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More on Connexions Between Subdifferential and Weak Compactness of Sublevels

Pedro Pérez-Aros
Instituto de Ciencias de la Ingeniería, Universidad de O'Higgins, Rancagua, Chile
pedro.perez@uoh.cl

Lionel Thibault
Institut Montpelliérain Alexander Grothendieck, Université de Montpellier, France; and: Centro de Modelamiento Matematico, Universidad de Chile, Santiago, Chile
lionel.thibault@umontpellier.fr

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We show how our previous paper "Weak compactness of sublevel sets in complete locally convex spaces" [J. Convex Analysis 26/3 (2019) 739–751] allows to derive with quite direct arguments a large number of recent results established in Banach spaces for connexions between subdifferential and weak compactness of sublevel sets. In fact, the arguments extend the results to locally convex spaces.

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1. Introduction

Given a real Hausdorff complete locally convex space (X,θ) with topological dual X^* and given a set $K \subset X^*$ with $0 \in K$, define $\mathcal{E}(K)$ as the class of all functions $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ such that

(i) $\varphi^* : X^* \to \mathbb{R} \cup \{+\infty\}$ with $\varphi^*(0) = 0$;

(ii) $K_\infty \subset \operatorname{dom} \varphi^* \subset K$;

(iii) $\sup\{\varphi^*(\eta x^*) : \eta > 0, \eta x^* \in \operatorname{dom} \varphi^*\} = +\infty$, for all $x^* \in K \setminus K_\infty$;

(iv) $(\partial\varphi)^{-1}(x^*) \neq \emptyset$ for all $x^* \in \operatorname{dom} \varphi^*$.

In Theorem 3.2 of our 2019 published paper [8] we proved the following:

Theorem 1.1. ([8]) *With the Hausdorff complete locally convex space (X,θ), let $K \subset X^*$ with $0 \in K$, $\mathbb{R}_+ K = X^*$ and $\mathcal{E}(K) \neq \emptyset$, and let $g : X \to \mathbb{R} \cup \{+\infty\}$ be a function such that $(\partial g)^{-1}(x^*) \neq \emptyset$ for all $x^* \in K$. Then for every $\gamma \in \mathbb{R}$ and every $\varphi \in \mathcal{E}(K)$ the sets*

$$\begin{aligned}
\operatorname{Ep}_g^\varphi(\gamma) &:= \{(x,\alpha) \in X \times \mathbb{R} : (x,\alpha) \in \operatorname{Epi} g + \operatorname{Epi}_\varphi \text{ and } \alpha \leq \gamma\},\\
S_{g\Box\varphi}(\gamma) &:= \{(x,\alpha) \in X \times \mathbb{R} : g\Box\varphi(x) \leq \alpha \leq \gamma\},\\
S_g(\gamma) &:= \{(x,\alpha) \in \times\mathbb{R} : g(x) \leq \alpha \leq \gamma\},\\
\operatorname{lev}_g^{\leq}(\gamma) &:= \{x \in X : g(x) \leq \gamma\}
\end{aligned}$$

are all relatively weakly compact.

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Concerning notation and concepts above, $\mathbb{R}_+$ stands for the interval $[0, +\infty[$ and we recall that, for a set K in the topological dual Y^* of a (real) Hausdorff locally convex space (Y, θ_Y) with $0 \in K$, a function $g : U \to \mathbb{R} \cup \{-\infty, +\infty\}$ on a convex set U of Y, and functions $g_1, g_2 : Y \to \mathbb{R} \cup \{+\infty\}$, one defines the *directional asymptotic cone* as

$$K_\infty := \{y^* \in Y^* : \forall \lambda \geq 0,\ \lambda y^* \in K\},$$

the *infimal convolution* $g_1 \Box g_2$ by

$$g_1 \Box g_2(y) := \inf\{g_1(y_1) + g_2(y_2) : y_1, y_2 \in Y,\ y_1 + y_2 = y\},$$

the *epigraph* of g by $\operatorname{Epi} g := \{(y, \alpha) \in U \times \mathbb{R} : g(y) \leq \alpha\}$;

the γ*-sublevel set* of g as given above with $\gamma \in \mathbb{R}$ by

$$\operatorname{lev}_g^{\leq}(\gamma) := \{x \in U : g(x) \leq \gamma\},$$

the *effective domain* of g by $\operatorname{dom} g := \{y \in U : g(y) < +\infty\}$,

the *Legendre-Fenchel conjugate* of g by

$$g^*(y^*) := \sup\{\langle y^*, y\rangle - g(y) : y \in U\} \quad \text{for all } y^* \in Y^*,$$

and the *Moreau-Rockafellar subdifferential* multimapping $\partial g : U \rightrightarrows Y^*$ by

$$\partial g(y) := \{y^* \in Y^* : \langle y^*, z - y\rangle + g(y) \leq g(z),\ \forall z \in U\} \text{ if } |g(y)| < +\infty$$

and $\partial g(y) = \emptyset$ if $|g(y)| = +\infty$. The *inverse multimapping* $(\partial g)^{-1} : Y^* \rightrightarrows U \subset Y$ is given by

$$y \in (\partial g)^{-1}(y^*) \iff y^* \in \partial g(y).$$

It is worth noticing that

$$(\partial g)^{-1}(y^*) \neq \emptyset \iff g - \langle y^*, \cdot\rangle \text{ attains its minimum on } U. \tag{1}$$

For $\varepsilon > 0$ we will also employ the ε*-approximate subdifferential*

$$\partial_\varepsilon g(y) := \{y^* \in Y^* : \langle y^*, z - y\rangle + g(y) \leq g(z) + \varepsilon,\ \forall z \in U\} \text{ if } |g(y)| < +\infty$$

and $\partial_\varepsilon g(y) = \emptyset$ if $|g(y)| = +\infty$.

The *indicator function* δ_C of a set C of Y will also be utilized, that is, the function from Y into $\mathbb{R} \cup \{+\infty\}$ defined by

$$\delta_C(y) = 0 \quad \text{if } y \in C \quad \text{and} \quad \delta_C(y) = +\infty \quad \text{if } y \in Y \setminus C.$$

The Legendre-Fenchel conjugate δ_C^* of δ_C coincides with the support function σ_C defined on Y^* by $\sigma_C(y^*) := \sup_{y \in C}\langle y^*, y\rangle$ for all $y^* \in Y^*$. The *polar* and *strict polar* of C are the sets

$$C^\circ := \{y^* \in Y^* : \sigma_C(y^*) \leq 1\} \quad \text{and} \quad C_s^\circ := \{y^* \in Y^* : \sigma_C(y^*) < 1\}.$$

Suppose that g is defined on the whole space Y. The θ_Y closed convex hull $\overline{\text{co}}^{\theta_Y}\, g$ of the function g is the supremum of all (extended real-valued) lower θ_Y semicontinuous convex functions on Y majorized by g, so $\overline{\text{co}}^{\theta_Y}\, g \le g$. One knows that

$$\operatorname{Epi}(\overline{\text{co}}^{\theta_Y}\, g) = \operatorname{cl}_{\theta_Y \times \mathcal{O}_{\mathbb{R}}}\big(\operatorname{Epi} g\big) \ \text{ and } \ \big(\overline{\text{co}}^{\theta_Y}\, g\big)^* = g^*, \tag{2}$$

where $\theta_Y \times \mathcal{O}_{\mathbb{R}}$ is the product topology of θ_Y and the canonical topology $\mathcal{O}_{\mathbb{R}}$ of $\mathbb{R}$. When there is no risk of ambiguity for the topology θ_Y one writes $\overline{\text{co}}\, g$.

We need to state another result established as Theorem 3.9 of our paper [8]. Recall that an extended real-valued function is *infcompact* for a topology if all its sublevel sets are compact for this topology.

Theorem 1.2. ([8]) *Keeping (X, θ) as a Hausdorff complete locally convex space, let U be a nonempty open set in X^* with respect to the Mackey topology and let $g : X \to \mathbb{R} \cup \{+\infty\}$ be a function such that $(\partial g)^{-1}(x^*) \neq \emptyset$ for every $x^* \in U$. Then for every $x^* \in U$ and every $\gamma \in \mathbb{R}$ the set*

$$\{x \in X : g(x) - \langle x^*, x\rangle \le \gamma\}$$

is relatively weakly compact in X.

We will also utilize the following lemma which is a direct consequence of the proof of the second part of Lemma 3.5 in our paper [8].

Lemma 1.3. *Let (Y, θ) be a Hausdorff locally convex space and K be a relatively weakly compact symmetric convex set in Y. Then there exists a weakly inf-compact convex function $\varphi \in \mathcal{E}(K_s^\circ)$ with $\varphi^* \ge 0$ and* $\operatorname{dom}\varphi^* = K_s^\circ$.

Our aim in this note is to illustrate how Theorem 1.1 and Theorem 1.2 allow to derive many recent results in the literature on weak compactness in a Banach space X of sublevel sets and on the inclusion in X of subdifferentials of functions defined on open convex sets of the topological dual Banach space X^*. The results will even be extended to Hausdorff complete locally convex spaces.

2. Preliminaries

Throughout this section and the next ones unless otherwise stated, (X, θ) is a real Hausdorff complete locally convex space and X^* is its topological dual. The duality bracket $\langle \cdot, \cdot \rangle : X^* \times X \to \mathbb{R}$ is given by $\langle x^*, x\rangle = x^*(x)$. In X^* we will consider the Mackey topology $\tau(X^*, X)$ (resp. the strong topology $\beta(X^*, X)$), that is, the topology in X^* of uniform convergence on convex symmetric weakly compact (resp. on convex symmetric bounded) subsets of X. The topological dual of $(X^*, \beta(X^*, X))$ is the topological bidual denoted (classically) by X^{**}. The space (X, θ) is embedded in X^{**} through the mapping $\widehat{\ } : X \to X^{**}$ with $\widehat{x} : X^* \to \mathbb{R}$ given by

$$\widehat{x}(x^*) = \langle x^*, x\rangle \quad \text{for all } x^* \in X^*,$$

so $\langle \widehat{x}, x^*\rangle = \langle x^*, x\rangle$. Now, for a set $C \subset X$, we define $\widehat{C} := \{\widehat{x} : x \in C\}$.

As usual, the *weak topology* on X is denoted by $w(X, X^*)$ and the *weak* topology* on X^* by $w(X^*, X)$. The topology $w(X^{**}, X^*)$ is called the weak** topology on X^{**}.

It is known (and not difficult to see) that the topology induced on X by $w(X^{**}, X^*)$ coincides with the weak topology $w(X, X^*)$.

Given a function $f : U \to \mathbb{R} \cup \{+\infty\}$ on a convex set U of X^*, unless otherwise stated the approximate subdifferentials and Legendre-Fenchel conjugate of f are taken for X^* endowed with the strong $\beta(X^*, X)$ topology, so one has $\partial f : U \rightrightarrows X^{**}$ and $\partial_\varepsilon f : U \rightrightarrows X^{**}$, that is,

$$\partial f(x^*) \subset X^{**} \text{ and } \partial_\varepsilon f(x^*) \subset X^{**} \quad \text{for all } x^* \in U. \tag{3}$$

The *Legendre-Fenchel conjugate* of f is then the function $f^* : X^{**} \to \mathbb{R}\cup\{-\infty, +\infty\}$ with

$$f^*(x^{**}) = \sup\{\langle x^{**}, x^*\rangle - f(x^*) : x^* \in U\} \quad \text{for all } x^{**} \in X^{**}.$$

We establish a lemma which will be utilized in our development. To state the lemma, let us recall that given a set $A \subset X$ one denotes its *algebraic interior* by

$$A^i := \{x^* \in X^* : \forall y^* \in X^*, \exists \eta > 0 \text{ such that } x^* + \lambda y^* \in A,\ \forall \lambda \in [0, \eta]\}.$$

We say that A is an *algebraic open set* provided that $A^i = A$.

Remark 2.1. Here, it is important to mention that for any set $A \subseteq X$, we have that $\operatorname{Int}_\theta(A) \subseteq A^i$, where θ is a Hausdorff locally convex topology on X^*. Furthermore, when A is convex we have that $A^i = \operatorname{Int}_\theta(A)$ provided that $\operatorname{Int}_{\theta_Y}(A)$ is nonempty. Indeed, the inclusion $\operatorname{Int}_\theta(A) \subseteq A^i$ follows directly from the definition of algebraic interior. The converse inclusion can be found in [9, Theorem 1.1.2].

Lemma 2.2. *Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be a function such that $\varphi^*(0) = 0$, $\operatorname{dom} \varphi^*$ is an algebraic open set and $(\partial\varphi)^{-1}(x^*) \neq \emptyset$ for all $x^* \in \operatorname{dom}\varphi^*$. Then $\varphi \in \mathcal{E}(\operatorname{dom}\varphi^*)$.*

Proof. Let us denote $K := \operatorname{dom}\varphi^*$. It is clear that $X^* = \mathbb{R}_+K$. Moreover, take any $x^* \in K\backslash K_\infty$, and denote $\overline{\eta} := \sup\{\eta > 0 : \eta x^* \in \operatorname{dom}\varphi^*\}$. It follows from the algebraic openness of $\operatorname{dom}\varphi^*$ that $\overline{\eta}x^* \notin \operatorname{dom}\varphi^*$, so by lower semicontinuity of φ^* we have that

$$+\infty = \varphi^*(\overline{\eta}x^*) \leq \sup\{\varphi^*(\eta x^*) : \eta > 0, \eta x^* \in \operatorname{dom}\varphi^*\},$$

and this concludes the proof. $\square$

Remark 2.3. Let X be a Hausdorff complete locally convex space and let $K \subset X^*$ with $0 \in K$ and $\mathbb{R}_+K = X^*$. It is important to notice by our Theorem 1.1 that every function $\varphi \in \mathcal{E}(K)$ is $w(X, X^*)$-infcompact, provided that it is proper, convex and lower semicontinuous. To see that, it is enough to apply Theorem 1.1 to the function $g = \delta_{\{0\}}$ for $S_{g\Box\varphi}(\gamma)$. So, by Moreau's result [7, §8.f, p. 49] the function φ^* is Mackey continuous at zero. Particularly, this implies that $0 \in \operatorname{Int}_{\tau(X^*,X)}(\operatorname{dom}\varphi^*)$, so the interior of $\operatorname{dom}\varphi^*$ with respect to the Mackey topology is nonempty.

We claim that $\operatorname{dom}\varphi^*$ is open with respect to the Mackey topology $\tau(X^*, X)$ of X^*. Indeed, suppose the contrary, i.e., there is a point $x^* \in \operatorname{dom}\varphi^* \setminus \operatorname{Int}_{\tau(X^*,X)}(\operatorname{dom}\varphi^*)$. Then $\sup\{\eta > 0 : \eta x^* \in \operatorname{dom}\varphi^*\} = 1$, otherwise by the inclusion $x^* \in \operatorname{dom}\varphi^*$ there would exist some real $\eta_0 > 1$ with $\eta_0 x^* \in \operatorname{dom}\varphi^*$, hence $[0, \eta_0 x^*[\subset \operatorname{Int}_{\tau(X^*,X)}(\operatorname{dom}\varphi^*)$, which would give the contradiction $x^* \in \operatorname{Int}_{\tau(X^*,X)}(\operatorname{dom}\varphi^*)$.

Now, note that the equality $\sup\{\eta > 0 : \eta x^* \in \operatorname{dom}\varphi^*\} = 1$ tells us that $\{\eta \geq 0 : \eta x^* \in \operatorname{dom}\varphi^*\} = [0,1]$ (since $0, x^* \in \operatorname{dom}\varphi^*$), so by continuity of the restriction of φ^* to $[0, x^*]$ we have $\sup\{\varphi^*(\eta x^*) : \eta > 0, \eta x^* \in \operatorname{dom}\varphi^*\} < +\infty$. But, the equality $\{\eta \geq 0 : \eta x^* \in \operatorname{dom}\varphi^*\} = [0,1]$ also says that $x^* \notin K_\infty$ (since K_∞ is a cone included in $\operatorname{dom}\varphi^*$ by property (ii)), hence by property (iii) in the introduction we have $\sup\{\varphi^*(\eta x^*) : \eta > 0, \eta x^* \in \operatorname{dom}\varphi^*\} = +\infty$, which is in contradiction with the latter inequality saying this supremum is finite. This confirms that $\operatorname{dom}\varphi^*$ is $\tau(X^*, X)$ open in X^*.

On the other hand, when X is a Banach space, it has been proved by Moors [5, Lemma 5] that a function φ satisfying properties (i), (ii) and (iii) in the introduction has $\operatorname{dom}\varphi^*$ open in $(X^*, \|\cdot\|_*)$. Here, it is important to mention that the proofs of Proposition 4 and Lemma 5 in [5] still hold when X^* is a Fréchet (metrizable) space for the strong topology $\beta(X^*, X)$. Nevertheless, the properties (i), (ii) and (iii) in the introduction, as far as we know, does not imply the openness of $\operatorname{dom}\varphi^*$ with respect to the Mackey topology of X^* only assuming that X is a Fréchet space.

3. Variational characterizations of compactness of sets

The first theorem in this section corresponds to an extension of [2, Theorem 1] to Hausdorff complete locally convex spaces. It also provides a new short proof of the Banach setting result in [2, Theorem 1].

Theorem 3.1. *Let (X, θ) be a Hausdorff complete locally convex space. Let C be a nonempty closed, convex (not necessarily bounded) subset of X for which there is a nonempty open subset Λ of X^* for the Mackey topology such that the supremum $\sup\{\langle x^*, c\rangle : c \in C\}$ is attained for every $x^* \in \Lambda$. Then for any $x_0^* \in \Lambda$ and any $\alpha \in \mathbb{R}$, the set*

$$\{x \in C : -\langle x_0^*, x\rangle \leq \alpha\}$$

is weakly compact in X. Moreover, the set $\widehat{C}$ is $w(X^{**}, X^*)$ closed in the bidual space X^{**}.

Proof. Let any $x_0^* \in \Lambda$ and any $\alpha \in \mathbb{R}$, we claim that the set

$$C(x_0^*, \alpha) := \{x \in C : -\langle x_0^*, x\rangle \leq \alpha\}$$

is weakly compact in X, i.e., $w(X, X^*)$ compact. Indeed, considering the function $g := \delta_C$, by a direct application of Theorem 1.2, we have that $C(x_0^*, \alpha)$ is relatively weakly compact, hence $w(X, X^*)$ compact according to the $w(X, X^*)$ closedness of C. It ensues that $\widehat{C}(x_0^*, \alpha) := \{\widehat{x} : x \in C(x_0^*, \alpha)\}$ is also $w(X^{**}, X^*)$ compact in X^{**}.

To prove that $\widehat{C}$ is $w(X^{**}, X^*)$ closed, we consider any net $(\widehat{x}_j)_{j\in J}$ in $\widehat{C}$ which $w(X^{**}, X^*)$ converges to some x^{**} in X^{**}. There are some real M and some $j_0 \in J$ such that $-\langle x_0^*, x_j\rangle \leq M$ for all $j \succeq j_0$ in J. Then $x_j \in C(x_0^*, M)$ for all $j \succeq j_0$ in J, so by $w(X^{**}, X^*)$ compactness of $\widehat{C}(x_0^*, M)$ there is a subnet of $(\widehat{x}_j)_{j\in J}$ $w(X^{**}, X^*)$ converging to some $\widehat{x} \in \widehat{C}(x_0^*, M)$. By uniqueness of limit, we obtain $x^{**} = \widehat{x}$, which confirms that $\widehat{C}$ is $w(X^{**}, X^*)$ closed in X^{**}. $\square$

The next theorem presents an extension of [2, Theorem 2] and [1, Theorem 3].

Before presenting this result we need the following lemma.

Lemma 3.2. *Let (X,θ) be a Hausdorff locally convex space. Let $g : X^* \to \mathbb{R}$ be a weak* lower semicontinuous convex function such that $g(0) < 0$, let $\varepsilon > 0$ and let $\rho : X^* \to \mathbb{R}_+$ be a seminorm which is continuous for the Mackey topology. Assume*

$$\partial g(x^*) \cap \widehat{X} \neq \emptyset, \text{ for all } x^* \in K := \{x^* \in X^* : \rho(x^*) < \varepsilon \text{ and } g(x^*) < 0\}.$$

Then, $\mathcal{E}(K) \neq \emptyset$; in fact there exists $\varphi \in \mathcal{E}(K)$ such that φ^* is bounded from below.

Proof. On the one hand, let us consider the function $\Psi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ given by

$$\Psi(\alpha) := \begin{cases} -\frac{1}{\alpha} + \frac{1}{g(0)} & \text{if } \alpha < 0 \\ +\infty & \text{if } \alpha \geq 0 \end{cases},$$

and consider $\varphi_1(x) := (\Psi \circ g)^*(x)$ for all $x \in X$. We note for $r \in \mathbb{R}$ that

$$\{x^* \in X^* : \Psi \circ g(x^*) \leq r\} = \begin{cases} \emptyset \text{ if } r \leq 1/g(0) \\ \{x^* \in X^* : g(x^*) \leq -\left(r - \frac{1}{g(0)}\right)^{-1}\} \text{ if } r > 1/g(0), \end{cases}$$

so the proper convex function $\Psi \circ g$ is $w(X^*, X)$ lower semicontinuous, hence we have $\varphi_1^* = \Psi \circ g$ on X^*. This equality tells us that $\operatorname{dom} \varphi_1^* = \{x^* \in X^* : g(x^*) < 0\}$, $\varphi_1^*(0) = 0$ and $\varphi_1^*(x^*) \geq \frac{1}{g(0)}$ for all $x^* \in X^*$. Further, for any $x^* \in K$ choosing $x \in X$ with $\widehat{x} \in \widehat{X} \cap \partial g(x^*)$ (by the assumption on g) we have

$$\Psi'(g(x))\widehat{x} \in \widehat{X} \cap \partial \varphi_1^*(x^*). \tag{4}$$

On the other hand, by Lemma 1.3 it is possible to consider

$$\varphi_2 \in \mathcal{E}(\{x^* \in X^* : \rho(x^*) < \varepsilon\}) \text{ with } \varphi_2^* \geq 0 \text{ and } \operatorname{dom} \varphi_2^* = \{x^* \in X^* : \rho(x^*) < \varepsilon\}.$$

Now, let us define $\varphi(x) := (\varphi_1^* + \varphi_2^*)^*(x)$ for all $x \in X$. Since $\varphi_1^* + \varphi_2^*$ is proper, convex and $w(X^*, X)$ lower semicontinuous, we have that $\varphi^* = (\varphi_1^* + \varphi_2^*)^{**} = \varphi_1^* + \varphi_2^*$ on X^*. Let us notice that $\varphi^*(0) = 0$ and $\operatorname{dom} \varphi^* = K$ is an algebraic open set. Moreover, for all $x^* \in K$ we have that $\partial \varphi_1^*(x^*) + \partial \varphi_2^*(x^*) \subseteq \partial \varphi^*(x^*)$ along with $\widehat{X} \cap \partial \varphi_1^*(x^*) \neq \emptyset$ by (4) and $\widehat{X} \cap \partial \varphi_2^*(x^*) \neq \emptyset$ since $\varphi_2 \in \mathcal{E}(\{u^* \in X^* : \rho(u^*) < \varepsilon\})$. It ensues that $\widehat{X} \cap \partial \varphi^*(x^*) \neq \emptyset$, so $(\partial \varphi)^{-1}(x^*) \neq \emptyset$. Then, using Lemma 2.2 we conclude that $\varphi \in \mathcal{E}(K)$. □

We can now show the locally convex version of [2, Theorem 2] and [1, Theorem 3].

Notice that our proof is quite short.

Theorem 3.3. *Let (X,θ) be a Hausdorff complete locally convex space. Let A be a nonempty, closed convex, bounded and non weakly compact subset of X with $0 \notin A$, and let $x^* \in X^*$ with $\sigma_A(x^*) < 0$. Let D be a nonempty convex and weakly compact subset of X.*
Then, given $\varepsilon > 0$ there exists $z^* \in X^*$ with $\sup\{|\langle x^* - z^*, x\rangle| : x \in D\} < \varepsilon$ and $\sigma_A(z^*) < 0$ such that z^* does not attain its supremum on A.

Proof. We may assume that D is a balanced set of X, otherwise we can take $D_0 := \mathrm{co}(D \cup -D)$, which is a convex balanced set. Moreover, noting that $D_0 = q\big([0,1] \times D \times (-D)\big)$ with $q(t,x,y) = tx + (1-t)y$ we see that D_0 is also weakly compact. Let us denote $\alpha := \sup\{-\langle x^*, x\rangle : x \in A\}$, which is finite due to the fact that A is bounded.
We now consider the Mackey continuous seminorm $\rho(u^*) := \sup\{|\langle u^*, w\rangle| : w \in D\}$ and the functions $f(x) := \delta_A(x) - \langle x^*, x\rangle$ and $g := \sigma_A(x^* + \cdot)$, so $f^* = g$. Now, let us suppose by contradiction that for all $\zeta^* \in K := \{u^* \in X^* : \rho(u^*) < \varepsilon \text{ and } g(u^*) < 0\}$ the function $x^* + \zeta^*$ attains its supremum on A. The subdifferential of g (with respect to the dual pair (X^*, X)) is given by

$$\partial g(\zeta^*) = \{x \in A : \sigma_A(x^* + \zeta^*) = \langle x^* + \zeta^*, x\rangle\}.$$

Then, by our assumption, we get $\partial g(\zeta^*) \cap X \neq \emptyset$ for every $\zeta^* \in K$. Hence by Lemma 3.2 we obtain $\mathcal{E}(K) \neq \emptyset$. Since $0 \in K$ and $\mathbb{R}_+ K = X^*$ as easily seen, we can apply Theorem 1.1 to the function f. Then, the set $\{x \in A : -\langle x^*, x\rangle \leq \alpha\}$ is weakly compact. Nevertheless, by the choice of α we obtain $A = \{x \in A : -\langle x^*, x\rangle \leq \alpha\}$, which furnishes the contradiction that A is weakly compact. This completes the proof. □

The next theorem can be seen as a functional counterpart of Theorem 3.1 and Theorem 3.3 above.

Theorem 3.4. *Let (X, θ) be a Hausdorff complete locally convex space and let $g\colon X \to \mathbb{R} \cup \{+\infty\}$ be a function such that $\mathrm{dom}\, g^*$ is a nonempty algebraic open set for which $(\partial g)^{-1}(x^*) \neq \emptyset$, for all $x^* \in \mathrm{dom}\, g^*$. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function such that $(\partial f)^{-1}(x^*) \neq \emptyset$ for all ${}^* \in \mathrm{dom}\, g^*$.*
Then, for every $x^* \in (\mathrm{dom}\, f^*)^i$ and every $\gamma \in \mathbb{R}$, the set

$$\{x \in X : f(x) - \langle x^*, x\rangle \leq \gamma\}$$

is relatively $w(X, X^*)$ compact in X. Furthermore, for every $x^* \in X^*$ the $w(X^{**}, X^*)$ closure in X^{**} of the set $\{\widehat{x} \in X^{**} : x \in X,\ f(x) - \langle x^*, x\rangle \leq \gamma\}$ is included in $\widehat{X}$.

Proof. Let $x^* \in \mathrm{dom}\, g^*$. Considering the functions

$$\tilde{f} := \overline{\mathrm{co}}\, f - \langle x^*, \cdot\rangle \quad \text{and} \quad \tilde{g} := g - \langle x^*, \cdot\rangle + g^*(x^*),$$

we may suppose that f is proper, convex and lower semicontinuous, $x^* = 0$ and $g^*(0) = 0$. We have $\mathcal{E}(\mathrm{dom}\, g^*) \neq \emptyset$ by Lemma 2.2, so applying Theorem 1.1 we get that $\{x \in X : f(x) \leq \gamma\}$ is relatively $w(X, X^*)$ compact for every $\gamma \in \mathbb{R}$. Hence, by Moreau's result [7, §8.f, p. 49] the function f^* is Mackey continuous at zero, and by convexity it is continuous at every point of the interior of its domain, which is nonempty. Particularly, $\mathrm{Int}_{\tau(X^*,X)}(\mathrm{dom}\, f^*) = (\mathrm{dom}\, f^*)^i$ (see Remark 2.1). Hence, Moreau's result [7, §8.f, p. 49] implies the first part of the proof.

Now, let $x^* \in X^*$ be arbitrary but fixed, and $x_0^* \in \mathrm{dom}\, g^* = (\mathrm{dom}\, g^*)^i$. Take any net $(\widehat{x}_j)_{j \in J}$ in $\{\widehat{x} \in X^{**} : x \in X, f(x) - \langle x^*, x\rangle \leq \gamma\}$ such that $\widehat{x}_j \to x^{**}$ with respect to the $w(X^{**}, X^*)$ topology. Consider $j_0 \in J$ and $\alpha \in \mathbb{R}$ such that $|\langle x^* - x_0^*, x_j\rangle| \leq \alpha$ for all $j \succeq j_0$. Then, we have that $x_j \in \{x \in X : f(x) - \langle x_0^*, x\rangle \leq \alpha + \gamma\}$ for every $j \succeq j_0$. By the first part, it follows that $(x_j)_{j \in J}$ has a cluster point x in X, so by uniqueness of the limit, it ensues that $x^{**} = \widehat{x} \in \widehat{X}$, and this finishes the proof. □

The following corollary considers the assumption $\widehat{X} \cap \partial f^*(x^*) \neq \emptyset$ instead of the condition $(\partial f)^{-1}(x^*) \neq \emptyset$. The result of the corollary has been established in [4, Theorem 2.5] and [5, Theorem 2] in Banach spaces. The corollary extends the result to locally convex spaces with a different approach which is quite short.

Corollary 3.5. *Let (X, θ) be a Hausdorff complete locally convex space and let $g\colon X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function such that $\operatorname{dom} g^*$ is an algebraic open set for which $\widehat{X} \cap \partial g^*(x^*) \neq \emptyset$ for all $x^* \in \operatorname{dom} g^*$. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function such that $\widehat{X} \cap \partial f^*(x^*) \neq \emptyset$ for all $x^* \in \operatorname{dom} g^*$. Then for every $x^* \in (\operatorname{dom} f^*)^i$ and every $\gamma \in \mathbb{R}$, the set*

$$\{x \in X : f(x) - \langle x^*, x\rangle \le \gamma\}$$

is relatively $w(X, X^*)$ compact in X. Furthermore, for every $x^* \in X^*$ the $w(X^{**}, X^*)$ closure in X^{**} of the set $\{\widehat{x} \in X^{**} : x \in X,\ f(x) - \langle x^*, x\rangle \le \gamma\}$ is included in $\widehat{X}$.

Proof. Denote by f_0 the closed convex hull $\overline{co}^\theta f$ of f, so $f_0 \le f$ and $f^* = f_0^*$ (see (2)). According to the assumption on f, for any $x^* \in \operatorname{dom} g^*$ one has $\widehat{X} \cap \partial f_0^*(x^*) \neq \emptyset$, which is equivalent to $(\partial f_0)^{-1}(x^*) \neq \emptyset$. Then, given any $x^* \in (\operatorname{dom} f^*)^i = (\operatorname{dom} f_0^*)^i$ (resp. any $x^* \in X^*$) Theorem 3.4 tells us that the set $\{x \in X : f_0(x) - \langle x^*, x\rangle \le \gamma\}$ is relatively $w(X, X^*)$ compact in X (resp. the set $\{\widehat{x} \in X^{**} : x \in X,\ f_0(x) - \langle x^*, x\rangle \le \gamma\}$ has its $w(X^{**}, X^*)$ closure included in $\widehat{X}$). Since

$$\{x \in X : f(x) - \langle x^*, x\rangle \le \gamma\} \subset \{x \in X : f_0(x) - \langle x^*, x\rangle \le \gamma\}$$

according to the inequality $f_0 \le f$, the corollary follows. □

The next Theorem corresponds to an extension of [2, Theorems 3 and 17]. The proof of this result can be derived from Theorem 3.4 and Lemma 1.3. However, we provide an independent proof in order to illustrate the potential applications of Theorem 1.2.

Theorem 3.6. *Let (X, θ) be a Hausdorff complete locally convex space and let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a proper function such that $\partial f(X)$ has nonempty interior with respect to the Mackey topology $\tau(X^*, X)$. Then for any $x^* \in (\operatorname{dom} f^*)^i$ and any $\gamma \in \mathbb{R}$ the set*

$$\{x \in X : f(x) - \langle x^*, x\rangle \le \gamma\}$$

is relatively $w(X, X^*)$ compact in X. Furthermore, for every $x^* \in X^*$ the $w(X^{**}, X^*)$ closure of the set $\{\widehat{x} \in X^{**} : x \in X,\ f(x) - \langle x^*, x\rangle \le \gamma\}$ is included in $\widehat{X}$. In addition, if for some $x_0^* \in X^*$ and $\gamma_0 \in \mathbb{R}$ the set $\{x \in X : f(x) - \langle x_0^*, x\rangle \le \gamma_0\}$ is bounded, we have that it is relatively $w(X, X^*)$ compact in X.

Proof. As in the proof of Theorem 3.4 we may suppose that f is proper, convex and lower semicontinuous. Let us notice that for every $x^* \in \operatorname{Int}_{\tau(X^*,X)}\big(\partial f(X)\big)$ the function $f - \langle x^*, \cdot\rangle$ attains its minimum, so $(\partial f)^{-1}(x^*) \neq \emptyset$. Then, applying

Theorem 1.2 we get the first part of the result for $x^* \in \operatorname{Int}_{\tau(X^*,X)}\big(\partial f(X)\big)$. Moreover $\operatorname{Int}_{\tau(X^*,X)}\big(\partial f(X)\big) = (\operatorname{dom} f^*)^i$. Indeed, we clearly have the first inclusion $\operatorname{Int}_{\tau(X^*,X)}\big(\operatorname{dom} f^*\big) \supseteq \operatorname{Int}_{\tau(X^*,X)}\big(\partial f(X)\big) \neq \emptyset$. Further, since f^* is Mackey continuous on the interior of its domain (see [7, §8.f, p. 49]), we also have

$$\partial f(X) \supseteq \operatorname{Int}_{\tau(X^*,X)}\big(\operatorname{dom} f^*\big).$$

It ensues that $\operatorname{Int}_{\tau(X^*,X)}\big(\partial f(X)\big) = \operatorname{Int}_{\tau(X^*,X)}\big(\operatorname{dom} f^*\big)$. Finally, Remark 2.1 yields that $\operatorname{Int}_{\tau(X^*,X)}\big(\partial f(X)\big) = (\operatorname{dom} f^*)^i$ as desired. Now, the second part follows from arguments similar to the ones given in Theorem 3.4. For the final part let us suppose that $\operatorname{lev}^{\leq}_{f-x_0^*}(\gamma_0) := \{x \in X : f(x) - \langle x_0^*, x\rangle \leq \gamma_0\}$ is bounded. We fix $a^* \in \operatorname{Int} \partial f(X)$, so that the quantity

$$\gamma := \gamma_0 + \sup\{|\langle a^* - x_0^*, x\rangle| : x \in \operatorname{lev}^{\leq}_{f-x_0^*}(\gamma_0)\}$$

is finite due to the boundedness of $\operatorname{lev}^{\leq}_{f-x_0^*}(\gamma_0)$. Then, the set $\operatorname{lev}^{\leq}_{f-x_0^*}(\gamma_0)$ is contained in $\{x \in X : f(x) - \langle a^*, x\rangle \leq \gamma\}$, which ends the proof according to what precedes. $\square$

Adapting the arguments employed in Corollary 3.5 the following corollary of Theorem 3.6 is easily derived. An earlier Banach space version was established in [2, Theorem 17] and [5, Corollary 3].

Corollary 3.7. *Let (X, θ) be a Hausdorff complete locally convex space and let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a proper function for which there exists a nonempty set $\Lambda \subset X^*$ open with respect to the Mackey topology $\tau(X^*, X)$ such that*

$$\widehat{X} \cap \partial f^*(x^*) \neq \emptyset \quad \textit{for all}\, x^* \in \Lambda.$$

Then for any $x^* \in (\operatorname{dom} f^*)^i$ and any $\gamma \in \mathbb{R}$ the set

$$\{x \in X : f(x) - \langle x^*, x\rangle \leq \gamma\}$$

is relatively $w(X, X^*)$ compact in X. Furthermore, for every $x^* \in X^*$ the $w(X^{**}, X^*)$ closure of the set $\{\widehat{x} \in X^{**} : x \in X,\ f(x) - \langle x^*, x\rangle \leq \gamma\}$ is included in $\widehat{X}$.

In addition, if for some $x_0^* \in X^*$ and $\gamma_0 \in \mathbb{R}$ the set $\{x \in X : f(x) - \langle x_0^*, x\rangle \leq \gamma_0\}$ is bounded, we have that it is relatively $w(X, X^*)$ compact in X.

Remark 3.8. (Necessity of linear translations) It is important to notice that the translation of the function f with respect to x^* is necessary in Theorem 3.6 even in the finite dimensional setting. Indeed, let us consider the continuous convex function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(t) = \begin{cases} t - \frac{1}{2} & \text{if } t \leq 1 \\ \frac{1}{2}t^2 & \text{if } t > 1. \end{cases}$$

It follows that $\partial f(\mathbb{R}) = [1, +\infty)$. Nevertheless, for all $\alpha \in \mathbb{R}$ the set $\{t : f(t) \leq \alpha\}$ is unbounded. Furthermore, for $x^* = 1$, we have that $\{t : f(t) - t \leq 0\}$ is also unbounded, which shows that the conclusion of Theorem 3.6 cannot be extended to all $x^* \in \partial f(X)$.

On the other hand, the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \frac{1}{2}t^2 & \text{if } t > 0 \end{cases}$$

satisfies that $\partial f(\mathbb{R}) = [0, +\infty)$ and f is bounded from below. Nevertheless, the set $\{t : f(t) \leq 1\}$ is unbounded. This shows that the property of relative $w(X, X^*)$ compactness of $\{x \in X : f(x) \leq \gamma\}$ fails when the function f is only assumed to be bounded from below (even if $0 \in \partial f(X)$).

4. Range of the subdifferential

As recalled in Section 2, given a Hausdorff locally convex space X, its topological bidual is denoted by X^{**}, and defined as the $X^{**} = (X^*, \beta(X^*, X))^*$. Now, given a function $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ its Legendre-Fenchel biconjugate function $f^{**} : X^{**} \to \mathbb{R} \cup \{-\infty, +\infty\}$ is defined by $f^{**} := (f^*)^*$, that is,

$$f^{**}(x^{**}) = \sup \{\langle x^{**}, x^* \rangle - f^*(x^*) : x^* \in X^*\} \quad \text{for all } x^{**} \in X^{**}.$$

For the given function $f\colon X \to \mathbb{R} \cup \{-\infty, +\infty\}$ we define $\widehat{f}\colon X^{**} \to \mathbb{R} \cup \{-\infty, +\infty\}$, its extension to X^{**}, by

$$\widehat{f}(x^{**}) := \begin{cases} f(x), & \text{if } x^{**} = \widehat{x} \text{ for some } x \in X \\ +\infty & \text{otherwise.} \end{cases}$$

Finally, according to Section 1 we denote $\overline{\text{co}}^{w^*} \widehat{f} : X^{**} \to \mathbb{R} \cup \{-\infty, +\infty\}$ the $w(X^{**}, X^*)$ closed convex hull of $\widehat{f}$, which is the function such that

$$\operatorname{Epi} \overline{\text{co}}^{w^*} \widehat{f} = \overline{\text{co}}^{w^*} \operatorname{Epi} \widehat{f}.$$

The following lemma is an extension of [3, Proposition 7.31].

Lemma 4.1. *Let X be a Hausdorff locally convex space, and let*

$$f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$$

be a function such that f^* is proper. Then $f^{**} = \overline{\text{co}}^{w^*} \widehat{f}$, where w^* stands for the $w(X^{**}, X^*)$ topology on X^{**}.

Proof. Let us consider the dual pair $(X^*, \beta(X^*, X), X^{**}, w(X^{**}, X^*))$. Then, it is easy to see that

$$f^*(x^*) = \sup \left\{ \langle x^*, x^{**} \rangle - \widehat{f}(x^{**}) : x^{**} \in X^{**} \right\},$$

that is, f^* is the conjugate function of $\widehat{f}$ in the sense of the above dual pair. Moreover, the function $\widehat{f}$ has nonempty domain. Now, by [9, Theorem 2.3.4, part i)] we conclude, that in this setting, $f^{**} = \overline{\text{co}}^{w^*} \widehat{f}$, which ends the proof. $\square$

The next theorem establishes an extension of [6, Theorem 2.13] to locally convex spaces. Our proof is another direct application of Theorem 1.1. We continue with the context (see (3)) that $\partial_\varepsilon f^* : X^* \rightrightarrows X^{**}$ for any function $f : X \to \mathbb{R} \cup \{+\infty\}$ on a locally convex space X.

Theorem 4.2. *Let (X, θ) be a Hausdoff complete locally convex space and let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a function such that* $\operatorname{dom} f^*$ *is a nonempty algebraic open set in X^*, and such that $(\partial f)^{-1}(x^*) \neq \emptyset$ for all $x^* \in \operatorname{dom} f^*$. Then, for any real $\varepsilon \geq 0$ one has $\partial_\varepsilon f^*(x^*) \subseteq \widehat{X}$ for all $x^* \in X^*$.*

Proof. First, let us notice that by (2) we may suppose that f is proper, convex and lower semicontinuous. Otherwise, we consider $\tilde{f} := \overline{\mathrm{co}} f$. Let us consider $x^* \in \operatorname{dom} f^*$ and $\varepsilon \geq 0$. We claim that $\partial_\varepsilon f^*(x^*) \subseteq \widehat{X}$. Without loss of generality we may assume that $x^* = 0$ and $f^*(0) = 0$. By Lemma 2.2 the set $\mathcal{E}(\operatorname{dom} f^*)$ is nonempty. Consequently, applying Theorem 1.1 we get that $\operatorname{lev}^{\leq}_{f}(\gamma)$ is $w(X, X^*)$ compact in X for all $\gamma \in \mathbb{R}$.

Finally, let us suppose by contradiction that there exists a point $x^{**} \in X^{**} \backslash \widehat{X}$ with $x^{**} \in \partial_\varepsilon f^*(x^*)$. Hence, by Lemma 4.1 there exists a net $(x_j, \alpha_j)_{j \in J}$ in $\operatorname{Epi} f \subset X \times \mathbb{R}$ such that $x_j \to x^{**}$ $w(X^{**}, X^*)$ and $f(x_j) \to f^{**}(x^*)$ (recall we supposed that f is convex). Therefore, we have that there exists $j_0 \in J$ such that for all $j \succeq j_0$

$$f(x_j) - \langle \widehat{x_j}, x^* \rangle \leq -f^*(x^*) + \varepsilon + 1.$$

Since $\langle \widehat{x_j}, x^* \rangle \to \langle x^{**}, x^* \rangle$, there is a real constant γ_0 and some $j_1 \succeq j_0$ in J such that $f(x_j) \leq \gamma_0$ for all $j \succeq j_1$ in J. It ensues that $x_j \in \operatorname{lev}^{\leq}_{f}(\gamma_0)$ for all $j \succeq j_0$ in J, and consequently $(x_j)_{j \in J}$ has a cluster point in X, which is a contradiction. □

Remark 4.3. Let us consider a Banach space $(X, \|\cdot\|)$ and the function $f(x) = \|x\|$. It follows that $f^* = \delta_{\mathbb{B}_{X^*}}$, and for every $x^* \in \operatorname{dom} f^*$ we have that

$$\partial f^*(x^*) \supseteq \{0\} \subseteq X.$$

Nevertheless, $\partial_\varepsilon f^*(0) = \varepsilon \mathbb{B}_{X^{**}}$. Therefore, Theorem 4.2 cannot be extended to a function f whose conjugate function f^* has no algebraic open domain.

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