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Rotundity Properties, and Non-Extendability of Lipschitz Quasiconvex Functions

Carlo Alberto De Bernardi
Dip. di Matematica per le Scienze Economiche, Finanziarie ed Attuariali,
Università Cattolica del Sacro Cuore, Milano, Italy
carloalberto.debernardi@unicatt.it, carloalberto.debernardi@gmail.com

Libor Veselý
Dip. di Matematica "F. Enriques", Università degli Studi di Milano, Italy
libor.vesely@unimi.it

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Let A be an open convex subset of a real normed linear space X. We prove that if the boundary of A contains a non-LUR point then there exists a Lipschitz quasiconvex function $f\colon A \to \mathbb{R}$ not admitting any continuous quasiconvex extension to the whole X.

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1. Introduction

Let A be a convex set in a nontrivial real normed linear space X. A function $f\colon A \to \mathbb{R}$ is called quasiconvex if its lower level sets $\{x \in A : f(x) < \beta\}$ $(\beta \in \mathbb{R})$ are convex. Quasiconvex functions are a natural generalization of convex functions and play a crucial role in mathematical programming, in economics, and in many other areas of mathematical analysis (see [1, 2, 15, 17] and the references therein).

In the last two decades, several authors studied the problem of extending real valued continuous convex functions from a subspace Y of X to the whole X (see e.g. [3, 4, 5, 9, 10, 18] and the references therein), obtaining positive results under suitable hypotheses involving certain separability conditions. In a recent paper [6], the first named author started to study the corresponding problem for continuous *quasiconvex* functions, obtaining some positive results essentially in the finite-dimensional case. Subsequently, in our paper [12], these results were generalized to the infinite-dimensional separable case by a detailed study of relations between extendability of continuous convex and that of continuous quasiconvex functions. Moreover, in [13] we obtained some positive results on extending Lipschitz quasiconvex functions from subspaces by proving that every quasiconvex L-Lipschitz (i.e., with a Lipschitz constant $L > 0$) function defined on a subspace of a normed space X admits an L-Lipschitz quasiconvex extension defined on the whole X.

Here we are interested in a related problem of extendability of Lipschitz quasiconvex functions from a convex set $A \subset X$ to the whole X. It is a well-known fact that if

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$g\colon A \to \mathbb{R}$ is an L-Lipschitz convex function then it admits an L-Lipschitz convex extension G to the whole X; moreover, such an extension G can be defined by the infimal-convolution formula

$$G(x) = \inf_{y\in A}\big[g(y) + L\|x-y\|\big], \qquad x \in X.$$

Thus it seems natural to ask to what extent a similar result holds for Lipschitz quasiconvex functions. Let us observe that the same formula does not work in the quasiconvex case: it is an easy exercise to find an example of a quasiconvex L-Lipschitz function $f\colon A \to \mathbb{R}$ such that the (L-Lipschitz) extension $F(x) = \inf_{y\in A}[f(y) + L\|x-y\|]$, $x \in X$, is not quasiconvex.

The aim of the present paper is to show that, in general, the problem mentioned above has a negative answer, even in a strong sense described below. In Section 2, after some notations and preliminaries, we present Example 2.10 which is a starting point of our work: an elementary example of a quasiconvex Lipschitz function defined on an open half-plane $A \subset \mathbb{R}^2$ that cannot be extended to a continuous quasiconvex function defined on any open half-plane properly containing A. The geometric construction used in this example and a certain characterization of strongly exposing functionals (Lemma 2.3 below) are the basic ingredients for Theorem 3.2 which is the first main result of the present paper asserting that:

> *If $A \subset X$ is an open convex set such that some $x_0 \in \partial A$ is not a strongly exposed point of $\overline{A}$, then there exists a Lipschitz quasiconvex function $f\colon A \to \mathbb{R}$ not admitting quasiconvex continuous extensions to any open convex subset of X containing A and x_0.*

Then, by combining this result and a certain property of non LUR points, stated in terms of support functionals, we prove the following result (see Theorem 3.3):

> *If X is a Banach space and $A \subset X$ is an open convex set such that some $x_0 \in \partial A$ is not a LUR point of $\overline{A}$, then there exists a Lipschitz quasiconvex function $f\colon A \to \mathbb{R}$ not admitting any quasiconvex continuous extension to X.*

These last two results should be compared with our paper [13] in which it is proved that each uniformly continuous quasiconvex function defined on a uniformly rotund body $A \subset X$ admits a uniformly continuous quasiconvex extension to the whole X. This comparison leads to a couple of problems considered in Section 4. The first one asks whether a similar extension result holds for the class of Lipschitz quasiconvex functions. This is solved in negative in Example 4.2 by providing a Lipschitz quasiconvex function on a circle in the Euclidean plane that does not admit any Lipschitz quasiconvex extension to the whole plane. The second problem asks whether the result in [13] about uniformly continuous quasiconvex functions can be reversed, that is, whether extendability of uniformly continuous quasiconvex functions from an open convex set A to the whole space implies that the set A is uniformly rotund. This last problem remains open.

2. Notation, preliminaries, and an elementary example

We consider only real normed linear spaces of dimension at least two. If X is a normed space then X^* is its dual Banach space, and U_X, B_X and S_X are the open unit ball, the closed unit ball and the unit sphere of X, respectively. Other notation is standard, and various topological notions refer to the norm topology of X, unless specified otherwise. In particular, given sets $E \subset Z$ in X, the symbols $\overline{E}^Z$, $\partial_Z(E)$ and $\operatorname{int}_Z(E)$ denote the relative closure, boundary and interior, respectively, of E in Z.

All functions we consider are real-valued. Let us recall that, if A is a convex subset of X, a function $f\colon A \to \mathbb{R}$ is called *quasiconvex* if all its strict sub-level sets

$$[f < \beta] := \{x \in A : f(x) < \beta\} \qquad (\beta \in \mathbb{R})$$

are convex; equivalently, if all its sub-level sets

$$[f \leq \beta] := \{x \in A : f(x) \leq \beta\} \qquad (\beta \in \mathbb{R})$$

are convex.

A set $B \subset X$ will be called a *body* if it is closed, convex and has nonempty interior. For $x, y \in X$, $[x, y]$ denotes the closed segment in X with endpoints x and y, and $(x, y) = [x, y] \setminus \{x, y\}$ is the corresponding relatively open segment; similarly, we define $[x, y) = [x, y] \setminus \{y\}$.

A *slice* of a set $K \subset X$ is a set of the form

$$S(x^*, \alpha, K) = \{x \in K : x^*(x) > \sup x^*(K) - \alpha\},$$

where $\alpha > 0$ and $x^* \in X^* \setminus \{0\}$ is bounded above on K. If $z \in K \subset X$, we denote the set of all *support functionals* for K at z by

$$\Sigma_K(z) := \{x^* \in X^* \setminus \{0\} : x^*(z) = \sup x^*(K)\}.$$

We shall need the following easy lemma which asserts that the supporting functionals of a body B at a point depend only on the portion of B contained in a neighborhood of that point. Its easy standard proof is left to the reader.

Lemma 2.1. *Let A be an open convex set and B a body in X. If $x_0 \in A \cap \partial B$ then $\Sigma_{A\cap B}(x_0) = \Sigma_B(x_0)$.*

For $x^* \in S_{X^*}$ and $\alpha \in (0, 1)$, we denote

$$C(x^*, \alpha) = \{x \in X : x^*(x) \geq \alpha\|x\|\},\ C^\circ(x^*, \alpha) = \{x \in X : x^*(x) > \alpha\|x\|\}.$$

It is easy to see that $C(x^*, \alpha)$ is a nonempty closed convex cone (with vertex at the origin), and $C^\circ(x^*, \alpha) = \operatorname{int} C(x^*, \alpha)$ is a nonempty open convex cone.

Let us recall the notion of a strongly exposed point and a strongly exposing functional.

Definition 2.2. (see, e.g., [14, Definition 7.10]) Let K be a nonempty subset of a normed space X, $a \in K$, and $f \in X^* \setminus \{0\}$. We say that f *strongly exposes* K at a if $f \in \Sigma_K(a)$, and $x_n \to a$ whenever $\{x_n\}$ is a sequence in K such that $\lim_n f(x_n) = \sup f(K)$. If this is the case, we say that a is a *strongly exposed* point of K.

It is elementary to see that f strongly exposes K at a if and only if $f(a) = \sup f(K)$ and $\operatorname{diam}\big(S(f,\alpha,K)\big) \to 0$ as $\alpha \to 0^+$. The next lemma gives a characterization of strongly exposing functionals of a convex set C in terms of containment of C in certain translates of cones of the form $C(f,\alpha)$. It essentially coincides with [7, Lemma 2.9]; we include a proof for the sake of completeness.

Lemma 2.3. *Let C be a convex set in X such that $0 \in C$. Let $f \in S_{X^*}$ be such that $f(0) = \inf f(C)$ and let $x_0 \in S_X$ be such that $\theta := f(x_0) > 0$. Let us consider the function*

$$\varepsilon\colon (0,1) \to [0,\infty], \quad \varepsilon(\alpha) := \inf\{\lambda > 0 : C \subset C(f,\alpha) - \lambda x_0\},$$

with the usual convention that $\inf \emptyset = \infty$*. Then $(-f)$ strongly exposes C at 0 if and only if $\varepsilon(\alpha)/\alpha \to 0$ as $\alpha \to 0^+$.*

Remark 2.4. Observe that if $\alpha \in (0,1)$ and $\varepsilon(\alpha)$ is finite then the infimum in the definition of the function ε above is actually a minimum. Hence, in this case, we have that $C \subset C(f,\alpha) - \varepsilon(\alpha)x_0$. Notice that, in the notation of Lemma 2.3, $\varepsilon(\alpha)$ is always finite whenever $\alpha \in (0,\theta)$ and C is bounded.

Proof of Lemma 2.3. First suppose that $\varepsilon(\alpha)/\alpha$ does not tend to 0 as $\alpha \to 0^+$. Then there exist $M > 0$ and $\alpha_n \to 0^+$ such that $\varepsilon(\alpha_n) > M\alpha_n$. For $n \in \mathbb{N}$, let $z_n \in C \setminus [C(f,\alpha_n) - M\alpha_n x_0]$ and observe that

$$f(z_n) + M\theta\alpha_n = f(z_n + M\alpha_n x_0) < \alpha_n\|z_n + M\alpha_n x_0\|.$$

Hence we have

$$0 \le f(z_n) < \alpha_n\|z_n + M\alpha_n x_0\| - M\theta\alpha_n = \alpha_n(\|z_n + M\alpha_n x_0\| - M\theta).$$

Since $\alpha_n > 0$, from the previous inequality we have $\|z_n + M\alpha_n x_0\| > M\theta$. Hence, by the continuity of the norm, eventually it holds $\|z_n\| > M\theta/2$. So we have

$$0 \le f\Big(\frac{z_n}{\|z_n\|}\Big) < \alpha_n \frac{\|z_n + M\alpha_n x_0\| - M\theta}{\|z_n\|} \le \alpha_n \frac{\|z_n\| + M\alpha_n - M\theta}{\|z_n\|} \le \alpha_n.$$

In particular, we have $f\big(\frac{M\theta z_n}{2\|z_n\|}\big) \to 0^+$ as $n \to \infty$. Since C is convex, $0 \in C$, and eventually $\frac{M\theta}{2\|z_n\|} < 1$, we have that eventually $\frac{M\theta z_n}{2\|z_n\|} \in C$. It follows that $(-f)$ does not strongly expose C at 0.

For the other implication, suppose that $\varepsilon(\alpha)/\alpha \to 0$ as $\alpha \to 0^+$. Then $\varepsilon(\alpha)$ is eventually (for $\alpha \to 0^+$) finite, and $C \subset C(f,\alpha) - \varepsilon(\alpha)x_0$ by Remark 2.4. Let $x \in C \cap \{z \in X;\ f(z) \le \alpha^2\}$, then eventually

$$\alpha\|x + \varepsilon(\alpha)x_0\| \le f(x + \varepsilon(\alpha)x_0) = f(x) + \varepsilon(\alpha)f(x_0) \le \alpha^2 + \varepsilon(\alpha)$$

and hence

$$\|x\| \le \|x + \varepsilon(\alpha)x_0\| + \|\varepsilon(\alpha)x_0\| = \|x + \varepsilon(\alpha)x_0\| + \varepsilon(\alpha) \le \alpha + \frac{\varepsilon(\alpha)}{\alpha} + \varepsilon(\alpha).$$

This proves that $\operatorname{diam}\big(S(-f,\alpha^2,C)\big) \to 0$ as $\alpha \to 0^+$, that is, $(-f)$ strongly exposes C at 0. $\square$

Definition 2.5. (see, e.g., [11] or [16, Definition 1.3]) Let $A \subset X$ be a body. We say that $x \in \partial A$ is an *LUR (locally uniformly rotund) point* of A if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $y \in A$ and $\|x - y\| \geq \varepsilon$ then

$$\operatorname{dist}\left(\frac{x+y}{2}, \partial A\right) \geq \delta. \tag{1}$$

We say that A is an *LUR body* if each point in ∂A is an LUR point of A.

The next lemma is well-known in the case the body is a ball (see, e.g. [14, Exercise 8.27]) and in the general case the proof is similar. For a proof see, e.g., [8, Lemma 4.3].

Lemma 2.6. *Let A be a body in X and suppose that $a \in \partial A$ is an LUR point of A. If $f \in \Sigma_A(a)$ then f strongly exposes A at a.*

Recall that a body is said to be *strictly convex* (or *rotund*) if its boundary does not contain any non-degenerate line segment.

Observation 2.7. Let A be a strictly convex body in a Banach space X and let $x_0 \in \partial A$ be a non LUR point of A. Then there exist $\varepsilon > 0$ and a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \partial A$ such that:

- $\operatorname{dist}\left(\frac{x_0+x_n}{2}, \partial A\right) \to 0$;
- $\|x_n - x_m\| \geq \varepsilon$, whenever $n, m \in \mathbb{N}$, $n \neq m$;
- $\|x_n - x_0\| \geq \varepsilon$, whenever $n \in \mathbb{N}$.

Proof. There exist $\varepsilon_1 > 0$ and a sequence $\{x_n\} \subset \partial A$ such that $\operatorname{dist}(\frac{x_n+x_0}{2}) \to 0$ and $\|x_n - x_0\| \geq \varepsilon_1$ for each n. The strict convexity of A implies that $\{x_n\}$ cannot contain any convergent subsequence, and hence by completeness of X, this sequence is not totally bounded. By passing to a subsequence, we can (and do) assume that there is $\varepsilon_2 > 0$ such that $\|x_n - x_m\| \geq \varepsilon_2$ whenever $n \neq m$. Putting $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$ completes the proof. □

In the sequel we shall need the following lemma.

Lemma 2.8. *Let A be an open convex subset of a Banach space X, $x_0 \in \partial A$, and let $\{x_n\} \subset \partial A$ be such that:*

- $\operatorname{dist}\left(\frac{x_0+x_n}{2}, \partial A\right) \to 0$;
- $\|x_n - x_m\| \geq 9$ *whenever* $n, m \in \mathbb{N}$, $n \neq m$;
- $\|x_n - x_0\| \geq 9$ *whenever* $n \in \mathbb{N}$.

Then there exist sequences $\{z_n\} \subset \partial A$ and $\{f_n\} \subset S_{X^*}$ such that:

- $f_n \in \Sigma_{\overline{A}}(z_n)$;
- $f_n(z_n) - f_n(x_0) \to 0$;
- $\|z_n - z_m\| \geq 4$ *whenever* $n, m \in \mathbb{N}$, $n \neq m$;
- $\|z_n - x_0\| \geq 4$ *whenever* $n \in \mathbb{N}$.

Proof. Let $\{z_n\} \subset \partial A$ be such that $d_n := \|z_n - \frac{x_0+x_n}{2}\| \to 0$. Suppose without any loss of generality that $d_n < \frac{1}{4}$, and observe that $\|z_n - x_0\| \geq \|\frac{x_0-x_n}{2}\| - d_n \geq 4$ for each $n \in \mathbb{N}$. Similarly, $\|z_n - z_m\| \geq 4$ whenever $n, m \in \mathbb{N}$, $n \neq m$.

By the Hahn-Banach theorem there exist functionals $f_n \in S_{X^*}$ ($n \in \mathbb{N}$) such that $f_n(z_n) = \sup f_n(A)$. For $n \in \mathbb{N}$, since clearly $f_n(\frac{x_0+x_n}{2}) \geq \sup f_n(A) - d_n$ and $f_n(x_n) \leq \sup f_n(A)$, we have that $f_n(\frac{x_n-x_0}{2}) \leq d_n$ and hence

$$0 < f_n(z_n) - f_n(x_0) = f_n(z_n - \tfrac{x_0+x_n}{2}) + f_n(\tfrac{x_n-x_0}{2}) \leq 2d_n.$$

The proof is complete since $d_n \to 0$. □

In the sequel we shall need the following lemma containing a geometrical construction that allows us to easily define Lipschitz quasiconvex functions. Given sets $D_n \subset A$ ($n \in \mathbb{N}$), the symbol $D_n \nearrow A$ means that the sequence $\{D_n\}$ is monotone nondecreasing and its union is A. By an *L-Lipschitz function* we mean a function which is Lipschitz with a Lipschitz constant L.

Lemma 2.9. *Let X be a normed space, $A \subset X$ an open convex set, and $L > 0$. Let $\{\nu_n\} \subset (1, \infty)$ and $\{D_n\}_{n\in\mathbb{N}}$ be a sequence of open convex sets in X such that:*

(i) $\frac{1}{L}U_X \subset D_1$;

(ii) $D_n \nearrow A$;

(iii) $A \cap \nu_{n+1}D_n \subset D_{n+1}$ *whenever* $n \in \mathbb{N}$.

Let us define $\alpha_1 = 1$ and $\alpha_n = 1 + \sum_{k=2}^{n}(\nu_k - 1)$ ($n > 1$). Let $f\colon A \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \mu_{D_1}(x) & x \in D_1, \\ \alpha_n - 1 + \mu_{D_n}(x) & x \in (A \cap \nu_{n+1}D_n) \setminus D_n,\ n \in \mathbb{N}, \\ \alpha_{n+1} & x \in D_{n+1} \setminus \nu_{n+1}D_n,\ n \in \mathbb{N}, \end{cases}$$

where μ_D denotes the Minkowski gauge of D. Then the function f is L-Lipschitz and quasiconvex, and

(I) $f(A) = [0, 1 + \sum_{k=2}^{\infty}(\nu_k - 1))$;

(II) *if* $\alpha \in (0, 1]$, *we have* $[f < \alpha] = \alpha D_1$;

(III) *if* $\alpha_n < \alpha \leq \alpha_{n+1}$, *for some* $n \in \mathbb{N}$, *we have*

$$[f < \alpha] = A \cap (1 + \alpha - \alpha_n)D_n.$$

Proof. It is easy to see that (I), (II), and (III) easily follow by the definition of f. In particular, f is quasiconvex. Let us prove that

$$|f(x) - f(y)| \leq L\|x - y\| \tag{2}$$

whenever $x, y \in A$. We claim that (2) holds if one of the following conditions is satisfied:

(a) $x, y \in \overline{D}_1^A$;

(b) for some $n \in \mathbb{N}$, $x, y \in D_{n+1} \setminus D_n$;

(c) for some $n \in \mathbb{N}$, $x \in \partial_A D_n$ and $y \in \partial_A D_{n+1}$.

Indeed, cases (a) and (b) follow by the definition of f and by the fact that μ_D is L-Lipschitz, whenever $\frac{1}{L}U_X \subset D$. If (c) holds then

$$0 < f(y) - f(x) = \alpha_{n+1} - \alpha_n = \nu_{n+1} - 1 \leq L\mathrm{dist}(y, D_n) \leq L\|x - y\|,$$

and the claim is proved. If $x, y \in A$, then (2) holds by the fact that each interval in A is the union of finitely many non-overlapping intervals whose endpoints satisfy one of the conditions (a)–(c). □

The next result is the starting point of our paper and provides an elementary example of a Lipschitz quasiconvex function defined on an open half-plane $A \subset \mathbb{R}^2$ not admitting quasiconvex extension to any half-space properly containing A. Observe that the continuity of the extension is not required in the statement of the result.

Example 2.10. Let X denote $\mathbb{R}^2$ equipped with the Euclidean norm, and let $A \subset X$ be the open half-plane
$$A = \{(x, y) \in X : y > -1\}.$$
Then there exists a Lipschitz quasiconvex function $f \colon A \to \mathbb{R}$ not admitting quasiconvex extension to any open half-space of X properly containing A.

Proof. Let $\{\varepsilon_n\} \subset (0, 1)$ be a null sequence. Define inductively positive numbers m_n and q_n ($n \in \mathbb{N}$) such that $m_1 = 1$, $q_1 = 1$ and:

(a) $q_{n+1} = \frac{1}{m_n} + 2q_n$;

(b) $m_{n+1}q_{n+1} < \varepsilon_n$.

For $n \in \mathbb{N}$, let $D_n = \{(x, y) \in X : y > -1,\ y > -1 + m_n(|x| - q_n)\}$.

Then (a) and (b) easily imply that $q_n \nearrow \infty$ and $m_n \searrow 0$, and hence $D_n \nearrow A$. Moreover, it is clear that $U_X \subset D_1$. For each $n \in \mathbb{N}$, put $\nu_n = 2$, and observe that (a) implies that
$$A \cap \nu_{n+1}D_n = \{(z, w) \in X : w > -1,\ w > -1 + m_n(|z| - q_{n+1})\} \subset D_{n+1}.$$
Let $\{\alpha_n\} \subset \mathbb{R}$ and $f \colon A \to \mathbb{R}$ be defined as in Lemma 2.9. Then $\alpha_n = n$ ($n \in \mathbb{N}$) and f is a 1-Lipschitz quasiconvex function such that:

(I) $f(A) = [0, \infty)$;

(II) if $\alpha \in (0, 1]$ then $[f < \alpha] = \alpha D_1$;

(III) if $n < \alpha \le n + 1$ for some $n \in \mathbb{N}$ then
$$[f < \alpha] = A \cap (1 + \alpha - n)D_n.$$
In particular, $[f \le n] = \bigcap_{\alpha > n}[f < \alpha] = \overline{D_n}^A$ whenever $n \in \mathbb{N}$.

Now, suppose on the contrary that there exists H, an open half-plane of X properly containing A, and a quasiconvex extension $F \colon H \to \mathbb{R}$ of f. Since
$$[F \le n] \cap A = [f \le n] = \overline{D_n}^A,$$
Lemma 2.1 shows that
$$[F \le n] \subset \{(z, w) \in X;\ w > -1 + m_n(|z| - q_n)\} =: E_n.$$
Since $m_n q_n \to 0$, we have that $\bigcap_{n \ge n_0} E_n \subset \overline{A}^X$ for each $n_0 \in \mathbb{N}$. Hence for each $n_0 \in \mathbb{N}$ we have
$$[F \le n_0] = \bigcap_{n \ge n_0}[F \le n] \subset \bigcap_{n \ge n_0} E_n \subset \overline{A}^X.$$
A contradiction since F is finite on H. □

3. Main results

Roughly speaking, the aim of the present section is to generalize the construction contained in Example 2.10 in order to obtain, under suitable geometrical assumption on the set A, existence of a Lipschitz quasiconvex function $f\colon A \to \mathbb{R}$ not admitting any continuous quasiconvex extension to the whole space. Let us start with the following technical, but quite elementary lemma.

Fact 3.1. *Let $x^* \in S_{X^*}$ and let $x_0 \in S_X$ be such that $x^*(x_0) = \theta > 0$. Define $H = \{x \in X;\ x^*(x) > 0\}$ and let $\mu > 0$, $0 < \alpha_2 < \alpha_1 < 1$, $\beta_i > 0$ ($i = 1, 2$).*

If $$\theta\tfrac{\beta_1}{\alpha_1} + \beta_1 + \mu < \theta\tfrac{\beta_2}{\alpha_2} - \beta_2 - \tfrac{\mu}{\alpha_1},$$

then $$[\mu B_X + C^\circ(x^*, \alpha_1) - \beta_1 x_0] \cap H \subset C^\circ(x^*, \alpha_2) - \beta_2 x_0.$$

Proof. Given $x \in X$, it is elementary to see that:

- if $x + \beta_1 x_0 \in C^\circ(x^*, \alpha_1)$ then $\|x\| < \frac{x^*(x)}{\alpha_1} + \theta\frac{\beta_1}{\alpha_1} + \beta_1$;
- if $\|x\| < \frac{x^*(x)}{\alpha_2} + \theta\frac{\beta_2}{\alpha_2} - \beta_2$ then $x + \beta_2 x_0 \in C^\circ(x^*, \alpha_2)$.

Now, if $x \in C^\circ(x^*, \alpha_1) - \beta_1 x_0$ and $z \in [x + \mu B_X] \cap H$, we have

$$\begin{aligned}\|z\| \le \mu + \|x\| &< \mu + \tfrac{x^*(x)}{\alpha_1} + \theta\tfrac{\beta_1}{\alpha_1} + \beta_1 = \tfrac{x^*(z)}{\alpha_1} + \tfrac{x^*(x-z)}{\alpha_1} + \theta\tfrac{\beta_1}{\alpha_1} + \beta_1 + \mu \\ &< \tfrac{x^*(z)}{\alpha_1} + \tfrac{\mu}{\alpha_1} + \theta\tfrac{\beta_2}{\alpha_2} - \beta_2 - \tfrac{\mu}{\alpha_1} \le \tfrac{x^*(z)}{\alpha_2} + \theta\tfrac{\beta_2}{\alpha_2} - \beta_2.\end{aligned}$$

Hence, $z \in C^\circ(x^*, \alpha_2) - \beta_2 x_0$ and the proof is complete. □

We are now ready for the first main result of the present paper.

Theorem 3.2. *Let A be an open convex set in a normed space X. Suppose that $0 \in \partial A$ and that there exists $x^* \in S_{X^*}$ such that*

(i) $x^*(0) = \inf f(A)$;

(ii) $(-x^*)$ *does not strongly expose* $\overline{A}$ *at* 0.

Then there exists a Lipschitz quasiconvex function $f\colon A \to \mathbb{R}$ not admitting quasiconvex continuous extension to any open convex subset of X containing A and the origin.

Proof. Let $x_0 \in A$ and $\theta = x^*(x_0) > 0$. Define $H = \{x \in X : x^*(x) > 0\}$. By homogeneity, we can (and do) assume that $\|x_0\| = 1$. By Lemma 2.3, there exist $K > 0$, $\alpha_n \searrow 0$, and $\{x_n\} \subset A$ such that

$$x_n \notin C(x^*, \alpha_n) - K\alpha_n x_0 \qquad (n \in \mathbb{N}). \tag{3}$$

Since, for $n \in \mathbb{N}$ and $\mu > 0$, we have

$$0 < 1 - \frac{\theta - \alpha_{n+1} - (2\mu/K\alpha_n)}{\theta + \alpha_n + (2\mu/K)} < \frac{2\alpha_n + (2\mu/K)(1 + (1/\alpha_n))}{\theta},$$

it is clear that, by passing to a suitable subsequence of $\{\alpha_n\}$, we can (and do) suppose that $\{\alpha_n\} \subset (0, \theta)$, and that there exists a sequence $\{\mu_n\}$ of positive numbers such that $\frac{2\mu_n}{K\alpha_n} < \theta - \alpha_{n+1}$ ($n \in \mathbb{N}$) and

$$\textstyle\prod_{n=1}^{\infty} \frac{\theta - \alpha_{n+1} - \frac{2\mu_n}{K\alpha_n}}{\theta + \alpha_n + \frac{2}{K}\mu_n} > \frac{2}{3}. \tag{4}$$

Define $\delta_1 = \frac{3}{2}$, and for $n \in \mathbb{N}$ put $\delta_{n+1} = \delta_n \frac{\theta - \alpha_{n+1} - \frac{2\mu_n}{K\alpha_n}}{\theta + \alpha_n + \frac{2}{K}\mu_n}$.

Observe that

(a) since $\delta_1 = \frac{3}{2}$ and by (4), we have that $\delta_n > 1$, whenever $n \in \mathbb{N}$;

(b) since $\frac{\theta - \alpha_{n+1} - \frac{2\mu_n}{K\alpha_n}}{\theta + \alpha_n + \frac{2}{K}\mu_n} < 1$, we have that $\delta_n < 2$, whenever $n \in \mathbb{N}$.

By (b) and by the definition of δ_n ($n \in \mathbb{N}$), we have

$$\delta_{n+1} < \delta_n \frac{\theta - \alpha_{n+1} - \frac{\delta_{n+1}\mu_n}{K\alpha_n}}{\theta + \alpha_n + \frac{\delta_n}{K}\mu_n} \qquad (n \in \mathbb{N}),$$

that is, $$\theta\frac{K}{\delta_n} + \frac{K}{\delta_n}\alpha_n + \mu_n < \theta\frac{K}{\delta_{n+1}} - \frac{K}{\delta_{n+1}}\alpha_{n+1} - \frac{\mu_n}{\alpha_n} \qquad (n \in \mathbb{N}).$$

Define $$D_n = [C^\circ(x^*, \alpha_n) - \tfrac{K}{\delta_n}\alpha_n x_0] \cap A \qquad (n \in \mathbb{N}).$$

Fix $n \in \mathbb{N}$. We claim that if $\nu_{n+1} \in (1, \infty)$ is such that

$$\Omega_n := (\nu_{n+1} - 1)(1 + \tfrac{K}{\delta_n}\alpha_n) < \mu_n,$$

then $$[x_0 + \nu_{n+1}(D_n - x_0)] \cap A \subset D_{n+1}.$$

Let us prove the claim. By Lemma 3.1 and (3), we have

$$[\mu_n B_X + C^\circ(x^*, \alpha_n) - \tfrac{K}{\delta_n}\alpha_n x_0] \cap H \subset C^\circ(x^*, \alpha_{n+1}) - \tfrac{K}{\delta_{n+1}}\alpha_n x_0.$$

By the previous inclusion and $A \subset H$, we have

$$\begin{aligned} [x_0 + \nu_{n+1}(D_n - x_0)] \cap A &\subset [x_0 + \nu_{n+1}\big(C^\circ(x^*, \alpha_n) - \tfrac{K}{\delta_n}\alpha_n x_0 - x_0\big)] \cap A \\ &= [C^\circ(x^*, \alpha_n) - \tfrac{K}{\delta_n}\alpha_n x_0 - \Omega_n x_0] \cap A \\ &\subset [C^\circ(x^*, \alpha_n) - \tfrac{K}{\delta_n}\alpha_n x_0 + \mu_n B_X] \cap A \\ &\subset [C^\circ(x^*, \alpha_{n+1}) - \tfrac{K}{\delta_{n+1}}\alpha_n x_0] \cap A = D_{n+1}, \end{aligned}$$

and the claim is proved.

Observe that since $x^*(0) = \inf x^*(A)$ and $\alpha_n \to 0$, we have that $D_n \nearrow A$, indeed

$$\begin{aligned} A = A \cap H &= A \cap \textstyle\bigcup_{n\in\mathbb{N}} C^\circ(x^*, \alpha_n) \\ &\subset A \cap \textstyle\bigcup_{n\in\mathbb{N}} [C^\circ(x^*, \alpha_n) - \tfrac{K}{\delta_n}\alpha_n x_0] = \textstyle\bigcup_{n\in\mathbb{N}} D_n \subset A. \end{aligned}$$

Let $L > 0$ be such that $x_0 + \frac{1}{L}B_X \subset D_1$. Define $\gamma_1 = 1$, $\gamma_n = 1 + \sum_{k=2}^{n}(\nu_k - 1)$ ($n > 1$), and $\gamma_\infty := \sup_{n\in\mathbb{N}} \gamma_n$. Up to a translation, we can apply Lemma 2.9 in order to obtain an L-Lipschitz quasiconvex function f on A such that:

(I) $f(A) = [0, \gamma_\infty)$;

(II) if $\alpha \in (0, 1]$, we have $[f < \alpha] = x_0 + \alpha(D_1 - x_0)$;

(III) if $\gamma_n < \alpha \le \gamma_{n+1}$ for some $n \in \mathbb{N}$ then

$$[f < \alpha] = A \cap \big[x_0 + (1 + \alpha - \gamma_n)(D_n - x_0)\big].$$

Then $[f \leq \gamma_n] = \bigcap_{\alpha \in (\gamma_n, \gamma_{n+1})} [f < \alpha] = A \cap \overline{D_n} = A \cap \big[C(x^*, \alpha_n) - \frac{K}{\delta_n} \alpha_n x_0\big]$ for each $n \in \mathbb{N}$.

Now, proceeding by contradiction, suppose that there exist an open convex set $B \subset X$ containing $A \cup \{0\}$, and a continuous quasiconvex extension $F \colon B \to \mathbb{R}$ of f. Since $(0, x_0] \subset D_1$, we have that $F(0) \leq 1$ $(< \gamma_\infty)$. Our proof will be done once we show that

$$F(-tx_0) \geq \gamma_\infty, \quad \text{for each } t > 0 \text{ such that } -tx_0 \in B, \tag{5}$$

since this will contradict the continuity of F at 0.

Let us show (5). Assume the contrary, that is, there exists $t > 0$ such that $-tx_0 \in B$ and $F(-tx_0) < \gamma_\infty$. Since $\alpha_n \searrow 0$ and $\gamma_n \nearrow \gamma_\infty$, there exists $n \in \mathbb{N}$ such that $t > K\alpha_n$ and $F(-tx_0) \leq \gamma_n$. Then $t > \frac{K}{\delta_n} \alpha_n$ since $\delta_n > 1$. For brevity, let us denote

$$P := [F \leq \gamma_n] \quad \text{and} \quad \Gamma := C(x^*, \alpha_n) - \tfrac{K}{\delta_n} \alpha_n x_0.$$

Then P is a convex body, Γ is a closed convex cone with vertex at $-\frac{K}{\delta_n} \alpha_n x_0$, and x_0 is an interior point of both P and Γ. Notice that $P \cap A = [f \leq \gamma_n] = \Gamma \cap A \neq A$ by our construction. Thus there exists $z \in \partial\Gamma \cap A$. Fix some $g \in \Sigma_{\Gamma \cap A}(z)$. By Lemma 2.1, g belongs to both $\Sigma_\Gamma(z)$ and $\Sigma_P(z)$ and, by a well-known property of cones, also to $\Sigma_\Gamma(-\frac{K}{\delta_n} \alpha_n x_0)$. So, denoting $\sigma := g(-\frac{K}{\delta_n} \alpha_n x_0)$, we have that $g(x_0) < \sigma$. Since $t > \frac{K}{\delta_n} \alpha_n$ we must have $g(-tx_0) > \sigma = \sup g(P)$. But this implies that $-tx_0 \notin P$, which is a contradiction that completes the proof of (5). We are done. $\square$

The above theorem will now be used to prove our second main result.

Theorem 3.3. *Let A be an open convex subset of a Banach space X. Suppose that ∂A contains a non-LUR point for $\overline{A}$. Then there exists a Lipschitz quasiconvex function $f \colon A \to \mathbb{R}$ not admitting quasiconvex continuous extension to the whole X.*

Proof. We can (and do) assume that each point of ∂A is strongly exposed for $\overline{A}$ by any of its support functionals; indeed, otherwise we are done by Theorem 3.2. Notice that then $\overline{A}$ is strictly convex.

Let $x_0 \in \partial A$ be a non LUR point of $\overline{A}$. By Observation 2.7, there exist $\varepsilon > 0$ and a sequence $\{x_n\} \subset \partial A$ such that:

- $\operatorname{dist}\big(\frac{x_0 + x_n}{2}, \partial A\big) \to 0$;
- $\|x_n - x_m\| \geq \varepsilon$, whenever $n, m \in \mathbb{N}$, $n \neq m$;
- $\|x_n - x_0\| \geq \varepsilon$, whenever $n \in \mathbb{N}$.

By homogeneity, we can (and do) suppose that $\varepsilon = 9$. By Lemma 2.8, there exist sequences $\{z_n\} \subset \partial A$ and $\{f_n\} \subset S_{X^*}$ such that:

- $\sup f_n(A) = f_n(z_n)$;
- $f_n(z_n) - f_n(x_0) \to 0$;
- $\|z_n - z_m\| \geq 4$, whenever $n, m \in \mathbb{N}$, $n \neq m$;
- $\|z_n - x_0\| \geq 4$, whenever $n \in \mathbb{N}$.

Moreover, by translation we can also suppose that $0 \in A$ and $\|x_0\| \leq 1$. Hence, $\|z_n\| \geq 3$, whenever $n \in \mathbb{N}$.

Now, since by our assumption f_n strongly exposes $\overline{A}$ at z_n ($n \in \mathbb{N}$), there exists $\delta_n > 0$ such that

$$\operatorname{diam}[S(f_n, \delta_n, \overline{A})] \leq 1.$$

So, the convex set $W = A \setminus \bigcup_{n\in\mathbb{N}} S(f_n, \delta_n, \overline{A})$, contains $A \cap 2U_X$. Let $f\colon A \to \mathbb{R}$ be the function defined as follows

$$f(x) = \begin{cases} \|x\| & x \in A \cap 2U_X \\ 2 & x \in W \setminus 2U_X \\ 2 + \operatorname{dist}(x, W) & x \in A \setminus W \end{cases}.$$

Then it is clear that f is quasiconvex and Lipschitz. Suppose on the contrary that there exists a continuous quasiconvex function $F\colon X \to \mathbb{R}$ extending f, then

$$[F \leq 2] \cap A = [f \leq 2] = W.$$

Observe that, by our construction, the sets $S(f_n, \delta_n, \overline{A})$ ($n \in \mathbb{N}$) are pairwise disjoint. In particular, for each $n \in \mathbb{N}$, there exists $t_n \in A \cap \partial W$ such that $f_n(t_n) = f_n(z_n) - \delta_n$. Hence, by Lemma 2.1, the set $[F \leq 2]$ is contained in the body

$$\Omega = \bigcap_{n\in\mathbb{N}} \{x \in X;\, f_n(x) \leq \sup f_n(W)\} = \bigcap_{n\in\mathbb{N}} \{x \in X;\, f_n(x) \leq f_n(z_n) - \delta_n\}.$$

Since $f_n(z_n) - f_n(x_0) \to 0$ and since by our construction $f_n(z_n) - f_n(x_0) \geq \delta_n$, we have that $\delta_n \to 0$, and hence that $f_n(x_0) - \sup f_n(W) \to 0$. This proves that $x_0 \in \partial\Omega$ and hence there exists a sequence $\{y_n\} \subset X$ such that $y_n \to x_0$ and such that $F(y_n) > 2$. Since $f(z) \leq 1$, whenever z belongs to the segment $[0, x_0)$, we get a contradiction by the continuity of F. □

4. Final remarks and an open problem

In the paper [13], the authors proved the following result on extendability of uniformly continuous quasiconvex functions.

Theorem 4.1. *Let X be a normed space, $A \subset X$ a bounded open convex set which is uniformly rotund (that is, uniformly convex), and $f\colon A \to \mathbb{R}$ a uniformly continuous quasiconvex function. Then there exists a uniformly continuous quasiconvex function $F\colon X \to \mathbb{R}$ that extends f and satisfies $F(X) \subset \overline{f(A)}$.*

The following 2-dimensional example shows that a similar result does not hold for the class of Lipschitz quasiconvex functions.

Example 4.2. There exists a Lipschitz quasiconvex function f on the open circle $A := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ such that f admits no Lipschitz quasiconvex extension defined on an open convex set containing $\overline{A}$.

Proof. Fix an increasing sequence $\{b_n\}_{n\in\mathbb{N}} \subset (1,\infty)$ such that $b_n/b_{n+1} \to 0$. Notice that $b_n \nearrow \infty$. Then fix a decreasing sequence $\{t_n\}_{n\in\mathbb{N}} \subset (0,1/2]$ such that $t_1 = 1/2$ and $b_{n+1}t_n \to 0$. Clearly, $t_n \searrow 0$.

For each $n \in \mathbb{N}$ consider:

- the point $w_n := (\sqrt{1-t_n^2}\,, t_n)$ of ∂A;
- the open half-plane $H_n := \{(x,y) \in \mathbb{R}^2 : y - t_n - b_n(x - \sqrt{1-t_n^2}) > 0\}$;
- the line $L_n := \partial H_n$ that passes through w_n and whose equation is clearly $y - t_n - b_n\big(x - \sqrt{1-t_n^2}\big) = 0$;
- the open convex set $D_n := A \cap H_n$.

Since $b_1\sqrt{1-t_1^2} > \sqrt{1-t_1^2} > t_1$, the set D_1 contains the origin. So there exists $L > 0$ such that $(1/L)U_{\mathbb{R}^2} \subset D_1$. The monotonicity of $\{b_n\}$ and $\{t_n\}$ and an easily sketched diagram show that, for each n, $D_n \subset D_{n+1}$ and even there exists $\nu_{n+1} \in (1,\infty)$ such that $A \cap \nu_{n+1}D_n \subset D_{n+1}$. Moreover, the greatest such ν_{n+1} is such that $w_{n+1} \in \nu_{n+1}L_n$, that is, $\frac{1}{\nu_{n+1}}w_{n+1} \in L_n$. An easy calculation gives that

$$\nu_{n+1} = \frac{b_n\sqrt{1-t_{n+1}^2} - t_{n+1}}{b_n\sqrt{1-t_n^2} - t_n}\,.$$

Also notice that $D_n \nearrow A$.

By Lemma 2.9, there exists an L-Lipschitz function $f\colon A \to [0,\infty)$ satisfying the properties (II), (III) therein. Then for each $n \in \mathbb{N}$,

$$[f \le \alpha_n] = A \cap \bigcap_{\alpha\in(\alpha_n,\alpha_{n+1})} [f < \alpha] = A \cap \bigcap_{\alpha\in(\alpha_n,\alpha_{n+1})} (1+\alpha-\alpha_n)D_n = \overline{D}_n^A, \tag{6}$$

where the numbers α_n are as in Lemma 2.9. Let us show that this function f has the required property.

Proceeding by contradiction, assume that there exists a Lipschitz extension F of f, defined on $B := (1+\eta)A$ for some $\eta > 0$. Let $M > 0$ be a Lipschitz constant of F. It is easy to see that there exists $\bar{n} \in \mathbb{N}$ such that the point $L_n \cap L_{n+1}$ belongs to B whenever $n \ge \bar{n}$.

In what follows, the sublevel sets of f and F (in our simplified notation) are considered in A and B, respectively. Consider the convex, relatively closed subsets $E_n := [F \le \alpha_n]$ of B, $n \in \mathbb{N}$. Since by (6), $E_n \cap A = [f \le \alpha_n] = \overline{D}_n^A$, by convexity we must have $\overline{E}_n \subset \overline{H}_n$ for each $n \in \mathbb{N}$. Observe that if $x, y \in B$ and $\alpha, \beta \in \mathbb{R}$ are such that $F(x) \le \alpha < \beta < F(y)$, then $\beta - \alpha \le F(y) - F(x) \le M\|y - x\|$. It follows that

$$\operatorname{dist}\big([F \le \alpha], [F > \beta]\big) \ge \frac{1}{M}(\beta - \alpha) \quad \text{whenever } \alpha < \beta. \tag{7}$$

So by (7), we must have for $n \ge \bar{n}$

$$\frac{1}{M} \le \frac{\operatorname{dist}(E_n, B \setminus E_{n+1})}{\alpha_{n+1} - \alpha_n} \le \frac{\operatorname{dist}(w_n, B \setminus \overline{H}_{n+1})}{\alpha_{n+1} - \alpha_n} = \frac{\operatorname{dist}(w_n, L_{n+1})}{\alpha_{n+1} - \alpha_n} =: \Delta_n. \tag{8}$$

The proof by contradiction will be complete once we show that $\Delta_n \to 0$.

First notice that

$$0 < \sqrt{1 - t_{n+1}^2} - \sqrt{1 - t_n^2} = \frac{(t_n - t_{n+1})(t_n + t_{n+1})}{\sqrt{1 - t_{n+1}^2} + \sqrt{1 - t_n^2}} \leq \frac{(t_n - t_{n+1})\, t_n}{\sqrt{1 - t_1^2}}. \tag{9}$$

In what follows, the notation $u_n \sim s_n$ means that $u_n/s_n \to 1$ as $n \to \infty$. Now, a well-known formula from analytic geometry, our choice of $\{b_n\}, \{t_n\}$, and (9) together imply that

$$\operatorname{dist}(w_n, L_{n+1}) = \frac{(t_n - t_{n+1}) + b_{n+1}\big(\sqrt{1 - t_{n+1}^2} - \sqrt{1 - t_n^2}\,\big)}{\sqrt{1 + b_{n+1}^2}} \sim \frac{t_n - t_{n+1}}{b_{n+1}}\,,$$

$$\alpha_{n+1} - \alpha_n = \nu_{n+1} - 1 = \frac{b_n\big(\sqrt{1 - t_{n+1}^2} - \sqrt{1 - t_n^2}\big) + (t_n - t_{n+1})}{b_n\sqrt{1 - t_n^2} - t_n} \sim \frac{t_n - t_{n+1}}{b_n}.$$

Therefore $\Delta_n \sim \frac{b_n}{b_{n+1}} \to 0$, and we are done. □

The results contained in the previous section show in particular that if the boundary of an open convex set $A \subset X$ contains a non-LUR point then there exists a uniformly quasiconvex function defined on A not admitting quasiconvex uniformly continuous extension to the whole X. Hence it seems natural to ask whether Theorem 4.1 can be reversed, in the sense formulated in the following open problem.

Problem 4.3. *Let X be a Banach space and $A \subset X$ a bounded open convex set such that each uniformly continuous quasiconvex function $f\colon A \to \mathbb{R}$ admits a uniformly continuous quasiconvex extension $F\colon X \to \mathbb{R}$. Is then the set A uniformly rotund?*

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