymmetric (nonsymmetrizable) spaces, $r(\,\cdot\,, M)$ can be discontinuous even for a singleton M).

Proof of Theorem 4.2. Let $x \in X$, $y \in \mathrm{F}_M x$ be a point of max-attraction from $\mathrm{F}_M x$ for a point x (i.e., for x and y condition (7) from the definition of a max-attractor at a point x is satisfied). We set

$$\delta_0 := \sup \bigl\{\delta \mid y \in \mathrm{F}_M z \text{ for the point } z = x + \delta(y - x)\bigr\}. \tag{11}$$

According to (7), $\delta_0 > 0$. We set $x_{\delta_0} := x + \delta_0(y - x)$. If $x_{\delta_0} = y$, then $M = \{y\}$ and there is nothing to prove. So, in what follows, we will assume that $x_{\delta_0} \neq y$. If $z_\lambda := \lambda x_{\delta_0} + (1 - \lambda)x$, $0 \le \lambda \le 1$, then by (11)

$$y \in \mathrm{F}_M z_\lambda \quad \text{for all } \ 0 \le \lambda < 1. \tag{12}$$

Since on the interval $[x, x_{\delta_0}]$ the topologies τ and τ_{sym} are equivalent (as on a subset of a finite-dimensional space), and since in view of (4) the max-distance function F_M is continuous with respect to the symmetrization norm $\|\cdot\|_{\mathrm{sym}}$, we have $y \in \mathrm{F}_M z_1$ $(= \mathrm{F}_M x_{\delta_0})$. Therefore,

$$y \in \mathrm{F}_M z_\lambda \quad \text{for all } \ 0 \le \lambda \le 1 \ \text{ (and, in particular } y \in \mathrm{F}_M x_{\delta_0}). \tag{13}$$

As a corollary,

$$\|y - x\,| = \|y - z_\lambda\,| + \|z_\lambda - x\,| \quad \text{for all } \ 0 \le \lambda \le 1. \tag{14}$$

By definition (11) of the number δ_0, at the point x_{δ_0} the "max-attraction" switches to some other farthest point $y_1 \in \mathrm{F}_M x_{\delta_0}$. It is clear that in this case we also have $y \in \mathrm{F}_M x_{\delta_0}$. For this point x_{δ_0} and the new attraction point y_1, we choose in accordance with (7) a number δ_0 such that $y_1 \in \mathrm{F}_M z^{(1)}_\lambda$ for all $0 \le \lambda < 1$, where $z^{(1)}_\lambda = \lambda x_\delta + (1 - \lambda) z_{\lambda_0}$, $x_\delta := z_{\lambda_0} + \delta(y_1 - z_{\lambda_0})$. Similarly to (11), we set

$$\delta_1 := \sup\big\{\delta > 0 \mid y_1 \in \mathrm{F}_M z \text{ for the point } z = x_{\delta_0} + \delta(y_1 - x_{\delta_0})\big\}.$$

As in (12), since F_M is continuous, we find that $y_1 \in \mathrm{F}_M x_{\delta_1}$.
As a corollary, $y_1 \in \mathrm{F}_M z^{(1)}_\lambda$ for all $0 \le \lambda \le 1$.
Assuming that $\|y - x_{\delta_1}\,| < \|y_1 - x_{\delta_1}\,|$, we readily get the contradiction:

$$\begin{aligned}\|y - z_{\lambda_0}\,| &\le \|x_{\delta_1} - z_{\lambda_0}\,| + \|y - x_{\delta_1}\,| < \|x_{\delta_1} - z_{\lambda_0}\,| + \|y_1 - x_{\delta_1}\,| \\ &= \|y_1 - z_{\lambda_0}\,| \overset{(13)}{=} \|y - z_{\lambda_0}\,|.\end{aligned}$$

Hence
$$\|y - z^{(1)}_\lambda\,| = \|y_1 - z^{(1)}_\lambda\,| \qquad \text{for all } \ 0 \le \lambda \le 1, \tag{15}$$

which gives
$$y \in \mathrm{F}_M z^{(1)}_\lambda \qquad \text{for all } \ 0 \le \lambda \le 1. \tag{16}$$

Next, we act similarly, constructing for $i \ge 2$ the point x_{δ_i} and an attraction point $y_i \in M$ for x_{δ_i}. If, for some i, we have $y_i = y$, then the process of construction of points x_{δ_i} and y_i terminates, and $M = \{y\}$, the result required. Now let the number of such i's be infinite. Similarly to (14) and (15), for any point $z \in [x_{\delta_i}, x_{\delta_{i+1}}]$ for any i we have

$$\|y - x\,| = \|y - z\,| + \|z - x\,|. \tag{17}$$

From (16) for any i we have $y \in \mathrm{F}_M x_{\delta_i}$, and, as a corollary, $y \in \mathrm{F}_M z$ for any point $z \in [x_{\delta_i}, x_{\delta_{i+1}}]$. By the construction, from (17) we have

$$\|y - x\,| = \|y - x_{\delta_{i+1}}\,| + \|x_{\delta_{i+1}} - x_{\delta_i}\,| + \|x_{\delta_i} - x\,|, \tag{18}$$

which gives
$$\|x_{\delta_{i+1}} - x\,| = \|x_{\delta_{i+1}} - x_{\delta_i}| + \|x_{\delta_i} - x\,|. \tag{19}$$

The sequence $\|x_{\delta_i} - x\,|$ is monotone increasing and is bounded by (18). Hence it converges. By (19), $\|x_{\delta_{i+1}} - x_{\delta_i}\,| \to 0$. By the assumption, the space X is complete[1], and hence, there exists a point $\widehat{x} \in X$ such that $\|x_{\delta_n} - \widehat{x}\,| \to 0$. From (17) we have

$$\|y - x\,| = \|y - x_{\delta_i}\,| + \|x_{\delta_i} - x\,|. \tag{20}$$

Letting $i \to \infty$ in (20), we have $\|y - x\,| = \|y - \widehat{x}\,| + \|\widehat{x} - x\,|$, which gives $y \in \mathrm{F}_M\widehat{x}$ by Lemma 4.1.

If $\widehat{x} = y$, then y is a farthest point from M for $\widehat{x}$, from which we readily get the required result: $M = \{y\}$. If $\widehat{x} \neq y$, then by the above $y \in \mathrm{F}_M\widehat{x}$, and we can move "further away" towards y, thereby obtaining a new limit point $\widehat{x}$, for which we again have $y \in \mathrm{F}_M\widehat{x}$. As a result, after a finite or infinite number of constructions of such limit points, we obtain $\widehat{x} = y$, giving the required result: $M = \{y\}$. Theorem 4.2 is proved. □

Definition 4.5. A set M will be called a max-*Chebyshev set* if, for any point $x \in X$ such that $r(x, M) > r(M)$, a farthest point from M for x is unique. □

In fact, in the proof of Theorem 4.2 we established the following result, which holds in arbitrary asymmetric spaces.

Corollary 4.6. *If a set is a local* max-*sun and a* max-*Chebyshev set, then this set is a* max-*sun.*

Theorem 4.7. *Let $M \neq \emptyset$ be a* max-*sun. Then:*

(1) *if the set of Chebyshev centers* $\mathrm{Z}(M)$ *for M is empty, then M is a singleton.*

(2) *if the set of Chebyshev centers* $\mathrm{Z}(M)$ *for M is a singleton $\{z\}$, then*

$$M \supset S(z, r(M)), \tag{21}$$

where $r(M)$ *is the Chebyshev radius of the set M (the inclusion $M \subset B(z, r(M))$ always holds).*

Remark 4.8. (1) The condition $\operatorname{card}\mathrm{Z}(M) = 1$ in assertion (2) of Theorem 4.7 is essential. For example, if M is the rectangle with vertices $(2, \pm 1)$, $(\pm 2, -1)$ in ℓ^∞_2, then the set of Chebyshev centers $\mathrm{Z}(M)$ of the set M is not a singleton, M is a max-sun, but for M (21) is not satisfied.

(2) Note that if x is a max-solar point of the set M, $y \in M$ is the corresponding max-attraction point from M for x, and

$$\text{\textit{if the half-open interval } } (y, x] \text{ \textit{does not contain a Chebyshev center of} } M\text{\textit{, then} } M \text{ \textit{is a singleton.}} \tag{22}$$

In particular,

$$\text{\textit{any} max-\textit{sun has a Chebyshev center.}} \tag{23}$$

Indeed, if a max-sun M would have no Chebyshev center, then by (22) M would be a singleton, but a singleton M always has a Chebyshev center. (However, not each

[1] By the hypothesis, the space is symmetrizable, and hence the definitions of its completeness with respect to the topology τ, the mirror topology, or the symmetrization topology are equivalent.

set admitting a unique Chebyshev center is necessary a max-sun – see Example 4.19 below). It follows that

> *if M is a nonsingleton max-sun, $x \in X$, and $y \in M$ is a* max-*luminosity point from M for x (in the sense of Definition* 3.4(4)), *then the interval $[y, x]$ contains a Chebyshev center of the set M.* (24)

Proof of Theorem 4.7. Assume on the contrary that there exists some element $y \in S(x_0, r) \setminus M$, where $\mathrm{Z}(M) = \{x_0\}$, r is the Chebyshev radius of the set M

Case 1: y is a limit point of the set $M \cap (y, x_0]$: $y = \lim v_n$, where $v_n \in M \cap (y, x_0]$. For a sufficiently small $\delta > 0$, we set $x := x_0 + \delta(x_0 - y)$. Let y_x be a max-luminosity point from M for x. According to (24), $(y_x, x) \ni x_0$. Since $v_n \to y$, we have $y_x \notin (y, x_0)$. But the case $y_x = \lambda y + (1 - \lambda)x_0$, $\lambda \geq$, i.e., when it lies further on the ray $[\overrightarrow{x_0, y}) \setminus [x_0, y)$, is also impossible, because for $\lambda = 1$ we get $y = y_x$, which is a contradiction, since $y \notin M$, $y_x \in M$, and since for $\lambda > 1$ the point y_x lies outside the ball $B(x_0, r)$, which is also impossible, inasmuch as $r(M) = r$.

Case 2: $y \notin M$ and there exists $\mathscr{O}_\delta(y) := \mathring{B}(y, \delta)$ such that $\emptyset = (\mathscr{O}_\delta(y) \cap [y, x_0] \cap M)$. Let $y' := \mathscr{O}_\delta(y) \cap [y, x_0]$. Then the function $r(\,\cdot\,, M)$ is continuous on the interval $[y, 2x_0 - y]$. Hence there exists a point $x \in (x_0, 2x_0 - y]$ which is sufficiently close to x_0 and such that $|r(x, M) - r(x_0, M)| < \delta/2$. Let y_x be a max-luminosity point from M for x_0. Arguing as above, we see that $y_x \in [y', x_0)$. But in this case, we will easily get a contradiction to the previous inequality. This contradiction completes the proof of Theorem 4.7. □

Recall that a metrizable space is called *acyclic* if its Čech cohomology group with coefficients over the rationals is trivial (see [3, § 6], [5, § 10.4.2]).

We need the following extension of the classical Himmelberg's fixed point theorem (see, for example, [22, Theorem 4.2]).

Theorem 4.9. *Let C be a nonempty convex subset of a locally convex Hausdorff topological space. Then any upper semicontinuous mapping $f : C \to 2^C$ with compact acyclic values and such that $f(C)$ is precompact in C has a fixed point.*

In the case of classical min-approximation, it is well known that a boundedly compact P-acyclic set [2] with upper semicontinuous metric projection is a sun (see, for example, [3, Remark 9.3], [5, § 10.4.3]). Let us give an analogue of this result for the case of max-approximation (see also Theorem 1 in [1]).

Theorem 4.10. *Let X be a finite-dimensional asymmetric space, $M \subset X$, $x_0 \in X$, and let U be a neighborhood of x_0. Assume that on U the* max-*projection operator* F_M *is upper semi-continuous and has nonempty compact acyclic values. Then x_0 is a local* max-*solar point of the set M.*

Definition 4.11. A set $\emptyset \neq M \subset X$ is called max-*approximatively compact* if, for any $x \in X$, any maximizing sequence (y_n) from M for x (i.e., $\|y_n - x| \to r(x, M)$) contains a subsequence (y_{n_k}) such that $\|y_{n_k} - y| \to 0$, $y \in M$.

[2] A set M is called *P-acyclic* if, for any point x, the set $P_M x$ of nearest points from M for x is acyclic.

In the symmetrical setting, max-approximatively compact sets were studied (under different names) by J. Blatter, B. Panda, O. Kapoor, and others (see, for example, [1]).

Remark 4.12. (1) The conclusion of Theorem 4.10 also holds if M is a precompact max-approximatively compact subset of a symmetrizable asymmetric space (in particular, if M is a bounded closed set in a finite-dimensional asymmetric space) and if the operator F_M has acyclic values. In this case, by a well-known result of J. Blatter (see, for example, [1]), the operator F_M is upper semicontinuous (Blatter's result is formulated in the normed space setting, but it can be easily carried over to the case of symmetrizable asymmetric spaces).

(2) It is well known that there exists a Banach space (see, for example, [4, Example 4.1]) containing a three-point set that does not admit a Chebyshev center. Moreover, S. V. Konyagin showed that any nonreflexive Banach space X can be (equivalently) renormed so that the resulting space contains a three-point set without a Chebyshev center (see, for example, [4, Chap. I, § 4]). According to Remark 4.12(a), such a set is not a max-Chebyshev set.

Proof of Theorem 4.10. Let $x \notin M$, $r > 0$. Consider the set-valued mapping

$$\varphi(z) = \varphi_{x,r}(z) := x + \frac{r}{r + r(x, M)}(\mathrm{F}_M z - z), \qquad z \in X.$$

We will first show that if z is a fixed point of the mapping φ, then there exists a point $y \in \mathrm{F}_M z$ such that

$$z \in (x, y). \tag{25}$$

Indeed, setting $\alpha := r(r + r(x, M))^{-1}$, we have $z = x + \alpha(y - z)$ for some $y \in \mathrm{F}_M z$. As a corollary, $z = (1 + \alpha)^{-1}x + a(1 + \alpha)^{-1}\, y$, which proves (25).

Next, we assume without loss of generality that $x = 0$, $r(0, M) = 1$. Let $r > 0$. For simplicity, we set $B_r := B(0, r)$. We claim that

$$\varphi(B_r) \subset B_r. \tag{26}$$

Indeed, for $v \in \varphi(B_r)$ we have $v = \alpha(y - z)$, where $z \in B_r$, $y \in \mathrm{F}_M z$, and $\alpha := r(r + 1)^{-1}$. It is clear that $r(u, M) \le \|v - u\,| + r(v, M)$ for all $u, v \in X$. Hence $\|y - z\,| = r(z, M) \le \|z\,| + r(0, M) \le r + 1$. Since $v = \alpha(y - z)$, we have $\|v\,| \le r$, proving (26). By the assumption, F_M has compact acyclic values on U. Hence φ also has acyclic values on U, because a homeomorphic (in particular, homothetic) image of an acyclic set is acyclic. By Theorem 4.9, the mapping φ has a fixed point on B_r. In view of (25) this implies that x_0 is a local max-solar point for M. Theorem 4.10 is proved. □

Motzkin, Valentine, and Straus [19] established that if M is a compact subset of a normed linear space and if the max-projection operator F_M is single-valued on some set T, then the operator F_M is continuous on T. Let us establish a stronger result.

Theorem 4.13. *Let X be a symmetrizable asymmetric space, M be a* max-*approximatively compact set. Assume that the* max-*projection operator* F_M *is single-valued on some set T. Then the operator* F_M *is continuous on T.*

Proof of Theorem 4.13. Let $x_n, x \in T$, $\|x - x_n| \to 0$ and let $\mathrm{F}_M x_n = \{y_n\}$, $\mathrm{F}_M x = \{y\}$. We have

$$\|y - x_n| \le \|y_n - x_n| \le \|y_n - x| + \|x - x_n| \le \|y - x| + \|x - x_n|. \tag{27}$$

By the assumption the space X is symmetrizable, and hence the asymmetric norm is continuous. Therefore we have $\|y - x_n| \to \|y - x|$, and so from (27) we conclude $\|y_n - x| \to \|y - x| = r(x, M)$, i.e., (y_n) is a maximizing sequence from M for x. The set M is max-approximatively compact, and hence there exist a subsequence $(y_{n_k}) \subset (y_n)$ and $z \in M$ such that $\|y_{n_k} - z| \to 0$. We have $\|x_n - y_n| \to \|x - y|$ (by continuity of the max-distance function $r(\,\cdot\,, M)$) and $\|x_{n_k} - y_{n_k}| \le \|x_{n_k} - z| + \|z - y_{n_k}|$, and hence letting $k \to \infty$ in this inequality and using the continuity of the (symmetrizable) asymmetric norm $\|\cdot|$, we get $\|x - y| \le \|x - z|$, which then implies $z \in \mathrm{F}_M x$ since $y \in \mathrm{F}_M x$. Next by the assumption $|\mathrm{F}_M(x)| = 1$, and so $y = z$, which gives $\|y_{n_k} - y| \to 0$. But since every subsequence of (y_n) has a convergent subsequence, this obviously implies that the sequence itself converges to y (see, for example, [14, § X.6.1]). Theorem 4.14 is proved. □

The following fact (which is well known in the symmetric case) is an easy consequence of Theorem 4.14:

in any finite-dimensional asymmetrically normed space any closed absolute max-*Chebyshev set is a singleton* (28)

(see also Remark 4.4). This fact can also be easily derived from classical Brouwer's fixed point theorem with the help of the following simple result (which is a direct corollary to Theorem 4.14):

in any finite-dimensional asymmetric space, the max-projection onto any closed absolute max-*Chebyshev set is continuous.* (29)

However, in the general case it is unknown (see, for example, [11, § 2]) whether (28) and (29) are true for nonclosed sets.

Let us prove (28) and (29) for strictly convex asymmetric spaces. To this end, we will establish the following more general result (the finite-dimensional analogue of Theorem 4.14 for not necessarily closed sets).

Theorem 4.14. *Let X be a finite-dimensional strictly convex asymmetric space, let $U \subset X$ be an open set, and let $M \subset X$. Assume that the* max-*projection operator F_M is single-valued on U. Then F_M is continuous on U.*

Proof of Theorem 4.14. Assume on the contrary that there exist $x \in U$ and $x_n \to x$, $x_n \in U$ such that $y_n \not\to y$, where $\mathrm{F}_M x = \{y\}$, $\mathrm{F}_M x_n = \{y_n\}$. We can assume without loss of generality that $x = 0$, $r(0, M) = 1$. By a compactness argument, $y_{n_k} \to z \in S(0,1)$. Setting $v_\lambda := -\lambda z$, $\lambda > 0$, we have

$$M \subset B(0,1) \subset B(v_\lambda, \|z - v_\lambda|)$$

and

$$B(0,1) \cap S(v_\lambda, \|z - v_\lambda|) = \{z\} \text{ for all } \lambda > 0, \tag{30}$$

where the first inclusion follows from the equality $r(0, M) = 1$ and (10), and the second inclusion is secured by strict convexity of the space (see, for example, [11, Proposition 1.2.2]). We have $y_{n_k} \to z$, $y_{n_k} \in M$, and so $r(v_\lambda, M) = \|z - x_\lambda|$. Let $\mathrm{F}_M z_\lambda = \{\widehat{z}\}$. Then $\widehat{z} \in S(v_\lambda, \|z - v_\lambda|)$, which in view of (30) gives $\widehat{z} = z$, but this is impossible, because $z \notin M$. This contradiction proves Theorem 4.14. $\square$

As a corollary, from Theorem 4.14 we have

Corollary 4.15. *In any finite-dimensional strictly convex asymmetric space, any absolute* max-*Chebyshev set is a singleton.*

The next result follows from Theorem 4.10 and Corollary 4.6 (for the case of normed spaces, see [11, Theorem 2.1.17]).

Corollary 4.16. *In any finite-dimensional asymmetric space X, any* max-*Chebyshev set with continuous* max-*projection onto $X \setminus \mathrm{Z}(M)$ is a* max-*sun in X.*

Remark 4.17. The conditions of Corollary 4.16 are satisfied, in particular, if M is nonempty and closed – in this case, the set M is max-approximatively compact, and hence this result is secured by Theorem 4.14.

Corollary 4.18. *If M ia a* max-*Chebyshev set M in a finite-dimensional asymmetric strictly convex space X, then M contains the Chebyshev sphere, i.e.,*

$$M \supset S(x_0, r(M)),$$

where x_0 is a (*unique*) *Chebyshev center of the set M.*

Example 4.19. Let us construct an example of a two-dimensional asymmetric space X and a max-Chebyshev set $M \subset X$ such that the max-projection F_M onto it is not continuous on $X \setminus \mathrm{Z}(M)$.

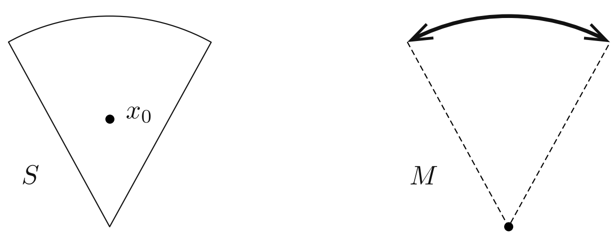

Fig. 1. A max-Chebyshev set $M \subset X$ from Example 4.19.

The required set M is constructed in the space X, whose unit sphere S with center at the point x_0 is depicted in Fig. 1. The set M itself consists of the open arc (the strictly convex part of the sphere) and an isolated point (see Fig. 1). It is easily seen that M is a max-Chebyshev set (the set $\mathrm{F}_M x$ is nonempty and is a singleton for all $x \in X \setminus \{x_0\}$, where x_0 is the (unique) Chebyshev center of the set M). If the max-projection operator F_M were continuous on $X \setminus \mathrm{Z}(M)$, then by Corollary 4.16 the set M would be a max-sun, and, therefore, by Theorem 4.7 we would have the inclusion $S(x_0, r(M)) \subset M$ (the inclusion $B(x_0, r(M)) \supset M$ is straightforward), where $r(M)$ is the Chebyshev radius of M. However, it is clear that this inclusion $S(x_0, r(M)) \subset M$ does not hold.

Remark 4.20. (1) The set M constructed in Example 4.19 also gives an example of a max-Chebyshev set which is not a local max-sun (and, of course, which is not a max-sun – this follows from Theorem 4.7, because the Chebyshev center for M is unique).

(2) The closure of the set M from Example 4.19 is a local max-sun, but not a max-sun.

Example 4.21. (A nonsingleton absolute max-Chebyshev set) The asymmetric norm on the two-dimensional space X, in which a nonsingleton absolute max-Chebyshev set M will be constructed, is defined from the unit sphere S depicted in Fig. 2 (the sphere S is a circular sector of opening $< 90°$). We change the set from Example 4.19 as follows: now the set M consists of the isolated point and the open arc of an ellipse such that, at the left-end point, the arc of the ellipse and the circle have the same tangent line and equal curvatures, and at the other points of the arc its curvature is greater than that of the circle S (see Fig. 2). It is easily seen that the resulting set M, which consists of an isolated point and the open arc, gives an example of a nonsingleton absolute max-Chebyshev set (note that the max-projection F_M onto the set M has points of discontinuity not only at the (unique) Chebyshev center of the set M).

Fig. 2. A nonsingleton absolute max-Chebyshev set M from Example 4.21.

5. Max-δ-suns

In the classical theory of min-approximation, δ-suns appear naturally in many problems (see [5]). For example, it is well known that any Chebyshev set with continuous metric projection is a δ-sun (but it is unknown in general wether such a set is a sun). The property of δ-solarity is also closely related to the important concept of almost convex sets (see, for example, [3], [5, § 10.2]).

Let us give define an analogue of a δ-sun for max-approximation problems. We first need an auxiliary result, which will be required in the proof of Theorem 5.3 on max-δ-suns.

Lemma 5.1. *Let $M \subset X$, $x \in X$, and $y, y' \in M$, $z \in [x, y]$ be such that*

$$r(x, M) = 1, \quad \big|\, \|y - x\,| - 1\big| < \varepsilon\delta, \quad \|z - x\,| = \delta,$$
$$\|y' - z\,| < \varepsilon\delta, \quad \|y - y'\|_{\mathrm{sym}} < \varepsilon.$$

Then
$$\Big|\frac{r(x, M) - r(z, M)}{\delta} - 1\Big| \le \frac{3\varepsilon}{1 - \delta\varepsilon}.$$

Proof. To begin with, we note that

$$\|y' - x\,| \le r(x,M) = 1 < \|y - x\,| + \varepsilon\delta,$$

and

$$\|y - z\,| \le r(z,M) < \|y' - z\,| + \varepsilon\delta.$$

Let $w \in [x, y']$ and let $\overrightarrow{zw} \parallel \overrightarrow{yy'}$. Then

$$\|w - z\| < \frac{\varepsilon\|z - x\,|}{\|y - x\,|} \le \frac{\varepsilon\delta}{1 - \varepsilon\delta}.$$

For brevity, we set $\|\cdot\| = \|\cdot\|_{\mathrm{sym}}$. We also note that

$$\big|\|w - x\,| - \|z - x\,|\big| \le \|w - z\| \quad \text{and} \quad \big|\|y' - w\,| - \|y' - z\,|\big| \le \|w - z\,|$$

Hence

$$\begin{aligned}\|y' - x\,| &= \|y' - w\,| + \|w - x\,| \le \|y' - z\,| + \|w - x\,| + \frac{\varepsilon\delta}{1 - \varepsilon\delta}\\ &\le r(z,M) + \frac{2\varepsilon\delta}{1 - \varepsilon\delta} + \|w - x\,|\\ &\le r(z,M) + \frac{3\varepsilon\delta}{1 - \varepsilon\delta} + \|z - x\,| \le r(z,M) + \delta + \frac{3\varepsilon\delta}{1 - \varepsilon\delta}.\end{aligned}$$

and

$$\begin{aligned}\|y' - x\,| = \|w - x\,| + \|y' - w\,| &\ge \|z - x\,| - \|w' - z\| + \|y' - z\,| - \|w - z\|\\ &\ge \delta + r(z,M) - \frac{3\varepsilon\delta}{1 - \varepsilon\delta}.\end{aligned}$$

As a corollary, $\big|\|y' - x\,| - (\delta + r(z,M))\big| \le \dfrac{3\varepsilon\delta}{1 - \varepsilon\delta}$.

We have $\|y' - x\,| \le r(x,M)$, and so

$$\frac{r(x,M) - r(z,M)}{\delta} \ge 1 - \frac{3\varepsilon\delta}{1 - \varepsilon\delta}.$$

Moreover,

$$r(x,M) - \delta - \frac{3\varepsilon\delta}{1 - \varepsilon\delta} \le \|y - z\,| \le r(z,M),$$

and, therefore,

$$\frac{r(x,M) - r(z,M)}{\delta} \le 1 + \frac{3\varepsilon}{1 - \varepsilon\delta}. \qquad \square$$

In the following definition, we introduce the max-analogue of a δ-sun (more precisely, a max-δ-solar point; see, for example, [3], [5, § 10.2]) from the theory of min-approximation.

Definition 5.2. Let $M \subset X$. We say that x is a max-δ-*solar point* for M if there exists a sequence (z_n) such that $\|z_n - x\,| \to 0$, $z_n \ne x$, and

$$\frac{r(x,M) - r(z_n,M)}{\|z_n - x\,|} \to 1.$$

The next result follows from Lemma 5.1.

We set $\mathscr{O}_\sigma(x) := \{z \in X \mid \|x - z\|_{\mathrm{sym}} \le \sigma\}$.

Theorem 5.3. *Let $x \in X$ and let, for some $\sigma_0 > 0$ and arbitrary $\sigma \in (0, \sigma_0)$, there exist a continuous mapping $\varphi : \mathscr{O}_\sigma(x) \to M$ (where in the range we consider the symmetrization topology) such that*

$$\|\varphi(z) - z \,| \ge r(z, M) + o(\sigma) \quad \textit{for all} \;\; z \in \mathscr{O}_\sigma(x).$$

Then x is a max-*δ-solar point for M.*

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