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Properties of the Level Sets of Some Products of Functions

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We are interested about pairs (f, g) of C^2-smooth functions $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ with bounded Hess^+ complements such that their product preserves this property as well. Recall that $\text{Hess}^+(f)$ stands for the set of all points $p \in \mathbb{R}^n$ such that the Hessian matrix $H_p(f)$ of the C^2-smooth function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is positive definite. In this paper we consider two pairs of real-valued functions with empty Hess^+ complements whose products happen to have bounded Hess^+ complements.

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1. Introduction

We continue the program initiated in [5] concerning the analysis of the Hess^+ regions and their complements along with its application to the study of the (sub)level sets. If in [5] we restricted ourselves to the study of the $\text{Hess}^+(\varphi_a)$ region and the subsequent analysis of the (sub)level sets of the polynomial function $\varphi_a : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $\varphi_a(x, y) = (x^2 + y^2)^2 - 2a^2(x^2 - y^2)$, in [4] we investigated the same issues for the function $\psi : \mathbb{R}^3 \longrightarrow \mathbb{R}$, $\psi(x, y, z) = (x^2 + y^2 + z^2)^2 - 8(x^2 - y^2 - z^2)$. Moreover, the latter work concerns some theoretical statements both on the Hess^+ complements and on level sets as well. Among them we recall the closedness property with respect to the product operation, of some family of functions with bounded Hess^+ complements, along with the connectedness and convexity of the (sub)level sets for sufficiently high levels.

We analyze the Hess^+ regions both for the product of squared distance functions $d_p^2 d_q^2$, where $d_p : \mathbb{R}^n \longrightarrow \mathbb{R}$, $d_p(z) = \|z - p\|$, and for the product $f_p f_q$, where $f_p = \sqrt{1 + d_p^2}$. The $\text{Hess}^+(d_p^2 d_q^2)$ and $\text{Hess}^+(f_p f_q)$ complements are shown to be bounded and a subsequent analysis on the (sub)level sets of the two products is done, partly as an application of the earlier proved boundedness property of the Hess^+ complements. The terminology of level and sublevel sets at the level $y \in \mathbb{R}$ is used for the inverse images of types $f^{-1}(y)$ and $f^{-1}((-\infty, y])$ of real-valued functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, respectively. If the function f is additionally convex, then its sublevel sets are obviously convex. The level curves are understood to be convex whenever they bound convex sublevel sets [6] (see also [2, p. 175]). Another way to consider convexity for regular curves of $\mathbb{R}^2$ consists in their quality to stay on the same side of all of their tangent lines [2, p. 174], [1, p. 37]. In order to select the *a priori* strictly

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convex level sets, among the connected regular ones, we still use, as we did in [4] and [5], the positivity test on the curvature function

$$\rho_C = -\frac{\begin{vmatrix} H(f) & (\nabla f)^T \\ \nabla f & 0 \end{vmatrix}}{\|\nabla f\|^3}$$

of $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ along a level set $C = f^{-1}(c)$. In order to find the Hess$^+$ regions of some products fg of functions we will be using the positivity test both for $\mathrm{Tr}(H(fg))$ and for $\det H(fg)$, where $H(fg)$ stands for the Hessian matrix valued function of the product fg. While the trace of $H(fg)$ has a simple formula, we are going to prove a more sophisticated formula for $\det H(fg)$ in order to systematize our computations.

2. Critical points of real-valued functions

For a real valued function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ the critical set of f is the vanishing set of its gradient ∇f. Since $d_p(fg) = f(p)(dg)_p + g(p)(df)_p$ it follows immediately that $\nabla_p(fg) = f(p)\nabla_p g + g(p)\nabla_p f$, where $\nabla_p h$ stands for the gradient of $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ at $p \in \mathbb{R}^n$, i.e. $\nabla_p h = (h_{x_1}(p), \ldots, h_{x_n}(p))$.

Remark 2.1. If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a C^1-smooth convex function, then its critical set $C(f)$ is convex. Indeed the critical points of f coincide with the global minimum points of f (see [7, Theorem 2.5.7]). In other words

$$C(f) = \{p \in \mathbb{R}^n \ : \ f(p) \le f(x),\ \forall x \in \mathbb{R}^n\}.$$

Note that $C(f) = f^{-1}(-\infty, f(p)]$ for every $p \in C(f)$, i.e. $C(f)$ is a sublevel set of the convex function f in this particular case. Therefore $C(f)$ is convex. $\square$

By using Remark 2.1 we shall provide examples of nonconvex functions which have, or are good candidates to have, bounded Hess$^+$ complements.

Example 2.2. **(1)** The product $d_p^2 d_q^2$ of two squared distance functions, where $d_p : \mathbb{R}^n \longrightarrow \mathbb{R}$, $d_p(x) = \|x - p\|$ is not convex if $p \neq q$, although both d_p^2 and d_q^2 are strictly convex. Indeed,

$$\begin{aligned} \nabla_x(d_p^2 d_q^2) = 0 &\Leftrightarrow d_p^2(x)(\nabla_x d_q^2) + d_q^2(x)(\nabla_x d_p^2) = 0 \\ &\Leftrightarrow \|x-p\|^2(x-q) + \|x-q\|^2(x-p) = 0, \end{aligned}$$

as $(\nabla_x d_p^2) = 2(x-p)$ and $\nabla_x d_q^2 = 2(x-q)$. Moreover

$$\begin{aligned} &\|x-p\|^2(x-q) + \|x-q\|^2(x-p) = 0 \\ &\Longleftrightarrow \|x-p\|^2(x-q) = -\|x-q\|^2(x-p) \\ &\Longrightarrow \|x-p\|^2\|x-q\| = \|x-q\|^2\|x-p\| \\ &\Longleftrightarrow \|x-p\| \cdot \|x-q\|(\|x-p\| - \|x-q\|) = 0 \\ &\Longleftrightarrow \|x-p\| \cdot \|x-q\| = 0 \text{ or } \|x-p\| = \|x-q\| \\ &\Longleftrightarrow x = p \text{ or } x = q \text{ or } \|x-p\| = \|x-q\|. \end{aligned}$$

These necessary conditions for x to be a critical point of $d_p^2 d_q^2$ combined with the necessary and sufficient one

$$\|x-p\|^2(x-q)+\|x-q\|^2(x-p)=0$$

show that the critical set of $d_p^2 d_q^2$ is the discrete set

$$C(d_p^2 d_q^2)=\left\{p, \frac{p+q}{2}, q\right\}.$$

Therefore the product $d_p^2 d_q^2$ is not a convex function via Remark 2.1 and its set of critical values is

$$B(d_p^2 d_q^2)=\left\{0, \frac{\|p-q\|^4}{16}\right\}.$$

Note that zero is the common value of $d_p^2 d_q^2$ applied to p and q and

$$\frac{\|p-q\|^4}{16}=(d_p^2 d_q^2)\left(\frac{p+q}{2}\right).$$

(2) Also the product $f_p^2 f_q^2$, where $f_p=\sqrt{1+d_p^2}$, is not convex if $\|p-q\|>2$, although each factor is a strictly convex function. Indeed,

$$\begin{aligned}
&\nabla_x(f_p^2 f_q^2)=0 \Leftrightarrow (1+d_p^2(x))(\nabla_x d_q^2)+(1+d_q^2(x))(\nabla_x d_p^2)=0\\
&\Leftrightarrow (1+\|x-p\|^2)(x-q)+(1+\|x-q\|^2)(x-p)=0, \qquad (1)\\
&\Leftrightarrow x=(1-t)p+tq, \text{ where } t=\frac{1+\|x-p\|^2}{2+\|x-p\|^2+\|x-q\|^2}.
\end{aligned}$$

By replacing x with $(1-t)p+tq$ in (1) and taking into account that $x-p=t(q-p)$ and $x-q=(1-t)(p-q)$, we obtain

$$\begin{gathered}
(1-2t)(\|p-q\|^2t^2-\|p-q\|^2t+1)=0, \text{ i.e.}\\
t_1=\frac{1}{2} \text{ and } t_{2,3}=\frac{\|p-q\|\pm\sqrt{\|p-q\|^2-4}}{2\|p-q\|}.
\end{gathered}$$

Thus, the critical set $C(f_p^2 f_q^2)$ is the discrete set

$$\left\{\frac{p+q}{2}-\sqrt{1-\frac{4}{\|p-q\|^2}}\cdot\frac{p-q}{2}, \frac{p+q}{2}, \frac{p+q}{2}+\sqrt{1-\frac{4}{\|p-q\|^2}}\cdot\frac{p-q}{2}\right\}.$$

and the set of critical values is

$$B(f_p^2 f_q^2)=\left\{\|p-q\|^2, \left(1+\frac{\|p-q\|^2}{4}\right)^2\right\}.$$

Note that $\|p-q\|^2$ is the common value of $f_p^2 f_q^2$ applied to

$$\frac{p+q}{2}-\sqrt{1-\frac{4}{\|p-q\|^2}}\cdot\frac{p-q}{2} \text{ and } \frac{p+q}{2}+\sqrt{1-\frac{4}{\|p-q\|^2}}\cdot\frac{p-q}{2}$$

and

$$\left(1+\frac{\|p-q\|^2}{4}\right)^2=(f_p^2 f_q^2)\left(\frac{p+q}{2}\right).$$

For $\|p-q\| \leq 2$ the product $f_p^2 f_q^2$ has only one critical point and the multiple connectedness structure of the critical set is not an obstruction any longer for the convexity of the product $f_p^2 f_q^2$, which might be convex when $\|p-q\| \leq 2$. This is actually the case, at least when $n=2$. Indeed the product $f_p^2 f_q^2$ is a strictly convex function, when $n=2$ and $\|p-q\| \leq 2$, as we shall see later in Remark 3.14. Note that $C(f_p^2 f_q^2)=C(f_p g_q)$.

3. On the Hess^+ of some products of functions

For $n=2$ we particularly obtain

$$H(fg)=\begin{pmatrix} fg_{xx}+gf_{xx}+2f_x g_x & fg_{xy}+gf_{xy}+f_x g_y+f_y g_x \\ fg_{xy}+gf_{xy}+f_x g_y+f_y g_x & fg_{yy}+gf_{yy}+2f_y g_y \end{pmatrix}. \tag{2}$$

Since the positive semi-definiteness of $H(fg)$ can be expressed by means of $\mathrm{Tr}(H(fg))$ and $\det(H(fg))$ we also observe that

$$\mathrm{Tr}(H(fg))=f\Delta g+g\Delta f+2\langle\nabla f,\nabla g\rangle, \tag{3}$$

where Δ stands for the Laplace operator. The formula (3) follows immediately by inspecting the Hessian matrix (2) of the product fg. In order to provide a formula for the determinant of the Hessian matrix of a product fg of two C^2-smooth functions, we first define the vector-valued functions $f\oplus g, D(g,f):\mathbb{R}^2\longrightarrow\mathbb{R}^2$, associated to the real-valued functions $f,g:\mathbb{R}^2\longrightarrow\mathbb{R}$ by $(f\oplus g)(x,y)=(f(x,y),g(x,y))$ and

$$\begin{aligned} D(g,f) &= f_x g_y(g_{xy}\oplus f_{xy})+f_y g_x(g_{xy}\oplus f_{xy})-f_x g_x(g_{yy}\oplus f_{yy})-f_y g_y(g_{xx}\oplus f_{xx}) \\ &= \begin{vmatrix} g_{xx}\oplus f_{xx} & g_{xy}\oplus f_{xy} & f_x \\ g_{yx}\oplus f_{yx} & g_{yy}\oplus f_{yy} & f_y \\ g_x & g_y & 0 \end{vmatrix}, \end{aligned}$$

respectively. Note that $f\oplus g$ and $g\oplus f$ are usually not equal. In fact

$$f\oplus g=r_b\circ(g\oplus f),$$

where $r_b:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ stands for the reflection about the first bisector $b:x=y$. Also $f\oplus g$ is differentiable whenever f and g are differentiable and $d_p(f\oplus g)=d_p f\oplus d_p g$ for every $p\in\mathbb{R}^2$.

Proposition 3.1. *If $f,g:\mathbb{R}^2\longrightarrow\mathbb{R}$ are C^2-smooth functions, then*

$$\begin{aligned} \det(H(fg)) &= f^2\det(H(g))+g^2\det(H(f))-\det{}^2 J(f\oplus g) \\ &\quad + fg\left(\det J(f_x\oplus g_y)+\det J(g_x\oplus f_y)\right)-2\left\langle f\oplus g, D(g,f)\right\rangle. \end{aligned} \tag{4}$$

Proof. The formula (4) can be deduced in the following way:

$$\begin{aligned}
\det(H(fg)) &= (fg_{xx} + gf_{xx} + 2f_xg_x)(fg_{yy} + gf_{yy} + 2f_yg_y) \\
&\quad - (fg_{xy} + gf_{xy} + f_xg_y + f_yg_x)^2 \\
&= f^2 \det(H(g)) + g^2 \det(H(f)) - \det{}^2 J(f \oplus g) \\
&\quad + fg\,(\det J(f_x \oplus g_y) + \det J(g_x \oplus f_y)) \\
&\quad + 2\,\langle f \oplus g, f_xg_x(g_{yy} \oplus f_{yy}) + f_yg_y(g_{xx} \oplus f_{xx})\rangle \\
&\quad + 2f(g_{xx}f_yg_y + g_{yy}f_xg_x) + 2g(f_{xx}f_yg_y + f_{yy}f_xg_x) \\
&\quad - 2\langle \nabla f, r_b \circ \nabla g\rangle \,\langle f \oplus g, (g_{xy} \oplus f_{xy})\rangle\,, \\
&= f^2 \det(H(g)) + g^2 \det(H(f)) - \det{}^2 J(f \oplus g) \\
&\quad + fg\,(\det J(f_x \oplus g_y) + \det J(g_x \oplus f_y)) \\
&\quad + 2\,\langle f \oplus g, f_xg_x(g_{yy} \oplus f_{yy}) + f_yg_y(g_{xx} \oplus f_{xx})\rangle \\
&\quad - 2\,\langle f \oplus g, f_xg_y(g_{xy} \oplus f_{xy}) + f_yg_x(g_{xy} \oplus f_{xy})\rangle \\
&= f^2 \det(H(g)) + g^2 \det(H(f)) - \det{}^2 J(f \oplus g) \\
&\quad + fg\,(\det J(f_x \oplus g_y) + \det J(g_x \oplus f_y)) - 2\,\langle f \oplus g, D(g, f)\rangle\,,
\end{aligned}$$

which completes the proof. □

Example 3.2. Consider the functions $d_p^2, d_q^2 : \mathbb{R}^2 \longrightarrow \mathbb{R}$, where $p, q \in \mathbb{R}^2$,

$$d_p(z) = \|z - p\|,\ d_q(z) = \|z - q\|,$$

and where $z = (x, y)$, $p = (x_0, y_0)$ and $q = (x_1, y_1)$. In other words,

$$d_p^2 = (x - x_0)^2 + (y - y_0)^2,\ d_q^2 = (x - x_1)^2 + (y - y_1)^2.$$

One can easily see that

$$\begin{aligned}
(\nabla d_p^2)(x, y) &= 2\,(x - x_0, y - y_0)\,,\ \text{i.e. } (\nabla d_p^2)(z) = 2(z - p) \\
(\nabla d_q^2)(x, y) &= 2\,(x - x_1, y - y_1)\,,\ \text{i.e. } (\nabla d_q^2)(z) = 2(z - q)
\end{aligned}$$

and

$$H_{(x,y)}(d_p^2) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2I_2 = H_{(x,y)}(d_q^2)$$

$$\det H_{(x,y)}(d_p^2) = \det H_{(x,y)}(d_q^2) = 4.$$

Note that $\|(\nabla d_p^2)(z)\| = 2\|z - p\| = 2d_p(z)$ and $\|(\nabla d_q^2)(z)\| = 2\|z - q\| = 2d_q(z)$. Therefore

$$\operatorname{Tr}(H(d_p^2 d_q^2)) = d_p^2 \Delta d_q^2 + d_q^2 \Delta d_p^2 + 2\langle \nabla d_p^2, \nabla d_q^2\rangle = 4\|2z - p - q\|^2 \geq 0. \qquad (5)$$

For the computation of $\det(H(d_p^2 d_q^2))$ we first recall that

$$\det(H(d_p^2)) = \det(H(d_q^2)) = 4,$$

and observe that

$$\begin{gathered}
\det{}^2 J(d_p^2 \oplus d_q^2)(x, y) = 16[(x - x_0)(y - y_1) - (x - x_1)(y - y_0)]^2 \\
= \|\nabla d_p^2 \times \nabla d_q^2\|^2(x, y) \det J((d_p^2)_x \oplus (d_p^2)_y) + \det J((d_q^2)_x \oplus (d_p^2)_y) = 4 + 4 = 8 \\
D(d_q^2, d_p^2)(x, y) = -[8(x - x_0)(x - x_1) + 8(y - y_0)(y - y_1)] = -2\langle \nabla d_p^2, \nabla d_q^2\rangle(x, y).
\end{gathered}$$

Consequently

$$\begin{aligned}
\det(H(d_p^2 d_q^2)) &= d_p^4 \det(H(d_q^2)) + d_q^4 \det(H(d_p^2)) - \det{}^2 J(d_p^2 \oplus d_q^2) \\
&\quad + d_p^2 d_q^2 \left(\det J((d_p^2)_x \oplus (d_q^2)_y) + \det J((d_q^2)_x \oplus (d_p^2)_y)\right) - 2\left\langle d_p^2 \oplus d_q^2, D(d_q^2, d_p^2)\right\rangle \\
&= 4(d_p^4 + d_q^4) - \|\nabla d_p^2 \times \nabla d_q^2\|^2 + 8 d_p^2 d_q^2 + 4\langle \nabla d_p^2, \nabla d_q^2\rangle (d_p^2 + d_q^2) \\
&= [2(d_p + d_q)^2 + \langle \nabla d_p^2, \nabla d_q^2\rangle] \times [2(d_p - d_q)^2 + \langle \nabla d_p^2, \nabla d_q^2\rangle].
\end{aligned} \tag{6}$$

Remark 3.3. The following statements are equivalent:

(1) $\det(H(d_p^2 d_q^2)) > 0$;

(2) $2(d_p - d_q)^2 + \langle \nabla d_p^2, \nabla d_q^2\rangle > 0$. □

Corollary 3.4. *The product $d_p^2 d_q^2$ is a Morse function.*

Proof. According to Example 2.2(1), the critical set of $d_p^2 d_q^2$ is

$$C(d_p^2 d_q^2) = \left\{p, \frac{p+q}{2}, q\right\}.$$

Since $\det(H_p(d_p^2 d_q^2)) = \det(H_q(d_p^2 d_q^2)) = 4\|q-p\|^4 > 0$ and

$$\det(H_{(p+q)/2}(d_p^2 d_q^2)) = -\|q-p\|^2 < 0$$

it follows that the critical points are indeed non-degenerate. □

Remark 3.5. The following relations hold:

(1) $\mathrm{Hess}^+(d_p^2 d_q^2) = \{z \in \mathbb{R}^2 : 2(d_p - d_q)^2 + \langle \nabla d_p^2, \nabla d_q^2\rangle > 0\}$.

(2) $\partial\mathrm{Hess}^+(d_p^2 d_q^2) = \{z \in \mathbb{R}^2 \ : \ 2(d_p - d_q)^2 + \langle \nabla d_p^2, \nabla d_q^2\rangle = 0\}$. □

Remark 3.6. $\mathrm{Hess}^+(d_p^2 d_q^2)$ is unbounded and

$$\mathbb{R}^2 \setminus \mathrm{cl}B\left(\frac{p+q}{2}, \frac{\|p-q\|}{2}\right) \subseteq \mathrm{Hess}^+(d_p^2 d_q^2).$$

Indeed if $\langle \nabla d_p^2, \nabla d_q^2\rangle > 0$, then $\det(H(d_p^2 d_q^2)) > 0$. On the other hand

$$\begin{aligned}
\|q-p\|^2 &= \|q - z + z - p\|^2 = \|q-z\|^2 + \|z-p\|^2 + 2\langle q - z, z - p\rangle \\
&= 2\|z\|^2 - 2\langle z, p+q\rangle + \|p\|^2 + \|q\|^2 - \frac{1}{2}\langle \nabla d_p^2, \nabla d_q^2\rangle
\end{aligned}$$

which shows that $\langle \nabla d_p^2, \nabla d_q^2\rangle > 0$ if and only if

$$\begin{aligned}
& 2\|z\|^2 - 2\langle z, p+q\rangle + \|p\|^2 + \|q\|^2 > \|q-p\|^2 \\
\iff\ & \|z\|^2 - \langle z, p+q\rangle + \frac{\|p\|^2 + \|q\|^2}{2} > \frac{\|q-p\|^2}{2} \\
\iff\ & \left\|z - \frac{p+q}{2}\right\|^2 > \frac{\|q-p\|^2}{4}.
\end{aligned}$$

□

According to (5) we have $\mathrm{Tr}(H((d_p^2 d_q^2)) > 0$ everywhere except at

$$\frac{p+q}{2} \quad \text{as} \quad \mathrm{Tr}(H_{(p+q)/2}(d_p^2 d_q^2)) = 0.$$

In particular $\mathrm{Tr}(H(d_p^2 d_q^2)) > 0$ over $\mathbb{R}^2 \setminus \mathrm{cl}B\Big(\frac{p+q}{2}, \frac{\|p-q\|}{2}\Big)$ which combined with the above arguments on

$$\det(H(d_p^2 d_q^2)) > 0 \quad \text{over} \quad \mathbb{R}^2 \setminus \mathrm{cl}B\left(\frac{p+q}{2}, \frac{\|p-q\|}{2}\right)$$

show that $H(d_p^2 d_q^2)$ is positive definite over

$$\mathbb{R}^2 \setminus \mathrm{cl}B\left(\frac{p+q}{2}, \frac{\|p-q\|}{2}\right)$$

and the statement is now completely justified. Moreover

$$\begin{aligned}
D_3(d_p^2 d_q^2) &= \begin{vmatrix} (d_p^2 d_q^2)_{xx} & (d_p^2 d_q^2)_{xy} & (d_p^2 d_q^2)_x \\ (d_p^2 d_q^2)_{xy} & (d_p^2 d_q^2)_{yy} & (d_p^2 d_q^2)_y \\ (d_p^2 d_q^2)_x & (d_p^2 d_q^2)_y & 0 \end{vmatrix} \\
&= 4(mv - nu)^2 d_p^2 d_q^2 - d_p^2 d_q^2 (d_p^2 + d_q^2)(d_p^2 + d_q^2 + 2(mu + nv)) \\
&= 8 d_p^2 d_q^2 (-s^2 - 2st + 4 d_p^2 d_q^2 - 4t^2),
\end{aligned}$$

where $m = x - x_0, n = y - y_0, u = x - x_1, v = y - y_1$, $s = d_p^2 + d_q^2 = m^2 + n^2 + u^2 + v^2$ and $t = mu + nv$. Since we are interested about values of k for which $D_3(d_p^2 d_q^2)$ keeps constant sign along the curve $d_p^2 d_q^2 = k$, we shall find some useful information on the local extrema of the restriction of the function $h := -s^2 - 2st + 4d_p^2 d_q^2 - 4t^2$ to the curve $d_p^2 d_q^2 = k$. Note that such level curves of the product $d_p^2 d_q^2$ are convex and do not contain critical points of h as $D_3(d_p^2 d_q^2)$ vanishes at such points. They are also compact due to the norm-coercivity of the product $d_p^2 d_q^2$ itself, i.e.

$$\lim_{\|z\| \to \infty} (d_p^2(z) d_q^2(z)) = \infty.$$

Remark 3.7. If $k < \|q-p\|^4/16$, then the determinant $D_3(d_p^2 d_q^2)$ is negative over the k-level curve $d_p^2 d_q^2 = k$. Indeed, one can easily see that the critical set of the convex function $s = d_p^2 + d_q^2$ reduces to the the midpoint $(q+p)/2$ of the segment $[pq]$, which is actually its global minimum point. In other words

$$s(z) \geq s\left(\frac{q+p}{2}\right) = \frac{\|q-p\|^2}{2}.$$

Therefore
$$\begin{aligned}
h = -s^2 - 2st + 4d_p^2 d_q^2 - 4t^2 &\leq -\frac{\|q-p\|^4}{4} - 2st + 4k - 4t^2 \\
&< -\frac{\|q-p\|^4}{4} - 2st + \frac{\|q-p\|^4}{4} - 4t^2 = -2st - 4t^2 \leq 0,
\end{aligned}$$

which shows that $D_3(d_p^2 d_q^2) = 8 d_p^2 d_q^2 h < 0$ over the k-level curve of $d_p^2 d_q^2$ whenever

$$k < \|q-p\|^4/16. \qquad \square$$

Theorem 3.8. *The level curves of the product $d_p^2 d_q^2$ have the properties:*

(1) *For $k < \|p-q\|^4/16$ the level curve $d_p^2 d_q^2 = k$ is compact but not connected. However, each of its connected components is convex.*

(2) *For $\|p-q\|^4/16 < k < \|p-q\|^4/4$ the level curve $d_p^2 d_q^2 = k$ is a compact connected regular curve, diffeomorphic with the circle S^1.*

(3) *For $k \geq \|p-q\|^4/4$ the level curve $d_p^2 d_q^2 = k$ is compact and convex.*

Proof. The compactness of all level sets of the product $d_p^2 d_q^2$ follows due to its norm-coercivity. We start our arguments by finding the local extrema of h subject to some level curve $d_p^2 d_q^2 = k$ constraint. To this end, we shall use the standard technique of Lagrange multipliers via the Lagrange function

$$\mathcal{L}(x,y,\lambda) = 4d_p^2 d_q^2 - (s^2 + 2st + 4t^2) - \lambda(d_p^2 d_q^2 - k).$$

Note that $h(x,y) = \mathcal{L}(x,y,0)$ and therefore

$$\operatorname{sgn}(D_3(d_p^2 d_q^2)) = \operatorname{sgn}(\mathcal{L}(x,y,0)).$$

The local extrema of h subject to the constraint $d_p^2 d_q^2 = k$ are part of the solutions of the equation $\nabla\mathcal{L}(x,y,\lambda) = 0$ which is equivalent to the following system

$$\begin{cases} (2x - x_0 - x_1)(2s + s + 2t + 4t) + (\lambda - 4)((x - x_0)d_q^2 + (x - x_1)d_p^2) = 0 \\ (2y - y_0 - y_1)(2s + s + 2t + 4t) + (\lambda - 4)((y - y_0)d_q^2 + (y - y_1)d_q^2) = 0 \\ [(x - x_0)^2 + (y - y_0)^2][(x - x_1)^2 + (y - y_1)^2] = k, \end{cases}$$

i.e.

$$\begin{cases} 3(s + 2t)(m + u) + (\lambda - 4)(md_q^2 + ud_p^2) = 0 \\ 3(s + 2t)(n + v) + (\lambda - 4)(nd_q^2 + vd_p^2) = 0 \\ [(x - x_0)^2 + (y - y_0)^2][(x - x_1)^2 + (y - y_1)^2] = k. \end{cases}$$

Multiplying the first equation with $nd_q^2 + vd_p^2$ and the second with $-(md_q^2 + ud_q^2)$ and adding them afterwards, we obtain

$$\begin{aligned} (s + 2t)[(m + u)(nd_q^2 + vd_p^2) - (n + v)(md_q^2 + ud_q^2)] = 0 \Leftrightarrow \\ \Leftrightarrow (s + 2t)(mv - nu)(d_p^2 - d_q^2) = 0 \\ \Leftrightarrow s + 2t = 0 \text{ or } mv = nu \text{ or } d_p^2 = d_q^2. \\ \Leftrightarrow x = \frac{x_0 + x_1}{2},\ y = \frac{y_0 + y_1}{2} \text{ or } mv = nu \text{ or } d_p = d_q. \end{aligned}$$

Note that the midpoint $(p+q)/2$ of the segment $[pq]$, is a critical point of h.

For the second restriction we have

$$\begin{aligned} mv = nu &\Leftrightarrow (x - x_0)(y - y_1) = (x - x_1)(y - y_0) \\ &\Leftrightarrow (y_0 - y_1)x + (x_1 - x_0)y + x_0 y_1 - x_1 y_0 = 0. \end{aligned} \tag{7}$$

Note that equation (7) represents the straight line pq. Moreover on the intersection of pq with the k-level curve of $d_p^2 d_q^2$, we have

$$k = d_p^2 d_q^2 = (mu + nv)^2 + (mv - nu)^2 = t^2 + 0^2 = t^2$$

and, in this case, we get

$$\begin{aligned}\mathcal{L}(x,y,0) &= 4d_p^2d_q^2 - s^2 - 2st - 4t^2 = 4k - s(s+2t) - 4k \\ &= -s[(m+u)^2 + (n+v)^2] \le 0. \end{aligned} \tag{8}$$

Note that $\mathcal{L}(x,y,0) = 0$ if and only if $x = (x_0+x_1)/2$ and $y = (y_0+y_1)/2$, and $k = \|q-p\|^4/16$ in that case. The intersection of the straight line pq and the k-level curve of the product $d_p^2d_q^2$ is not empty unless the k-level set of $d_p^2d_q^2$ is empty, i.e. $k < 0$. This intersection can be easily found by using the parametrization

$$z = \frac{p+q}{2} + t(q-p),\ t \in \mathbb{R}$$

of the line pq. Combining this equation with $d_p^2d_q^2 = k \Leftrightarrow \|z-p\|^2\|z-q\|^2 = k$ we obtain

$$t^2 = \frac{1}{4} \pm \frac{\sqrt{k}}{\|q-p\|^2}, \tag{9}$$

which provides at least two points in the intersection, away from the midpoint of $[pq]$, which correspond to the values

$$\pm\sqrt{\frac{1}{4} + \frac{\sqrt{k}}{\|q-p\|^2}}$$

of the parameter t. At these points we have $\mathcal{L}(x,y,0) < 0$, i.e. $D_3(d_p^2d_q^2) < 0$ there (see (8)). For the third restriction we have

$$\begin{aligned} d_p^2 = d_q^2 &\Leftrightarrow (x-x_0)^2 + (y-y_0)^2 = (x-x_1)^2 + (y-y_1)^2 \\ &\Leftrightarrow 2(x_1-x_0)x + 2(y_1-y_0)y + x_0^2 + y_0^2 - x_1^2 - y_1^2 = 0 \end{aligned} \tag{10}$$

and equation (10) represents the perpendicular bisector, say b, of the segment $[pq]$. Moreover on the intersection of b with the k-level curve of $d_p^2d_q^2$, we have $f = g = \sqrt{k}$, and $s = 2\sqrt{k}$. Consequently, for $(x,y) \in b \cap (d_p^2d_q^2)^{-1}(k)$ we have

$$\begin{aligned}\mathcal{L}(x,y,0) &= 4d_p^2d_q^2 - s^2 - 2st - 4t^2 = 4k - 4k - 2t(s+2t) \\ &= -(s - \|p-q\|^2)(2s - \|p-q\|^2),\end{aligned}$$

as $\|p-q\|^2 = (m-u)^2 + (n-v)^2 = m^2+n^2+u^2+v^2 - 2(mu+nv) = s - 2t$. Since

$$s = ||z-p||^2 + ||z-q||^2 \ge \frac{(||z-p|| + ||z-q||)^2}{2} \ge \frac{\|p-q\|^2}{2},$$

it follows that $D_3(d_p^2d_q^2)$ is negative on the intersection of the perpendicular bisector b of the segment $[pq]$ with the k-level curve of $d_p^2d_q^2$ if and only if

$$2\sqrt{k} = s > \|p-q\|^2 \text{ i.e. } k > \frac{\|p-q\|^4}{4}.$$

Note that this intersection starts to be nonempty with

$$k = \frac{\|p-q\|^4}{16} = (d_p^2d_q^2)\left(\frac{p+q}{2}\right),$$

which is a critical value of the product $d_p^2 d_q^2$, and keeps being nonempty for all greater values of k. Indeed, the perpendicular bisector b of $[pq]$ can be parameterized by the equation

$$z = \frac{p+q}{2} + tR_{\pi/2}(q-p), \ t \in \mathbb{R},$$

where $R_{\pi/2} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ stands for a rotation of angle $\pi/2$ around the origin, and it punctures the level set $d_p^2 d_q^2 = k$ in the points which correspond to the values

$$t^2 = -\frac{1}{4} + \frac{\sqrt{k}}{\|p-q\|^2}.$$

of the parameter t. Therefore the intersection between the line b and the k-level curve of the product $d_p^2 d_q^2$ is, indeed, not empty if and only if

$$\frac{\sqrt{k}}{\|p-q\|^2} \geq \frac{1}{4} \ \Leftrightarrow \ k \geq \frac{\|p-q\|^4}{16}.$$

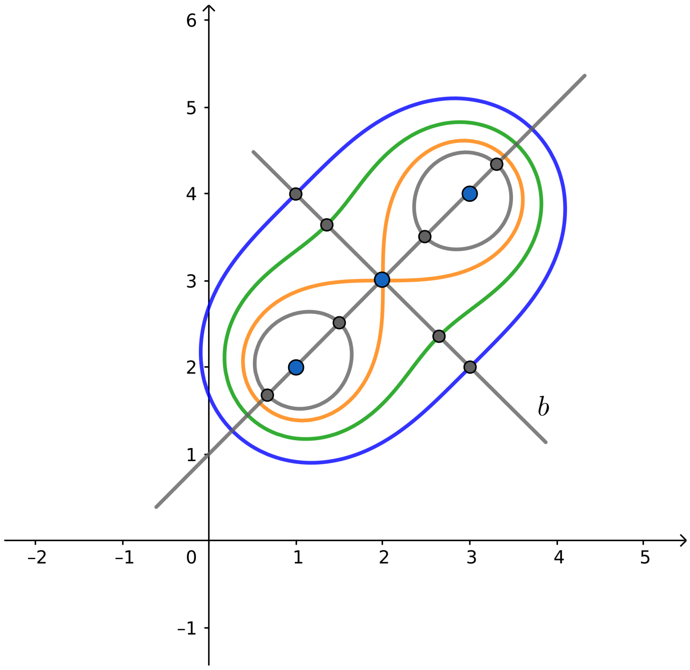

Figure 3.1: A non-connected negative regular level of the product $d_p^2 d_q^2$ with two components, a critical level and two connected regular levels, one of which is non-convex and the other one is convex.

For $k \geq \|p-q\|^4/16$, these intersection points correspond to the values

$$\pm\sqrt{-\frac{1}{4} + \frac{\sqrt{k}}{\|p-q\|^2}}$$

of the parameter t. Since the local extrema of h, including the global ones, subject to the constraint $d_p^2 d_q^2 = k$, are contained in the union of the lines

$$pq, \ b \text{ and the level set } d_p^2 d_q^2 = k \tag{11}$$

and $D_3(d_p^2 d_q^2) < 0$ on (11) if and only if $k > \|p-q\|^4/4$, it follows that $D_3(d_p^2 d_q^2) < 0$ on the whole k-level curve of $d_p^2 d_q^2$ if and only if $k > \|p-q\|^4/4$.

Thus, for $k < \|p-q\|^4/16$ the k-level curve of the product $d_p^2 d_q^2$ is not connected as its intersection with the line b is empty and its intersections with the opposite perpendicular half lines, of the line pq, starting at the midpoint of $[pq]$ are both nonempty and these intersection points are located on both sides of b, as they correspond to the values (9) of the parameter t. On the other hand the determinant $D_3(d_p^2 d_q^2)$ is, according to Remark 3.7, negative over all these k-level curves of the product $d_p^2 d_q^2$, which means that all of their connected components are convex.

By combining the equation

$$z = \frac{p+q}{2} + t\mathbf{v},\ t \in \mathbb{R}$$

of the line through the midpoint of $[pq]$ in the direction of an arbitrary unit vector $\mathbf{v} \in S^1$ with the equation $d_p^2 d_q^2 = k \iff \|z-p\|^2\|z-q\|^2 = k$ one can show that the intersection of each half line starting at the midpoint of $[pq]$ with the k-level curve of the product $d_p^2 d_q^2$ is a singleton if $k > \|p-q\|^4/16$. The intersection point of such a half line

$$z = \frac{p+q}{2} + t\mathbf{v},\ t \geq 0$$

and the k-level curve of the product $d_p^2 d_q^2$ corresponds to the value

$$\frac{1}{\sqrt{2}}\sqrt{\langle q-p, \mathbf{v}\rangle^2 - \frac{\|q-p\|^2}{2} + \sqrt{4k + \langle q-p, \mathbf{v}\rangle^4 - \|q-p\|^2\langle q-p, \mathbf{v}\rangle^2}}$$

of the parameter t and can be used to provide a diffeomorphism between the circle S^1 and the k-level curve of the product $d_p^2 d_q^2$ whenever $k \geq \|p-q\|^4/16$. This shows that all these level curves of the product $d_p^2 d_q^2$ are connected.

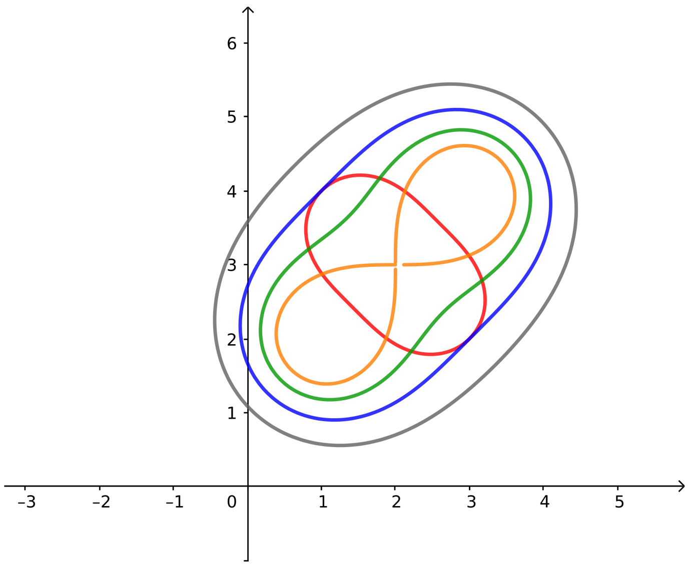

Figure 3.2: The nontrivial critical level curve of $d_p^2 d_q^2$, a regular level curve of $d_p^2 d_q^2$ partially contained in $\mathrm{Hess}^+(d_p^2 d_q^2)$ and partially in its complement, the boundary of $\mathrm{Hess}^+(d_p^2 d_q^2)$, the first convex regular level curve of $d_p^2 d_q^2$ completely contained in the closure of $\mathrm{Hess}^+(d_p^2 d_q^2)$, and a regular level curve of $d_p^2 d_q^2$ completely contained in $\mathrm{Hess}^+(d_p^2 d_q^2)$ ($p = (1, 2)$, $q = (3, 4)$).

According to Example 2.2(1), the critical values of the product $d_p^2 d_q^2$ are

$$(d_p^2 d_q^2)(p) = (d_p^2 d_q^2)(q) = 0 \text{ and } (d_p^2 d_q^2)\left(\frac{p+q}{2}\right) = \frac{\|q-p\|^4}{16}$$

which shows that the set of taken regular values of the product are

$$\left(0, \frac{\|q-p\|^4}{16}\right) \cup \left(\frac{\|q-p\|^4}{16}, \infty\right).$$

In particular the k-level curves of the product are compact connected regular curves if $k \geq \|p-q\|^4/4$, diffeomorphic with the circle S^1. Since the determinant $D_3(d_p^2 d_q^2)$ keeps constant sign along these k-level curves when $k > \|p-q\|^4/4$ it follows that these k-level curves are also convex. For

$$\frac{\|q-p\|^4}{16} < k < \frac{\|q-p\|^4}{4}$$

the k-level curves of the product $d_p^2 d_q^2$ are still diffeomorphic with the circle S^1, due to the classification theorem of 1-dimensional manifolds and the Non-Critical Neck Principle [3, p. 194]. However they are not convex any longer, as the determinant $D_3(d_p^2 d_q^2)$ changes its sign along them, being positive over the intersection of b with the level curve $(d_p^2 d_q^2)(z) = k$ and negative over the intersection of the line pq with the k-level curve of $d_p^2 d_q^2$. □

Remark 3.9. The level curve

$$d_p^2 d_q^2 = \frac{\|p-q\|^4}{4} \tag{12}$$

remains convex. Indeed, the convexity of (12) follows immediately out of its representation

$$(d_p^2 d_q^2)^{-1}\left(-\infty, \frac{\|p-q\|^4}{4}\right] = \bigcap_{n\geq 1} (d_p^2 d_q^2)^{-1}\left(-\infty, \frac{\|p-q\|^4}{4} + \frac{1}{n}\right],$$

and the convexity of all sublevel sets

$$(d_p^2 d_q^2)^{-1}\left(-\infty, \frac{\|p-q\|^4}{4} + \frac{1}{n}\right], \; n \geq 1. \qquad \square$$

In fact (12) is the first convex level curve of the product $d_p^2 d_q^2$ completely contained in cl $\text{Hess}^+(d_p^2 d_q^2)$. It is tangent to the boundary of $\text{Hess}^+(d_p^2 d_q^2)$, whose equation is

$$2(d_p - d_q)^2 + \left\langle \nabla d_p^2, \nabla d_q^2 \right\rangle = 0.$$

In order to completely justify this, we need to show (see Remark 3.5) that

$$2(d_p - d_q)^2 + \left\langle \nabla d_p^2, \nabla d_q^2 \right\rangle \geq 0$$

over the level curve $d_p^2 d_q^2 = \|p-q\|^4/4$ of the product $d_p^2 d_q^2$.

It suffices to show that whenever $2d_p d_q = \|q-p\|^2$, we have

$$2(d_p^2 + d_q^2) + 4\langle z-p, z-q\rangle \geq 4d_p d_q \Leftrightarrow \|2z-p-q\|^2 = s + 2t \geq \|p-q\|^2.$$

The relation $2d_pd_q = \|q-p\|^2$ gives us

$$\|q-p\|^2 = 2d_pd_q \leq d_p^2 + d_q^2 = \|z-p\|^2 + \|z-q\|^2$$

and subsequently $\|2z-p-q\|^2 = 2(\|z-p\|^2 + \|z-q\|^2) - \|p-q\|^2 \geq \|p-q\|^2$, which fully justifies our claim.

Example 3.10. Consider the functions $f_p, f_q : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$f_p(z) = \sqrt{1+\|z-p\|^2},\ f_q(z) = \sqrt{1+\|z-q\|^2},$$

where $z=(x,y)$, $p=(x_0,y_0)$ and $q=(x_1,y_1)$. In other words,

$$f_p(x,y) = \sqrt{1+(x-x_0)^2+(y-y_0)^2}, f_q(x,y) = \sqrt{1+(x-x_1)^2+(y-y_1)^2}.$$

One can easily see that

$$(\nabla f_p)(x,y) = \left(\frac{\partial f_p}{\partial x}(x,y), \frac{\partial f_p}{\partial y}(x,y)\right) = \left(\frac{x-x_0}{f_p(x,y)}, \frac{y-y_0}{f_p(x,y)}\right),$$

$$(\nabla f_q)(x,y) = \left(\frac{\partial f_q}{\partial x}(x,y), \frac{\partial f_q}{\partial y}(x,y)\right) = \left(\frac{x-x_1}{f_q(x,y)}, \frac{y-y_1}{f_q(x,y)}\right),$$

$$\text{i.e. } (\nabla f_p)(z) = \frac{z-p}{f_p(z)} \text{ and } (\nabla f_q)(z) = \frac{z-q}{f_q(z)}$$

as well as

$$H_{(x,y)}(f_p) = \begin{pmatrix} \dfrac{\partial^2 f_p}{\partial x^2} & \dfrac{\partial^2 f_p}{\partial x \partial y} \\ \dfrac{\partial^2 f_p}{\partial x \partial y} & \dfrac{\partial^2 f_p}{\partial y^2} \end{pmatrix} = \begin{pmatrix} \dfrac{1+(y-y_0)^2}{f_p^3(x,y)} & -\dfrac{(x-x_0)(y-y_0)}{f_p^3(x,y)} \\ -\dfrac{(x-x_0)(y-y_0)}{f_p^3(x,y)} & \dfrac{1+(x-x_0)^2}{f_p^3(x,y)} \end{pmatrix},$$

$$H_{(x,y)}(f_q) = \begin{pmatrix} \dfrac{\partial^2 f_q}{\partial x^2} & \dfrac{\partial^2 f_q}{\partial x \partial y} \\ \dfrac{\partial^2 f_q}{\partial x \partial y} & \dfrac{\partial^2 f_q}{\partial y^2} \end{pmatrix} = \begin{pmatrix} \dfrac{1+(y-y_1)^2}{f_q^3(x,y)} & -\dfrac{(x-x_1)(y-y_1)}{f_q^3(x,y)} \\ -\dfrac{(x-x_1)(y-y_1)}{f_q^3(x,y)} & \dfrac{1+(x-x_1)^2}{f_q^3(x,y)} \end{pmatrix}.$$

Therefore

$$\begin{aligned}
\mathrm{Tr}(H(f_pf_q)) &= f_p\Delta f_q + f_q \Delta f_p + 2\langle \nabla f_p, \nabla f_q\rangle \\
&= f\frac{1+f_q^2}{f_q^3} + g\frac{1+f^2}{f_p^3} + \frac{2}{f_pf_q}\langle z-p, z-q\rangle \\
&= \frac{f_p}{f_q^3} + \frac{f_q}{f_p^3} + \frac{f_p}{f_q} + \frac{f_q}{f_p} + \frac{f_p^2+f_q^2-2-\|p-q\|^2}{f_pf_q} \\
&= \frac{(f^2-f_q^2)^2}{f_p^3f_q^3} + 2\left(\frac{f_p}{f_q}+\frac{f_q}{f_p}\right) - \frac{\|p-q\|^2}{f_pf_q} \geq 2\left(\frac{f_p}{f_q}+\frac{f_q}{f_p}\right) - \frac{\|p-q\|^2}{f_pf_q}.
\end{aligned}$$

Note that

$$\begin{aligned} 2\left(\frac{f_p}{f_q}+\frac{f_q}{f_p}\right)-\frac{\|p-q\|^2}{f_pf_q}>0 &\Leftrightarrow 2(f_p^2+f_q^2)>\|p-q\|^2\\ &\Leftrightarrow 2(1+\|z-p\|^2+1+\|z-q\|^2)>\|q-p\|^2\\ &\Leftrightarrow 4+2(\|z-p\|^2+\|z-q\|^2)>\|q-p\|^2. \end{aligned}$$

But $\|q-p\|\leq\|z-q\|+\|z-p\|$ which implies

$$\|q-p\|^2\leq\|z-p\|^2+\|z-q\|^2+2\|z-p\|\cdot\|z-q\|$$

and the inequality

$$\|z-p\|^2+\|z-q\|^2+2\|z-p\|\cdot\|z-q\|<4+2(\|z-p\|^2+\|z-q\|^2)$$

is equivalent with $4+(\|z-q\|-\|z-p\|)^2>0$, which is obvious.

For the calculation of $\det(H(f_pf_q))$ we first observe that

$$\det(H(f_p))=\frac{1}{f_p^4},\quad \det(H(f_q))=\frac{1}{f_q^4},$$

$$\det J(f_p\oplus d_q)=\frac{1}{f_pf_q}[(x-x_0)(y-y_1)-(x-x_1)(y-y_0)]$$

$$\det J\left(\frac{\partial f_p}{\partial x}\oplus\frac{\partial f_q}{\partial y}\right)+\det J\left(\frac{\partial f_q}{\partial x}\oplus\frac{\partial f_p}{\partial y}\right)=\frac{1}{f_pf_q}\left(\frac{f_p^2+f_q^2}{f_p^2f_q^2}+\det{}^2 J(f_p\oplus f_q)\right)$$

$$\begin{aligned} -D(f_q,f_p)&=\frac{\partial f_p}{\partial x}\cdot\frac{\partial f_q}{\partial x}\left(\frac{\partial^2 f_q}{\partial y^2}\oplus\frac{\partial^2 f_p}{\partial y^2}\right)+\frac{\partial f_p}{\partial y}\cdot\frac{\partial f_q}{\partial y}\left(\frac{\partial^2 f_q}{\partial x^2}\oplus\frac{\partial^2 f_p}{\partial x^2}\right)\\ &\quad-\frac{\partial f_p}{\partial x}\cdot\frac{\partial f_q}{\partial y}\left(\frac{\partial^2 f_q}{\partial x\partial y}\oplus\frac{\partial^2 f_p}{\partial x\partial y}\right)-\frac{\partial f_p}{\partial y}\cdot\frac{\partial f_q}{\partial x}\left(\frac{\partial^2 f_q}{\partial x\partial y}\oplus\frac{\partial^2 f_p}{\partial x\partial y}\right)\\ &=\frac{(y-y_0)(y-y_1)}{f_pf_q}\left(\frac{1+(y-y_1)^2}{f_q^3},\frac{1+(y-y_0)^2}{f_p^3}\right)\\ &\quad+\frac{(x-x_0)(x-x_1)}{f_pf_q}\left(\frac{1+(x-x_1)^2}{f_q^3},\frac{1+(x-x_0)^2}{f_p^3}\right)\\ &\quad+\frac{(x-x_0)(y-y_1)}{f_pf_q}\left(\frac{(x-x_1)(y-y_1)}{f_q^3},\frac{(x-x_0)(y-y_0)}{f_p^3}\right)\\ &\quad+\frac{(x-x_1)(y-y_0)}{f_pf_q}\left(\frac{(x-x_1)(y-y_1)}{f_q^3},\frac{(x-x_0)(y-y_0)}{f_p^3}\right)\\ &\quad-2\left\langle f_p\oplus f_q, D(f_q,f_p)\right\rangle\\ &=2[(x-x_0)(x-x_1)+(y-y_0)(y-y_1)]\Big(\frac{1}{f^2}+\frac{1}{f_q^2}\Big). \end{aligned}$$

Therefore

$$\det(H(f_pf_q)) = f_p^2\det(H(f_q)) + f_q^2\det(H(f_p)) - \det{}^2 J(f_p\oplus f_q)$$
$$+ f_pf_q\left(\det J\left(\frac{\partial f_p}{\partial x}\oplus\frac{\partial f_q}{\partial y}\right) + \det J\left(\frac{\partial f_q}{\partial x}\oplus\frac{\partial f_p}{\partial y}\right)\right) - 2\left\langle f_p\oplus f_q, D(f_q,f_p)\right\rangle$$
$$= \frac{f^2}{f_q^4} + \frac{f_q^2}{f_p^4} + \frac{f_p^2+f_q^2}{f_p^2f_q^2}\times\left[\|z-p\|^2 + \|z-q\|^2 + 1 - \|p-q\|^2\right] \tag{13}$$
$$= \frac{f_p^2}{f_q^4} + \frac{f_q^2}{f_p^4} + 2\frac{f_p^2+f_q^2}{f_p^2f_q^2}\times\left[\left\|z-\frac{p+q}{2}\right\|^2 - \frac{\|p-q\|^2}{4} + \frac{1}{2}\right]. \tag{14}$$

Remark 3.11. The version (13) of $\det H(f_pf_q)$ shows that $H(f_pf_q)$ is everywhere positive definite whenever $\|p-q\| < \sqrt{2}$ as $\det H(f_pf_q)$ is positive in such a case and $\mathrm{Tr}H(f_pf_q)$ is everywhere positive anyway. The upper-bound $\sqrt{2}$ for the distance between p and q can be improved up to 2 by keeping the everywhere positive definiteness of $H(f_pf_q)$, as we will see in Proposition 3.13. If $\|p-q\| \geq \sqrt{2}$, then the version (14) of $\det H_z(f_pf_q)$ shows that $H(f_pf_q)$ is positive definite when

$$\left\|z-\frac{p+q}{2}\right\|^2 \geq \frac{\|p-q\|^2}{4}, \quad \text{i.e.} \quad z\in\mathbb{R}^2\setminus B\left(\frac{p+q}{2},\frac{\|p-q\|}{2}\right).$$

In other words, $\mathrm{Hess}^+(f_pf_q)$ is unbounded and

$$\mathbb{R}^2\setminus B\left(\frac{p+q}{2},\frac{\|p-q\|}{2}\right) \subseteq \mathrm{Hess}^+(f_pf_q).$$

Note that $\det H_{(p+q)/2}(f_pf_q) = \frac{16-4\|p-q\|^2}{4+\|p-q\|^2} < 0$, which shows that

$$\det H_{(p+q)/2}(f_pf_q) < 0 \quad \text{whenever} \quad \|p-q\| > 2 \tag{15}$$

and therefore $H(f_pf_q)$ is not definite on a certain open neighbourhood O of $(p+q)/2$ when $\|p-q\| > 2$. In other words

$$\mathbb{R}^2\setminus B\left(\frac{p+q}{2},\frac{\|p-q\|}{2}\right) \subseteq \mathrm{Hess}^+(f_pf_q) \subseteq \mathbb{R}^2\setminus O. \qquad \square$$

Remark 3.12. The product f_pf_q is a Morse function if $\|p-q\| > 2$. Indeed, in this case we have $\det\left(H_{(p+q)/2}(f_pf_q)\right) < 0$, due to (15) and for the critical points

$$z = \frac{p+q}{2} \pm \sqrt{1-\frac{4}{\|p-q\|^2}}\cdot\frac{p-q}{2} = \frac{p+q}{2} \pm \sqrt{\frac{\|p-q\|^2}{4}-1}\cdot\frac{p-q}{\|p-q\|} \tag{16}$$

of f_pf_q, we have, according to (14),

$$\det(H_z(f_pf_q)) = \left(\frac{f_p^2}{f_q^4} + \frac{f_q^2}{f_p^4} - \frac{f_p^2+f_q^2}{f_p^2f_q^2}\right)(z) = \frac{(f_p^2+f_q^2)(f_p^2-f_q^2)^2}{f_p^4f_q^4}(z) > 0. \qquad \square$$

Therefore the critical points (16) are local minima of the product f_pf_q and the midpoint of the segment $[pq]$ is a critical point of index one of f_pf_q. In fact the critical points (16) are global minima of the product f_pf_q.

Proposition 3.13. *If* $\|p-q\|<2$, *then* $\mathrm{Hess}^+(f_pf_q)=\mathbb{R}^2$.

Proof. In order to simplify the computations we first observe that

$$\det(H_z(f_pf_q)) \geq \frac{f_p^2+f_q^2}{f_p^2f_q^2}\Big(2+2\Big\|z-\frac{p+q}{2}\Big\|^2-\frac{\|p-q\|^2}{2}\Big),$$

with equality if and only if $f=g$ as

$$\frac{f_p^2}{f_q^4}+\frac{f_q^2}{f_p^4}\geq\frac{1}{f_p^2}+\frac{1}{f_q^2}\Leftrightarrow\frac{f_p^2}{f_q^4}+\frac{f_q^2}{f_p^4}-\frac{1}{f_p^2}-\frac{1}{f_q^2}\Leftrightarrow\frac{(f_p^2+f_q^2)(f_p^2-f_q^2)^2}{f_p^4f_q^4}\geq 0.$$

In particular, for every $u\in S^1$, we have

$$\begin{aligned}
&\left(1+\left\|\frac{q-p}{2}+tu\right\|^2\right)\left(1+\left\|\frac{p-q}{2}+tu\right\|^2\right)\det(H_{(p+q)/2+tu}(f_pf_q))\geq\\
&\geq\left(2+\left\|\frac{q-p}{2}+tu\right\|^2+\left\|\frac{p-q}{2}+tu\right\|^2\right)\left(2+2t^2-\frac{\|p-q\|^2}{2}\right)\\
&=\left(2+\frac{\|p-q\|^2}{2}+2t^2\right)\left(2+2t^2-\frac{\|p-q\|^2}{2}\right)\\
&=4(1+t^2)^2-\frac{\|p-q\|^4}{4}=\frac{16(1+t^2)^2-\|p-q\|^4}{4}\geq\frac{16-\|p-q\|^4}{4}>0,
\end{aligned}$$

for every $u\in S^1$ and $\|p-q\|<2$. If $z\in\mathbb{R}^2\setminus\{(p+q)/2\}$ is an arbitrary point, then

$$z=\frac{p+q}{2}+tu,\quad\text{where}\quad u=\frac{z-\frac{p+q}{2}}{\left\|z-\frac{p+q}{2}\right\|}\quad\text{and}\quad t=\left\|z-\frac{p+q}{2}\right\|.$$

Therefore

$$\det H_z(f_pf_q)>0\quad\text{and}\quad\det H_{(p+q)/2}(f_pf_q)=\frac{16-4\|p-q\|^2}{4+\|p-q\|^2}>0,$$

whenever $\|p-q\|<2$. □

Remark 3.14. If $\|p-q\|<2$, then f_pf_q is a strictly convex function as we have $\mathrm{Hess}^+(f_pf_q)=\mathbb{R}^2$ in such a case. The product f_pf_q remains strictly convex even when $\|p-q\|=2$, although

$$\mathrm{Hess}^+(f_pf_q)=\mathbb{R}^2\setminus\left\{\frac{p+q}{2}\right\}$$

in such a case. Indeed the restriction of f_pf_q to every segment of $\mathbb{R}^2$ has everywhere positive second order derivative except for at most one point where it could be zero.

The boundary of $\mathrm{Hess}^+(f_pf_q)$ is the curve $\det H(f_pf_q)=0$, i.e.

$$\begin{aligned}\frac{f_p^2}{f_q^4}+\frac{f_q^2}{f_p^4}+\frac{f_p^2+f_q^2}{f_p^2f_q^2}&\times[1+2(x-x_0)(x-x_1)+2(y-y_0)(y-y_1)]=0\\ \Leftrightarrow\ f_p^4+f_q^4&+2f_p^2f_q^2[(x-x_0)(x-x_1)+(y-y_0)(y-y_1)]=0. \qquad (17)\end{aligned}$$

The unbounded component of its complement is precisely $\mathrm{Hess}^+(f_pf_q)$. □

Remark 3.15. When $\|p-q\|>2$, the convex level sets of the product f_pf_q will be selected among those of $f_p^2f_q^2=1+d_p^2+d_q^2+d_p^2d_q^2$ as the two products f_pf_q and $f_p^2f_q^2$ have the same level sets. Indeed $(f_pf_q)^{-1}(c)=(f_p^2f_q^2)^{-1}(c^2)$. Moreover

$$\begin{aligned}\mathrm{Tr}(H_z(f_p^2f_q^2))&=\mathrm{Tr}(H_z(d_p^2+d_q^2))+\mathrm{Tr}(H_z(d_p^2d_q^2))\\&=8+4\|2z-p-q\|^2>0,\ \forall z\in\mathbb{R}^2.\end{aligned}$$

We also have

$$\begin{aligned}\det(H_z(f_p^2f_q^2))&=\det(4I_2+H_z(d_p^2d_q^2))=4[4+4(d_p^2+d_q^2)+2\left\langle\nabla d_p^2,\nabla d_q^2\right\rangle]\\&\quad+[2(d_p+d_q)^2+\left\langle\nabla d_p^2,\nabla d_q^2\right\rangle]\times[2(d_p-d_q)^2+\left\langle\nabla d_p^2,\nabla d_q^2\right\rangle]\\&=[\left\langle\nabla d_p^2,\nabla d_q^2\right\rangle+4(d_p-d_q)^2+4][\left\langle\nabla d_p^2,\nabla d_q^2\right\rangle+4(d_p+d_q)^2+4].\end{aligned}$$

Thus, $H_z(f_p^2f_q^2)$ is positive definite whenever

$$\left\langle\nabla d_p^2,\nabla d_q^2\right\rangle+4(d_p-d_q)^2+4>0.$$

Therefore $2(d_p-d_q)^2+\left\langle\nabla d_p^2,\nabla d_q^2\right\rangle>0$ implying $\left\langle\nabla d_p^2,\nabla d_q^2\right\rangle+4(d_p-d_q)^2+4>0$, which shows, via Remark 3.5(1), that $\mathrm{Hess}^+(d_p^2d_q^2)\subseteq\mathrm{Hess}^+(f_p^2f_q^2)$. □

The convex levels of $f_p^2f_q^2$ are those along which the determinant

$$\begin{aligned}\frac{1}{8}\,D_3(f_p^2f_q^2)&=\frac{1}{8}\begin{vmatrix}(f_p^2f_q^2)_{xx} & (f_p^2f_q^2)_{xy} & (f_p^2f_q^2)_x\\ (f_p^2f_q^2)_{xy} & (f_p^2f_q^2)_{yy} & (f_p^2f_q^2)_y\\ (f_p^2f_q^2)_x & (f_p^2f_q^2)_y & 0\end{vmatrix}\\&=4(d_p^2d_q^2-c^2)(1+d_p^2)(1+d_q^2)\\&\quad-(2+d_p^2+d_q^2)[2c(1+d_p^2)(1+d_q^2)+d_p^4d_q^2+d_p^2d_q^4+4d_p^2d_q^2+d_p^2+d_q^2]\end{aligned}$$

keeps constant sign, where $m=x-x_0, n=y-y_0, u=x-x_1, v=y-y_1$ and $c=mu+nv=\langle z-p,z-q\rangle$. In this respect we shall find some useful information on the local extrema of the restriction of the function

$$D:=\frac{1}{8}D_3(f_p^2f_q^2)$$

to the curve $f_p^2f_q^2=k$. Note that such level curves of the product $f_p^2f_q^2$ are convex and do not contain critical points of D as $D_3(f_p^2f_q^2)$ vanishes at such points. They are also compact due to the norm-coercivity of the product $d_p^2d_q^2$ itself, i.e.

$$\lim_{\|z\|\to\infty}(f_p^2f_q^2)(z)=\infty.$$

Theorem 3.16. *If we have $\|p-q\|>2$, then the the level curves of the product $f_p^2 f_q^2=(1+d_p^2)(1+d_q^2)$ have the following properties:*

(1) *For $k\in\left(\|p-q\|^2,\left(1+\frac{\|p-q\|^2}{4}\right)^2\right)$ the level curve $f_p^2 f_q^2=k$ is compact but not connected.*

(2) *For $k\in\left(\left(1+\frac{\|p-q\|^2}{4}\right)^2,\frac{\|p-q\|^4}{4}\right)$, every connected component of the level curve $f_p^2 f_q^2=k$ is compact and convex.*

(3) *For $k>\left(1+\frac{\|p-q\|^2}{4}\right)^2$ the level curve $f_p^2 f_q^2=k$ is a compact connected regular curve, diffeomorphic with the circle S^1.*

(4) *For $k\geq\|p-q\|^4/4$ the level curve $d_p^2 d_q^2=k$ is compact and convex.*

Proof. The compactness of all level sets of the product $d_p^2 d_q^2$ follows due to its norm-coercivity.

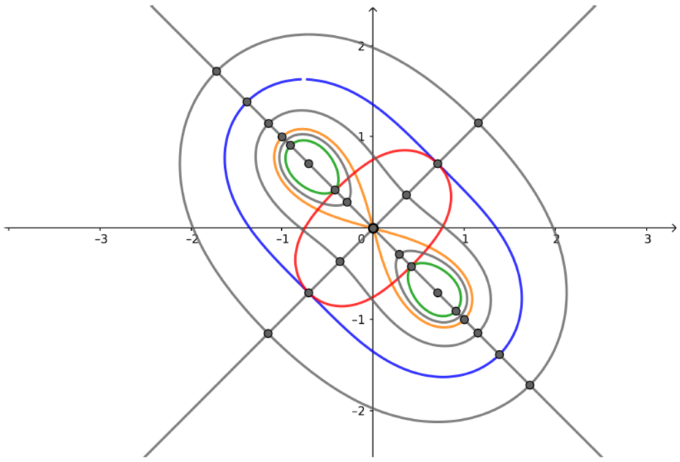

Figure 3.3: Two non-connected negative regular levels of the product $f_p f_q$ with two components each, a nontrivial critical level curve of $f_p^2 f_q^2$, a nonconvex connected regular level, the first convex regular level and another convex connected regular level. The boundary of the region $\mathrm{Hess}^+(f_p f_q)$ for $p=(1,-1)$, $q=(-1,1)$.

We start our arguments by finding the constrained local extrema of D subject to some level curve $f_p^2 f_q^2=k$ constraint. We shall use the standard technique of Lagrange multipliers via the Lagrange function

$$\mathcal{L}(x,y,\lambda)=D(x,y)-\lambda(f_p^2(x,y)f_q^2(x,y)-k).$$

Note that $D(x,y)=\mathcal{L}(x,y,0)$ and therefore

$$\operatorname{sgn}(D_3(f_p^2 f_q^2))=\operatorname{sgn}(D)=\operatorname{sgn}(\mathcal{L}(x,y,0)).$$

The local extrema of D subject to the constraint $f_p^2 f_q^2=k$ are part of the solutions of the equation $\nabla\mathcal{L}(x,y,\lambda)=0$ which is equivalent to the following system

$$\begin{cases} D_x-2\lambda(md_q^2+ud_p^2+m+u)=0\\ D_y-2\lambda(nd_q^2+vd_p^2+n+v)=0\\ f_p^2 f_q^2=k,\end{cases}$$

By multiplying the first equation with $nd_v^2 + qd_p^2 + n + v$ and the second one with $-(md_q^2 + vd_p^2 + m + u)$ and adding them afterwards, we obtain

$$\begin{aligned}
&D_x(nd_q^2 + vd_p^2 + n + v) - D_y(md_q^2 + ud_p^2 + m + u) = 0 \\
\iff\ & (d_p^2 - d_q^2)(mv - nu)[-8kc - 16k - 4ck - 2d_p^2 d_q^2 s \\
&\quad - 8d_p^2 d_q^2 + 12s + 16 - 4sk + 3s^2] = 0, \\
\iff\ & mv = nu \ \text{ or } d_p = d_q \ \text{ or} \\
&\quad - 8kc - 16k - 4ck - 2d_p^2 d_q^2 s - 8d_p^2 d_q^2 + 12s + 16 - 4sk + 3s^2] = 0
\end{aligned}$$

where $s = d_p^2 + d_q^2$. The last factor

$$2(2s^2 + 7s - d_p^2 d_q^2 s - 2ks - 6ck - 4d_p^2 d_q^2 - 8k + 8)$$

vanishes only when

$$2s^2+7s-(k-s-1)s-2ks-6ck-4(k-s-1)-8k+8 = 0 \ \Leftrightarrow\ c = \frac{(s+2)^2 - k(s+4)}{2k}.$$

Along the curve defined by the last factor we have

$$\begin{aligned}
D &= 4k\left[k - s - 1 - \left(\frac{(s+2)^2 - k(s+4)}{2k}\right)^2\right] \\
&\quad - (2+s)[(s+2)^2 - k(s+4) + (s+4)(k-s-1) + s] \\
&= \frac{(s-k+2)^2[4k - (s+2)^2]}{4k} \le 0.
\end{aligned}$$

For $mv = nu$ we have $c^2 = d_p^2 d_q^2 - (mv - nu)^2 = k - s - 1$, namely

$$\begin{aligned}
\mathcal{L}(x,y,0) &= -(2+s)[2ck + 4c^2 + (c^2+1)s] \\
&= -(2+s)(2c^3 + 2cs + 2c + 4c^2 + c^2 s + s) = -(s+2)(2c+s)(c+1)^2.
\end{aligned}$$

We are now looking for the extreme points of the intersection between the straight line $pq : mv = nu$ and the level curve $f_p^2 f_q^2 = k$, i.e.

$$(1 + d_p^2)(1 + d_q^2) = k.$$

On the points of the line pq, parameterized by the equation

$$z = \frac{p+q}{2} + t(q-p),\ \ t \in \mathbb{R},$$

we have
$$d_p^2\left(\frac{p+q}{2} + t(q-p)\right) = \left(t + \frac{1}{2}\right)^2 \|q-p\|^2$$

and
$$d_q^2\left(\frac{p+q}{2} + t(q-p)\right) = \left(t - \frac{1}{2}\right)^2 \|q-p\|^2.$$

We also have

$$c = \langle z-p, z-q\rangle = \left\langle t(q-p)+\frac{q-p}{2}, t(q-p)-\frac{q-p}{2}\right\rangle$$
$$= t^2\|q-p\|^2 - \frac{\|q-p\|^2}{4} = \left(t^2-\frac{1}{4}\right)\|q-p\|^2 \qquad (18)$$

which shows that $\dfrac{2c+s}{\|q-p\|^2} = 4t^2$, namely $s+2>0$ and $2c+s \geq 0$ and therefore D is nonpositive.

We are next looking for the extreme points of the intersection between the perpendicular bisector of the segment $[pq] : d_p = d_q$ and the level curve

$$f_p^2 f_q^2 = k \Longleftrightarrow (1+d_p^2)(1+d_q^2) = k, \ \textit{i.e.}\ d_p = d_q = \sqrt{k}-1.$$

But for $d_p = d_q = \sqrt{k}-1$ we have

$$\mathcal{L}(x,y,0) = 4k[(\sqrt{k}-1)^2 - c^2] - 2\sqrt{k}[2ck + (\sqrt{k}-1)^2(2+2\sqrt{k}) + 2\sqrt{k}-2]$$
$$= -4k(c+1)(c+\sqrt{k}-1).$$

The extreme points are located at the intersection between the level curve

$$(1+d_p^2)(1+d_q^2) = k$$

and the perpendicular bisector $d_p = d_q$ of the segment $[pq]$, which is parameterized by the equation

$$z = \frac{p+q}{2} + tR_{\pi/2}(q-p), \ t \in \mathbb{R},$$

where $R_{\pi/2} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ stands for the rotation of angle θ and is an isometry of $\mathbb{R}^2$. One can easily see that the values of t which correspond to the intersection points between the level curve

$$(1+d_p^2)(1+d_q^2) = k \Longleftrightarrow (1+\|z-p\|^2)(1+\|z-q\|^2) = k$$

and the perpendicular bisector of the segment $[pq]$ are the solutions of the equation

$$t^2 = \frac{\sqrt{k}-1}{\|q-p)\|^2} - \frac{1}{4}.$$

It follows that c at the intersection points between the perpendicular bisector of the segment $[pq] : d_p = d_q$ and the curve $(1+d_p^2)(1+d_q^2) = k$ is

$$c = \langle z-p, z-q\rangle = \left\langle tR_{\pi/2}(q-p)+\frac{q-p}{2}, tR_{\pi/2}(q-p)-\frac{q-p}{2}\right\rangle$$
$$= t^2\|R_{\pi/2}(q-p)\|^2 - \frac{\|q-p\|^2}{4} = \left(t^2-\frac{1}{4}\right)\|q-p\|^2$$
$$= \left(\frac{\sqrt{k}-1}{\|q-p)\|^2} - \frac{1}{2}\right)\|q-p\|^2 = \sqrt{k}-1-\frac{\|q-p\|^2}{2}. \qquad (19)$$

Therefore $\quad \mathcal{L}(x,y,0) = -8k\left(\sqrt{k} - \frac{\|p-q\|^2}{2}\right)\left(\sqrt{k} - 1 - \frac{\|p-q\|^2}{4}\right).$

If $\|q-p\| > 2$, then for the values of D over the intersection points between the perpendicular bisector of the segment $[pq] : d_p = d_q$ and the curve $(1+d_p^2)(1+d_q^2) = k$ we have:

(1) For $k \in \left(1, \left(1 + \frac{\|p-q\|^2}{4}\right)^2\right)$, these values of D are negative.

(2) For $k \in \left(\left(1 + \frac{\|p-q\|^2}{4}\right)^2, \frac{\|p-q\|^4}{4}\right)$ these values of D are positive.

(3) For $k \in \left(\frac{\|p-q\|^4}{4}, \infty\right)$, these values of D are negative.

By using a similar approach to the one which we used when we studied the level curves of the product $d_p^2 d_q^2$ one can show that the k-level curves of the product $f_p^2 f_q^2 = (1+d_p^2)(1+d_q^2)$ are not connected for

$$k \in \left(\|p-q\|^2, \left(1 + \frac{\|p-q\|^2}{4}\right)^2\right), \tag{20}$$

and convex connected for

$$k \in \left(\frac{\|p-q\|^4}{4}, \infty\right). \tag{21}$$

Note that the inverse image of $\|p-q\|^2$ through $f_p^2 f_q^2 = (1+d_p^2)(1+d_q^2)$ reduces to its global minima, i.e. the critical points

$$z = \frac{p+q}{2} \pm \sqrt{1 - \frac{4}{\|p-q\|^2}} \cdot \frac{p-q}{2} = \frac{p+q}{2} \pm \sqrt{\frac{\|p-q\|^2}{4} - 1} \cdot \frac{p-q}{\|p-q\|}.$$

The constant sign of D over the k-levels (20) of $f_p^2 f_q^2 = (1+d_p^2)(1+d_q^2)$, which are nonconnected, show the convexity of every connected component of such a level curve. Note that for

$$k \in \left(\left(1 + \frac{\|p-q\|^2}{4}\right)^2, \frac{\|p-q\|^4}{4}\right) \tag{22}$$

the k-level of the product $f_p^2 f_q^2 = (1+d_p^2)(1+d_q^2)$ remains connected, via the Non-Critical Neck Principle (see e.g. [3, p. 194]), but not convex as D changes its sign over it, being positive on its intersection with the perpendicular bisector of the segment $[pq]$ and negative over the intersection of the level curve $(1+d_p^2)(1+d_q^2) = k$ and the line pq. On the other hand the connected k-levels (21) of the product $f_p^2 f_q^2 = (1+d_p^2)(1+d_q^2)$ are also convex. The level curve

$$(1+d_p^2)(1+d_q^2) = \frac{\|p-q\|^4}{4} \tag{23}$$

remains convex as the argument of Remark 3.9 works to justify this statement as well. In fact (23) is the first convex level curve of the product $f_p^2 f_q^2 = (1+d_p^2)(1+d_q^2)$ completely contained in $\mathrm{cl}\,\mathrm{Hess}^+(f_p f_q)$. In order to completely justify this, we need to show (see (17)) that

$$f_p^4 + f_q^4 + 2f_p^2 f_q^2[(x-x_0)(x-x_1) + (y-y_0)(y-y_1)] \geq 0$$

over the level curve

$$f_p^2 f_q^2 = \frac{\|p-q\|^4}{4}$$

of the product $f_p^2 f_q^2$. In this respect we only need to show that

$$(x-x_0)(x-x_1)+(y-y_0)(y-y_1) \geq -1 \tag{24}$$

over the level curve

$$f_p^2 f_q^2 = \frac{\|p-q\|^4}{4} \tag{25}$$

of the product $f_p^2 f_q^2$. Indeed, in such a case

$$f_p^4 + f_q^4 + 2f_p^2 f_q^2 \langle z-p, z-q\rangle \geq (f_p^2 - f_q^2)^2 \geq 0.$$

In order to solve the constrained problem (24)+(25), we shall use the standard Lagrange multipliers technique for the Lagrange function

$$\mathcal{L}(x,y,\lambda) = (x-x_0)(x-x_1)+(y-y_0)(y-y_1) - \lambda\Big(f_p^2(x,y)f_q^2(x,y) - \frac{\|p-q\|^4}{4}\Big).$$

Indeed, the constrained local extrema of the function

$$c = \langle z-p, z-q\rangle = (x-x_0)(x-x_1)+(y-y_0)(y-y_1)$$

subject to constraint (25) are part of the solution set of the equation $\nabla\mathcal{L}(x,y,\lambda)=0$ which is equivalent to the following system

$$\begin{cases} x - \dfrac{x_0+x_1}{2} - \lambda\left(f_p f_q^2 \dfrac{\partial f_p}{\partial x} + f_q f_p^2 \dfrac{\partial f_q}{\partial x}\right) = 0 \\ y - \dfrac{y_0+y_1}{2} - \lambda\left(f_p f_q^2 \dfrac{\partial f_p}{\partial y} + f_q f_p^2 \dfrac{\partial f_q}{\partial y}\right) = 0 \\ f_p^2(x,y) f_q^2(x,y) = \dfrac{\|p-q\|^4}{4}. \end{cases} \tag{26}$$

By multiplying the first and second equation with

$$f_p f_q^2 \frac{\partial f_p}{\partial y} + f_q f_p^2 \frac{\partial f_q}{\partial y} = \frac{1}{2}\left(f_q^2 \frac{\partial f_p^2}{\partial y} + f_p^2 \frac{\partial f_q^2}{\partial y}\right) = (y-y_0)f_q^2 + (y-y_1)f_p^2 \tag{27}$$

and

$$-f_p f_q^2 \frac{\partial f_p}{\partial x} - f_q f_p^2 \frac{\partial f_q}{\partial x} = \frac{1}{2}\left(f_q^2 \frac{\partial f_p^2}{\partial x} + f_p^2 \frac{\partial f_q^2}{\partial x}\right) = (x-x_0)f_q^2 + (x-x_1)f_p^2$$

respectively, and then adding them we obtain

$$(y_0-y_1)x + (x_1-x_0)y + x_0y_1 - x_1y_0 = 0 \Longleftrightarrow d_p(x,y) = d_q(x,y).$$

Thus, the constrained local extrema of the function $c = \langle z-p, z-q\rangle$ subject to the constraint

$$f_p^2(x,y)f_q^2(x,y) = \frac{\|p-q\|^4}{4}$$

are at the intersection between the line pq with the level curve (25) of the product $f_p^2 f_q^2$ as well as at the intersection between the perpendicular bisector of the segment $[pq] : d_p = d_q$ with the level curve (25) of the product $f_p^2 f_q^2$. The value of $c = \langle z-p, z-q\rangle$ over the intersection of the perpendicular bisector of the segment $[pq]$ with the level curve (25) of $f_p^2 f_q^2$ is -1, according to (19), and the intersection between the line pq with the level curve (25) of the product $f_p^2 f_q^2$ can be found by combining (18) with the equation (25) itself and by taking into account that the line pq can be parameterized by the equation

$$pq: \; z = \frac{p+q}{2} + t(q-p).$$

Therefore we have

$$\begin{aligned}
f_p^2 f_q^2 = \frac{\|p-q\|^4}{4} &\iff (1+d_p^2)(1+d_q^2) = \frac{\|p-q\|^4}{4} \\
&\iff (1+\|z-p\|^2)(1+\|z-q\|^2) = \frac{\|p-q\|^4}{4} \\
&\iff \left(t^2 - \frac{1}{4}\right)^2 \|q-p\|^4 + 2\left(t^2-\frac{1}{4}\right)\|q-p\|^2 + 1 + \|q-p\|^2 - \frac{\|p-q\|^4}{4} = 0 \qquad (28) \\
&\iff c^2 + 2c + 1 + \|q-p\|^2 = \frac{\|p-q\|^4}{4} \\
&\iff c+1 = \pm\frac{\|q-p\|\sqrt{\|q-p\|^2-4}}{2}.
\end{aligned}$$

The version with the minus sign is unacceptable as its counterpart of the solution for t^2 of the equation (28), leads us to

$$t^2 = \frac{1}{4} - \frac{1 + \|q-p\|\sqrt{\frac{\|q-p\|^2}{4} - 1}}{\|q-p\|^2},$$

whose right hand side is less than zero for $\|q-p\| > 2$. Therefore the value of c over the intersection between the line pq with the level curve (25) is

$$c = -1 + \frac{\|q-p\|\sqrt{\|q-p\|^2-4}}{2} > -1.$$

Thus, the maximum of the function

$$c = \langle z-p, z-q\rangle = (x-x_0)(x-x_1) + (y-y_0)(y-y_1)$$

subject to the constraint (25) is

$$-1 + \frac{\|q-p\|\sqrt{\|q-p\|^2-4}}{2}.$$

and its minimum is -1, i.e. the constrained inequality (24) is now completely justified. $\square$

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