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On Concentration Behavior and Multiplicity of Solutions for a System in $\mathbb{R}^N$

Segundo Manuel A. Salirrosas
Universidade de Brasília, Dep. de Matemática, Brasília, Brazil
semaarsa@gmail.com

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We describe a result on the asymptotic behavior of the solutions of a system with two elliptic equations in $\mathbb{R}^N$ involving a small parameter. More precisely, we study the system

$$\begin{cases} -\varepsilon^2 \mathrm{div}(a(x)\nabla u) + u = Q_u(u,v) + \dfrac{\gamma}{2^*} K_u(u,v) \text{ in } \mathbb{R}^N, \\ -\varepsilon^2 \Delta v + b(x) v = Q_v(u,v) + \dfrac{\gamma}{2^*} K_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N),\ u(x),\ v(x) > 0 \text{ for each } x \in \mathbb{R}^N, \end{cases}$$

where $2^* = 2N/(N-2)$, $N \geq 3$, $\varepsilon > 0$, a and b are positive continuous potentials, and Q and K are homogeneous functions with K having critical growth. We use the penalization method for system introduced by C. O. Alves [*Local mountain pass for a class of elliptic system*, J. Math. Analysis Appl. 335 (2007) 135–150] in order to find a family of solutions $(u_\varepsilon, v_\varepsilon)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ such that, if $\Pi_{\varepsilon,a}$ and $\Pi_{\varepsilon,b}$ are maximum points of u_ε and v_ε respectively, then

$$\lim_{\varepsilon \to 0^+} a(\Pi_{\varepsilon,a}) = \inf_{x \in \mathbb{R}^N} a(x) \quad \text{and} \quad \lim_{\varepsilon \to 0^+} b(\Pi_{\varepsilon,b}) = \inf_{x \in \mathbb{R}^N} b(x).$$

Moreover, we relate the number of solutions with the topology of the set where the potentials a and b attain their minima. We consider the subcritical case $\gamma = 0$ and the critical case $\gamma = 1$.

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1. Introduction

Motivated by Floer and Weinstein [19] and Oh [25], [26] and [27], Rabinowitz shows in [29] existence of a family of solutions u_ε of the problem

$$(R) \qquad -\varepsilon^2 \Delta u + V(x) u = f(u) \text{ in } \mathbb{R}^N,$$

where $\varepsilon > 0$ is a small parameter. In [30] Wang showed that the family of solutions found by Rabinowitz concentrates around a local minimum of the potential V, when ε converges to zero. Wang also noted that the concentration of any family of solutions with energy uniformly bounded can only occur in a critical point of V. Del Pino and Felmer [13] were the first to study (R) from the local point of view. In [13] a penalization method was created to find a family of mountain pass solutions that concentrates around a local minimum of V.

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Equations in this way appear in different physical and biological models, in which the presence of a small $\varepsilon > 0$ diffusion parameter is natural. For example, in the study of the existence of standing waves, that is, solutions of the form

$$\psi(x,t) = \exp\left(\frac{-iEt}{\varepsilon}\right) u(x)$$

for non-linear Schrödinger equations of the type

$$i\varepsilon\frac{\partial\psi}{\partial t} = -\varepsilon^2\Delta\psi + V(x)\psi - f(\psi),$$

the problem is reduced to an equation similar to (R).

On the other hand, Caldiroli and Musina [9] studied the problem

$$(CM) \qquad -\mathrm{div}(a(x)\nabla u) = g(x,u) \;\text{ in }\; \Omega, \quad u = 0 \;\text{ on }\; \partial\Omega,$$

where a is measurable, nonnegative and can be vanishing at some finite number of points if Ω is bounded and it can be vanishing at infinity number of points if Ω is unbounded. The authors proved the existence of solutions by variational methods. In [28] Passaseo considers problem (CM), where g has a powerlike behaviour and a is bounded. Here the case $\inf a = 0$ is considered, so that the equation is degenerate, and standard variational techniques do not apply; on the contrary, some concentration phenomena arise, similar to those occurring with critical exponent. The author proves first a very general identity (similar to the Pokhozhaev identity), from which he deduces a nonexistence result in starshaped domains. Then he gives a condition on a, sufficient to ensure the existence and multiplicity of nonnegative solutions, and shows that this condition is optimal. Finally, he studies the effect of the topology of the vanishing set of the function a on the number of positive solutions of uniformly elliptic problems, which approximate the one given. In [10] Chabrowski showed an existence result, where a is a positive function on the outside of a ball.

Lazzo in [22] and Figueiredo and Furtado in [14, 15, 16] showed multiplicity of solution for problem (CM) using category theory. However, they do not show concentration results.

Concentration phenomena, as described above, for elliptical systems are also known in the literature. In the spirit of problem (R), Alves and Soares [4] showed concentration and existence of solutions for the system

$$(AS) \qquad \begin{cases} -\varepsilon^2\Delta u + W(x)u = Q_u(u,v) \text{ in } \mathbb{R}^N, \\ -\varepsilon^2\Delta v + V(x)v = Q_v(u,v) \text{ in } \mathbb{R}^N, \\ u,v \in H^1(\mathbb{R}^N), \quad u(x),v(x) > 0 \text{ for each } x \in \mathbb{R}^N. \end{cases}$$

The system (AS) in the spirit of problem with techniques of Del Pino and Felmer [13] was studied in [1], [20], [21] and [24].

Alves, Figueiredo and Furtado in [2] and in [3], Ambrosio in [7] and [6] showed multiplicity of solution for system (AS) using category theory. However, they do not show concentration results.

Concentration and multiplicity of solutions using category theory for the system

$$\begin{cases} -\varepsilon^2 \text{div}(a(x)\nabla u) + u = Q_u(u,v) + \frac{1}{2^*}K_u(u,v) \text{ in } \mathbb{R}^N, \\ -\varepsilon^2 \text{div}(b(x)\nabla v) + v = Q_v(u,v) + \frac{1}{2^*}K_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), u(x), v(x) > 0 \ \text{ for each } x \in \mathbb{R}^N \end{cases}$$

was showed by the authors of this paper in [18].

Motivated by the results just described, a natural question is whether the same phenomenon of concentration and multiplicity results involving category theory occurs for the following class of gradient systems that now we describe below.

In the first part of this paper we are concerned with the existence, multiplicity and concentration of positive solutions for the following system given by

$$(S_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \text{div}(a(x)\nabla u) + u = Q_u(u,v) \text{ in } \mathbb{R}^N, \\ \quad -\varepsilon^2 \Delta v + b(x)v \ \ = Q_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \ \ u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N, \end{cases}$$

where $\varepsilon > 0$, $N \geq 3$, $2^* = \frac{2N}{N-2}$ and a, b are continuous potentials. The hypotheses on the functions a and b are the following:

(ab_1) There are $a_0 > 0$ and $b_0 > 0$ such that

$$a_0 \leq a(x) \ \text{ and } \ b_0 \leq b(x) \ \text{ for all } \ x \in \mathbb{R}^N.$$

(ab_2) There exists a bounded domain $\Lambda \subset \mathbb{R}^N$ such that

$$a_0 = \inf_{x\in\Lambda} a(x) < \inf_{x\in\partial\Lambda} a(x) \ \text{ and } \ b_0 = \inf_{x\in\Lambda} b(x) < \inf_{x\in\partial\Lambda} b(x).$$

Setting $\mathbb{R}^2_+ := [0,\infty) \times [0,\infty)$, for any given $q \geq 1$ we denote by $\mathcal{H}^q$ the collection of all functions $F \in C^2(\mathbb{R}^2_+, \mathbb{R})$ satisfying the following properties.

$(\mathcal{H}_0^q)$ F is q-homogeneous; that is

$$F(\lambda s, \lambda t) = \lambda^q F(s,t), \ \text{ for each } \ \ \lambda > 0 \ \text{ and } \ (s,t) \in \mathbb{R}^2_+.$$

$(\mathcal{H}_1^q)$ There exists $c_1 > 0$ such that

$$|F_s(s,t)| + |F_t(s,t)| \leq c_1 \left(s^{q-1} + t^{q-1}\right) \ \text{ for each } (s,t) \in \mathbb{R}^2_+.$$

$(\mathcal{H}_2)$ $F(s,t) > 0$ for each $s, t > 0$.

$(\mathcal{H}_3)$ $\nabla F(1,0) = \nabla F(0,1) = (0,0)$.

$(\mathcal{H}_4)$ $F_s(s,t), F_t(s,t) \geq 0$ for each $(s,t) \in \mathbb{R}^2_+$.

We relate the number of solutions of (S_ε) with the topology of the set of minima of the potentials a and b. In order to present our result we introduce the following set:

$$M = \{x \in \mathbb{R}^N : a(x) = a_0 \ \text{ and } \ b(x) = b_0\} \neq \emptyset.$$

We recall that, if Y is a closed set of a topological space X, $\text{cat}_X(Y)$ is the Ljusternik-Schnirelmann category of Y in X, namely the least number of closed and contractible sets in X which cover Y.

We denote by $M_\delta := \{x \in \mathbb{R}^N : \mathrm{dist}(x, M) < \delta\} \subset \Lambda$, the closed δ-neighborhood of M, and we shall prove the following result.

Theorem 1.1. *Suppose that a and b are continuous potentials satisfying (ab_1) and (ab_2). Suppose also that $Q \in \mathcal{H}^p$ for any $2 < p < 2^*$. Then,*

(i) *For all $\varepsilon > 0$, system (S_ε) has a ground state positive solution.*

(ii) *For any $\delta > 0$ there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the system (S_ε) has at least $\mathrm{cat}_{M_\delta}(M)$ positive solutions.*

(iii) *If $(u_\varepsilon, v_\varepsilon)$ is a solution for (S_ε) and if $\Pi_{\varepsilon,a}$ and $\Pi_{\varepsilon,b}$ are maximum points of u_ε and v_ε respectively, then $\Pi_{\varepsilon,a}, \Pi_{\varepsilon,b} \in \Lambda$, $\lim\limits_{\varepsilon \to 0^+} a(\Pi_{\varepsilon,a}) = a_0$ and $\lim\limits_{\varepsilon \to 0^+} b(\Pi_{\varepsilon,b}) = b_0$.*

(iv) *Each solution $(u_\varepsilon, v_\varepsilon) \in C^{2,\lambda}(\mathbb{R}^N)$, for some $\lambda \in (0, 1)$.*

In the second part of the paper we deal with a critical version of (S_ε), namely the system

$$(CS_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \mathrm{div}(a(x)\nabla u) + u = Q_u(u, v) + \frac{1}{2^*} K_u(u, v) \text{ in } \mathbb{R}^N, \\ \quad -\varepsilon^2 \Delta v + b(x) v \;= Q_v(u, v) + \frac{1}{2^*} K_v(u, v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \quad u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N. \end{cases}$$

In order to deal with the critical growth of the nonlinearity we assume the following hypotheses on the functions Q and K:

(A_1) $K \in \mathcal{H}^{2^*}$ and $Q \in \mathcal{H}^p$ for some $2 < p < 2^*$.

(A_2) The 1-homogeneous function $G : \mathbb{R}^2_+ \to \mathbb{R}$ given by $G(s^{2^*}, t^{2^*}) := K(s, t)$ is concave.

(A_3) There exists $\sigma^* > 0$ such that $Q(s, t) \geq \dfrac{\sigma}{p_1} s^\lambda t^\beta$, for all $(s, t) \in \mathbb{R}^2_+$, $\lambda, \beta > 1$, $\lambda + \beta =: p_1 \in (2, 2^*)$, for all $\sigma > \sigma^*$.

The hypothesis (A_2) appeared for the first time in [12] and it will be used in Proposition 6.2.

The constant σ^* will also appear naturally in Proposition 6.2.

The critical version of Theorem 1.1 can be stated as follows.

Theorem 1.2. *Suppose that a and b are continuous potentials satisfying (ab_1) and (ab_2). Suppose also that Q and K satisfy $(A_1) - (A_3)$. Then,*

(i) *There exist $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$ the system (CS_ε) has a ground state positive solution.*

(ii) *For any $\delta > 0$ there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the system (CS_ε) has at least $\mathrm{cat}_{M_\delta}(M)$ positive solutions.*

(iii) *If $(u_\varepsilon, v_\varepsilon)$ is a solution for (CS_ε) and if $\Pi_{\varepsilon,a}$ and $\Pi_{\varepsilon,b}$ are maximum points of u_ε and v_ε respectively, then $\Pi_{\varepsilon,a}, \Pi_{\varepsilon,b} \in \Lambda$, $\lim\limits_{\varepsilon \to 0^+} a(\Pi_{\varepsilon,a}) = a_0$ and $\lim\limits_{\varepsilon \to 0^+} b(\Pi_{\varepsilon,b}) = b_0$.*

(iv) *Each solution $(u_\varepsilon, v_\varepsilon) \in C^{2,\lambda}(\mathbb{R}^N)$, for some $\lambda \in (0, 1)$.*

In several results mentioned above, the existence of solutions is related to the existence of a minimum of the potentials. Therefore, it seems natural to ask if concentration behavior and multiplicity results hold for the systems (S_ε) and (CS_ε). The answer for this question is positive.

The present work is strongly influenced by the articles above. Below we list what we believe that are the main contributions of our paper.

(1) We give results of concentration and multiplicity for a new class of system that, at least to our knowledge, had never been studied before.

(2) Since the potentials are in different positions in the equations, some results need to be adapted, such as Lemma 2.1.

The paper is organized as follows. In order to overcome the lack of compactness, in Section 2 we make a penalization of the nonlinearity using arguments that can be found in [1]. In Section 3 we show existence of solution for the auxiliary system introduced in Section 2. In Section 4 we obtain uniform estimates in order to show that the solution of the auxiliary system is a solution of the original system. The proof of the main result in the subcritical case is in Section 5. The critical case is studied in Section 6.

2. Variational framework and a modified system

Since we are interested in positive solutions we extend the function Q and K to the whole $\mathbb{R}^2$ by setting $Q(u,v) = K(u,v) = 0$ if $u \leq 0$ or $v \leq 0$. We also note that for any function $F \in \mathcal{H}^q$, we can use the homogeneity condition $(\mathcal{H}_0^q)$ to conclude that

$$qF(s,t) = sF_s(s,t) + tF_t(s,t) \tag{1}$$

and
$$q(q-1)F(s,t) = s^2 F_{ss}(s,t) + t^2 F_{tt}(s,t) + 2st F_{st}(s,t) \tag{2}$$

for any $(s,t) \in \mathbb{R}^2$.

Hereafter, we will work with the following system equivalent to (S_ε).

$$(\widehat{S}_\varepsilon) \qquad \begin{cases} -\operatorname{div}(a(\varepsilon x)\nabla u) + u = Q_u(u,v) \text{ in } \mathbb{R}^N, \\ \quad -\Delta v + b(\varepsilon x) v \;\; = Q_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \quad u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N. \end{cases}$$

In order to overcome the lack of compactness originated by the unboundedness of $\mathbb{R}^N$ we use a penalization method. Such kind of idea has first appeared in the paper of Del Pino and Felmer [13]. Here, we use an adaptation for systems introduced in [1].

We start by choosing $\alpha > 0$ and considering $\eta : \mathbb{R} \to \mathbb{R}$ a non-increasing function of class C^2 such that

$$\eta \equiv 1 \text{ on } (-\infty, \alpha], \quad \eta \equiv 0 \text{ on } [5\alpha, +\infty), \quad |\eta'(s)| \leq \frac{C}{\alpha} \text{ and } |\eta''(s)| \leq \frac{C}{\alpha^2} \tag{3}$$

for each $s \in \mathbb{R}$ and for some positive constant $C > 0$. Using the function η, we define $\widehat{Q} : \mathbb{R}^2 \to \mathbb{R}$ by

$$\widehat{Q}(s,t) := \eta(|(s,t)|)Q(s,t) + (1 - \eta(|(s,t)|))A(s^2 + t^2),$$

where
$$A := \max\left\{ \frac{Q(s,t)}{s^2 + t^2} : (s,t) \in \mathbb{R}^2, \ \alpha \leq |(s,t)| \leq 5\alpha \right\}.$$

Notice that, since $A > 0$ tends to zero as $\alpha \to 0^+$, we may suppose that $A \in (0, \mu/4)$ where $\mu = \max\{1, 1/b_0\}^{-1}$.

Finally, denoting by I_Λ the characteristic function of the set Λ, we define the function $H : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ by setting

$$H(x,s,t) := I_\Lambda(x)Q(s,t) + (1 - I_\Lambda(x))\widehat{Q}(s,t). \tag{4}$$

For any $\alpha > 0$ small and $(s,t) \in \mathbb{R}^2$ we have the following result.

Lemma 2.1. *The function H satisfies the following estimates:*

(H_1) $pH(x,s,t) = sH_s(x,s,t) + tH_t(x,s,t)$, *for each* $x \in \Lambda$.

(H_2) $2H(x,s,t) \le sH_s(x,s,t) + tH_t(x,s,t)$, *for each* $x \in \mathbb{R}^N \setminus \Lambda$.

(H_3) *For α small we have for each $x \in \mathbb{R}^N \setminus \Lambda$:*

$$sH_s(x,s,t) + tH_t(x,s,t) \le \frac{1}{4}\left(s^2 + b(x)t^2\right).$$

(H_4) *For α small we have* $\dfrac{|H_s(x,s,t)|}{\alpha}, \dfrac{|H_t(x,s,t)|}{\alpha} \le \dfrac{\mu}{4}$ *for each* $x \in \mathbb{R}^N \setminus \Lambda$.

Proof. Since $H(x,s,t) = Q(s,t)$ on the set Λ, we can use (1) to get

$$pH(x,s,t) = sH_s(x,s,t) + tH_t(x,s,t)$$

for all $x \in \Lambda$. This proves (H_1).

In what follows we denote $|z| := \sqrt{s^2 + t^2}$. Notice that $H(x,s,t) = \widehat{Q}(s,t)$ for all $x \in \mathbb{R}^N \setminus \Lambda$, consequently

$$H_s = \eta' \frac{s}{|z|} Q + \eta Q_s - \eta' \frac{s}{|z|} A(s^2 + t^2) + 2A(1-\eta)s$$

and

$$H_t = \eta' \frac{t}{|z|} Q + \eta Q_t - \eta' \frac{t}{|z|} A(s^2 + t^2) + 2A(1-\eta)t$$

so, $$sH_s + tH_t = \eta'|z|\left[Q - A(s^2+t^2)\right] + \eta\left[sQ_s + tQ_t\right] + 2A(1-\eta)(s^2+t^2). \tag{5}$$

Notice that, in view of the definition of A, we have $Q(s,t) - A|z|^2 \le 0$, for all x belonging to the support of η'. Hence recalling that $\eta' \le 0$, we can use the above estimate, (5), (1) and the fact $2 < p < 2^*$, to obtain

$$sH_s + tH_t \ge p\eta Q + 2A(1-\eta)|z|^2 \ge 2\left[\eta Q + A(1-\eta)|z|^2\right] = 2H$$

for all $x \in \mathbb{R}^N \setminus \Lambda$. Thus, (H_2) holds.

Since η is smooth and supp $\eta' \subset [\alpha, 5\alpha]$, we can use (5), $(\mathcal{H}_1^p)$, (3) and the definition of A, to get

$$\begin{aligned}
\frac{sH_s + tH_t}{s^2+t^2} &= \eta'|z|\left[\frac{Q(s,t)}{s^2+t^2} - A\right] + \frac{\eta}{|z|^2}\left[sQ_s + tQ_t\right] + 2A(1-\eta) \\
&\le \eta'|z|\left[\frac{Q(s,t)}{s^2+t^2} - A\right] + 2c_1\frac{\eta}{|z|^2}(s^p + t^p) + 2A(1-\eta) \\
&\le |\eta'||z|\left|\frac{Q(s,t)}{s^2+t^2} - A\right| + 4c_1|z|^{p-2} + 4A \\
&\le \frac{C}{\alpha}\cdot 5\alpha \cdot 2A + 4c_1(5\alpha)^{p-2} + 4A.
\end{aligned}$$

Then, for α sufficiently small we have that

$$sH_s + tH_t \leq \frac{\mu}{4}(s^2 + t^2).$$

Thus, for this choice of α, we can use the above estimate and (ab_1) to obtain, for each $x \in \mathbb{R}^N \setminus \Lambda$,

$$sH_s + tH_t \leq \frac{1}{4}\left(s^2 + b(x)t^2\right)$$

showing that (H_3) holds.

Since $H(x,s,t) = \widehat{Q}(s,t)$ for all $x \in \mathbb{R}^N \setminus \Lambda$, from definition of $\widehat{Q}$, supp $\eta' \subset [\alpha, 5\alpha]$, $(\mathcal{H}_1^p)$ and (3)

$$\begin{aligned}
|H_s(x,s,t)| &= \left|\eta' \frac{s}{|z|} Q + \eta Q_s - \eta' \frac{s}{|z|} A(s^2+t^2) + 2A(1-\eta)s\right| \\
&\leq |\eta'| \frac{Q(s,t)}{s^2+t^2}|z|^2 + |\eta| c_1(s^{p-1} + t^{p-1}) + |\eta'||A|z|^2 + 4A|z| \\
&\leq |\eta'| \frac{Q(s,t)}{s^2+t^2}|z|^2 + |\eta| 2c_1|z|^{p-1} + |\eta'||A|z|^2 + 4A|z| \\
&\leq \frac{C}{\alpha} \cdot A \cdot 25\alpha^2 + 2c_1(5\alpha)^{p-1} + \frac{C}{\alpha} \cdot A \cdot 25\alpha^2 + 20\alpha A.
\end{aligned}$$

Then, for α sufficiently small we have $\dfrac{|H_s(x,s,t)|}{\alpha} \leq \dfrac{\mu}{4}$.

Using similar arguments, it is possible to prove $\dfrac{|H_t(x,s,t)|}{\alpha} \leq \dfrac{\mu}{4}$, proving (H_4). □

In view of definition (4), we deal in the sequel with the modified system

$$(S_{\varepsilon,aux}) \qquad \begin{cases} -\mathrm{div}(a(\varepsilon x)\nabla u) + u = H_u(\varepsilon x, u, v) \text{ in } \mathbb{R}^N, \\ \quad -\Delta v + b(\varepsilon x)v \quad = H_v(\varepsilon x, u, v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \quad u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N, \end{cases}$$

and we will look for solutions $(u_\varepsilon, v_\varepsilon)$ verifying

$$|(u_\varepsilon(\varepsilon x), v_\varepsilon(\varepsilon x))| \leq \alpha \quad \text{for each } x \in \mathbb{R}^N \setminus \Lambda_\varepsilon,$$

where $\Lambda_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \Lambda\}$.

For each $\varepsilon > 0$ we denote by X_ε the Hilbert space

$$X_\varepsilon := \left\{(u,v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} \left(a(\varepsilon x)|\nabla u|^2 + b(\varepsilon x)|v|^2\right) dx < \infty\right\}$$

endowed with the norm $\|\cdot\|$ given by

$$\|(u,v)\|_\varepsilon^2 := \int_{\mathbb{R}^N} \left[a(\varepsilon x)|\nabla u|^2 + |\nabla v|^2 + |u|^2 + b(\varepsilon x)|v|^2\right] dx.$$

The conditions (H_3) and $(\mathcal{H}_1^p)$ imply that the critical points of the C^1-functional $J_\varepsilon : X_\varepsilon \to \mathbb{R}$ given by

$$J_\varepsilon(u,v) = \frac{1}{2}\int_{\mathbb{R}^N} \left[a(\varepsilon x)|\nabla u|^2 + |\nabla v|^2 + |u|^2 + b(\varepsilon x)|v|^2\right] dx - \int_{\mathbb{R}^N} H(\varepsilon x, u, v) dx$$

are weak solutions of $(S_{\varepsilon,aux})$. We recall that these critical points belong to the Nehari manifold of J_ε, namely on the set

$$\mathcal{N}_\varepsilon := \{(u,v) \in X_\varepsilon \setminus \{(0,0)\} : J'_\varepsilon(u,v)(u,v) = 0\}.$$

It is a well-known fact that, for any nontrivial element $(u,v) \in X_\varepsilon$ the function $t \mapsto J_\varepsilon(tu,tv)$, for $t \geq 0$, achieves its maximum value at a unique point $t_{u,v}(u,v) \in \mathcal{N}_\varepsilon$. We define the number b_ε by setting

$$b_\varepsilon := \inf_{(u,v)\in\mathcal{N}_\varepsilon} J_\varepsilon(u,v). \tag{6}$$

3. Existence of a ground state solution for the modified system $(S_{\varepsilon,aux})$

We start the section by defining the Palais-Smale compactness condition. A sequence $((u_n,v_n)) \subset X_\varepsilon$ is a Palais-Smale sequence at level c_ε for the functional J_ε if

$$J_\varepsilon(u_n,v_n) \to c_\varepsilon$$

and
$$\|J'_\varepsilon(u_n,v_n)\| \to 0 \quad \text{in} \quad (X_\varepsilon)',$$

where
$$c_\varepsilon = \inf_{\eta\in\Gamma}\max_{t\in[0,1]} J_\varepsilon(\eta(t)) > 0$$

and
$$\Gamma := \{\eta \in C([0,1], X_\varepsilon) : \eta(0) = (0,0),\ J_\varepsilon(\eta(1)) < 0\}.$$

If every Palais-Smale sequence of J_ε has a strong convergent subsequence, then one says that J_ε satisfies the Palais-Smale condition ((PS) for short).

In order to show existence of a ground state solution for the modified system $(S_{\varepsilon,aux})$, we use the Mountain Pass Theorem [5].

Lemma 3.1. *The functional J_ε satisfies the following conditions*

(i) *There exists $C, \rho > 0$, such that $J_\varepsilon(u,v) \geq C$, if $\|(u,v)\|_\varepsilon = \rho$.*

(ii) *For any $(\phi,\psi) \in C_0^\infty(\Lambda_\varepsilon) \times C_0^\infty(\Lambda_\varepsilon)$ with $\phi,\psi \geq 0$, we have*

$$\lim_{t\to+\infty} J_\varepsilon(t\phi,t\psi) = -\infty.$$

Proof. By using Lemma 2.1 and $(\mathcal{H}_1^p)$, we have

$$J_\varepsilon(u,v) \geq \frac{1}{2}\|(u,v)\|_\varepsilon^2 - \frac{2c_1}{p}\int_{\Lambda_\varepsilon} (|u|^p + |v|^p)\, dx - \frac{1}{8}\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} \left(|u|^2 + b(\varepsilon x)|v|^2\right) dx.$$

By Sobolev embeddings, there exists $C > 0$ such that

$$J_\varepsilon(u,v) \geq \frac{3}{8}\|(u,v)\|_\varepsilon^2 - \frac{C}{p}\|(u,v)\|_\varepsilon^p$$

and the proof of item (i) is over.

Now, by definition of H and $(\mathcal{H}_0^p)$, we get

$$J_\varepsilon(t\phi, t\psi) = \frac{t^2}{2}\|(\phi,\psi)\|_\varepsilon^2 - t^p \int_{\Lambda_\varepsilon} Q(\phi,\psi)dx,$$

and the proof of item (ii) is over. □

Hence, there exists a Palais-Smale sequence $((u_n, v_n)) \subset X_\varepsilon$ at level c_ε. Using $(\mathcal{H}_0^p)$, it is possible to prove that

$$c_\varepsilon = b_\varepsilon = \inf_{(u,v)\in X_\varepsilon\setminus\{(0,0)\}} \sup_{t\geq 0} J_\varepsilon(tu, tv), \tag{7}$$

where b_ε was defined in (6).

In order to prove the Palais-Smale condition, we need to prove the next lemma.

Lemma 3.2. *Let $((u_n, v_n))$ be a $(PS)_d$ sequence for J_ε. Then for each $\xi > 0$, there exists $R = R(\xi)$ such that*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R(0)} \left[a(\varepsilon x)|\nabla u_n|^2 + |\nabla v_n|^2 + |u_n|^2 + b(\varepsilon x)|v_n|^2\right] dx < \xi.$$

Proof. Let $\eta_R \in C^\infty(\mathbb{R}^N)$ such that $\eta_R(x) = 0$ if $x \in B_{R/2}(0)$ and $\eta_R(x) = 1$ if $x \notin B_R(0)$, with $0 \leq \eta_R(x) \leq 1$ and $|\nabla \eta_R| \leq \frac{C}{R}$, where C is constant independent of R. Since that the sequence $((u_n\eta_R, v_n\eta_R))$ is bounded in X_ε, fixing $R > 0$ such that $\Lambda_\varepsilon \subset B_{R/2}(0)$ and by definition of the functional J_ε, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left[a(\varepsilon x)|\nabla u_n|^2 + |\nabla v_n|^2 + |u_n|^2 + b(\varepsilon x)|v_n|^2\right] \eta_R dx \\
&= J_\varepsilon'(u_n, v_n)(u_n\eta_R, v_n\eta_R) + \int_{\mathbb{R}^N} \left[u_n H_u(\varepsilon x, u_n, v_n) + v_n H_v(\varepsilon x, u_n, v_n)\right] \eta_R dx \\
&\quad - \int_{\mathbb{R}^N} \left[a(\varepsilon x) u_n \nabla u_n + v_n \nabla v_n\right] \nabla \eta_R dx.
\end{aligned}$$

Using (H_3), we get the estimate

$$\begin{aligned}
&\frac{3}{4}\int_{\mathbb{R}^N\setminus B_R(0)} \left[a(\varepsilon x)|\nabla u_n|^2 + |\nabla v_n|^2 + |u_n|^2 + b(\varepsilon x)|v_n|^2\right] dx \\
&\leq \int_{\mathbb{R}^N} \left[a(\varepsilon x)|u_n||\nabla u_n| + |v_n||\nabla v_n|\right] |\nabla \eta_R| dx + o_n(1).
\end{aligned}$$

Since $((u_n, v_n))$ is bounded in X_ε and $|\nabla \eta_R| \leq \frac{C}{R}$, exists $C_1 > 0$ such that

$$\int_{\mathbb{R}^N\setminus B_R(0)} \left[a(\varepsilon x)|\nabla u_n|^2 + |\nabla v_n|^2 + |u_n|^2 + b(\varepsilon x)|v_n|^2\right] dx \leq \frac{C_1}{R} + o_n(1)$$

proving the lemma. □

Lemma 3.3. *The functional J_ε satisfies the Palais-Smale condition at any level c.*

Proof. Let $((u_n, v_n)) \subset X_\varepsilon$ such that $J_\varepsilon(u_n, v_n) \to c$ and $J_\varepsilon'(u_n, v_n) \to 0$. Standard calculations show that $((u_n, v_n))$ is bounded in X_ε.

Then, up to a subsequence, we may suppose that,

$$\begin{aligned} &(u_n, v_n) \rightharpoonup (u, v) \quad \text{weakly in} \quad X_\varepsilon, \\ &u_n \to u, \ v_n \to v \quad \text{strongly in} \quad L^s_{\text{loc}}(\mathbb{R}^N), \quad \text{for any} \quad 2 \le s < 2^*, \\ &u_n(x) \to u(x), \ v_n(x) \to v(x) \quad \text{for a.e.} \quad x \in \mathbb{R}^N. \end{aligned} \tag{8}$$

Now using a density argument, we can conclude that (u, v) is a critical point of J_ε.

Hence
$$\|(u,v)\|^2_\varepsilon = \int_{\mathbb{R}^N} [uH_u(\varepsilon x, u, v) + vH_v(\varepsilon x, u, v)]\, dx. \tag{9}$$

On the other hand, we have

$$\|(u_n,v_n)\|^2_\varepsilon = \int_{\mathbb{R}^N} [u_nH_u(\varepsilon x, u_n, v_n) + v_nH_v(\varepsilon x, u_n, v_n)]\, dx + o_n(1). \tag{10}$$

From Lemma 3.2, for any $\xi > 0$ given, there exists $R > 0$ such that $\Lambda_\varepsilon \subset B_R(0)$ and

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N \setminus B_R(0)} \left[a(\varepsilon x)|\nabla u_n|^2 + |\nabla v_n|^2 + |u_n|^2 + b(\varepsilon x)|v_n|^2\right] dx < \xi.$$

This inequality, (H_3) and the Sobolev embedding imply that, for n large enough, there holds

$$\int_{\mathbb{R}^N \setminus B_R(0)} [u_nH_u(\varepsilon x, u_n, v_n) + v_nH_v(\varepsilon x, u_n, v_n)]\, dx \le C_1 \frac{1}{4}\xi, \tag{11}$$

where C_1 is positive constant. On the other hand, taking R large enough, we can suppose that

$$\left| \int_{\mathbb{R}^N \setminus B_R(0)} [uH_u(\varepsilon x, u, v) + vH_v(\varepsilon x, u, v)]\, dx \right| < \xi. \tag{12}$$

Then, by (11) and (12), we conclude that

$$\begin{aligned} &\int_{\mathbb{R}^N \setminus B_R(0)} [u_nH_u(\varepsilon x, u_n, v_n) + v_nH_v(\varepsilon x, u_n, v_n)]\, dx \\ &= \int_{\mathbb{R}^N \setminus B_R(0)} [uH_u(\varepsilon x, u, v) + vH_v(\varepsilon x, u, v)]\, dx + o_n(1). \end{aligned} \tag{13}$$

Since the set $B_R(0) \cap (\mathbb{R}^N \setminus \Lambda_\varepsilon)$ is bounded, we can use (H_3), (8) and Lebesgue's theorem to conclude that

$$\begin{aligned} &\lim_{n\to\infty} \int_{B_R(0)\cap(\mathbb{R}^N \setminus \Lambda_\varepsilon)} [u_nH_u(\varepsilon x, u_n, v_n) + v_nH_v(\varepsilon x, u_n, v_n)]\, dx \\ &= \int_{B_R(0)\cap(\mathbb{R}^N \setminus \Lambda_\varepsilon)} [uH_u(\varepsilon x, u, v) + vH_v(\varepsilon x, u, v)]\, dx. \end{aligned} \tag{14}$$

Using $(\mathcal{H}_1^p)$, (8) and Lebesgue's theorem again, we obtain

$$\begin{aligned} &\lim_{n\to\infty} \int_{\Lambda_\varepsilon} [u_nH_u(\varepsilon x, u_n, v_n) + v_nH_v(\varepsilon x, u_n, v_n)]\, dx \\ &= \int_{\Lambda_\varepsilon} [uH_u(\varepsilon x, u, v) + vH_v(\varepsilon x, u, v)]\, dx. \end{aligned} \tag{15}$$

From (13), (14) and (15) we get

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}^N}&[u_nH_u(\varepsilon x,u_n,v_n)+v_nH_v(\varepsilon x,u_n,v_n)]\,dx\\ &=\int_{\mathbb{R}^N}[uH_u(\varepsilon x,u,v)+vH_v(\varepsilon x,u,v)]\,dx.\end{aligned}$$

This equation in combination with (9) and (10) imply that $\|(u_n,v_n)\|_\varepsilon^2\to\|(u,v)\|_\varepsilon^2$. In consequence $(u_n,v_n)\to(u,v)$ in X_ε. □

4. Multiple solutions for the modified system $(S_{\varepsilon,aux})$

In order to prove the multiplicity result, we consider the following autonomous system associated to (S_0), namely

$$(S_0)\qquad\begin{cases}-a_0\Delta u+u=Q_u(u,v)\text{ in }\mathbb{R}^N,\\ -\Delta v+b_0v\ =Q_v(u,v)\text{ in }\mathbb{R}^N,\\ u,v\in H^1(\mathbb{R}^N),\quad u(x),v(x)>0\text{ for each }x\in\mathbb{R}^N.\end{cases}$$

In view of conditions (ab_1) and $(\mathcal{H}_1^p)$, the above system has a variational structure and the associated functional is given by

$$I_0(u,v):=\frac{1}{2}\int_{\mathbb{R}^N}\left[a_0|\nabla u|^2+|\nabla v|^2+|u|^2+b_0|v|^2\right]dx-\int_{\mathbb{R}^N}Q(u,v)dx$$

is well defined for $(u,v)\in E_0:=H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. We denote the norm in E_0 by

$$\|(u,v)\|^2:=\int_{\mathbb{R}^N}\left[a_0|\nabla u|^2+|\nabla v|^2+|u|^2+b_0|v|^2\right]dx.$$

We can show that I_0 has the Mountain Pass geometry and therefore we can set the minimax level c_0 in the following way

$$c_0:=\inf_{\eta\in\Gamma}\max_{t\in[0,1]}I_0(\gamma(t)),$$

where $\Gamma:=\{\gamma\in C([0,1],E_0):\gamma(0)=(0,0),\ I_0(\gamma(1))<0\}$. Moreover, c_0 can be further characterized as

$$c_0=\inf_{(u,v)\in\mathcal{M}_0}I_0(u,v)\tag{16}$$

with $\mathcal{M}_0$ being the Nehari manifold of I_0, that is

$$\mathcal{M}_0:=\{(u,v)\in E_0\setminus\{(0,0)\}:I_0'(u,v)(u,v)=0\}.$$

Lemma 4.1. *Let $((u_n,v_n))\subset\mathcal{M}_0$ be a sequence such that $I_0(u_n,v_n)\to c_0$. Then there are a sequence $(y_n)\subset\mathbb{R}^N$ and constants $R,\eta>0$ such that*

$$\liminf_{n\to\infty}\int_{B_R(y_n)}\left(|u_n|^2+|v_n|^2\right)dx\geq\eta.\tag{17}$$

Proof. Suppose that (17) is not satisfied. Since $((u_n, v_n))$ is bounded in the space $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, then, by [23, Lemma l.1], we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^s dx = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^s dx = 0,$$

for all $s \in (2, 2^*)$. Thus, from $(\mathcal{H}_1^p)$, we conclude

$$\int_{\mathbb{R}^N} [u_n Q_u(u_n, v_n) + v_n Q_v(u_n, v_n)]\, dx = o_n(1).$$

Since $I_0'(u_n, v_n)(u_n, v_n) = 0$, we obtain $\|(u_n, v_n)\| = o_n(1)$, which implies $c_0 = 0$, which is a contradiction. $\square$

The next result allows to show that system (S_0) has a solution that reaches c_0.

Lemma 4.2. (A Compactness Lemma) *Let $((u_n, v_n)) \subset \mathcal{M}_0$ be a sequence satisfying $I_0(u_n, v_n) \to c_0$. Then there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that, up to a subsequence, $(w_n(x), z_n(x)) = (u_n(x + \widetilde{y}_n), v_n(x + \widetilde{y}_n))$ converges strongly in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. In particular, there exists a minimizer for c_0.*

Proof. Applying Ekeland's Variational Principle [31, Theorem 8.5], we may suppose that $((u_n, v_n))$ is a $(PS)_{c_0}$ for I_0. Since $((u_n, v_n))$ is bounded in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, we have that $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$.

Then, $\|(u, v)\|^2 \le \liminf\limits_{n\to\infty} \|(u_n, v_n)\|^2$. We are going to prove that

$$\|(u, v)\|^2 = \lim_{n\to\infty} \|(u_n, v_n)\|^2. \tag{18}$$

Suppose, by contradiction, that (18) does not hold. Then, by $(\mathcal{H}_3)$, we can consider $(u, v) \neq (0, 0)$, using a density argument we have that $I_0'(u, v)(u, v) = 0$, where we conclude that $(u, v) \in \mathcal{M}_0$. Using (1), we obtain

$$\begin{aligned} c_0 \le I_0(u, v) &= I_0(u, v) - \frac{1}{p} I_0'(u, v)(u, v) = \left(\frac{1}{2} - \frac{1}{p}\right) \|(u, v)\|^2 \\ &< \left(\frac{1}{2} - \frac{1}{p}\right) \liminf_{n\to\infty} \|(u_n, v_n)\|^2 = \liminf_{n\to\infty} \left[I_0(u_n, v_n) - \frac{1}{p} I_0'(u_n, v_n)(u_n, v_n)\right] = c_0 \end{aligned}$$

which is a contradiction. Hence, $(u_n, v_n) \to (u, v)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Consequently, $I_0(u, v) = c_0$ and the sequence $(\widetilde{y}_n)$ is the sequence null.

If $(u, v) \equiv (0, 0)$, then in this case we cannot have $(u_n, v_n) \to (u, v)$ strongly in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ because $c_0 > 0$. Hence, using the Lemma 4.1, there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that

$$(w_n, z_n) \rightharpoonup (w, z) \quad \text{in} \quad H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N),$$

where $w_n(x) = u_n(x + \widetilde{y}_n)$ and $z_n(x) = v_n(x + \widetilde{y}_n)$. Therefore, $((w_n, z_n))$ is also $(PS)_{c_0}$ sequence of I_0 and $(w, z) \not\equiv (0, 0)$. It follows from above arguments that, up to a subsequence, $((w_n, z_n))$ converges strongly in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ and the proof of the lemma is complete. $\square$

In order to prove the multiplicity results, we need of the following the abstract results that involve category theory.

Theorem 4.3. *Let I be a C^1-functional defined on a C^1-Finsler manifold $\mathcal{V}$. If I is bounded from below and satisfies the Palais-Smale condition, then I has at least* $\operatorname{cat}_{\mathcal{V}}(\mathcal{V})$ *distinct critical points.*

The following result, whose proof is similar to that presented in [8, Lemma 4.3], will be used.

Lemma 4.4. *Let Γ, Ω^+, Ω^- be closed sets with $\Omega^- \subset \Omega^+$. Let $\beta : \Gamma \to \Omega^+$, $\Phi : \Omega^- \to \Gamma$ be two continuous maps such that $\beta \circ \Phi$ is homotopically equivalent to the embedding $\iota : \Omega^- \to \Omega^+$. Then* $\operatorname{cat}_\Gamma(\Gamma) \geq \operatorname{cat}_{\Omega^+}(\Omega^-)$.

4.1. The Palais-Smale condition in the Nehari manifold associated to J_ε

From Lemma 3.3, the unconstrained functional satisfies $(PS)_c$ for each $c \in \mathbb{R}$. Nevertheless, to get multiple critical points, we need to work with the functional J_ε constrained to $\mathcal{N}_\varepsilon$. In order to prove the desired compactness result we shall first present some properties of $\mathcal{N}_\varepsilon$. The proofs of the next three results follows by using the same arguments employed in [2, Lemma 2.2, Lemma 2.3 and Proposition 2.4] for other class of system. For the sake of completeness, we sketch here.

Lemma 4.5. *There exist positive constants α_1, δ_1, C such that, for each $\alpha \in (0, \alpha_1)$, $(u, v) \in \mathcal{N}_\varepsilon$, the following inequalities hold*

$$\int_{\Lambda_\varepsilon} Q(u,v)dx \geq \delta_1 \tag{19}$$

and

$$\int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} \left(u^2 + b(\varepsilon x)v^2\right) dx \leq C \int_{\Lambda_\varepsilon} Q(u,v)dx. \tag{20}$$

Proof. Since H has subcritical growth, it is easy to obtain $\widehat{\delta} > 0$ such that

$$\|(u,v)\|_\varepsilon \geq \widehat{\delta} \quad \text{for each} \quad (u,v) \in \mathcal{N}_\varepsilon.$$

Thus, we can use (1) and (H_3) to get

$$\begin{aligned} \widehat{\delta}^2 \leq \|(u,v)\|_\varepsilon^2 &\leq \int_{\Lambda_\varepsilon} [uQ_u(u,v) + vQ_v(u,v)]\, dx + \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} [uH_u + vH_v]\, dx \\ &\leq p \int_{\Lambda_\varepsilon} Q(u,v)dx + \frac{1}{4} \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} (u^2 + b(\varepsilon x)v^2)dx \end{aligned}$$

and therefore $\frac{3}{4}\widehat{\delta} \leq \frac{3}{4}\|(u,v)\|_\varepsilon^2 \leq p \int_{\Lambda_\varepsilon} Q(u,v)dx$ which implies (19) with $\delta_1 = \frac{3\widehat{\delta}^2}{4p}$.
By using (1) and (H_3) again, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} \left(u^2 + b(\varepsilon x)v^2\right) dx \leq \|(u,v)\|_\varepsilon^2 \\ &\qquad \leq p \int_{\Lambda_\varepsilon} Q(u,v)dx + \frac{1}{4} \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} \left(u^2 + b(\varepsilon x)v^2\right) dx \end{aligned}$$

from which (20) follows. The lemma is proved. □

The following technical results are the key stones in our compactness result.

Lemma 4.6. *Let $\phi_\varepsilon : X_\varepsilon \to \mathbb{R}$ be given by*

$$\phi_\varepsilon(u,v) := \|(u,v)\|_\varepsilon^2 - \int_{\mathbb{R}^N} [uH_u(\varepsilon x,u,v) + vH_v(\varepsilon x,u,v)]\,dx.$$

Then there exist $\alpha_2, \widetilde{M} > 0$ such that, for each $\alpha \in (0,\alpha_2)$,

$$\phi'_\varepsilon(u,v)(u,v) \le -\widetilde{M} < 0 \quad \textit{for each} \quad (u,v) \in \mathcal{N}_\varepsilon. \tag{21}$$

Proof. Given $(u,v) \in \mathcal{N}_\varepsilon$, we can use the definition of H, (1) and (2) to get

$$\begin{aligned}\phi'_\varepsilon(u,v)(u,v) &= \int_{\Lambda_\varepsilon} \left[(uQ_u + vQ_v) - (u^2Q_{uu} + v^2Q_{vv} + 2uvQ_{uv})\right] dx \\ &\quad + \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} [uH_u + vH_v]\,dx - \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} \left[u^2H_{uu} + v^2H_{vv} + 2uvH_{uv}\right] dx \\ &= -p(p-2)\int_{\Lambda_\varepsilon} Q(u,v)dx + \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} [D_1 - D_2]\,dx\end{aligned}$$

with $D_1 := uH_u + vH_v$ and $D_2 := u^2H_{uu} + v^2H_{vv} + 2uvH_{uv}$.
Arguing as in the proof of [2, Lemma 2.3], and using (ab_1), we have

$$\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} [D_1 - D_2]\,dx \le o(1)\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} (u^2+v^2)dx \le o(1)\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} (u^2 + b(\varepsilon x)v^2)dx,$$

where $o(1) \to 0$ as $\alpha \to 0^+$.
Now we can use Lemma 4.5 to obtain, for α small enough

$$\phi'_\varepsilon(u,v)(u,v) \le [-p(p-2) + o(1)]\int_{\Lambda_\varepsilon} Q(u,v)dx \le -\frac{p(p-2)}{2}\delta_1 = -\widetilde{M} < 0.$$

The lemma is proved. □

Proposition 4.7. *The functional J_ε restricted to $\mathcal{N}_\varepsilon$ satisfies $(PS)_c$ for each $c \in \mathbb{R}$.*

Proof. Let $((u_n,v_n)) \subset \mathcal{N}_\varepsilon$ be such that

$$J_\varepsilon(u_n,v_n) \to c \quad \text{and} \quad \|J'_\varepsilon(u_n,v_n)\|_* = o_n(1),$$

where $o_n(1)$ approaches zero as $n \to \infty$. Then there exists $(\lambda_n) \subset \mathbb{R}$ satisfying

$$J'_\varepsilon(u_n,v_n) = \lambda_n \phi'_\varepsilon(u_n,v_n) + o_n(1) \tag{22}$$

with ϕ_ε as in Lemma 4.6. Since $(u_n,v_n) \in \mathcal{N}_\varepsilon$ we have that

$$0 = J'_\varepsilon(u_n,v_n)(u_n,v_n) = \lambda_n\phi'_\varepsilon(u_n,v_n)(u_n,v_n) + o_n(1)\|(u_n,v_n)\|_\varepsilon.$$

Straightforward calculations show that $((u_n,v_n))$ is bounded. Moreover, in view of Lemma 4.6, we may suppose that $\phi'_\varepsilon(u_n,v_n)(u_n,v_n) \to l < 0$. Hence, the above expression shows that $\lambda_n \to 0$ and therefore we conclude that $J'_\varepsilon(u_n,v_n) \to 0$ in the dual space of X_ε. It follows from Lemma 3.3 that $((u_n,v_n))$ has a convergent subsequence. □

From now on we will denote by (w_1, w_2) the solution for the system (S_0) given by Lemma 4.2.

Let us consider $\delta > 0$ such that $M_\delta \subset \Lambda$ and $\psi \in C^\infty(\mathbb{R}^+, [0,1])$ a non-increasing function such that $\psi \equiv 1$ on $[0, \delta/2]$ and $\psi \equiv 0$ on $[\delta, \infty)$. For any $y \in M$, we define the function $\Psi_{i,\varepsilon,y} \in X_\varepsilon$ by setting

$$\Psi_{i,\varepsilon,y}(x) := \psi(|\varepsilon x - y|) w_i \left(\frac{\varepsilon x - y}{\varepsilon} \right), \quad i = 1, 2,$$

and denote by $t_\varepsilon > 0$ the unique positive number verifying

$$J_\varepsilon(t_\varepsilon(\Psi_{1,\varepsilon,y}, \Psi_{2,\varepsilon,y})) = \max_{t \geq 0} J_\varepsilon(t(\Psi_{1,\varepsilon,y}, \Psi_{2,\varepsilon,y})).$$

In view of the above remarks, it is well defined the function $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ given by

$$\Phi_\varepsilon(y) := t_\varepsilon(\Psi_{1,\varepsilon,y}, \Psi_{2,\varepsilon,y}).$$

In the next lemma we prove an important relationship between Φ_ε and the set M.

Lemma 4.8. *Uniformly for $y \in M$, we have*

$$\lim_{\varepsilon \to 0^+} J_\varepsilon(\Phi_\varepsilon(y)) = c_0,$$

where c_0 was given in (16).

Proof. Suppose, by contradiction, that the lemma is false. Then there exist $\delta > 0$, $(y_n) \subset M$ and $\varepsilon_n \to 0^+$ such that

$$|J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - c_0| \geq \delta > 0. \tag{23}$$

We notice that, if $z \in B_{\delta/\varepsilon_n}(0)$ then $\varepsilon_n z + y_n \in B_\delta(y_n) \subset M_\delta \subset \Lambda$. Thus recalling that $H \equiv Q$ in Λ and $\psi(s) = 0$ for $s \geq \delta$, we can use the change of variables $z \mapsto \frac{\varepsilon_n x - y_n}{\varepsilon_n}$ to write

$$\begin{aligned}
J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) &= \frac{1}{2} \|(t_{\varepsilon_n} \Psi_{1,\varepsilon_n,y}, t_{\varepsilon_n} \Psi_{2,\varepsilon_n,y})\|^2_{\varepsilon_n} - \int_{\mathbb{R}^N} H(\varepsilon_n x, t_{\varepsilon_n} \Psi_{1,\varepsilon_n,y}, t_{\varepsilon_n} \Psi_{2,\varepsilon_n,y}) dx \\
&= \frac{t^2_{\varepsilon_n}}{2} \int_{\mathbb{R}^N} a(\varepsilon_n z + y_n) |\nabla(\psi(|\varepsilon_n z|) w_1(z))|^2 dz + \frac{t_{\varepsilon_n}}{2} \int_{\mathbb{R}^N} |\nabla(\psi(|\varepsilon_n z|) w_2(z))|^2 dz \\
&\quad + \frac{t^2_{\varepsilon_n}}{2} \int_{\mathbb{R}^N} |\psi(|\varepsilon_n z|) w_1(z)|^2 dz + \frac{t^2_{\varepsilon_n}}{2} \int_{\mathbb{R}^N} b(\varepsilon_n z + y_n) |\psi(|\varepsilon_n z|) w_2(z)|^2 dz \\
&\quad - \int_{\mathbb{R}^N} Q(t_{\varepsilon_n} \psi(|\varepsilon_n z|) w_1(z), t_{\varepsilon_n} \psi(|\varepsilon_n z|) w_2(z)) dz.
\end{aligned}$$

Since Q is homogeneous, we have that $t_{\varepsilon_n} \to 1$. This and Lebesgue's theorem imply

$$\lim_{n \to \infty} \|(\Psi_{1,\varepsilon_n,y_n}, \Psi_{2,\varepsilon_n,y_n})\|^2_{\varepsilon_n} = \|(w_1, w_2)\|^2$$

and

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} Q(\Psi_{1,\varepsilon_n,y_n}, \Psi_{2,\varepsilon_n,y_n}) dx = \int_{\mathbb{R}^N} Q(w_1, w_2) dx.$$

Therefore $\lim_{n \to \infty} J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = I_0(w_1, w_2) = c_0$, which contradicts (23).

This proves the lemma. $\square$

Proposition 4.9. *Let $\varepsilon_n \to 0$ and $((u_n, v_n)) \subset \mathcal{N}_{\varepsilon_n}$ be such that $J_{\varepsilon_n}(u_n, v_n) \to c_0$. Then there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that the sequence*

$$(w_n(x), z_n(x)) := (u_n(x + \widetilde{y}_n), v_n(x + \widetilde{y}_n))$$

has a convergent subsequence in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

Moreover up to a subsequence, $y_n \to y \in M$, where $y_n = \varepsilon_n \widetilde{y}_n$.

Proof. Since $a_0 \leq a(x)$ and $b_0 \leq b(x)$ for $x \in \mathbb{R}^N$ and $c_0 > 0$, we can use $(\mathcal{H}_1^p)$, (H_3) and repeat the same arguments in Lemma 4.1 to conclude that there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ and constants $R, \eta > 0$ such that

$$\liminf_{n \to \infty} \int_{B_R(\widetilde{y}_n)} \left(|u_n|^2 + |v_n|^2\right) dx \geq \eta.$$

Thus, since $((u_n, v_n))$ is bounded in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, considering $(w_n(x), z_n(x))$ defined above, we have that, up to a subsequence, $w_n \rightharpoonup w \not\equiv 0$ in $H^1(\mathbb{R}^N)$ and $z_n \rightharpoonup z \not\equiv 0$ in $H^1(\mathbb{R}^N)$. Let $t_n > 0$ be such that

$$(\widetilde{w}_n, \widetilde{z}_n) = t_n(w_n, z_n) \in \mathcal{M}_0. \tag{24}$$

Then, $$c_0 \leq I_0(\widetilde{w}_n, \widetilde{z}_n) \leq J_{\varepsilon_n}(t_{\varepsilon_n}(u_n, v_n)) \leq J_{\varepsilon_n}(u_n, v_n) = c_0 + o_n(1) \tag{25}$$

which implies $I_0(\widetilde{w}_n, \widetilde{z}_n) \to c_0$ and $((\widetilde{w}_n, \widetilde{z}_n)) \subset \mathcal{M}_0$.

From the boundedness of $((w_n, z_n))$ and (25), we get that (t_n) is bounded. In consequence the sequence $((\widetilde{w}_n, \widetilde{z}_n))$ is also bounded in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, which implies, for some subsequence, $(\widetilde{w}_n, \widetilde{z}_n) \rightharpoonup (\widetilde{w}, \widetilde{z})$ weakly in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

We can assume that $t_n \to t_0 > 0$. Then, this limit implies $(\widetilde{w}, \widetilde{z}) = t_0(w, z) \not\equiv (0, 0)$. From Lemma 4.2, we conclude that $(\widetilde{w}_n, \widetilde{z}_n) \to (\widetilde{w}, \widetilde{z})$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, and as a consequence $(w_n, z_n) \to (w, z)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

Now, we consider $y_n = \varepsilon_n \widetilde{y}_n$. Our goal is to show that (y_n) has a subsequence, still denoted by (y_n), satisfying $y_n \to y$ for $y \in M$. First of all, we claim that (y_n) is bounded. Indeed, suppose that there exists a subsequence, still denoted by (y_n), verifying $|y_n| \to \infty$. Note that from (ab_1)

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left[a_0|\nabla w_n|^2 + |\nabla z_n|^2 + |w_n|^2 + b_0|z_n|^2\right] dx \\
\leq\ &\int_{\mathbb{R}^N} \left[a(\varepsilon_n x + y_n)|\nabla w_n|^2 + |\nabla z_n|^2 + |w_n|^2 + b(\varepsilon_n x + y_n)|z_n|^2\right] dx \\
=\ &\int_{\mathbb{R}^N} \left[a(\varepsilon_n z)|\nabla u_n(z)|^2 + |\nabla v_n(z)|^2 + |u_n|^2 + b(\varepsilon_n z)|v_n(z)|^2\right] dz \\
=\ &\int_{\mathbb{R}^N} \left[w_n H_w(\varepsilon_n x + y_n, w_n, z_n) + z_n H_z(\varepsilon_n x + y_n, w_n, z_n)\right] dx.
\end{aligned}$$

Fixing $R > 0$ such that $\Lambda \subset B_R(0)$, since $|\varepsilon_n x + y_n| \geq R$ and (H_3), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left[w_n H_w(\varepsilon_n x + y_n, w_n, z_n) + z_n H_z(\varepsilon_n x + y_n, w_n, z_n)\right] dx \\
\leq\ &\frac{1}{4}\int_{B_{R/\varepsilon_n}(0)} \left(|w_n|^2 + b(\varepsilon_n x + y_n)|z_n|^2\right) dx + o_n(1).
\end{aligned}$$

This implies that, $\frac{3}{4}\|(w_n, z_n)\|^2 \leq o_n(1)$. It follows that $(w_n, z_n) \to (0,0)$ in the $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, which is a contradiction because $c_0 > 0$. Hence (y_n) is bounded and, up to a subsequence, $y_n \to y \in \mathbb{R}^N$.

Arguing as above, if $y \notin \overline{\Lambda}$, we will obtain again $(w_n, z_n) \to (0,0)$ in the space $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, thus $y \in \overline{\Lambda}$.

Now we are going to show that $y \in M$. It is sufficient to show that $a(y) = a_0$ and $b(y) = b_0$. Supposing, by contradiction, that $a(y) > a_0$ or $b(y) > b_0$, we have

$$c_0 = I_0(\widetilde{w}, \widetilde{z}) < \frac{1}{2}\int_{\mathbb{R}^N} \left[a(y)|\nabla \widetilde{w}|^2 + |\nabla \widetilde{z}|^2 + |\widetilde{w}|^2 + b(y)|\widetilde{z}|^2\right] dx - \int_{\mathbb{R}^N} Q(\widetilde{w}, \widetilde{z}) dx.$$

Using again the fact that $(\widetilde{w}_n, \widetilde{z}_n) \to (\widetilde{w}, \widetilde{z})$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, from Fatou's lemma we get

$$\begin{aligned}
c_0 &< \liminf_{n\to\infty} \left\{ \frac{t_n^2}{2}\int_{\mathbb{R}^N} \left[a(\varepsilon_n z)|\nabla u_n|^2 + |\nabla v_n|^2 + |u_n|^2 + b(\varepsilon_n z)|v_n|^2\right] dz \right.\\
&\qquad \left. - \int_{\mathbb{R}^N} Q(t_n u_n, t_n v_n) dz \right\}\\
&\leq \liminf_{n\to\infty} \left\{ \frac{t_n^2}{2}\int_{\mathbb{R}^N} \left[a(\varepsilon_n z)|\nabla u_n|^2 + |\nabla v_n|^2 + |u_n|^2 + b(\varepsilon_n z)|v_n|^2\right] dz \right.\\
&\qquad \left. - \int_{\mathbb{R}^N} H(\varepsilon_n z, t_n u_n, t_n v_n) dz \right\}\\
&= \liminf_{n\to\infty} J_{\varepsilon_n}(t_n(u_n, v_n)) \leq \liminf_{n\to\infty} J_{\varepsilon_n}(u_n, v_n) = c_0
\end{aligned}$$

obtaining a contradiction. We conclude that $y \in M$. □

Let us consider $\rho = \rho_\delta > 0$ in such way that $M_\delta \subset B_\rho(0)$ and define $\Upsilon : \mathbb{R}^N \to \mathbb{R}^N$ by setting $\Upsilon(x) := x$ for $|x| < \rho$ and $\Upsilon(x) := \rho x/|x|$ for $|x| \geq \rho$. We also consider the barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ given by

$$\beta_\varepsilon(u, v) := \frac{\displaystyle\int_{\mathbb{R}^N} \Upsilon(\varepsilon x)\left(|u(x)|^2 + |v(x)|^2\right) dx}{\displaystyle\int_{\mathbb{R}^N} \left(|u(x)|^2 + |v(x)|^2\right) dx}.$$

Since $M \subset B_\rho(0)$, the definition of Υ and Lebesgue's theorem imply that

$$\lim_{\varepsilon\to 0} \beta_\varepsilon(\Phi_\varepsilon(y)) = y \quad \text{uniformly for} \quad y \in M. \tag{26}$$

Following [11], we introduce the set

$$\Sigma_\varepsilon := \left\{(u, v) \in \mathcal{N}_\varepsilon : J_\varepsilon(u, v) \leq c_0 + h(\varepsilon)\right\},$$

where $h : \mathbb{R}^+ \to \mathbb{R}^+$ is such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Given $y \in M$, we can use Lemma 4.8 to conclude that $h(\varepsilon) = |J_\varepsilon(\Phi_\varepsilon(y)) - c_0|$ satisfies $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.

Thus, $\Phi_\varepsilon(y) \in \Sigma_\varepsilon$ and therefore $\Sigma_\varepsilon \neq \emptyset$, for any $\varepsilon > 0$ small.

Lemma 4.10. *For any $\delta > 0$ we have*

$$\lim_{\varepsilon \to 0^+} \sup_{(u,v)\in\Sigma_\varepsilon} \operatorname{dist}\left(\beta_\varepsilon(u,v), M_\delta\right) = 0. \tag{27}$$

Proof. Let $(\varepsilon_n) \subset \mathbb{R}$ be such that $\varepsilon_n \to 0^+$. By definition, there exists a sequence $((u_n, v_n)) \subset \Sigma_{\varepsilon_n}$ such that

$$\operatorname{dist}\left(\beta_{\varepsilon_n}(u_n,v_n), M_\delta\right) = \sup_{(u,v)\in\Sigma_{\varepsilon_n}} \operatorname{dist}\left(\beta_{\varepsilon_n}(u,v), M_\delta\right) + o_n(1).$$

Thus, it suffices to find a sequence $(y_n) \subset M_\delta$ such that

$$|\beta_{\varepsilon_n}(u_n,v_n) - y_n| = o_n(1). \tag{28}$$

Thus, recalling that $((u_n,v_n)) \subset \Sigma_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$, we obtain

$$c_0 \le \max_{t\ge 0} I_0(tu_n, tv_n) \le \max_{t\ge 0} J_{\varepsilon_n}(tu_n, tv_n) = J_{\varepsilon_n}(u_n,v_n) \le c_0 + h(\varepsilon_n)$$

from which follows that $J_{\varepsilon_n}(u_n,v_n) \to c_0$. Thus, we may invoke Proposition 4.9 to obtain a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that $(y_n) := (\varepsilon_n \widetilde{y}_n) \subset M_\delta$, for n large. Hence,

$$\begin{aligned}
\beta_{\varepsilon_n}(u_n,v_n) &= \frac{\displaystyle\int_{\mathbb{R}^N} \Upsilon(\varepsilon_n x)\left(|u_n(x)|^2 + |v_n(x)|^2\right) dx}{\displaystyle\int_{\mathbb{R}^N} \left(|u_n(x)|^2 + |v_n(x)|^2\right) dx} \\
&= \frac{\displaystyle\int_{\mathbb{R}^N} \Upsilon(\varepsilon_n z + y_n)\left(|u_n(z+\widetilde{y}_n)|^2 + |v_n(z+\widetilde{y}_n)|^2\right) dz}{\displaystyle\int_{\mathbb{R}^N} \left(|u(z+\widetilde{y}_n)|^2 + |v_n(z+\widetilde{y}_n)|^2\right) dz} \\
&= y_n + \frac{\displaystyle\int_{\mathbb{R}^N} \left(\Upsilon(\varepsilon_n z + y_n) - y_n\right)\left(|u_n(z+\widetilde{y}_n)|^2 + |v_n(z+\widetilde{y}_n)|^2\right) dz}{\displaystyle\int_{\mathbb{R}^N} \left(|u(z+\widetilde{y}_n)|^2 + |v_n(z+\widetilde{y}_n)|^2\right) dz}.
\end{aligned}$$

Since $\varepsilon_n z + y_n \to y \in M$ and from strong convergence of $((u_n(\cdot + \widetilde{y}_n), v_n(\cdot + \widetilde{y}_n)))$, we have that $\beta_{\varepsilon_n}(u_n,v_n) = y_n + o_n(1)$ and therefore the sequence (y_n) satisfies (28), proving the lemma. □

Theorem 4.11. *Suppose that a and b are continuous potentials and satisfy (ab_1) and (ab_2). Suppose also that Q satisfies $(Q_0) - (Q_5)$. Then,*

(i) *For all $\varepsilon > 0$, system $(S_{\varepsilon,aux})$ has a ground state positive solution.*

(ii) *For any $\delta > 0$ there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the system $(S_{\varepsilon,aux})$ has at least $\operatorname{cat}_{M_\delta}(M)$ positive solutions.*

Proof. By using Lemma 3.1, Lemma 3.3, Mountain Pass Theorem [5] and of the characterization of minimax level c_ε given in (7) we conclude that the system $(S_{\varepsilon,aux})$ has a ground state positive solution.

Now, given $\delta > 0$ such that $M_\delta \subset \Lambda$, we can use (26), Lemma 4.8, (27) and argue as in [11, Section 6] to obtain $\widehat{\varepsilon}_\delta > 0$ such that, for any $\varepsilon \in (0, \widehat{\varepsilon}_\delta)$, the diagram

$$M \xrightarrow{\Phi_\varepsilon} \Sigma_\varepsilon \xrightarrow{\beta_\varepsilon} M_\delta$$

is well defined and $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopically equivalent to the embedding $\iota : M \to M_\delta$.

Thus
$$\mathrm{cat}_{\Sigma_\varepsilon}(\Sigma_\varepsilon) \geq \mathrm{cat}_{M_\delta}(M).$$

It follows from Proposition 4.7 and Theorem 4.3 that J_ε possesses at least $\mathrm{cat}_M(M_\delta)$ critical points on $\mathcal{N}_\varepsilon$. The same argument employed in the proof of Proposition 4.7 shows that each of these critical points is also a critical point of the unconstrained functional J_ε. Thus, we obtain $\mathrm{cat}_{M_\delta}(M)$ nontrivial solutions for $(S_{\varepsilon,aux})$. □

5. Proof of Theorem 1.1

Proof. Suppose that $\delta > 0$ is such that $M_\delta \subset \Lambda$. Arguing by contradiction we can use [17, Lemma 5.1] to get $\widetilde{\varepsilon}_\delta > 0$ such that, for any $0 < \varepsilon < \widetilde{\varepsilon}_\delta$ and any solution $(u_\varepsilon, v_\varepsilon) \in \Sigma_\varepsilon$ of the system $(S_{\varepsilon,aux})$ there holds

$$|(u_\varepsilon(\varepsilon x), v_\varepsilon(\varepsilon x))| \leq \alpha \quad \text{for each} \quad x \in \mathbb{R}^N \setminus \Lambda_\varepsilon. \tag{29}$$

Let $\widehat{\varepsilon}_\delta > 0$ be given as in the proof of item (ii) of Theorem 4.11. We now define $\varepsilon_\delta := \min\{\widehat{\varepsilon}_\delta, \widetilde{\varepsilon}_\delta\}$. We shall prove the theorem for this choice of ε_δ. Let $0 < \varepsilon < \varepsilon_\delta$ be fixed. By applying Theorem 4.11, we obtain $\mathrm{cat}_{M_\delta}(M)$ nontrivial solutions of the system $(S_{\varepsilon,aux})$. If $(u, v) \in X_\varepsilon$ is one of these solutions we have that $(u, v) \in \Sigma_\varepsilon$, and therefore we can use (29) and the definition of H to conclude $H(\cdot, u, v) \equiv Q(u, v)$. Hence, (u, v) is also a solution of the system $(\widehat{S}_\varepsilon)$. An easy calculation shows that $(\widehat{u}(x), \widehat{v}(x)) := (u(x/\varepsilon), v(x/\varepsilon))$ is a solution of the original system (S_ε). Then, (S_ε) has at least $\mathrm{cat}_{M_\delta}(M)$ nontrivial solutions.

We now consider $\varepsilon_n \to 0^+$ and take a sequence $(u_n, v_n) \in X_{\varepsilon_n}$ of solutions of the system $(\widehat{S}_{\varepsilon_n})$ as above. By applying [17, Lemma 5.1] again, we obtain $R > 0$ and $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that

$$\|u_n\|_{L^\infty(B_R(\widetilde{y}_n))^c} < \gamma \quad \text{and} \quad \|v_n\|_{L^\infty(B_R(\widetilde{y}_n))^c} < \gamma.$$

Up to a subsequence, we may also assume that

$$\|u_n\|_{L^\infty(B_R(\widetilde{y}_n))} \geq \gamma \tag{30}$$

and
$$\|v_n\|_{L^\infty(B_R(\widetilde{y}_n))} \geq \gamma. \tag{31}$$

Indeed, if this is not the case, we have $\|u_n\|_{L^\infty(\mathbb{R}^N)} < \gamma$ or $\|v_n\|_{L^\infty(\mathbb{R}^N)} < \gamma$, which is is a contradiction with (19). Thus, (30) and (31) hold.

By using (30) and (31) we conclude that the maximum point $\pi_{n,a} \in \mathbb{R}^N$ of u_n and the maximum point $\pi_{n,b} \in \mathbb{R}^N$ of v_n belong to $B_R(\widetilde{y}_n)$. Hence $\pi_{n,a} = \widetilde{y}_n + q_{n,a}$, for some $q_{n,a} \in B_R(0)$ and $\pi_{n,b} = \widetilde{y}_n + q_{n,b}$, for some $q_{n,b} \in B_R(0)$. Recalling that the associated solution of (S_{ε_n}) is of the form $(\widehat{u}_n(x), \widehat{v}_n(x)) = (u_n(x/\varepsilon_n), v_n(x/\varepsilon_n))$, we conclude that the maximum point $\Pi_{n,a}$ of $\widehat{u}_n$ and the maximum point $\Pi_{n,b}$ of v_n

are $\Pi_{n,a} := \varepsilon_n \widetilde{y}_n + \varepsilon_n q_{n,a}$ and $\Pi_{n,b} := \varepsilon_n \widetilde{y}_n + \varepsilon_n q_{n,b}$. Since $(q_{n,a}), (q_{n,b}) \subset B_R(0)$ are bounded and $\varepsilon_n \widetilde{y}_n \to y \in M$ (according to Proposition 4.9), we obtain

$$\lim_{n\to\infty} a(\Pi_{n,a}) = a(y) = a_0 \quad \text{and} \quad \lim_{n\to\infty} b(\Pi_{n,a}) = b(y) = b_0.$$

Now we prove the regularity of the solution. By using [17, Lemma 5.1], (30) and (31), we have $u_\varepsilon, v_\varepsilon \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. From the interpolation inequality, we get $(u_\varepsilon, v_\varepsilon) \in L^q(\mathbb{R}^N) \times L^q(\mathbb{R}^N)$, $\forall q \geq 2$. This implies $Q_u(u_\varepsilon, v_\varepsilon), Q_v(u_\varepsilon, v_\varepsilon) \in L^q(\mathbb{R}^N)$, $\forall q \geq 2$. From regularity elliptic theory, we get $(u_\varepsilon, v_\varepsilon) \in W^{2,q}(\mathbb{R}^N) \times W^{2,q}(\mathbb{R}^N)$, $\forall q \geq 2$. For q sufficiently large, we obtain $W^{2,q}(\mathbb{R}^N) \hookrightarrow C^{1,\lambda}(\mathbb{R}^N)$, for some λ with $0 < \lambda < 1$. Then $u_\varepsilon, v_\varepsilon \in C^{1,\lambda}(\mathbb{R}^N)$. Since $Q \in C^2(\mathbb{R}^N)$, we obtain that $u_\varepsilon, v_\varepsilon \in C^{2,\lambda}(\mathbb{R}^N)$, which concludes the proof of the theorem. $\square$

6. The critical case

In this section we present the proof of Theorem 1.2. Since many calculations are adaptations of those presented in the early section, we will emphasize only the differences between the subcritical and the critical case.

Hereafter, we will work with the following system equivalent to (CS_ε)

$$(C\widehat{S}_\varepsilon) \qquad \begin{cases} -\mathrm{div}(a(\varepsilon x)\nabla u) + u = Q_u(u,v) + \frac{1}{2^*} K_u(u,v) \text{ in } \mathbb{R}^N, \\ \quad -\Delta v + b(\varepsilon x) v \;= Q_v(u,v) + \frac{1}{2^*} K_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \quad u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N. \end{cases}$$

Using a function η given in (3), we define $\widehat{K} : \mathbb{R}^2 \to \mathbb{R}$ by

$$\widehat{K}(s,t) := \eta(|(s,t)|)\Big(Q(s,t) + \frac{1}{2^*} K(s,t)\Big) + (1 - \eta(|(s,t)|))\widetilde{A}(s^2 + t^2),$$

where $\quad \widetilde{A} := \max\Big\{ \dfrac{Q(s,t) + \frac{1}{2^*}K(s,t)}{s^2+t^2} : (s,t) \in \mathbb{R}^2,\ \alpha \leq |(s,t)| \leq 5\alpha \Big\}.$

Notice that, since $\widetilde{A} > 0$ tends to zero as $\alpha \to 0^+$, we may suppose that $\widetilde{A} \in (0, \mu/4)$.

We define $\widetilde{H} : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ by setting

$$\widetilde{H}(x,s,t) := I_\Lambda(x)\Big(Q(s,t) + \frac{1}{2^*}K(s,t)\Big) + (1 - I_\Lambda(x))\widehat{K}(s,t). \tag{32}$$

Using the fact that $Q \in \mathcal{H}^p$ and $K \in \mathcal{H}^{2^*}$, we can arguing as the proof of Lemma 2.1 to get

Lemma 6.1. *The function $\widetilde{H}$ satisfies the following estimates:*

$(\widetilde{H}_1)$ $p\widetilde{H}(x,s,t) \leq s\widetilde{H}_s(x,s,t) + t\widetilde{H}_t(x,s,t)$*, for each* $x \in \Lambda$.

$(\widetilde{H}_2)$ $2\widetilde{H}(x,s,t) \leq s\widetilde{H}_s(x,s,t) + t\widetilde{H}_t(x,s,t)$*, for each* $x \in \mathbb{R}^N \setminus \Lambda$.

$(\widetilde{H}_3)$ *For α small we have for each $x \in \mathbb{R}^N \setminus \Lambda$:*

$$s\widetilde{H}_s(x,s,t) + t\widetilde{H}_t(x,s,t) \leq \frac{1}{4}\left(s^2 + b(x)t^2\right).$$

$(\widetilde{H}_4)$ *For α small we have* $\dfrac{|\widetilde{H}_s(x,s,t)|}{\alpha}, \dfrac{|\widetilde{H}_t(x,s,t)|}{\alpha} \leq \dfrac{\mu}{4}$ *for each* $x \in \mathbb{R}^N \setminus \Lambda$.

Using the definition (32), we deal in the sequel with the modified system

$$(CS_{\varepsilon,aux}) \qquad \begin{cases} -\mathrm{div}(a(\varepsilon x)\nabla u) + u = \widetilde{H}_u(\varepsilon x, u, v) \text{ in } \mathbb{R}^N, \\ \quad -\Delta v + b(\varepsilon x)v \;\; = \widetilde{H}_v(\varepsilon x, u, v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \;\; u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N, \end{cases}$$

and we will look for solutions $(u_\varepsilon, v_\varepsilon)$ satisfying

$$|(u_\varepsilon(\varepsilon x), v_\varepsilon(\varepsilon x))| \leq \alpha \;\text{ for each } x \in \mathbb{R}^N \setminus \Lambda_\varepsilon.$$

The conditions $(\widetilde{H}_3)$ and (A_1) imply that the critical points of the C^1-functional $\widetilde{J}_\varepsilon : X_\varepsilon \to \mathbb{R}$ given by

$$\widetilde{J}_\varepsilon(u, v) = \frac{1}{2}\int_{\mathbb{R}^N} \left[a(\varepsilon x)|\nabla u|^2 + |\nabla v|^2 + |u|^2 + b(\varepsilon x)|v|^2\right] dx - \int_{\mathbb{R}^N} \widetilde{H}(\varepsilon x, u, v)dx$$

are weak solutions of $(CS_{\varepsilon,aux})$. We recall that these critical points belong to the Nehari manifold of $\widetilde{J}_\varepsilon$, namely on the set

$$\widetilde{\mathcal{N}}_\varepsilon := \left\{(u, v) \in X_\varepsilon \setminus \{(0, 0)\} : \widetilde{J}'_\varepsilon(u, v)(u, v) = 0\right\}$$

and we define the number $\widetilde{b}_\varepsilon$ by setting

$$\widetilde{b}_\varepsilon := \inf_{(u,v)\in\widetilde{\mathcal{N}}_\varepsilon} \widetilde{J}_\varepsilon(u, v). \tag{33}$$

In order to prove the multiplicity result for the system $(C\widehat{S}_\varepsilon)$, we consider the critical version of the problem (S_0), namely

$$(CS_0) \qquad \begin{cases} -a_0\Delta u + u = Q_u(u, v) + \frac{1}{2^*}K_u(u, v) \text{ in } \mathbb{R}^N, \\ -\Delta v + b_0 v \;= Q_v(u, v) + \frac{1}{2^*}K_v(u, v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \;\; u(x), v(x) > 0 \text{ for each } x \in \mathbb{R}^N. \end{cases}$$

In view of the conditions (ab_1), $(\mathcal{H}_1^p)$ and $(\mathcal{H}_1^{2^*})$, the above system has a variational structure and the associated functional is given by

$$\widetilde{I}_0(u, v) := \frac{1}{2}\int_{\mathbb{R}^N}\left[a_0|\nabla u|^2 + |\nabla v|^2 + |u|^2 + b_0|v|^2\right] dx - \int_{\mathbb{R}^N} Q(u, v)dx - \frac{1}{2^*}\int_{\mathbb{R}^N} K(u, v)dx$$

is well defined for $(u, v) \in E_0$.

Standard calculations show that $\widetilde{I}_0$ has the Mountain Pass geometry and therefore we can set the the minimax level $\widetilde{c}_0$ in the following way

$$\widetilde{c}_0 := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \widetilde{I}_0(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0, 1], E_0) : \gamma(0) = (0, 0), \; \widetilde{I}_0(\gamma(1)) < 0\}$.

Moreover, $\widetilde{c}_0$ can be further characterized as

$$\widetilde{c}_0 = \inf_{(u,v)\in\widetilde{\mathcal{M}}_0} \widetilde{I}_0(u,v), \tag{34}$$

with $\widetilde{\mathcal{M}}_0$ being the Nehari manifold of $\widetilde{I}_0$, that is

$$\widetilde{\mathcal{M}}_0 := \{(u,v) \in E_0 \setminus \{(0,0)\} : \widetilde{I}_0'(u,v)(u,v) = 0\}.$$

As usual, we denote by S the best constant of the embedding $W^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}$. To state the next result we need to define $\widetilde{S}_K$ the best constant of the immersion $D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N) \times L^{2^*}(\mathbb{R}^N)$, that is,

$$\widetilde{S}_K := \inf_{\substack{u,v\in D^{1,2}(\mathbb{R}^N)\\ u,v\neq 0}} \frac{\displaystyle\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)dx}{\left(\displaystyle\int_{\mathbb{R}^N} K(u,v)dx\right)^{2/2^*}}.$$

Proposition 6.2. *There exists $\sigma^* > 0$ such that for all $\sigma > \sigma^*$*

$$\widetilde{c}_0 < \frac{1}{N}\left(\min\{a_0, 1\}\widetilde{S}_K\right)^{N/2}.$$

Proof. By using $(\mathcal{H}_0^p)$ and $(\mathcal{H}_0^{2^*})$, it is possible to prove that

$$\widetilde{c}_0 = \inf_{(u,v)\in E_0\setminus\{(0,0)\}} \max_{t\geq 0} \widetilde{I}_0(tu,tv) > 0.$$

Thus, it suffices to obtain $(u,v) \in E_0$ such that

$$\max_{t\geq 0} I_0(tu,tv) < \frac{1}{N}\left(\min\{a_0, 1\}\widetilde{S}_K\right)^{N/2}.$$

We first recall that, for any $\delta > 0$ the function

$$w_\delta(x) := [\delta N(N-2)]^{(N-2)/4}(\delta + |x|^2)^{(2-N)/2}$$

satisfies

$$\int_{\mathbb{R}^N} |\nabla w_\delta|^2 dx = \int_{\mathbb{R}^N} |w_\delta|^{2^*} dx = S^{N/2}.$$

By [12, Lemma 3], there exist $A, B \in \mathbb{R}$ such that $\widetilde{S}_K$ is attainded by

$$\widetilde{S}_K = \frac{\int_{\mathbb{R}^N} \left(|\nabla(Aw_\delta)|^2 + |\nabla(Bw_\delta)|^2\right) dx}{\left(\int_{\mathbb{R}^N} K(Aw_\delta, Bw_\delta)dx\right)^{2/2^*}} = \frac{S^{N/2}(A^2+B^2)}{\left(\int_{\mathbb{R}^N} K(Aw_\delta, Bw_\delta)dx\right)^{2/2^*}}.$$

Let $\eta \in C_0^\infty(\mathbb{R}^N, [0,1])$ be such that $\eta \equiv 1$ on $B_1(0)$ and $\eta \equiv 0$ on $\mathbb{R}^N \setminus B_2(0)$.

Consider

$$\psi_\delta(x) := \frac{\eta(x)w_\delta(x)}{|\eta w_\delta|_{2^*}}.$$

By using the definition of ψ_δ, (A_3) and $(\mathcal{H}_0^{2^*})$ we get

$$\widetilde{I}_0(tA\psi_\delta, tB\psi_\delta) \leq \frac{t^2}{2}D_\delta(A^2+B^2) - \frac{t^{2^*}}{2^*}\int_{\mathbb{R}^{\mathbb{N}}} K(A\psi_\delta, B\psi_\delta)dx - \frac{\sigma}{p_1}t^{p_1}A^\lambda B^\beta \int_{B_2(0)} |\psi_\delta|^{p_1}dx,$$

where $p_1 \in (2, 2^*)$ is given by condition (A_3) and

$$D_\delta = \int_{\mathbb{R}^N} \max\{a_0, b_0, 1\} \left(|\nabla \psi_\delta|^2 + |\psi_\delta|^2\right) dx.$$

Thus

$$\max_{t\geq 0} \left\{ \frac{t^2}{2} D_\delta (A^2 + B^2) - \frac{t^{2^*}}{2^*} \int_{\mathbb{R}^N} K(A\psi_\delta, B\psi_\delta) dx - \frac{\sigma}{p_1} t^{p_1} A^\lambda B^\beta \int_{B_2(0)} |\psi_\delta|^{p_1} dx \right\}$$
$$\geq \widetilde{I}_0(tA\psi_\delta, tB\psi_\delta).$$

Straightforward calculations show that

$$\begin{aligned} \widetilde{I}_0(tA\psi_\delta, tB\psi_\delta) &\leq \frac{1}{\sigma^{2/(p_1-2)}} \left(\frac{1}{2} - \frac{1}{p_1}\right) \frac{(D_\delta(A^2+B^2))^{p_1/(p_1-2)}}{\left(A^\lambda B^\beta \int_{B_2(0)} |\psi_\delta|^{p_1} dx\right)^{2/(p_1-2)}} \\ &= \frac{1}{\sigma^{2/(p_1-2)}} C(a_0, b_0). \end{aligned}$$

Thus, $\max\limits_{t\geq 0} \widetilde{I}_0(tA\psi_\delta, tB\psi_\delta) < \frac{1}{N}\Big(\min\{a_0, 1\}\widetilde{S}_K\Big)^{N/2}$, for all $\sigma > \sigma^*$ where

$$\sigma^* := \left(\frac{C(a_0, b_0)}{\frac{1}{N}\big(\min\{a_0, 1\}\widetilde{S}_K\big)^{N/2}} \right)^{\frac{p_1-2}{2}}.$$

The proof is complete. □

Lemma 6.3. *Let* $((u_n, v_n)) \subset \widetilde{\mathcal{M}}_0$ *be a sequence such that* $\widetilde{I}_0(u_n, v_n) \to \widetilde{c}_0$ *with* $\widetilde{c}_0 < \frac{1}{N}\left(\min\{a_0, 1\}\widetilde{S}_K\right)^{N/2}$. *Then we have either*

(i) $\|(u_n, v_n)\| \to 0$, *or*

(ii) *there exists a sequence* $(y_n) \subset \mathbb{R}^N$ *and constants* $R, \eta > 0$ *such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} (|u_n|^2 + |v_n|^2) dx \geq \eta.$$

Proof. Suppose that (ii) does not hold. Since $((u_n, v_n))$ is bounded in the space $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, then, by in [23, Lemma I.1], we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^s dx = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^s dx = 0,$$

for all $s \in (2, 2^*)$. Thus, from $(\mathcal{H}_1^p)$, we conclude

$$\int_{\mathbb{R}^N} [u_n Q_u(u_n, v_n) + v_n Q_v(u_n, v_n)] dx = o_n(1).$$

Since $\widetilde{I}_0'(u_n, v_n)(u_n, v_n) = 0$, taking a subsequence, we obtain $l \geq 0$ such that

$$\|(u_n, v_n)\|^2 \to l \quad \text{and} \quad \int_{\mathbb{R}^N} K(u_n, v_n) dx \to l. \tag{35}$$

Since $\widetilde{I}_0(u_n, v_n) \to \widetilde{c}_0$, we can use (35) to conclude that $l = N\widetilde{c}_0$.

Recalling the definition of $\widetilde{S}_K$ we get

$$\begin{aligned}\|(u_n,v_n)\|^2 &= \int_{\mathbb{R}^N}\left[a_0|\nabla u_n|^2+|\nabla v_n|^2+|u_n|^2+b_0|v_n|^2\right]dx\\ &\geq \min\{a_0,1\}\int_{\mathbb{R}^N}\left[|\nabla u_n|^2+|\nabla v_n|^2\right]dx\\ &\geq \min\{a_0,1\}\widetilde{S}_K\left(\int_{\mathbb{R}^N}K(u_n,v_n)dx\right)^{2/2^*}.\end{aligned}$$

Taking the limit we conclude that $l \geq \min\{a_0,1\}\widetilde{S}_K l^{2/2^*}$. If $l>0$ we obtain immediately $N\widetilde{c}_0 = l \geq \big(\min\{a_0,1\}\widetilde{S}_K\big)^{N/2}$, which does not make sense. Hence $l=0$ and therefore (i) holds. □

By using Lemma 6.3, we can arguing as the proof of Lemma 4.2 and show that the system (CS_0) has a solution that reaches $\widetilde{c}_0$.

Lemma 6.4. (A Compactness Lemma) *Let $((u_n,v_n)) \subset \widetilde{\mathcal{M}}_0$ be a sequence satisfying $\widetilde{I}_0(u_n,v_n) \to \widetilde{c}_0$. Then, there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that, up to a subsequence, $(w_n(x),z_n(x)) = (u_n(x+\widetilde{y}_n), v_n(x+\widetilde{y}_n))$ converges strongly in the space $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. In particular, there exists a minimizer for $\widetilde{c}_0$.*

As in Lemma 3.1, the functional $\widetilde{J}_\varepsilon$ satisfies the mountain Pass Geometry. Hence there exists a Palais-Smale sequence $((u_n,v_n)) \subset X_\varepsilon$ at level $\widetilde{c}_\varepsilon$. Using $(\mathcal{H}_0^p)$ and $(\mathcal{H}_0^{2^*})$, it is possible to prove that

$$\widetilde{c}_\varepsilon = \widetilde{b}_\varepsilon = \inf_{(u,v)\in X_\varepsilon\setminus\{0\}} \sup_{t\geq 0} \widetilde{J}_\varepsilon(tu,tv), \tag{36}$$

where $\widetilde{b}_\varepsilon$ was defined in (33).

Lemma 6.5. *Any sequence $((u_n,v_n)) \subset X_\varepsilon$ such that*

$$\widetilde{J}_\varepsilon(u_n,v_n) \to c < \frac{1}{N}\left(min\{a_0,1\}\widetilde{S}_K\right)^{N/2} \quad \textit{and} \quad \widetilde{J}'_\varepsilon(u_n,v_n)\to 0$$

possesses a convergent subsequence.

Proof. Standart calculations show that $((u_n,v_n))$ is bounded in X_ε. Then, up to a subsequence, we may suppose that

$$\begin{aligned}&(u_n,v_n) \rightharpoonup (u,v) \ \text{ weakly in } \ X_\varepsilon,\\ &u_n\to u, v_n\to v \ \text{ strongly in } \ L^s_{loc}(\mathbb{R}^N), \ \text{ for any } \ 2\leq s<2^*,\\ &u_n(x)\to u(x), v_n(x)\to v(x) \ \text{ for a.e. } \ x\in\mathbb{R}^N.\end{aligned} \tag{37}$$

Now using a density argument, we can conclude that (u,v) is a critical point of $\widetilde{J}_\varepsilon$.

Hence

$$\|(u,v)\|_\varepsilon^2 = \int_{\mathbb{R}^N}[u\widetilde{H}_u(\varepsilon x,u,v)+v\widetilde{H}_v(\varepsilon x,u,v)]dx. \tag{38}$$

On the other hand, we have

$$\|(u_n,v_n)\|_\varepsilon^2 = \int_{\mathbb{R}^N}[u_n\widetilde{H}_u(\varepsilon x,u_n,v_n)+v_n\widetilde{H}_v(\varepsilon x,u_n,v_n)]dx + o_n(1). \tag{39}$$

Claim 1. $\lim_{n\to\infty}\int_{\Lambda_\varepsilon} K(u_n,v_n)dx = \int_{\Lambda_\varepsilon} K(u,v)dx.$

Since $((u_n,v_n))$ is bounded, we may suppose that

$$|\nabla u_n|^2 \rightharpoonup \mu, \quad |\nabla v_n|^2 \rightharpoonup \sigma \quad \text{and} \quad K(u_n,v_n) \rightharpoonup \nu \quad \text{(weak*-sense of measures).}$$

By [12, Lemma 6], we obtain an at most countable index set Γ, sequences $(x_i) \in \mathbb{R}^N$, (μ_i), (σ_i), $(\nu_i) \subset (0,\infty)$ such that

$$\begin{aligned} &\mu \geq |\nabla u|^2 + \sum_{i\in\Gamma}\mu_i\delta_{x_i}, \quad \sigma \geq |\nabla v|^2 + \sum_{i\in\Gamma}\sigma_i\delta_{x_i} \\ &\nu = K(u,v) + \sum_{i\in\Gamma}\nu_i\delta_{x_i} \quad \text{and} \quad \widetilde{S}_K\nu_i^{2/2^*} \leq \mu_i + \sigma_i \end{aligned} \tag{40}$$

for all $i \in \Gamma$, where δ_{x_i} is the Dirac mass at the point $x_i \in \mathbb{R}^N$.

Suppose that $\{x_i\}_{i\in\Gamma} \cap \Lambda_\varepsilon \neq \emptyset$, then exists $x_i \in \Lambda_\varepsilon$ for some $i \in \Gamma$. Define, for $\varrho > 0$, the function $\psi_\varrho(x) := \psi((x - x_i)/\varrho)$ where $\psi \in C_0^\infty(\mathbb{R}^N, [0,1])$ is such that $\psi \equiv 1$ on $B_1(0)$, $\psi \equiv 0$ on $\mathbb{R}^N \setminus B_2(0)$ and $|\nabla\psi|_\infty \leq 2$. We suppose that ϱ is chosen in such a way that the support of ψ_ϱ is contained in Λ_ε. Since $((\psi_\varrho u_n, \psi_\varrho v_n))$ is bounded, $\widetilde{J}'_\varepsilon(u_n,v_n)(\psi_\varrho u_n, \psi_\varrho v_n) = o_n(1)$. Then

$$\begin{aligned} &\int_{\mathbb{R}^N}[a(\varepsilon x)\psi_\varrho|\nabla u_n|^2 + \psi_\varrho|\nabla v_n|^2]dx \\ &\quad + \int_{\mathbb{R}^N}[a(\varepsilon x)u_n\nabla u_n\nabla\psi_\varrho + v_n\nabla v_n\nabla\psi_\varrho]dx + \int_{\mathbb{R}^N}[\psi_\varrho u_n^2 + b(\varepsilon x)\psi_\varrho v_n^2]dx \\ &= \int_{\mathbb{R}^N}[u_n\widetilde{H}_u(\varepsilon x,u_n,v_n) + v_n\widetilde{H}_v(\varepsilon x,u_n,v_n)]\psi_\varrho dx + o_n(1). \end{aligned}$$

Since $\operatorname{supp}(\psi_\varrho) \subset \Lambda_\varepsilon$, we can use definition of $\widetilde{H}$, (1) and (ab_1) to get

$$\begin{aligned} &\min\{a_0,1\}\int_{\mathbb{R}^N}[\psi_\varrho|\nabla u_n|^2 + \psi_\varrho|\nabla v_n|^2]dx \\ &\leq -\int_{\mathbb{R}^N}[a(\varepsilon x)u_n\nabla u_n\nabla\psi_\varrho + v_n\nabla v_n\nabla\psi_\varrho]dx \\ &\quad + p\int_{\mathbb{R}^N}Q(u_n,v_n)\psi_\varrho dx + \int_{\mathbb{R}^N}K(u_n,v_n)\psi_\varrho dx + o_n(1). \end{aligned}$$

Since Q has subcritical growth and ψ_ϱ has compact support, we can let $n \to \infty$, $\varrho \to 0$ and use (40) to conclude that

$$\min\{a_0,1\}(\mu_i + \sigma_i) \leq \nu_i.$$

As $\widetilde{S}_K\nu_i^{2/2^*} \leq \mu_i + \sigma_i$, we get $\quad \nu_i \geq \left(\min\{a_0,1\}\widetilde{S}_K\right)^{N/2}.$

By using Lemma 6.1, $p > 2$ and (1) we get

$$
\begin{aligned}
c &= \widetilde{J}_\varepsilon(u_n, v_n) - \frac{1}{2}\widetilde{J}'_\varepsilon(u_n, v_n)(u_n, v_n) + o_n(1) \\
&= \int_{\mathbb{R}^N \setminus \Lambda_\varepsilon} \left(\frac{1}{2}[u_n \widetilde{H}_u(\varepsilon x, u_n, v_n) + v_n \widetilde{H}_v(\varepsilon x, u_n, v_n)] - \widetilde{H}(\varepsilon x, u_n, v_n) \right) dx \\
&\quad + \int_{\Lambda_\varepsilon} \left(\frac{1}{2}[u_n Q_u(u_n, v_n) + v_n Q_v(u_n, v_n)] - Q(u_n, v_n) \right) dx \\
&\quad + \frac{1}{2^*} \int_{\Lambda_\varepsilon} \left(\frac{1}{2}[u_n K_u(u_n, v_n) + v_n K_v(u_n, v_n)] - K(u_n, v_n) \right) dx + o_n(1) \\
&\geq \frac{1}{N} \int_{\Lambda_\varepsilon} K(u_n, v_n) dx + o_n(1) \geq \frac{1}{N} \int_{\Lambda_\varepsilon} \psi_\varrho K(u_n, v_n) dx + o_n(1).
\end{aligned}
$$

By taking the limit and using (40) we get

$$
c \geq \frac{1}{N} \sum_{\{i \in \Gamma : x_i \in \Lambda_\varepsilon\}} \psi_\varrho(x_i)\nu_i = \frac{1}{N} \sum_{\{i \in \Gamma : x_i \in \Lambda_\varepsilon\}} \nu_i \geq \frac{1}{N} \left(\min\{a_0, 1\} \widetilde{S}_K \right)^{N/2}
$$

which does not make sense. Therefore $\{x_i\}_{i \in \Gamma} \cap \Lambda_\varepsilon = \emptyset$, this conclude the proof of the claim 1.

Claim 2.

$$
\int_{\mathbb{R}^N} [u_n \widetilde{H}_u(\varepsilon x, u_n, v_n) + v_n \widetilde{H}_v(\varepsilon x, u_n, v_n)] dx \to \int_{\mathbb{R}^N} [u \widetilde{H}_u(\varepsilon x, u, v) + v \widetilde{H}_v(\varepsilon x, u, v)] dx.
$$

Arguing as in the Lemma 3.2, for any $\xi >$ given, there exists $R > 0$ such that $\Lambda_\varepsilon \subset B_R(0)$ and

$$
\limsup_{n \to \infty} \int_{\mathbb{R}^N \setminus B_R(0)} \left[a(\varepsilon x)|\nabla u_n|^2 + |\nabla v_n|^2 + |u_n|^2 + b(\varepsilon x)|v_n|^2 \right] dx < \xi.
$$

This inequality, $(\widetilde{H}_3)$ and the Sobolev embeddings imply that, for n large enough,

$$
\int_{\mathbb{R}^N \setminus B_R(0)} [u_n \widetilde{H}_u(\varepsilon x, u_n, v_n) + v_n \widetilde{H}_v(\varepsilon x, u_n, v_n)] dx \leq C_1 \frac{1}{4} \xi, \tag{41}
$$

where C_1 is positive constant. On the other hand, taking R large enough, we can suppose that

$$
\left| \int_{\mathbb{R}^N \setminus B_R(0)} [u \widetilde{H}_u(\varepsilon x, u, v) + v \widetilde{H}_v(\varepsilon x, u, v)] dx \right| < \xi. \tag{42}
$$

Then, by (41) and (42), we can conclude

$$
\begin{aligned}
&\int_{\mathbb{R}^N \setminus B_R(0)} [u_n \widetilde{H}_u(\varepsilon x, u_n, v_n) + v_n \widetilde{H}_v(\varepsilon x, u_n, v_n)] dx \\
&= \int_{\mathbb{R}^N \setminus B_R(0)} [u \widetilde{H}_u(\varepsilon x, u, v) + v \widetilde{H}_v(\varepsilon x, u, v)] dx + o_n(1).
\end{aligned} \tag{43}
$$

On the other hand, since the set $B_R(0) \cap (\mathbb{R}^N \setminus \Lambda_\varepsilon)$ is bounded, we can use $(\widetilde{H}_3)$, (37) and Lebesgue's theorem to conclude that

$$\lim_{n\to\infty}\int_{B_R(0)\cap(\mathbb{R}^N\setminus\Lambda_\varepsilon)}[u_n\widetilde{H}_u(\varepsilon x,u_n,v_n)+v_n\widetilde{H}_v(\varepsilon x,u_n,v_n)]dx$$
$$=\int_{B_R(0)\cap(\mathbb{R}^N\setminus\Lambda_\varepsilon)}[u\widetilde{H}_u(\varepsilon x,u,v)+v\widetilde{H}_v(\varepsilon x,u,v)]dx. \tag{44}$$

By using Claim 1, $(\mathcal{H}_1^p)$, (37) and Lebesgue's theorem again, we obtain

$$\lim_{n\to\infty}\int_{\Lambda_\varepsilon}[u_n\widetilde{H}_u(\varepsilon x,u_n,v_n)+v_n\widetilde{H}_v(\varepsilon x,u_n,v_n)]dx$$
$$=\int_{\Lambda_\varepsilon}[u\widetilde{H}_u(\varepsilon x,u,v)+v\widetilde{H}_v(\varepsilon x,u,v)]dx. \tag{45}$$

From (43), (44) and (45) claim 2 is proved. By using (38), claim 2 and (39), we have $\|(u_n,v_n)\|_\varepsilon^2\to\|(u,v)\|_\varepsilon^2$. This implies $(u_n,v_n)\to(u,v)$ in X_ε. □

From Lemma 6.5, the unconstrained functional satisfies $(\mathrm{PS})_c$ for under the condition $c<\frac{1}{N}(\min\{a_0,1\}\widetilde{S}_K)^{N/2}$. Nevertheless, to get multiple critical points, we need to work with the functional $\widetilde{J}_\varepsilon$ constrained to $\widetilde{\mathcal{N}}_\varepsilon$. The proof the next three results follows by using the same arguments employed in Lemma 4.5, Lemma 4.6 and Proposition 4.7

Lemma 6.6. *There exist positive constants $\widetilde{\alpha}_1,\widetilde{\delta}_1,\widetilde{C}$ such that, for each $\alpha\in(0,\widetilde{\alpha}_1)$, $(u,v)\in\widetilde{\mathcal{N}}_\varepsilon$, we have*

$$\int_{\Lambda_\varepsilon}[pQ(u,v)+K(u,v)]dx\geq\widetilde{\delta}_1$$

and
$$\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}\left(u^2+b(\varepsilon x)v^2\right)dx\leq\widetilde{C}\int_{\Lambda_\varepsilon}[pQ(u,v)+K(u,v)]dx.$$

Lemma 6.7. *Let $\widetilde{\phi}_\varepsilon:X_\varepsilon\to\mathbb{R}$ be given by*

$$\widetilde{\phi}_\varepsilon(u,v):=\|(u,v)\|_\varepsilon^2-\int_{\mathbb{R}^N}\left[u\widetilde{H}_u(\varepsilon x,u,v)+v\widetilde{H}_v(\varepsilon x,u,v)\right]dx.$$

Then there exist $\widetilde{\alpha}_2,\widetilde{M}>0$ such that, for each $\alpha\in(0,\widetilde{\alpha}_2)$,

$$\widetilde{\phi}_\varepsilon'(u,v)(u,v)\leq-\widetilde{M}<0\quad\text{for each}\quad(u,v)\in\widetilde{\mathcal{N}}_\varepsilon.$$

Proposition 6.8. *The functional $\widetilde{J}_\varepsilon$ restricted to $\widetilde{\mathcal{N}}_\varepsilon$ satisfies $(PS)_c$ at any level $c<\frac{1}{N}\left(\min\{a_0,1\}\widetilde{S}_K\right)^{N/2}$.*

We also have the critical version of Proposition 4.9 and its proof is similar.

Proposition 6.9. *Let $\varepsilon_n\to0^+$ and $((u_n,v_n))\subset\widetilde{\mathcal{N}}_{\varepsilon_n}$ be such that $\widetilde{J}_{\varepsilon_n}(u_n,v_n)\to\widetilde{c}_0$. Then there exists a sequence $(\widetilde{y}_n)\subset\mathbb{R}^N$ such that*

$$(w_n(x),z_n(x)):=(u_n(x+\widetilde{y}_n),v_n(x+\widetilde{y}_n))$$

has a convergent subsequence in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. Moreover, up to a subsequence, $y_n\to y\in M$, where $y_n=\varepsilon_n\widetilde{y}_n$.

The proof of the next result is in the same spirit of [18, Lemma 4.14]. We omit the details.

Lemma 6.10. *The minimax level* $\widetilde{c}_\varepsilon$ *satisfies* $\limsup_{\varepsilon\to 0^+} \widetilde{c}_\varepsilon \leq \widetilde{c}_0$.

Theorem 6.11. *Suppose that* a *and* b *are continuous potentials and satisfy* (ab_1) *and* (ab_2). *Suppose also* $(A_1)-(A_3)$. *Then,*

(i) *There exists* $\varepsilon_1 > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_1)$ *the system* $(CS_{\varepsilon,aux})$ *has a positive ground state solution.*

(ii) *For any* $\delta > 0$ *there exists* $\varepsilon_\delta > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_\delta)$, *the system* $(CS_{\varepsilon,aux})$ *has at least* $\operatorname{cat}_{M_\delta}(M)$ *positive solutions.*

Proof. The demonstration of item (i) follows from the fact that $\widetilde{J}_\varepsilon$ satisfies the mountain Pass Geometry, Lemma 6.5, Mountain Pass Theorem [5], the characterization of minimax level c_ε given in (36) and Lemma 6.10.

Now we prove the item (ii). As in the Section 4, we set $\delta > 0$ such that $M_\delta \subset \Lambda$ and $\psi \in C^\infty(\mathbb{R}^+, [0,1])$ a non-increasing function such that $\psi(s) = 1$ if $0 \leq s \leq \delta/2$ and $\psi(s) = 0$ if $s \geq \delta$. Let $(\widetilde{w}_1, \widetilde{w}_2) \in E_0$ be a solution of (CS_0) given by Lemma 6.4 and define, for any $y \in M$

$$\widetilde{\Psi}_{i,\varepsilon,y}(x) := \psi(|\varepsilon x - y|)\widetilde{w}_i\left(\frac{\varepsilon x - y}{\varepsilon}\right), \quad i = 1, 2.$$

We introduce the map $\widetilde{\Phi}_\varepsilon : M \to \widetilde{\mathcal{N}}_\varepsilon$ by setting

$$\widetilde{\Phi}_\varepsilon(y) := \widetilde{t}_\varepsilon(\widetilde{\Psi}_{1,\varepsilon,y}, \widetilde{\Psi}_{2,\varepsilon,y}),$$

where $\widetilde{t}_\varepsilon$ is the unique positive number satisfying

$$\widetilde{J}_\varepsilon(\widetilde{t}_\varepsilon(\widetilde{\Psi}_{1,\varepsilon,y}, \widetilde{\Psi}_{2,\varepsilon,y})) = \max_{t\geq 0} \widetilde{J}_\varepsilon(t(\widetilde{\Psi}_{1,\varepsilon,y}, \widetilde{\Psi}_{2,\varepsilon,y})).$$

We conclude: $\lim_{\varepsilon\to 0^+} \widetilde{J}_\varepsilon(\widetilde{\Phi}_\varepsilon(y)) = \widetilde{c}_0$.

Let $\Upsilon : \mathbb{R}^N \to \mathbb{R}^N$ be a function defined in Section 4 and consider the barycenter map $\widetilde{\beta}_\varepsilon : \widetilde{\mathcal{N}}_\varepsilon \to \mathbb{R}^N$ given by

$$\widetilde{\beta}_\varepsilon(u, v) := \frac{\displaystyle\int_{\mathbb{R}^N} \Upsilon(\varepsilon x)\left(|u(x)|^2 + |v(x)|^2\right) dx}{\displaystyle\int_{\mathbb{R}^N} \left(|u(x)|^2 + |v(x)|^2\right) dx}.$$

As before we can check that

$$\lim_{\varepsilon\to 0^+} \widetilde{\beta}_\varepsilon(\widetilde{\Phi}_\varepsilon(y)) = y \quad \text{uniformly for } y \in M$$

and
$$\lim_{\varepsilon\to 0^+} \sup_{(u,v)\in\widetilde{\Sigma}_\varepsilon} \operatorname{dist}\left(\widetilde{\beta}_\varepsilon(u, v), M_\delta\right) = 0,$$

where
$$\widetilde{\Sigma}_\varepsilon := \left\{(u, v) \in \widetilde{\mathcal{N}}_\varepsilon : \widetilde{J}_\varepsilon(u, v) \leq \widetilde{c}_0 + \widetilde{h}(\varepsilon)\right\},$$

and $\widetilde{h} : [0, \infty) \to [0, \infty)$ satisfies $\widetilde{h}(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.

The above equations provide $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the diagram

$$M \xrightarrow{\widetilde{\Phi}_\varepsilon} \widetilde{\Sigma}_\varepsilon \xrightarrow{\widetilde{\beta}_\varepsilon} M_\delta$$

is well defined and $\widetilde{\beta}_\varepsilon \circ \widetilde{\Phi}_\varepsilon$ is homotopically equivalent to the embedding $\iota : M \to M_\delta$. Hence we conclude that $\mathrm{cat}_{\widetilde{\Sigma}_\varepsilon}(\widetilde{\Sigma}_\varepsilon) \geq \mathrm{cat}_{M_\delta}(M)$. It follows from Proposition 6.8 and Theorem 4.3 that $\widetilde{J}_\varepsilon$ possesses at least $\mathrm{cat}_{M_\delta}(M)$ critical points on $\widetilde{\mathcal{N}}_\varepsilon$. The same argument employed in the proof of Proposition 6.8 shows that each of these critical points is also a critical point of the unconstrained functional $\widetilde{J}_\varepsilon$. Thus, we obtain $\mathrm{cat}_{M_\delta}(M)$ nontrivial solutions for $(CS_{\varepsilon,aux})$. □

6.1. Proof of Theorem 1.2

Proof. By using [18, Lemma 5.1] and repeating the same arguments than Theorem 1.1 we get the result. □

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