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Extreme Points of Convex Sets

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Given a nonempty set $E \subset \mathbb{R}^n$, we provide necessary and sufficient conditions for the existence of a convex set $K \subset \mathbb{R}^n$ (possibly, nonclosed and unbounded) such that $\operatorname{ext} K = E$. Also, we describe a family of convex sets $K \subset \mathbb{R}^n$ satisfying the equality $K = \operatorname{conv}(\operatorname{ext} K)$, and, more general, $K = \operatorname{conv}(\operatorname{ext} K) + \operatorname{rec} K$, where $\operatorname{rec} K$ denotes the recession cone of K.

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1. Introduction

In a standard way, a point x of a convex set K in the Euclidean space $\mathbb{R}^n$ is called *extreme* provided the equality $x = (1-\lambda)y + \lambda z$, where $0 < \lambda < 1$ and $y, z \in K$, holds if and only if $x = y = z$. The sets of all extreme points of K is denoted $\operatorname{ext} K$. Extreme points serve as an important tool in the study of *closed* convex sets, and their properties are extensively studied in the literature (see, e. g., the survey [10] and further references on it). Essentially less is known about the extreme structure of *nonclosed* convex sets. Our paper partly covers this gap. We discuss below the following problems.

(P1) Given a set $E \subset \mathbb{R}^n$, find necessary and sufficient conditions for the existence of a convex set $K \subset \mathbb{R}^n$ of certain type (bounded, compact, unbounded, or closed) such that $\operatorname{ext} K = E$.

(P2) Describe those convex sets $K \subset \mathbb{R}^n$ which can be expressed as

$$K = \operatorname{conv}(\operatorname{ext} K) \tag{1}$$

and, more general, as
$$K = \operatorname{conv}(\operatorname{ext} K) + \operatorname{rec} K, \tag{2}$$

where $\operatorname{rec} K$ denotes the recession cone of K.

For the case of bounded or compact convex sets, some known results related to the problem (P1) are contained in the papers [1, 4]. Our Theorems 2.4 and 3.3 solve this problem for arbitrary convex sets (bounded or unbounded). Within the class of closed convex sets, Theorems 2.6 and 3.7 give a solution to this problem for compact and line-free closed convex sets.

The problem (P2) is well-studied for the case of closed convex sets. It is proved in [9, pp. 157–161] that a compact convex set $K \subset \mathbb{R}^n$ is the convex hull of its extreme points. If K is closed and unbounded, then $K = \operatorname{conv}(\operatorname{ext} K)$ if and only if K is

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line-free and has no extreme halflines (see [5] and [7]). As shown in [3, Section 2.5], a line-free closed convex set $K \subset \mathbb{R}^n$ satisfies the equality (2).

For the case of arbitrary bounded convex sets, the problem (P2) is solved in [12]: a bounded convex set $K \subset \mathbb{R}^n$ satisfies the equality (1) if and only if we have $\operatorname{ext}(\operatorname{cl} F) \subset \operatorname{cl}(\operatorname{ext} F)$ for every extreme face F of K. Our Theorem 4.5 expands this assertion in the following way: it describes all line-free convex sets $K \subset \mathbb{R}^n$ which satisfy the equality (2) and have the property $\operatorname{rec} K = \operatorname{rec}(\operatorname{cl} K)$. In particular, this description holds for the case of weakly Motzkin decomposable sets (see Section 4 for details).

We conclude this section with necessary definitions, notation, and results on convex sets in the n-dimensional Euclidean space $\mathbb{R}^n$ (see, e. g., [11] for details). The elements of $\mathbb{R}^n$ are called vectors (or points). Also, o stands for the zero vector of $\mathbb{R}^n$, and uv means the dot product of vectors u and v. Given distinct points $u, v \in \mathbb{R}^n$, we denote by $[u, v]$ and (u, v) the closed segment and the open segment with endpoints u and v. Also, $[u, v\rangle$ denotes the closed halfline through v with endpoint u.

As usual, $\operatorname{cl} X$ and $\operatorname{int} X$ stand for the closure and interior of a set $X \subset \mathbb{R}^n$, and $U_\rho(u)$ denotes the open ball with center $u \in \mathbb{R}^n$ and radius $\rho > 0$. The convex hull and affine span of X are denoted by $\operatorname{conv} X$ and $\operatorname{aff} X$, respectively, and $\dim X$ is defined as the dimension of the plane $\operatorname{aff} X$. The *direction space* of X, denoted $\operatorname{dir} X$, is the subspace of $\mathbb{R}^n$ which is a translate of $\operatorname{aff} X$. One has $\operatorname{dir} X = \operatorname{aff} X - u$ for any point $u \in X$.

A nonempty convex set $C \subset \mathbb{R}^n$ is called a *cone* (with apex o) provided $\lambda x \in C$ whenever $\lambda \geq 0$ and $x \in C$. Obviously, this definition implies the inclusion $o \in C$ (although a stronger condition $\lambda > 0$ can be beneficial; see, e. g., [6]). The cone C is called *pointed* if $C \cap (-C) = \{o\}$. It is known that a closed convex cone in $\mathbb{R}^n$ is pointed if and only if it is line-free (that is, contains no line). In what follows, a *ray* means a closed halfline originated at o.

The *recession cone* of a nonempty convex set $K \subset \mathbb{R}^n$ is defined by

$$\operatorname{rec} K = \{e \in \mathbb{R}^n : x + \lambda e \in K \text{ whenever } x \in K \text{ and } \lambda \geq 0\}.$$

One has $\operatorname{rec} K \subset \operatorname{rec}(\operatorname{cl} K) \subset \operatorname{dir} K$. Given a point $u \in \operatorname{rint} K$, the cone $\operatorname{rec}(\operatorname{cl} K)$ is the largest among all closed convex cones $C \subset \mathbb{R}^n$ satisfying the inclusion $u + C \subset \operatorname{rint} K$. In particular, the recession cone of a closed convex set is closed.

We recall that a convex subset F of a convex set $K \subset \mathbb{R}^n$ is an *extreme face* of K if either $F = \varnothing$, or $F \neq \varnothing$ and, for any pair of points $u, v \in K$, the condition $F \cap (u, v) \neq \varnothing$ implies that $u, v \in F$. Thus extreme points of K are its 0-dimensional extreme faces.

Lemma 1.1. ([11], Chapter 11) *For a convex set $K \subset \mathbb{R}^n$, the following conditions are equivalent.*

(a) *A convex subset F of K is its extreme face if and only if $K \cap \operatorname{aff} F = F$ and $K \setminus F$ is convex.*

(b) *If F is a nonempty extreme face of K and a convex subset G of K satisfies the condition $F \cap \operatorname{rint} G \neq \varnothing$, then $G \subset F$.* □

2. Bounded convex sets

Lemma 2.1. *For a nonempty set* $E \subset \mathbb{R}^n$ *and a point* $x \in \operatorname{conv} E$*, the following conditions are equivalent.*

(a) $x \in \operatorname{ext}(\operatorname{conv} E)$.

(b) *The set* $\operatorname{conv} E \setminus \{x\}$ *is convex.*

(c) $x \notin \operatorname{conv}(E \setminus \{x\})$.

Furthermore, each of the conditions (a)–(c) *implies the inclusion* $x \in E$.

Proof. (a) ⇒ (b) The inclusion $x \in \operatorname{ext}(\operatorname{conv} E)$ means that the singleton $\{x\}$ is an extreme face of $\operatorname{conv} E$. Lemma 1.1 implies that the set $\operatorname{conv} E \setminus \{x\}$ is convex.

(b) ⇒ (c) If the set $\operatorname{conv} E \setminus \{x\}$ is convex, then the obvious inclusion $E \setminus \{x\} \subset \operatorname{conv} E \setminus \{x\}$ gives

$$\operatorname{conv}(E \setminus \{x\}) \subset \operatorname{conv}(\operatorname{conv} E \setminus \{x\}) = \operatorname{conv} E \setminus \{x\}.$$

Since $x \notin \operatorname{conv} E \setminus \{x\}$, it follows that $x \notin \operatorname{conv}(E \setminus \{x\})$.

(c) ⇒ (a) Let $x \notin \operatorname{conv}(E \setminus \{x\})$. Assume for a moment that $x \notin \operatorname{ext}(\operatorname{conv} E)$. Then $x = (1-\lambda)u + \lambda v$ for suitable points u and v from $\operatorname{conv} E \setminus \{x\}$ and a scalar $0 < \lambda < 1$. Carathéodory's theorem (see [2] and [11], Theorem 4.3) implies that u and v can be written as positive convex combinations of points from E:

$$u = \mu_1 u_1 + \cdots + \mu_p u_p \quad \text{and} \quad v = \gamma_1 v_1 + \cdots + \gamma_q v_q.$$

Consequently, x is a positive convex combination,

$$x = (1-\lambda)\mu_1 u_1 + \cdots + (1-\lambda)\mu_p u_p + \lambda\gamma_1 v_1 + \cdots + \lambda\gamma_q v_q, \tag{3}$$

of points from E. Since $u \neq x \neq v$, at least of the points $u_1, \ldots, u_p, v_1, \ldots, v_q$ is not x. Combining like terms in the right-hand side of (3), we express x as a positive convex combination $x = \lambda_1 z_1 + \cdots + \lambda_r z_r$ of pairwise distinct points

$$z_1, \ldots, z_r \in \{u_1, \ldots, u_p, v_1, \ldots, v_q\} \subset E$$

such that at least one of $z_1, \ldots, z_r$ is not x.

If none of $z_1, \ldots, z_r$ were x, then $\{z_1, \ldots, z_r\} \subset E \setminus \{x\}$, implying that

$$x \in \operatorname{conv}\{z_1, \ldots, z_r\} \subset \operatorname{conv}(E \setminus \{x\}),$$

in contradiction with the choice of x. Hence one of the points $z_1, \ldots, z_r$, say z_1, should be x. Clearly we have $0 < \lambda_1 < 1$ due to $r \geq 2$. Rewriting the equality $x = \lambda_1 x + \lambda_2 z_2 + \cdots + \lambda_r z_r$ as

$$x = \frac{\lambda_2}{1-\lambda_1} z_2 + \cdots + \frac{\lambda_r}{1-\lambda_1} z_r,$$

we obtain that x is a positive convex combination of $z_2, \ldots, z_r \in E \setminus \{x\}$, again implying the impossible inclusion

$$x \in \operatorname{conv}\{z_2, \ldots, z_r\} \subset \operatorname{conv}(E \setminus \{x\}).$$

Summing up, $x \in \operatorname{ext}(\operatorname{conv} E)$.

For the last assertion, it suffices to show that condition (a) implies the inclusion $x \in E$. Indeed, assume for a moment that we have $x \notin E$. Since $x \in \operatorname{conv} E$, Carathéodory's theorem implies that x can be expressed as a positive convex combination $x = \mu_1 u_1 + \mu_2 u_2 + \cdots + \mu_p u_p$ of suitable points $u_1, \ldots, u_p \in E$. Thus x belongs to the relative interior of the convex set $G = \operatorname{conv}\{u_1, \ldots, u_p\} \subset \operatorname{conv} E$. This argument and Lemma 1.1 show that every extreme face of $\operatorname{conv} E$ which contains x should also contain G. Therefore, the singleton $\{x\}$ is not an extreme face of $\operatorname{conv} E$. Equivalently, $x \notin \operatorname{ext}(\operatorname{conv} E)$. □

Definition 2.2. We recall that a nonempty set $E \subset \mathbb{R}^n$ is *convexly independent* provided $x \notin \operatorname{conv}(E \setminus \{x\})$ for all $x \in E$. The empty set is assumed convexly independent.

Lemma 2.1 implies the following corollary.

Corollary 2.3. *For any set $E \subset \mathbb{R}^n$, one has $\operatorname{ext}(\operatorname{conv} E) \subset E$. Furthermore, $\operatorname{ext}(\operatorname{conv} E) = E$ if and only if E is convexly independent.* □

Theorem 2.4. *The following assertions hold.*

(a) *For a convex set $K \subset \mathbb{R}^n$, the set $\operatorname{ext} K$ is convexly independent.*

(b) *If $E \subset \mathbb{R}^n$ is a (bounded) convexly independent set, then the (bounded) convex set $K = \operatorname{conv} E$ satisfies the condition $\operatorname{ext} K = E$.*

Proof. (a) Since the case $\operatorname{ext} K = \varnothing$ is trivial, we may assume that $\operatorname{ext} K \neq \varnothing$. Choose a point $x \in \operatorname{ext} K$. Then $\operatorname{ext} K \setminus \{x\} \subset K \setminus \{x\}$, and Lemma 1.1 (with $F = \{x\}$) gives

$$\operatorname{conv}(\operatorname{ext} K \setminus \{x\}) \subset \operatorname{conv}(K \setminus \{x\}) = K \setminus \{x\}.$$

Since $x \notin K \setminus \{x\}$, it follows that $x \notin \operatorname{conv}(\operatorname{ext} K \setminus \{x\})$. Hence $\operatorname{ext} K$ is convexly independent.

(b) This part immediately follows from Corollary 2.3. Furthermore, K is bounded if and only if E is bounded. □

We observe that a convexly independent set $E \subset \mathbb{R}^n$ may be the extreme set of distinct convex sets. For instance, given the square $S = \{(x, y) : -1 \leq x, y \leq 1\}$ and the open disk $D = \{(x, y) : x^2 + y^2 < 2\}$, the set $D \cup S$ is convex and distinct from S, while $\operatorname{ext}(D \cup S) = \operatorname{ext} S$.

If a convexly independent set $E \subset \mathbb{R}^n$ is unbounded, then any convex set $K \subset \mathbb{R}^n$, with $\operatorname{ext} K = E$, must be unbounded as well. In view of part (b) of Theorem 2.4, the open problem here is whether, given a *bounded* convexly independent set E, there is an *unbounded* convex set K, with $\operatorname{ext} K = E$. For instance, if E is the vertex-set of a regular hexagon in $\mathbb{R}^2$, then an easy geometric argument shows that any convex set $K \subset \mathbb{R}^2$, with $\operatorname{ext} K = E$, should be bounded. On the other hand, if E is the vertex-set of a triangle in $\mathbb{R}^2$, then there is a variety of distinct unbounded convex sets $K \subset \mathbb{R}^2$, with $\operatorname{ext} K = E$. Our Theorems 3.3 and 3.7 are related to this problem.

Lemma 2.5. ([11], Theorem 4.16) *Assume $X \subset \mathbb{R}^n$. If X is bounded one has* $\operatorname{conv}(\operatorname{cl} X) \subset \operatorname{cl}(\operatorname{conv} X)$, *with* $\operatorname{conv}(\operatorname{cl} X) = \operatorname{cl}(\operatorname{conv} X)$. *Consequently,* $\operatorname{conv} X$ *is compact if X is compact.* □

Theorem 2.6. *The following assertions hold.*

(a) *For a (nonempty) compact convex set $K \subset \mathbb{R}^n$, the set* $\operatorname{ext} K$ *is (nonempty) bounded, and convexly independent, with* $\operatorname{cl}(\operatorname{ext} K) \subset \operatorname{conv}(\operatorname{ext} K)$.

(b) *Let a set $E \subset \mathbb{R}^n$ be bounded and convexly independent, with* $\operatorname{cl} E \subset \operatorname{conv} E$. *Then $K = \operatorname{conv} E$ is the unique compact convex set such that* $\operatorname{ext} K = E$.

Proof. (a) Let $K \subset \mathbb{R}^n$ be a compact convex set. If K is nonempty, then $\operatorname{ext} K \neq \varnothing$, as follows from [9, pp. 157–161]. By Theorem 2.4, $\operatorname{ext} K$ is convexly independent. Finally, $\operatorname{cl}(\operatorname{ext} K) \subset K = \operatorname{conv}(\operatorname{ext} K)$.

(b) By Theorem 2.4, the convex set $K = \operatorname{conv} E$ is bounded and $\operatorname{ext} K = E$. Furthermore, Lemma 2.5 gives

$$\operatorname{cl} K = \operatorname{cl}(\operatorname{conv} E) = \operatorname{conv}(\operatorname{cl} E) \subset \operatorname{conv}(\operatorname{conv} E) = \operatorname{conv} E = K.$$

Hence K is closed, and thus it is compact.

The uniqueness of K is obvious if $E = \varnothing$. Let $E \neq \varnothing$. Then $K = \operatorname{conv} E \neq \varnothing$. Suppose that $\operatorname{ext} M = E$ for a suitable compact convex set $M \subset \mathbb{R}^n$. Then $E \subset M$, implying that $K = \operatorname{conv} E \subset M$. For the opposite inclusion, assume, for contradiction, the existence of a point $z \in M \setminus K$. Choose a hyperplane $H \subset \mathbb{R}^n$ strictly separating z from K and express it as $H = \{x \in \mathbb{R}^n : x \cdot e = \gamma\}$, where e is a nonzero vector. Without loss of generality we may suppose that K is contained in the open halfspace $W = \{x \in \mathbb{R}^n : x \cdot e < \gamma\}$. Consequently, $z \cdot e > \gamma$. Let

$$\gamma' = \max\{x \cdot e : x \in M\} \quad \text{and} \quad H' = \{x \in \mathbb{R}^n : x \cdot e = \gamma'\}.$$

Clearly, H' supports M and is disjoint from K. The nonempty compact convex set $H' \cap M$ has and extreme point u, which also belongs to $\operatorname{ext} M = E$. On the other hand, $u \notin K$ and thus $u \notin E$, a contradiction. Summing up, $M \subset K$, as desired. □

An assertion, similar to part (b) of Theorem 2.6, was proved in [1]): E is the set of extreme points of a suitable compact convex set in $\mathbb{R}^n$ if and only if $\operatorname{cl} E$ is compact, $\operatorname{cl} E \subset \operatorname{conv} E$, and $E \cap \operatorname{conv} A \subset A$ for all $A \subset E$.

3. Unbounded convex sets

Our next result, Theorem 3.3, allows us to specify whether a convex set $K \subset \mathbb{R}^n$, with a given set $E = \operatorname{ext} K$, may have nonzero recession cone. The key ingredient here is the concept of *cone acceptability*, as defined below.

Definition 3.1. Given a convex cone $C \subset \mathbb{R}^n$, we will say that a nonempty set $E \subset \mathbb{R}^n$ is *C-acceptable* provided $(x - C) \cap \operatorname{conv} E = \{x\}$ for all $x \in E$. The empty set is assumed C-acceptable.

Clearly, our concept of cone acceptability is similar in the spirit to those of cone admissibility, cone efficiency, cone dominance, etc, widely used in the literature on Pareto optimality theory. Obviously, any set $E \subset \mathbb{R}^n$ is o-acceptable.

Lemma 3.2. *Given a nonempty set $E \subset \mathbb{R}^n$ and a convex cone $C \subset \mathbb{R}^n$, let $K = \operatorname{conv} E + C$. Then $\operatorname{ext} K \subset E$ and $C \subset \operatorname{rec} K$.*

Proof. For any convex sets K and M in $\mathbb{R}^n$, one has $\operatorname{ext}(K+M) \subset \operatorname{ext} K + \operatorname{ext} M$ (see [11], Theorem 11.17). Since $\operatorname{ext} C \subset \{o\}$, the above assertion and Corollary 2.3 give

$$\operatorname{ext} K = \operatorname{ext}(\operatorname{conv} E + C) \subset \operatorname{ext}(\operatorname{conv} E) + \operatorname{ext} C \subset E + \{o\} = E.$$

For the second inclusion, $C \subset \operatorname{rec} K$, it suffices to show (according to the definition of $\operatorname{rec} K$) that $x + C \subset K$ whenever $x \in K$. Indeed, let $x = y + z$, where $y \in \operatorname{conv} E$ and $z \in C$. Since $z + C \subset C$, we have

$$x + C = y + (z + C) \subset y + C \subset \operatorname{conv} E + C = K.$$

Theorem 3.3. *The following assertions hold.*

(a) *For any convex set $K \subset \mathbb{R}^n$, the set $\operatorname{ext} K$ is convexly independent and $\operatorname{rec} K$-acceptable. If, additionally, $\operatorname{ext} K \neq \varnothing$, then the cone $\operatorname{rec} K$ is pointed.*

(b) *Let a nonempty set $E \subset \mathbb{R}^n$ be convexly independent and C-acceptable, where $C \subset \mathbb{R}^n$ is a pointed convex cone. Then the convex set $K = \operatorname{conv} E + C$ satisfies the conditions $\operatorname{ext} K = E$ and $C \subset \operatorname{rec} K$.*

Proof. (a) The convex independence of $\operatorname{ext} K$ is proved in Theorem 2.4. Assume for a moment that $\operatorname{ext} K$ is not $\operatorname{rec} K$-acceptable. Then there is a point $x \in \operatorname{ext} K$ such that $(x - \operatorname{rec} K) \cap \operatorname{conv}(\operatorname{ext} K) \neq \{x\}$. Choose a point

$$z \in (x - \operatorname{rec} K) \cap \operatorname{conv}(\operatorname{ext} K) \setminus \{x\}.$$

Consequently, $z \in \operatorname{conv}(\operatorname{ext} K) \setminus \{x\} \subset K$. Furthermore,

$$z \in x - \operatorname{rec} K \quad \Leftrightarrow \quad -z \in \operatorname{rec} K - x \quad \Leftrightarrow \quad 2x - z \in x + \operatorname{rec} K.$$

Since $x + \operatorname{rec} K \subset K$, one has $2x - z \in K$. Thus

$$x = \tfrac{1}{2}(z + (2x - z)) \in (z, 2x - z) \subset K, \quad \text{where} \quad z \neq 2x - z,$$

contrary to the choice $x \in \operatorname{ext} K$. Hence $\operatorname{ext} K$ must be $\operatorname{rec} K$-acceptable.

Next, suppose that $\operatorname{ext} K \neq \varnothing$ and choose some point $x \in \operatorname{ext} K$. Let us assume for a moment that the cone $\operatorname{rec} K$ is not pointed. Then there is a nonzero vector $e \in \operatorname{rec} K \cap (-\operatorname{rec} K)$. Consequently, both points $x_1 = x + e$ and $x_2 = x - e$ belong to $x + \operatorname{rec} K \subset K$ and $x \in (x_1, x_2)$, contrary to the choice $x \in \operatorname{ext} K$. Hence the cone $\operatorname{rec} K$ must be pointed.

(b) Due to Lemma 3.2, it suffices to show that $E \subset \operatorname{ext} K$. Choose any point $x \in E$ and suppose that $x = (1-\lambda)x_1 + \lambda x_2$, where $x_1, x_2 \in K$ and $0 < \lambda < 1$. The equality $K = \operatorname{conv} E + C$ allows us to write $x_i = y_i + z_i$, where $y_i \in \operatorname{conv} E$ and $z_i \in C$, $i = 1, 2$. So,

$$x = ((1-\lambda)y_1 + \lambda y_2) + ((1-\lambda)z_1 + \lambda z_2). \tag{4}$$

By a convexity argument, $x = y + z$, where

$$y = (1-\lambda)y_1 + \lambda y_2 \in \operatorname{conv} E \quad \text{and} \quad z = (1-\lambda)z_1 + \lambda z_2 \in C.$$

Since $y = x - z \in x - C$, one has $y \in (x - C) \cap \operatorname{conv} E$, and the C-admissibility of E gives $y = x$. Thus $x = y = (1-\lambda)y_1 + \lambda y_2$, where $y_1, y_2 \in \operatorname{conv} E$. The inclusion $x \in E = \operatorname{ext}(\operatorname{conv} E)$ (see Corollary 2.3) gives $x = y_1 = y_2$. Consequently, (4) implies that $x = x + z$, or $z = o$. Because the cone C is pointed, the equality $o = (1-\lambda)z_1 + \lambda z_2$ is possible only if $z_1 = z_2 = o$. Summing up, $x_1 = x = x_2$, as desired. Hence, $x \in \operatorname{ext} K$. □

The example below shows that the inclusion $C \subset \operatorname{rec} K$ in Theorem 3.3 may be proper.

Example 3.4. Consider the convex set $K = \{(x, y) : x > 0,\ y \geq 1/x\}$. Clearly, $\operatorname{ext} K$ is the boundary of K and $\operatorname{rec} K$ is the negative quadrant of $\mathbb{R}^2$. Furthermore, $K = \operatorname{conv}(\operatorname{ext} K)$. If C is a ray contained in $\operatorname{rec} K$, then $\operatorname{ext} K$ is C-acceptable and $K = \operatorname{conv}(\operatorname{ext} K) + C$. □

We observe that Theorem 3.3 trivially holds in the case $C = \{o\}$. Nevertheless, the following example shows that in this case the convex set $K = \operatorname{conv} E$ still may be unbounded.

Example 3.5. Let $B = \{(x, y, 0) : x^2 + y^2 = 1\}$ be the unit circle in the coordinate xy-plane of $\mathbb{R}^3$. Choose a countably infinite set X of points $u_i = (a_i, b_i, 0) \in B$, $i \geq 1$, which is dense in B, and put $v_i = (a_i, b_i, i)$, $i \geq 1$. The set $E = X \cup \{v_1, v_2, \dots\}$ is unbounded and convexly independent. It is easy to see that the convex set $K = \operatorname{conv} E$ is the union of the one-way unbounded cylinder

$$Q = \{(x, y, z) : x^2 + y^2 < 1,\ z \geq 0\}$$

and all closed segments $[u_i, v_i]$, $i \geq 1$. By Theorem 2.4, $\operatorname{ext} K = E$. Furthermore, the trivial cone $\{o\}$ is the only convex cone $C \subset \mathbb{R}^3$ for which E is C-admissible. Indeed, assume the existence of a nonzero convex cone $C \subset \mathbb{R}^3$ such that E is C-admissible. An obvious argument shows that C should be the negative part of the z-axis. Consequently,

$$(v_i - C) \cap \operatorname{conv} E = [u_i, v_i] \neq \{v_i\}, \quad i \geq 1,$$

contrary to Definition 3.1. □

The next example shows that Theorem 2.6 cannot be routinely expanded to the case of unbounded closed convex sets in $\mathbb{R}^n$. Namely, the condition "E is convexly independent, with $\operatorname{cl} E \subset \operatorname{conv} E$," is necessary but not sufficient for the existence of a closed convex set $K \subset \mathbb{R}^n$, with $\operatorname{ext} K = E$. (Observe that the set $\operatorname{ext} K$ is Example 3.5 is not closed.)

Example 3.6. Let $B = \{(x, y, 0) : x^2 + y^2 = 1\}$ be the unit circle in the coordinate xy-plane of $\mathbb{R}^3$. Choose a countably infinite set of points $(a_i, b_i, 0) \in B$, $i \geq 1$, which is dense in B, and put $u_i = (a_i, b_i, (-1)^i i)$. The set $E = \{u_1, u_2, \dots\}$ is unbounded, closed, and convexly independent, and the set $K = \operatorname{conv} E$ is unbounded and nonclosed, with $\operatorname{ext} K = E$. Clearly, $K = E \cup U$, where U is the both-way infinite open cylinder $\{(x, y, z) : x^2 + y^2 < 1\}$. It is easy to see that $\operatorname{cl} K = \{(x, y, z) : x^2 + y^2 \leq 1\}$ is the only closed convex set containing E in its boundary. Furthermore, $\operatorname{ext}(\operatorname{cl} K) = \varnothing$. Hence E cannot be the extreme set of any closed convex set in $\mathbb{R}^3$. □

Theorem 3.7. *The following assertions hold.*

(a) *For a nonempty line-free closed convex set $K \subset \mathbb{R}^n$, the set* $\operatorname{ext} K$ *is nonempty and convexly independent, with* $\operatorname{cl}(\operatorname{ext} K) \subset \operatorname{conv}(\operatorname{ext} K) + \operatorname{rec} K$.

(b) *Let a nonempty set $E \subset \mathbb{R}^n$ be convexly independent and C-acceptable, where $C \subset \mathbb{R}^n$ is a line-free closed convex cone. If E is bounded and if we have* $\operatorname{cl} E \subset \operatorname{conv} E + C$, *then the set* $K = \operatorname{conv} E + C$ *is a line-free closed convex set such that* $\operatorname{ext} K = E$ *and* $C = \operatorname{rec} K$.

Proof. (a) Let $K \subset \mathbb{R}^n$ be a nonempty line-free closed convex set. Then we have $K = \operatorname{conv}(\operatorname{ext} K) + \operatorname{rec} K$, according to [3, Section 2.5]. By Theorem 2.4, $\operatorname{ext} K$ is convexly independent. Finally,

$$\operatorname{cl}(\operatorname{ext} K) \subset K = \operatorname{conv}(\operatorname{ext} K) + \operatorname{rec} K.$$

(b) Because the line-free closed convex cone C is pointed, Theorem 3.3 shows that $\operatorname{ext} K = E$ and $C \subset \operatorname{rec} K$. The set E is bounded; so, its convex hull $\operatorname{conv} E$ is also bounded. Consequently, K is line-free as the sum of a bounded set $\operatorname{conv} E$ and a line-free closed convex cone C (see [11], Theorem 7.22). Finally, Lemma 2.5 and the assumption $\operatorname{cl} E \subset \operatorname{conv} E + C$ give

$$\begin{aligned}\operatorname{cl} K &= \operatorname{cl}(\operatorname{conv} E + C) \subset \operatorname{cl}(\operatorname{conv} E) + C = \operatorname{conv}(\operatorname{cl} E) + C \\ &\subset \operatorname{conv}(\operatorname{conv} E + C) + C = \operatorname{conv}(\operatorname{conv} E) + C + C \\ &= \operatorname{conv} E + C = K.\end{aligned}$$

Hence the set K is closed, as desired. The equality $C = \operatorname{rec} K$ is proved in [8], Proposition 10. ☐

The example below shows that the conclusion $\operatorname{cl}(\operatorname{ext} K) \subset \operatorname{conv}(\operatorname{ext} K) + \operatorname{rec} K$ in part (a) of Theorem 3.7 cannot be sharpened by stating that $\operatorname{cl}(\operatorname{ext} K) \subset \operatorname{conv}(\operatorname{ext} K)$.

Example 3.8. Consider the convex set $K \subset \mathbb{R}^3$ given by

$$K = \operatorname{conv}(\{a, b\} \cup \{(x, y, z) : x \geq 0,\ x^2 + y^2 \leq 1,\ z \geq 1\}),$$

where $a = (0, 1, 0)$ and $b = (0, -1, 0)$. It is easy to see that K is closed and $\operatorname{ext} K$ is the union of $\{a, b\}$ and the open arc $\{(x, y, 1) : x > 0,\ x^2 + y^2 = 1\}$. Furthermore, the points $u = (0, 1, 1)$ and $v = (0, -1, 1)$ belong to $\operatorname{cl}(\operatorname{ext} K)$ but not to $\operatorname{conv}(\operatorname{ext} K)$. Clearly, $\operatorname{rec} K$ is the negative part of the z-axis and $u, v \in \{a, b\} + \operatorname{rec} K$. ☐

4. Sets satisfying condition (2)

In this section, we describe a class of line-free convex sets $K \subset \mathbb{R}^n$ satisfying the condition (2). We will need some auxiliary lemmas.

Lemma 4.1. *If $K \subset \mathbb{R}^n$ is a nonempty convex set and F is a nonempty extreme face of K, then* $\operatorname{rec} K \cap \operatorname{dir} F \subset \operatorname{rec} F$. *If, additionally,* $\operatorname{rec} K = \operatorname{rec}(\operatorname{cl} K)$, *then*

$$\operatorname{rec} K \cap \operatorname{dir} F = \operatorname{rec} F = \operatorname{rec}(\operatorname{cl} F).$$

Proof. Choose any vector $e \in \operatorname{rec} K \cap \operatorname{dir} F$ and a point $x \in F$. For any $\lambda \geq 0$,

$$x + \lambda e \in x + \operatorname{rec} K \subset K \quad \text{and} \quad x + \lambda e \in x + \operatorname{dir} F = \operatorname{aff} F.$$

By Lemma 1.1, $x + \lambda e \in K \cap \operatorname{aff} F = F$. Consequently, $e \in \operatorname{rec} F$.

Suppose, additionally, that $\operatorname{rec} K = \operatorname{rec}(\operatorname{cl} K)$. Choose a vector $e \in \operatorname{rec} F$ and a point $u \in F$. Then the halfline $[u, e + u\rangle$ is contained in F, and thus in K. Let $w \in \operatorname{rint} K$. Then $[w, e + w\rangle \subset \operatorname{rint} K$ (see [11], Theorem 6.10). Consequently, $e \in \operatorname{rec}(\operatorname{cl} K) = \operatorname{rec} K$. Since we have $\operatorname{rec} F \subset \operatorname{dir} F$, we obtain the desired inclusion $e \in \operatorname{rec} K \cap \operatorname{dir} F$. Summing up, $\operatorname{rec} F \subset \operatorname{rec} K \cap \operatorname{dir} F$.

Because $\operatorname{rec} F \subset \operatorname{rec}(\operatorname{cl} F)$, it suffices to prove the opposite inclusion. Choose a vector $e \in \operatorname{rec}(\operatorname{cl} F) \subset \operatorname{dir}(\operatorname{cl} F) = \operatorname{dir} F$ and a point $u \in \operatorname{rint} F$. As above,

$$[u, e + u\rangle \subset \operatorname{rint} F \subset F \subset K.$$

Let $w \in \operatorname{rint} K$. Then $[w, e + w\rangle \subset \operatorname{rint} K$. Hence $e \in \operatorname{rec}(\operatorname{cl} K) = \operatorname{rec} K$. By what proved above, $e \in \operatorname{rec} K \cap \operatorname{dir} F \subset \operatorname{rec} F$. Thus $\operatorname{rec}(\operatorname{cl} F) \subset \operatorname{rec} F$, as desired. $\square$

Lemma 4.2. *If a nonempty line-free convex set $K \subset \mathbb{R}^n$ satisfies the condition* (2), *then* $\operatorname{ext}(\operatorname{cl} K) \subset \operatorname{cl}(\operatorname{ext} K)$.

Proof. Assume for a moment that $\operatorname{ext}(\operatorname{cl} K) \not\subset \operatorname{cl}(\operatorname{ext} K)$. Choose some point $u \in \operatorname{ext}(\operatorname{cl} K) \setminus \operatorname{cl}(\operatorname{ext} K)$ and an open ball $U_\rho(u) \subset \mathbb{R}^n$ satisfying the condition $U_\rho(u) \cap \operatorname{cl}(\operatorname{ext} K) = \varnothing$. Since the set $\operatorname{cl} K$ is line-free, there is an open halfspace $W \subset \mathbb{R}^n$ such that $\operatorname{cl} K \cap W \subset U_\rho(u)$ (see [11], Theorem 11.19). Denote by V the complementary closed halfspace of $\mathbb{R}^n$: $V = \mathbb{R}^n \setminus W$. By the above argument,

$$(\operatorname{cl} K \setminus U_\rho(u)) \cap W = (\operatorname{cl} K \cap W) \setminus U_\rho(u) = \varnothing.$$

Hence

$$\operatorname{ext} K \subset \operatorname{cl}(\operatorname{ext} K) \subset \operatorname{cl} K \setminus U_\rho(u) \subset \mathbb{R}^n \setminus W = V.$$

Thus $\operatorname{conv}(\operatorname{ext} K) \subset K \cap V$ by the convexity of V.

Next, we assert that $v + \operatorname{rec} K \subset V$ for any point $v \in K \cap V$. Indeed, suppose that $v + \operatorname{rec} K \not\subset V$ for a suitable point $v \in K \cap V$. Then there is a vector $c \in \operatorname{rec} K$ such that the point $w = c + v$ belongs to W. It is well-known that W can be expressed as $W = \{x \in \mathbb{R}^n : x{\cdot}e < \gamma\}$, where e is a nonzero vector and $\gamma \in \mathbb{R}$. Clearly, $v{\cdot}e > \gamma$ and $w{\cdot}e < \gamma$. Therefore,

$$c{\cdot}e = (w - v){\cdot}e = w{\cdot}e - v{\cdot}e < 0.$$

For any scalar $\lambda > 1$,

$$(v + \lambda c){\cdot}e = (w + (\lambda - 1)c){\cdot}e = w{\cdot}e + (\lambda - 1)c{\cdot}e < \gamma + 0 = \gamma.$$

Thus $v + \lambda c \in W$ for all $\lambda > 1$. Furthermore,

$$\|u - (v + \lambda c)\| = \|\lambda c - (u - v)\| \geq \lambda\|c\| - \|u - v\| \to \infty \quad \text{as} \quad \lambda \to \infty.$$

Hence there is a scalar $\lambda_0 > 0$ such that $\|u - (v + \lambda_0 c)\| \geq \rho$. On the other hand, $v + \lambda_0 c \in v + \operatorname{rec} K \subset K$, in contradiction with $K \cap W \subset U_\rho(u)$.

Finally, the above argument and condition (2) give

$$K = \mathrm{conv}\,(\mathrm{ext}\,K) + \mathrm{rec}\,K \subset K \cap V + \mathrm{rec}\,K \subset V.$$

Consequently, $\mathrm{cl}\,K \subset V$, due to the closedness of V. Therefore, $\mathrm{cl}\,K \cap W = \varnothing$. On the other hand $u \in \mathrm{ext}\,(\mathrm{cl}\,K) \cap W \subset \mathrm{cl}\,K \cap W$, a contradiction. □

The following example shows that in Lemma 4.2 the inclusion $\mathrm{ext}\,(\mathrm{cl}\,K) \subset \mathrm{cl}\,(\mathrm{ext}\,K)$ does not imply the equality (2).

Example 4.3. Let $D = \{(x,y) : x^2 + y^2 \le 2, y > 0\}$ and $K = D \cup (a,b)$, where $a = (-1,0)$ and $b = (1,0)$. Clearly,

$$\mathrm{ext}\,K = \{(x,y) : x^2 + y^2 = 2, y > 0\} \ \text{ and } \ \mathrm{conv}\,(\mathrm{ext}\,K) = D \ne K.$$

On the other hand,

$$\mathrm{cl}\,K = \{(x,y) : x^2 + y^2 \le 2, y \ge 0\} \ \text{ and } \ \mathrm{ext}\,(\mathrm{cl}\,K) = \mathrm{cl}\,(\mathrm{ext}\,K).$$

In contrast, if $K' = D \cup [a,b]$, then

$$\mathrm{cl}\,K' = \mathrm{cl}\,K \ \text{ and } \ \mathrm{ext}\,K' = \mathrm{ext}\,K \cup \{a,b\}.$$

Consequently, $\mathrm{ext}\,(\mathrm{cl}\,K') \subset \mathrm{cl}\,(\mathrm{ext}\,K')$ and $K' = \mathrm{conv}\,(\mathrm{ext}\,K')$. □

Lemma 4.4. *Let a nonempty convex set $K \subset \mathbb{R}^n$ satisfy the condition (2). If F is a nonempty extreme face of K, then $F = \mathrm{conv}\,(\mathrm{ext}\,F) + \mathrm{rec}\,F$.*

Proof. Since the inclusion $\mathrm{conv}\,(\mathrm{ext}\,F) + \mathrm{rec}\,F \subset F$ is obvious, it suffices to prove the opposite one. For this, choose any point $x \in F$. Then $x \in K$ and the condition (2) implies that $x = u + e$, where $u \in \mathrm{conv}\,(\mathrm{ext}\,K)$ and $e \in \mathrm{rec}\,K$. By Carathéodory's theorem, u can be expressed as a positive convex combination of pairwise distinct points $u_1, \dots, u_r \in \mathrm{ext}\,K$. So, $u \in \mathrm{rint}\,(\mathrm{conv}\,\{u_1, \dots, u_r\})$ (see [11], Corollary 4.19). By Lemma 1.1, $\mathrm{conv}\,\{u_1, \dots, u_r\} \subset F$. In particular, $u_1, \dots, u_r \in F$. Because all these points belong to $\mathrm{ext}\,K$, they should belong to $\mathrm{ext}\,F$. Hence $u \in \mathrm{conv}\,\{u_1, \dots, u_r\} \subset \mathrm{conv}\,(\mathrm{ext}\,F)$.

Next, $u + 2e \in K$ because $2e \in \mathrm{rec}\,K$. Thus

$$x = \tfrac{1}{2}(u + (u + 2e)) \in (u, u + 2e) \subset K.$$

Since F is an extreme face, we obtain that $u, u + 2e \in F \subset \mathrm{aff}\,F$ from the above argument. Hence $2e \in \mathrm{aff}\,F - u = \mathrm{dir}\,F$. Thus $e \in \mathrm{dir}\,F$. According to Lemma 4.1, $e \in \mathrm{rec}\,K \cap \mathrm{dir}\,F \subset \mathrm{rec}\,F$.

Summing up, $x = u + e \in \mathrm{conv}\,(\mathrm{ext}\,F) + \mathrm{rec}\,F$, as desired. □

Theorem 4.5. *Let $K \subset \mathbb{R}^n$ be a nonempty line-free convex set, with the property $\mathrm{rec}\,K = \mathrm{rec}\,(\mathrm{cl}\,K)$. Then the following conditions are equivalent.*

(a) $K = \mathrm{conv}\,(\mathrm{ext}\,K) + \mathrm{rec}\,K$.

(b) $\mathrm{ext}\,(\mathrm{cl}\,F) \subset \mathrm{cl}\,(\mathrm{ext}\,F)$ *for every extreme face F of K.*

Proof. We will prove the assertion by induction on $r = \dim K$. The case $r = 0$ is trivial. If $r = 1$ then K satisfies any of the conditions (a) and (b) if and only if K is a closed segment or a closed halfline. Suppose that the equivalence of (a) and (b) holds for all $r \le m-1$, where $m \ge 2$, and let $K \subset \mathbb{R}^n$ be any line-free convex set of dimension m, with $\operatorname{rec} K = \operatorname{rec}(\operatorname{cl} K)$.

(a) $\Rightarrow$ (b) This part follows from Lemmas 4.2 and 4.4.

(b) $\Rightarrow$ (a) Since the inclusion $\operatorname{conv}(\operatorname{ext} K) + \operatorname{rec} K \subset K$ is obvious, it remains to prove the opposite one. For this, choose any point $x \in K$.

Assume first that $x \in K \cap \operatorname{rbd} K$ and denote by H the extreme face of K generated by x (H is the largest convex subset of K satisfying the condition $x \in \operatorname{rint} F$, see [11], Theorem 11.9). The inclusion $x \in K \cap \operatorname{rbd} K$ implies that

$$H \subset K \cap \operatorname{rbd} K \quad \text{and} \quad \dim H \le \dim K - 1 = m - 1$$

(see [11], Theorem 11.4). Furthermore, $\operatorname{rec} H = \operatorname{rec}(\operatorname{cl} H)$, according to Lemma 4.1. Since every extreme face of H is also an extreme face of K, condition (b) implies that every extreme face F of H has the property $\operatorname{ext}(\operatorname{cl} F) \subset \operatorname{cl}(\operatorname{ext} F)$. By the induction hypothesis, $H = \operatorname{conv}(\operatorname{ext} H) + \operatorname{rec} H$. Because $\operatorname{ext} H \subset \operatorname{ext} K$ and $\operatorname{rec} H = \operatorname{rec} K \cap \operatorname{dir} H$ (see Lemma 4.1), one has

$$x \in H = \operatorname{conv}(\operatorname{ext} H) + \operatorname{rec} H \subset \operatorname{conv}(\operatorname{ext} K) + \operatorname{rec} K.$$

It remains to consider the case $x \in \operatorname{rint} K$. Since the convex set $\operatorname{cl} K$ is line-free, it satisfies the equality (2). Furthermore, because K is a trivial extreme face of itself, it satisfies condition (b): $\operatorname{ext}(\operatorname{cl} K) \subset \operatorname{cl}(\operatorname{ext} K)$. Consequently, Lemma 2.5 gives

$$\operatorname{conv}(\operatorname{ext}(\operatorname{cl} K)) \subset \operatorname{conv}(\operatorname{cl}(\operatorname{ext} K)) \subset \operatorname{cl}(\operatorname{conv}(\operatorname{ext} K)) \subset \operatorname{cl} K.$$

Combined with (2), this argument gives

$$\begin{aligned} \operatorname{cl} K &= \operatorname{conv}(\operatorname{ext}(\operatorname{cl} K)) + \operatorname{rec}(\operatorname{cl} K) \subset \operatorname{cl}(\operatorname{conv}(\operatorname{ext} K)) + \operatorname{rec}(\operatorname{cl} K) \\ &\subset \operatorname{cl} K + \operatorname{rec}(\operatorname{cl} K) = \operatorname{cl} K. \end{aligned}$$

So, due to $\operatorname{rec} K = \operatorname{rec}(\operatorname{cl} K)$, we obtain

$$\operatorname{cl} K = \operatorname{cl}(\operatorname{conv}(\operatorname{ext} K)) + \operatorname{rec}(\operatorname{cl} K) = \operatorname{cl}(\operatorname{conv}(\operatorname{ext} K)) + \operatorname{rec} K.$$

Using the standard properties of the relative interior of convex sets (see, e. g., [11], Chapter 4), one has

$$\begin{aligned} x \in \operatorname{rint} K &= \operatorname{rint}(\operatorname{cl} K) = \operatorname{rint}(\operatorname{cl}(\operatorname{conv}(\operatorname{ext} K)) + \operatorname{rec} K) \\ &= \operatorname{rint}(\operatorname{cl}(\operatorname{conv}(\operatorname{ext} K))) + \operatorname{rint}(\operatorname{rec} K) = \operatorname{rint}(\operatorname{conv}(\operatorname{ext} K)) + \operatorname{rint}(\operatorname{rec} K) \\ &\subset \operatorname{conv}(\operatorname{ext} K)) + \operatorname{rec} K, \end{aligned}$$

as desired. Summing up,

$$K = (K \cap \operatorname{rbd} K) \cup \operatorname{rint} K \subset \operatorname{conv}(\operatorname{ext} K) + \operatorname{rec} K. \qquad \square$$

The next example shows that the condition $\operatorname{rec} K = \operatorname{rec}(\operatorname{cl} K)$ in Theorem 4.5 cannot be weakened by requiring the closedness of $\operatorname{rec} K$.

Example 4.6. Consider in $\mathbb{R}^2$ the unbounded convex set

$$K = \{(x, y) : x \geq 0,\ 0 \leq y < 1\} \cup \{(0, 1)\}.$$

It is easy to see that $\operatorname{rec} K = \{o\}$ and $\operatorname{ext} K = \{(0, 0), (1, 0)\}$. Hence we have $K \neq \operatorname{conv}(\operatorname{ext} K) + \operatorname{rec} K$. On the other hand, every extreme face F of K satisfies condition (b) of Theorem 4.5. □

We recall that a convex set in $\mathbb{R}^n$ is *weakly Motzkin decomposable* if it is the sum of a bounded convex set and a closed convex cone. If $K \subset \mathbb{R}^n$ is a weakly M-decomposable set, then, as proved in [8], the set $\operatorname{ext} K$ is bounded and we have $\operatorname{rec} K = \operatorname{rec}(\operatorname{cl} K)$. Consequently, Theorem 4.5 implies the following assertion.

Corollary 4.7. *If $K \subset \mathbb{R}^n$ is a nonempty line-free weakly Motzkin decomposable set, then conditions* (a) *and* (b) *of Theorem* 4.5 *are equivalent.* □

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