Journal of Convex Analysis
Volume 30 (2023), No. 1, 343–358

Diffeomorphism Groups of Convex Polytopes

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Received: March 22, 2022
Accepted: September 8, 2022

Let M be a convex polytope in $\mathbb{R}^n$, with non-empty interior. We turn the group $\mathrm{Diff}(M)$ of all C^∞-diffeomorphisms of M into a regular Lie group.

Keywords: Polytope, polyhedron, diffeomorphism group, infinite-dimensional Lie group, regularity, manifold with corners, local addition.

2020 Mathematics Subject Classification: Primary 58D05; secondary 22E65, 46T05, 46T10, 52B11, 52B15, 52B70.

1. Introduction and statement of results

Let E be a finite-dimensional real vector space and $M \subseteq E$ be the convex hull of a finite subset of E, with non-empty interior M^0. For each $x \in M$, there exists a smallest face $M(x)$ of M such that $x \in M(x)$; we set

$$E(x) := \mathrm{span}_{\mathbb{R}}(M(x) - x) \subseteq E.$$

For $k \in \mathbb{N}_0 \cup \{\infty\}$, endow the space $C^k(M, E)$ of all C^k-maps $f\colon M \to E$ with the compact-open C^k-topology (as in [15, Section 1.7]); then

$$C^k_{\mathrm{str}}(M, E) := \{f \in C^k(M, E)\colon (\forall x \in M)\ f(x) \in E(x)\}$$

is a closed vector subspace (on which the induced topology will be used). We regard the functions $f \in C^k(M, E)$ as C^k-vector fields on M; the elements $f \in C^k_{\mathrm{str}}(M, E)$ will be called *stratified* C^k-vector fields (cf. [9]). Let

$$\mathrm{Diff}^k(M) := \{\phi \in C^k(M, M)\colon (\exists \psi \in C^k(M, M))\colon \phi \circ \psi = \psi \circ \phi = \mathrm{id}_M\}$$

be the set of all C^k-diffeomorphisms $\phi\colon M \to M$, and abbreviate $\mathrm{Diff}(M) := \mathrm{Diff}^\infty(M)$. Then $\mathrm{Diff}^k(M)$ is a group, using composition of diffeomorphisms as the group multiplication. The neutral element is the identity map id_M. Our main goal is the following result:

Theorem 1.1. *The group* $\mathrm{Diff}(M)$ *admits a smooth manifold structure modeled on* $C^\infty_{\mathrm{str}}(M, E)$ *making it a Lie group.*

More details concerning the Lie group structure on $\mathrm{Diff}(M)$ are now described. If $P \subseteq E$ is a convex polytope, we call a homeomorphism $\phi\colon P \to P$ *face respecting* if $\phi(F) = F$ for each face F of P. Then also ϕ^{-1} is face respecting.

ISSN 0944-6532 / \$ 2.50

For each $k \in \mathbb{N} \cup \{\infty\}$, the set

$$\mathrm{Diff}^k_{\mathrm{fr}}(M) := \{\phi \in \mathrm{Diff}^k(M) \colon \phi \text{ is face respecting}\}$$

is a normal subgroup of finite index in $\mathrm{Diff}^k(M)$ (see Lemma 5.5) and

$$\Omega_k := \mathrm{Diff}^k_{\mathrm{fr}}(M) - \mathrm{id}_M \tag{1}$$

is an open 0-neighbourhood in $C^k_{\mathrm{str}}(M, E)$ (see Lemma 4.1). We give $\mathrm{Diff}^k_{\mathrm{fr}}(M)$ the smooth manifold structure modeled on $C^k_{\mathrm{str}}(M, E)$ making the bijective map

$$\Phi_k \colon \mathrm{Diff}^k_{\mathrm{fr}}(M) \to \Omega_k, \quad \phi \mapsto \phi - \mathrm{id}_M$$

a C^∞-diffeomorphism. Then the following holds, using $C^{k,\ell}$-maps as in [2]:

Proposition 1.2. *For all $k \in \mathbb{N} \cup \{\infty\}$ and $\ell \in \mathbb{N}_0 \cup \{\infty\}$, the map*

$$c_{k,\ell} \colon \mathrm{Diff}^{k+\ell}_{\mathrm{fr}}(M) \times \mathrm{Diff}^k_{\mathrm{fr}}(M) \to \mathrm{Diff}^k_{\mathrm{fr}}(M), \quad (\phi, \psi) \mapsto \phi \circ \psi$$

is $C^{\infty,\ell}$ (and thus C^ℓ); moreover, the map

$$\iota_{k,\ell} \colon \mathrm{Diff}^{k+\ell}_{\mathrm{fr}}(M) \to \mathrm{Diff}^k_{\mathrm{fr}}(M), \quad \phi \mapsto \phi^{-1}$$

is C^ℓ. Notably, $\mathrm{Diff}^\infty_{\mathrm{fr}}(M)$ is a smooth Lie group modeled on $C^\infty_{\mathrm{str}}(M, E)$.

To establish Theorem 1.1, we shall give $\mathrm{Diff}(M)$ a smooth Lie group structure modeled on $C^\infty_{\mathrm{str}}(M, E)$ which turns $\mathrm{Diff}^\infty_{\mathrm{fr}}(M)$ into an open subgroup.

Remark 1.3. The differentiability properties of the maps $c_{k,\ell}$ established in Proposition 1.2 show that $\mathrm{Diff}^\infty_{\mathrm{fr}}(M)$ fits into the framework of [17, pp. 630–631] (see [16, Remark 4.2 (b)] for more details). Thus $\mathrm{Diff}^\infty_{\mathrm{fr}}(M)$ (and hence also $\mathrm{Diff}(M)$) is an L^1-regular Lie group in the sense of [13, Definition 5.16], by [16, Theorem 1.2]. Notably, the Fréchet-Lie group $\mathrm{Diff}(M)$ is C^0-regular (as in [12]) by [13, Theorem A]. It is clear from the definition in [12] that every C^0-regular Lie group is C^∞-regular (i.e., regular in the sense of Milnor [20]). □

Remark 1.4. The easiest – but most relevant – case of our construction is the case of a cube $M := [0,1]^n \subseteq \mathbb{R}^n$. The corresponding diffeomorphism group is of interest in numerical mathematics [6]. Note that $[0,1]^n$ is a prime example of a smooth manifold with corners (as in [7, 8, 19]; cf. [18] for infinite-dimensional generalizations). Michor [19] discussed the diffeomorphism group of a paracompact, smooth manifold M with corners, but [19] contains a flaw: Contrary to claims in the book, local additions in Michor's sense never exist if $\partial M \neq \emptyset$, as we show in Appendix B. □

As before, let E be a finite-dimensional real vector space. If $M \subseteq E$ is any compact convex subset with non-empty interior, then a Lie group structure can be constructed on the group $\mathrm{Diff}_{\partial M}(M)$ of all C^∞-diffeomorphisms $\phi \colon M \to M$ such that $\phi(x) = x$ for all $x \in \partial M$ (see [14]). In the case that M is a polytope, our discussion of the Lie group structure on $\mathrm{Diff}^\infty_{\mathrm{fr}}(M)$ proceeds along similar lines. Then $\mathrm{Diff}_{\partial M}(M)$ is a closed normal subgroup and smooth submanifold of both $\mathrm{Diff}^\infty_{\mathrm{fr}}(M)$ and $\mathrm{Diff}(M)$ (see Remark 5.6). For numerical applications, it is useful to know a lower bound for the 0-neighbourhoods $\Omega_k \subseteq C^k_{\mathrm{str}}(M, E)$. Fixing any norm $\|\cdot\|$ on E to calculate operator norms, we show:

Proposition 1.5. *Let $M \subseteq E$ be a convex polytope with non-empty interior and $k \in \mathbb{N} \cup \{\infty\}$. Let U_k be the set of all $f \in C^k_{\mathrm{str}}(M, E)$ such that*

$$\|f\|_{\infty,\mathrm{op}} := \sup_{x \in M} \|f'(x)\|_{\mathrm{op}} < 1.$$

Then U_k is an open 0-neighbourhood in $C^k_{\mathrm{str}}(M, E)$ and $U_k \subseteq \Omega_k$ holds, for Ω_k as in (1).

For $E = \mathbb{R}^n$ and $M = [0,1]^n$, see already [6, Proposition 3.4]. An analogous result for $\mathrm{Diff}_{\partial M}(M)$ was also established in [6], for compact convex subsets $M \subseteq \mathbb{R}^n$ with non-empty interior (see [6, Lemma 3.1]).

Remark 1.6. Note that $\|f\|_{\infty,\mathrm{op}}$ is the smallest Lipschitz constant $\mathrm{Lip}(f)$ for f in the situation of Proposition 1.5 (see, e.g., [15, Lemma 1.5.3 (c)]).

2. Preliminaries and basic facts

We write $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The term "locally convex space" means a locally convex, Hausdorff topological vector space over the ground field $\mathbb{R}$. If $(E, \|\cdot\|)$ is a normed space, let

$$B^E_r(x) := \{y \in E \colon \|y - x\| < r\} \quad \text{and} \quad \overline{B}^E_r(x) := \{y \in E \colon \|y - x\| \leq r\}$$

for $x \in E$ and $r > 0$. For background concerning convex polytopes and their faces, the reader is referred to [5] and [22]. If E is a real vector space and $S \subseteq E$ a subset, we write $\mathrm{aff}(S)$ for the affine subspace of E generated by S. In this article, we are working in the setting of infinite-dimensional calculus known as Keller's C^k_c-theory (going back to [3]). Accordingly, the smooth manifolds and Lie groups we consider are modeled on locally convex spaces which need not have finite dimension; see [10], [15], and [21] for further information.

Let E and F be locally convex spaces. We recall from [15, Chapter 1]:

2.1. A map $f \colon U \to E$ on an open subset $U \subseteq E$ is called C^1 if f is continuous, the directional derivative

$$df(x, y) := (D_y f)(x) := \lim_{t \to 0} \frac{1}{t}\,(f(x + ty) - f(x))$$

exists in F for all $(x, y) \in U \times E$, and the map $df \colon U \times E \to F$ is continuous. □

Then $f'(x) := df(x, \cdot) \colon E \to F$ is linear for each $x \in U$.

2.2. Let $U \subseteq E$ be a subset which is locally convex (in the sense that each $x \in U$ has a convex neighbourhood in U) and whose interior U^0 is dense in U. A map $f \colon U \to F$ is called C^0 if f is continuous. If f is continuous, $f|_{U^0}$ is C^1 and $d(f|_{U^0}) \colon U^0 \times E \to F$ has a continuous extension

$$df \colon U \times E \to F,$$

then f is called C^1. Given $k \in \mathbb{N}$, we say that f is C^k if f is C^1 and df is C^{k-1}. If f is C^k for all $k \in \mathbb{N}_0$, then f is called a C^∞-map or also *smooth*. □

If E and F are locally convex spaces, we write $L(E, F)_b$ for the locally convex space of all continuous linear maps $\alpha \colon E \to F$, endowed with the topology of uniform convergence on bounded subsets of E. For finite-dimensional domains, the C^k-property can be reformulated in the expected way:

Lemma 2.3. *Let E be a finite-dimensional real vector space, $U \subseteq E$ be a locally convex subset with dense interior, F be a locally convex space, and $k \in \mathbb{N}$. Then the following conditions are equivalent for a continuous map $f \colon U \to F$:*

(a) *f is C^k;*

(b) *f is C^1 and $f' \colon U \to L(E,F)_b$, $x \mapsto f'(x) = df(x,\cdot)$ is C^{k-1}.*

Proof. To see that (a) implies (b), we may assume that $E \neq \{0\}$, excluding a trivial case. Let $b_1, \ldots, b_n$ be a basis for E. Then

$$\psi \colon L(E,F)_b \to F^n, \quad \alpha \mapsto (\alpha(b_1), \ldots, \alpha(b_n))$$

is an isomorphism of topological vector spaces. If f is C^k, then f is C^1 and

$$\psi \circ f' = (df(\cdot, b_1), \ldots, df(\cdot, b_n))$$

is C^{k-1} (by [15, Lemma 1.4.6 and Proposition 1.4.10]), entailing that the composition $f' = \psi^{-1} \circ (\psi \circ f')$ is C^{k-1}.

Conversely, assume that (b) holds. The projections $\mathrm{pr}_1 \colon U \times E \to U$, $(x,y) \mapsto x$ and $\mathrm{pr}_2 \colon U \times E \to E$, $(x,y) \mapsto y$ are smooth, being restrictions of continuous linear maps $E \times E \to E$. Since E is finite dimensional and thus normable, the evaluation map $\varepsilon \colon L(E,F)_b \times E \to F$, $(\alpha, y) \mapsto \alpha(y)$ is continuous and thus smooth, being bilinear. Hence $df = \varepsilon \circ (f' \circ \mathrm{pr}_1, \mathrm{pr}_2)$ is C^{k-1}, using [15, Lemma 1.4.6 and Proposition 1.4.10]. □

If $f \colon X \to Y$ is a function between metric spaces (X, d_X) and (Y, d_Y), we define

$$\mathrm{Lip}(f) := \sup \left\{ \frac{d_Y(f(x), f(y))}{d_X(x,y)} \colon x, y \in X \text{ with } x \neq y \right\} \in [0, \infty].$$

Thus f is Lipschitz continuous if and only if $\mathrm{Lip}(f) < \infty$, and $\mathrm{Lip}(f)$ is the smallest Lipschitz constant for f in this case. We shall use a version of the Lipschitz Inverse Function Theorem.

Lemma 2.4. *Let $(E, \|\cdot\|)$ be a normed space, $M \subseteq E$ a subset and $f \colon M \to E$ a Lipschitz continuous mapping such that* $\mathrm{Lip}(f) < 1$. *Then $\phi := \mathrm{id}_M + f \colon M \to E$ is injective and $\phi^{-1} \colon \phi(M) \to M$ is Lipschitz continuous with*

$$\mathrm{Lip}(\phi^{-1}) \ \leq \ \frac{1}{1 - \mathrm{Lip}(f)}. \tag{2}$$

If, moreover, $(E, \|\cdot\|)$ is a Banach space and M is open in E, then $\phi(M)$ is open in E.

Proof. For all $x, y \in M$,

$$\|\phi(x) - \phi(y)\| \geq \|x - y\| - \|f(x) - f(y)\| \geq (1 - \mathrm{Lip}(f))\|x - y\|.$$

Thus $\phi(x) = \phi(y)$ entails $x = y$, and also (2) follows. If $(E, \|\cdot\|)$ is a Banach space and M is open in E, then ϕ is an open map, as a consequence of [11, Theorem 5.3] (or Lemma A.1 in our Appendix A, applied with $A := \mathrm{id}_E$). Notably, $\phi(M)$ is open. □

2.5. Let E_1, E_2, and F be locally convex spaces, $U_1 \subseteq E_1$ as well as $U_2 \subseteq E_2$ be locally convex subsets with dense interior, and $r, s \in \mathbb{N}_0 \cup \{\infty\}$. Following [2], we say that a map $f \colon U_1 \times U_2 \to F$ is $C^{r,s}$ if f is continuous and, for all $i, j \in \mathbb{N}_0$ with $i \leq r$ and $j \leq s$, there exists a continuous map

$$d^{(i,j)} f \colon U_1 \times U_2 \times (E_1)^i \times (E_2)^j \to F$$

such that the iterated directional derivatives

$$(D_{(v_i,0)} \cdots D_{(v_1,0)} D_{(0,w_j)} \cdots D_{(0,w_1)} f)(x, y)$$

exist for all $x \in U_1^0$, $y \in U_2^0$, $v_1, \ldots, v_i \in E_1$, $w_1, \ldots, w_j \in E_2$, and coincide with $d^{(i,j)} f(x, y, v_1, \ldots, v_i, w_1, \ldots, w_j)$, using the interiors U_1^0 and U_2^0 (see also [15]). $\square$

We mention that $C^{r,s,t}$-maps $f \colon U_1 \times U_2 \times U_3 \to F$ can be defined analogously [1].

3. Proof of Proposition 1.5

By definition of the compact-open C^k-topology, the map

$$C^k_{\mathrm{str}}(M, E) \to C(M \times E, E), \quad f \mapsto df$$

is continuous if we endow $C(M \times E, E)$ with the compact-open topology. Hence

$$U_k = \{f \in C^k_{\mathrm{str}}(M, E) \colon df(M \times \overline{B}^E_1(0)) \subseteq B^E_1(0)\}$$

is open in $C^k_{\mathrm{str}}(M, E)$. Moreover, $0 \in U_k$. We prove the remaining assertion of Proposition 1.5 by induction on the dimension of E. The case $\dim(E) = 0$ is trivial, since $f = 0$ for all $f \in C^k_{\mathrm{str}}(M, E)$ in this case and thus $\mathrm{id}_M + f = \mathrm{id}_M \in \mathrm{Diff}^k_{\mathrm{fr}}(M)$. Let $\dim(E) > 0$ now and assume that the assertion has been established for all finite-dimensional vector spaces of dimension $< \dim(E)$, in place of E. Given $f \in U_k$, our goal is to show that $\phi := \mathrm{id}_M + f$ is a face-respecting C^k-diffeomorphism of M. As a first step, we show that $\phi(F) = F$ for each face $F \neq M$ of M. Pick an element x in the algebraic interior $\mathrm{algint}(F)$ of F (called the "relative interior" in [5, 22]). Then $F = M(x)$; we endow

$$E(x) = \mathrm{span}_{\mathbb{R}}(F - x)$$

with the norm induced by $\|\cdot\|$. For each $z \in \mathrm{algint}(F)$, we have

$$f(z) \in E(z) = \mathrm{aff}(F) - z = \mathrm{aff}(F) - x = E(x).$$

Then $f(z) \in E(x)$ for all $z \in F$, as f is continuous and $\mathrm{algint}(F)$ is dense in F. Since $F \neq M$, we have $\dim(E(x)) < \dim(E)$. Now $N := F - x$ is a polytope in $E(x)$, with non-empty interior relative $E(x)$. Moreover, the map

$$g \colon N \to E(x), \quad y \mapsto f(x + y)$$

is C^k (using [15, Lemma 1.4.16]) and Lipschitz with $\mathrm{Lip}(g) \leq 1$. If $y \in N$, let $N(y)$ be the smallest face of N such that $y \in N(y)$. Then $x + N(y)$ is the smallest face of F which contains $x + y$, and hence also the smallest face $M(x + y)$ of M which contains $x + y$. As a consequence,

$$\begin{aligned} g(y) = f(x + y) \in E(x + y) &= \mathrm{span}_{\mathbb{R}}(M(x + y) - (x + y)) \\ &= \mathrm{span}_{\mathbb{R}}(x + N(y) - (x + y)) = \mathrm{span}_{\mathbb{R}}(N(y) - y) \end{aligned}$$

holds, showing that $g \in C^k_{\mathrm{str}}(N, E(x))$.

By the inductive hypothesis, we have $(\mathrm{id}_N + g)(N) = N$. Since $(\mathrm{id}_M + f)(z) = z + f(z) = x + (z - x) + f(x + (z - x)) = x + (\mathrm{id}_N + g)(z - x)$ for all $z \in F$, we deduce that

$$(\mathrm{id}_M + f)(F) = x + (\mathrm{id}_N + g)(F - x) = x + (\mathrm{id}_N + g)(N) = x + N = F,$$

concluding the first step of the proof. As the second step, we prove $\phi(M) \subseteq M$. If this was false, we could find $x \in M$ such that $\phi(x) \notin M$. Now M being an intersection of half-spaces, we would find a linear functional $\lambda \colon E \to \mathbb{R}$ and $a \in \mathbb{R}$ such that

$$\lambda(M) \subseteq \,]-\infty, a] \quad \text{and} \quad \lambda(\phi(x)) > a.$$

Then $\lambda \neq 0$, entailing that λ is an open map. Since M is compact and $\lambda \circ \phi$ is continuous, after replacing x if necessary we may assume that

$$\lambda(\phi(x)) = \max \lambda(\phi(M)).$$

Since $\phi(F) = F \subseteq M$ for each proper face F of M by the first step, we must have $x \in M^0$. Since $\phi(M^0)$ is open in E by Lemma 2.4, $\lambda(\phi(M^0))$ is an open neighbourhood of $\lambda(\phi(x))$ in $\mathbb{R}$, contradicting the maximailty of $\lambda(\phi(x))$. Hence x cannot exist and $\phi(M) \subseteq M$ must hold.

As the third step, we show that $\phi(M) = M$. Since ∂M is the union of all proper faces F of M and $\phi(F) = F$ by Step 1, we have $\phi(\partial M) = \partial M$. Now M being closed in E, we have $M = M^0 \cup \partial M$ with $M^0 \cap \partial M = \emptyset$. We already observed that $\phi(M^0)$ is open in E, by Lemma 2.4; thus $\phi(M^0) \subseteq M^0$. Since $\phi(M)$ is compact and hence closed in M, the intersection

$$\phi(M) \cap M^0 = (\phi(M^0) \cup \phi(\partial M)) \cap M^0 = \phi(M^0)$$

is closed in M^0. But $\phi(M^0)$ is also open in E, and hence open in M^0. Since M^0 is connected and $\phi(M^0) \neq \emptyset$, we deduce that $M^0 = \phi(M^0)$. As a consequence, $\phi(M) = \phi(M^0 \cup \partial M) = M^0 \cup \partial M = M$.

By Lemma 2.4, $\phi \colon M \to M$ is a homeomorphism. By Steps 1 and 3, $\phi(F) = F$ for each face F of M, whence ϕ is face respecting.

The inversion map $j \colon \mathrm{GL}(E) \to \mathrm{GL}(E)$, $\alpha \mapsto \alpha^{-1}$ is smooth on the general linear group $\mathrm{GL}(E) := L(E,E)^\times$. For each $x \in M^0$, we have $\phi'(x) - \mathrm{id}_E = f'(x)$ with $\|f'(x)\|_{\mathrm{op}} < 1$, whence $\phi'(x) \colon E \to E$ is invertible (by means of Neumann's series). Thus $\phi|_{M^0}$ is a local C^k-diffeomorphism at x, by the Inverse Function Theorem. As a consequence, the bijection $\phi|_{M^0} \colon M^0 \to M^0$ is a C^k-diffeomorphism. Now $\phi^{-1} \colon M \to E$ is a continuous map and $\phi^{-1}|_{M^0} = (\phi|_{M^0})^{-1}$ is C^1 with

$$(\phi^{-1}|_{M^0})' = ((\phi|_{M^0})^{-1})' = j \circ (\phi|_{M_0})' \circ (\phi|_{M^0})^{-1}.$$

By the preceding, the map

$$g := j \circ \phi' \circ \phi^{-1} \colon M \to L(E,E)_b$$

is a continuous extension of $(\phi^{-1}|_{M^0})' \colon M^0 \to L(E,E)_b$. Then $g^\wedge \colon U \times E \to E$, $(x,y) \mapsto g(x)(y) = \varepsilon(g(x), y)$ is a continuous extension of $d(\phi^{-1}|_{M^0})$, using that the evaluation map $\varepsilon \colon L(E,E)_b \times E \to E$ is continuous. Hence ϕ^{-1} is C^1 with $d(\phi^{-1}) = g$ and thus

$$(\phi^{-1})' = j \circ \phi' \circ \phi^{-1}. \tag{3}$$

By induction on $\ell \in \mathbb{N}$ with $\ell \leq k$, we may assume that ϕ^{-1} is $C^{\ell-1}$. Then $(\phi^{-1})'$ is $C^{\ell-1}$, by (3), whence ϕ^{-1} is C^ℓ, by Lemma 2.3. Thus ϕ^{-1} is C^k, entailing that $\phi \in \mathrm{Diff}^k_{\mathrm{fr}}(M)$ and hence $f = \phi - \mathrm{id}_M \in \Omega_k$. Thus $U_k \subseteq \Omega_k$.

4. Proof of Proposition 1.2

We first show that Ω_k is open.

Lemma 4.1. *For each $k \in \mathbb{N} \cup \{\infty\}$, the set $\Omega_k := \{\phi - \mathrm{id}_M \colon \phi \in \mathrm{Diff}^k_{\mathrm{fr}}(M)\}$ is an open 0-neighbourhood in $C^k_{\mathrm{str}}(M, E)$.*

Proof. If $\phi \in \mathrm{Diff}^k_{\mathrm{fr}}(M)$, then $\phi - \mathrm{id}_M \colon M \to E$ is a C^k-map. For each $x \in M$, we have $\phi(x) - x \in \phi(M(x)) - x = M(x) - x \subseteq E(x)$, showing that $\phi - \mathrm{id}_M$ is a stratified vector field and thus $\phi - \mathrm{id}_M \in C^k_{\mathrm{str}}(M, E)$. By Proposition 1.5, Ω_k is a 0-neighbourhood in $C^k_{\mathrm{str}}(M, E)$. It remains to show that Ω_k is open. Let $V := C^k(M, E)$. For each $g \in \Omega_k$, the map

$$R_g \colon C^k(M, E) \to C^k(M, E), \quad f \mapsto f \circ (\mathrm{id}_M + g)$$

is continuous linear, by [15, Proposition 1.7.11]. Now $h := (\mathrm{id}_M + g)^{-1} - \mathrm{id}_M \in \Omega_k$. Since $\mathrm{id}_M + h = (\mathrm{id}_M + g)^{-1}$, we see that $R_h \circ R_g = R_g \circ R_h = \mathrm{id}_V$. Thus R_g is an isomorphism of topological vector spaces, with $(R_g)^{-1} = R_h$. Also

$$\tau \colon C^k(M, E) \to C^k(M, E), \quad f \mapsto \mathrm{id}_M + f$$

is a homeomorphism. We deduce that

$$r_g := \tau^{-1} \circ R_g \circ \tau \colon C^k(M, E) \to C^k(M, E), \quad f \mapsto g + f \circ (\mathrm{id}_M + g)$$

is a homeomorphism with $r_g^{-1} = \tau^{-1} \circ R_h \circ \tau = r_h$. Using that $\phi := \mathrm{id}_M + g \in \mathrm{Diff}^k_{\mathrm{fr}}(M)$ is face respecting, we now show that

$$r_g(f) \in C^k_{\mathrm{str}}(M, E) \quad \text{for each } f \in C^k_{\mathrm{str}}(M, E). \tag{4}$$

To this end, let $f \in C^k_{\mathrm{str}}(M, E)$. For $x \in M$, the image $\phi(M(x)) = M(x)$ is a face containing the element $\phi(x)$, whence $M(\phi(x)) \subseteq M(x)$. Replacing x with $\phi(x)$ and ϕ with its inverse, the same argument shows that $M(x) = M(\phi^{-1}(\phi(x))) \subseteq M(\phi(x))$. Thus $\phi(M(x)) = M(\phi(x))$. As a consequence, $\mathrm{aff}\, M(x) = \mathrm{aff}\, M(\phi(x))$ and hence

$$E(x) = (\mathrm{aff}\, M(x)) - x = (\mathrm{aff}\, M(\phi(x))) - \phi(x) = E(\phi(x)).$$

Now $r_g(f)(x) = g(x) + f(x + g(x)) = g(x) + f(\phi(x)) \in E(x) + E(\phi(x)) = E(x)$, establishing (4). By (4), the map r_g restricts to a self-map

$$\rho_g \colon C^k_{\mathrm{str}}(M, E) \to C^k_{\mathrm{str}}(M, E)$$

which is continuous as we endowed $W := C^k_{\mathrm{str}}(M, E)$ with the topology induced by $C^k(M, E)$. As $\rho_g \circ \rho_h = \rho_h \circ \rho_g = \mathrm{id}_W$, we deduce that ρ_g is a homeomorphism with $(\rho_g)^{-1} = \rho_h$. We claim that

$$\rho_g(\Omega_k) \subseteq \Omega_k.$$

If this is true, then Ω_k is a g-neighbourhood in $C^k_{\mathrm{str}}(M, E)$ (which completes the proof), as Ω_k is a 0-neighbourhood, ρ_g a homeomorphism, and $\rho_g(0) = g$.

To establish the claim, let $f \in \Omega_k$. Then $\mathrm{id}_M + f \in \mathrm{Diff}^k_{\mathrm{fr}}(M)$ and hence we have $(\mathrm{id}_M + f) \circ (\mathrm{id}_M + g) \in \mathrm{Diff}^k_{\mathrm{fr}}(M)$, the latter being closed under composition. Thus $\rho_g(f) = g + f \circ (\mathrm{id}_M + g) = (\mathrm{id}_M + f) \circ (\mathrm{id}_M + g) - \mathrm{id}_M \in \Omega_k$. □

We write $\mathrm{Diff}^r(K) := \{\phi \in C^r(K,K) \colon (\exists \psi \in C^r(K,K))\ \phi \circ \psi = \psi \circ \phi = \mathrm{id}_K\}$ in the next lemma.

Lemma 4.2. *Let F be a locally convex space, $U \subseteq F$ be a locally convex subset with dense interior, E be a finite-dimensional real vector space, $K \subseteq E$ be a compact convex subset with non-empty interior and $r \in \mathbb{N}_0 \cup \{\infty\}$. If a map $f \colon U \times K \to K \subseteq E$ is C^r and $f_x := f(x, \cdot) \in \mathrm{Diff}^r(K)$ for all $x \in U$, then also the following map is C^r:*

$$g \colon U \times K \to K, \quad (x,z) \mapsto (f_x)^{-1}(z).$$

Proof. If $r = 0$, then the graph of f is closed in $U \times K \times K$. As a consequence, the graph of g is closed in $U \times K \times K$ (being obtained from the former by flipping the second and third component). Since K is compact, continuity of g follows (see, e.g., [14, Lemma 2.1]). For $r \in \mathbb{N} \cup \{\infty\}$, we can repeat the proof of [14, Theorem C] without changes. □

Proof of Proposition 1.2. The evaluation map

$$\varepsilon_k \colon C^k(M,E) \times M \to E, \quad (f,x) \mapsto f(x)$$

is $C^{\infty,k}$ and thus $C^{\ell,k}$, by [2, Proposition 3.20]. Likewise, the evaluation map $\varepsilon_{k+\ell} \colon C^{k+\ell}(M,E) \times M \to E$ is $C^{\infty,k+\ell}$. Let us show that the map

$$H_{k,\ell} \colon C^{k+\ell}(M,E) \times \Omega_k \to C^k(M,E), \quad (f,g) \mapsto g + f \circ (\mathrm{id}_M + g)$$

is C^ℓ, for all $k \in \mathbb{N} \cup \{\infty\}$ and $\ell \in \mathbb{N}_0 \cup \{\infty\}$. It suffices to show that

$$\Gamma_{k,\ell} \colon C^{k+\ell}(M,E) \times \Omega_k \to C^k(M,E), \quad (f,g) \mapsto f \circ (\mathrm{id}_M + g)$$

is C^ℓ, since $H_{k,\ell}(f,g) = g + \Gamma_{k,\ell}(f,g)$. This will hold if we can show that $\Gamma_{k,\ell}$ is $C^{\infty,\ell}$ (see [2, Lemma 3.15]). By [1, Theorem 3.20], it suffices to show that

$$\Gamma^\wedge_{k,\ell} \colon C^{k+\ell}(M,E) \times \Omega_k \times M \to E, \quad (f,g,x) \mapsto f(x + g(x)) = \varepsilon_{k+\ell}(f, x + \varepsilon_k(g,x))$$

is $C^{\infty,\ell,k}$. Now $\Omega_k \times M \to M$, $(g,x) \mapsto x$ is a smooth map and hence $C^{\ell,k}$. As also ε_k is $C^{\ell,k}$, we find that

$$h_2 \colon \Omega_k \times M \to M, \quad (g,x) \mapsto x + g(x) = x + \varepsilon_k(g,x)$$

is $C^{\ell,k}$. The identity map $h_1 \colon C^{k+\ell}(M,E) \to C^{k+\ell}(M,E)$, $f \mapsto f$ is C^∞. Since $\varepsilon_{k+\ell}$ is $C^{\infty,k+\ell}$, the Chain Rule in the form [1, Lemma 3.16] shows that

$$\Gamma^\wedge_{k,\ell} = \varepsilon_{k+\ell} \circ (h_1 \times h_2)$$

is $C^{\infty,\ell,k}$. Thus $H_{k,\ell}$ is C^ℓ, whence also $H_{k,\ell}|_{\Omega_{k+\ell} \times \Omega_k}$ is C^ℓ. Now

$$H_{k,\ell}(f,g) = (\mathrm{id}_M + f) \circ (\mathrm{id}_M + g) - \mathrm{id}_M \in \Omega_k \subseteq C^k_{\mathrm{str}}(M,E)$$

for all $f \in \Omega_{k+\ell} \subseteq \Omega_k$ and $g \in \Omega_k$, since $\mathrm{Diff}^k_{\mathrm{fr}}(M)$ is closed under composition.

Hence $H_{k,\ell}|_{\Omega_{k+\ell}\times\Omega_k}$ co-restricts to a map

$$h_{k,\ell}\colon \Omega_{k+\ell}\times\Omega_k\to\Omega_k\subseteq C^k_{\mathrm{str}}(M,E)$$

which is C^ℓ as the vector subspace $C^k_{\mathrm{str}}(M,E)$ of $C^k(M,E)$ is closed (see [15, Lemma 1.3.19]). Then also $c_{k,\ell}=\Phi_k^{-1}\circ h_{k,\ell}\circ(\Phi_{k+\ell}\times\Phi_k)$ is C^ℓ.

For the discussion of the inversion maps, note that, for each $k\in\mathbb{N}\cup\{\infty\}$, $\mathrm{Diff}^k_{\mathrm{fr}}(M)$ is a smooth submanifold of $C^k(M,E)$ modeled on the closed vector subspace $C^k_{\mathrm{str}}(M,E)$ of $C^k(M,E)$. In fact, since Ω_k is an open subset of $C^k_{\mathrm{str}}(M,E)$ whose topology is induced by the compact-open C^k-topology on $C^k(M,E)$, we find an open subset $Q\subseteq C^k(M,E)$ with $Q\cap C^k_{\mathrm{str}}(M,E)=\Omega_k$. Then $P:=\mathrm{id}_M+Q=\{\mathrm{id}_M+f\colon f\in Q\}$ is an open subset of $C^k(M,E)$ and

$$\kappa\colon P\to Q,\quad \phi\mapsto\phi-\mathrm{id}_M$$

is a chart for $C^k(M,E)$ such that $\kappa(P\cap\mathrm{Diff}^k_{\mathrm{fr}}(M))=\Omega_k=Q\cap C^k_{\mathrm{str}}(M,E)$. As a consequence, for $k\in\mathbb{N}\cup\{\infty\}$ and $\ell\in\mathbb{N}_0\cup\{\infty\}$, the map $\iota_{k,\ell}$ will be C^ℓ to $\mathrm{Diff}^k_{\mathrm{fr}}(M)$ if we can show that $\iota_{k,\ell}$ is C^ℓ as a map to $C^k(M,E)$ (see [15, Lemma 3.2.7]). By [1, Theorem 3.25], the latter will hold if

$$\iota^\wedge_{k,\ell}\colon\ \mathrm{Diff}^{k+\ell}_{\mathrm{fr}}(M)\times M\to E,\quad (\phi,z)\mapsto\iota_{k,\ell}(\phi)(z)=\phi^{-1}(z)$$

is $C^{\ell,k}$. But $\iota^\wedge_{k,\ell}$ even is $C^{k+\ell}$; in fact,

$$f\colon\ \mathrm{Diff}^{k+\ell}_{\mathrm{fr}}(M)\times M\to E,\quad (\phi,x)\mapsto\phi(x)$$

is $C^{\infty,k+\ell}$ (and hence $C^{k+\ell}$) as the restriction to a submanifold of the evaluation mapping $\varepsilon_{k+\ell}\colon C^{k+\ell}(M,E)\times M\to E$, which is $C^{\infty,k+\ell}$. Since $\mathrm{Diff}^{k+\ell}_{\mathrm{fr}}(M)$ is C^∞-diffeomorphic to the open set $\Omega_{k+\ell}\subseteq C^{k+\ell}_{\mathrm{str}}(M,E)$, the $C^{k+\ell}$-property of $\iota^\wedge_{k,\ell}\colon(\phi,z)\mapsto f(\phi,\cdot)^{-1}(z)$ follows from Lemma 4.2.

Note that $c_{\infty,\infty}$ is the group multiplication of $\mathrm{Diff}^\infty_{\mathrm{fr}}(M)$ and $\iota_{\infty,\infty}$ the group inversion. As both of these mappings are smooth, $\mathrm{Diff}^\infty_{\mathrm{fr}}(M)$ is a Lie group. □

5. Proof of Theorem 1.1

Two lemmas will be useful for the proof of Theorem 1.1.

Lemma 5.1. *Let E be a finite-dimensional real vector space and $M\subseteq E$ be a convex polytope with non-empty interior. If $r>0$ and $\gamma\colon[0,r[\,\to M$ is a C^1-map, then there exists $\varepsilon>0$ such that $\gamma(0)+t\gamma'(0)\in M$ for all $t\in[0,\varepsilon]$.*

Proof. We may assume $E\neq\{0\}$, omitting a trivial case. There are a finite set $\Lambda\neq\emptyset$ of non-zero linear functionals $\lambda\colon E\to\mathbb{R}$ and numbers $a_\lambda\in\mathbb{R}$ such that

$$M=\{x\in E\colon(\forall\lambda\in\Lambda)\ \ \lambda(x)\leq a_\lambda\}. \tag{5}$$

Let $\Lambda_0:=\{\lambda\in\Lambda\colon\lambda(\gamma(0))=a_\lambda\}$. Then

$$\lambda(\gamma'(0))=(\lambda\circ\gamma)'(0)=\lim_{t\to0_+}\frac{\lambda(\gamma(t))-\lambda(\gamma(0))}{t}\leq 0$$

for each $\lambda\in\Lambda_0$, as $\lambda(\gamma(t))\leq a_\lambda$ and $\lambda(\gamma(0))=a_\lambda$. As a consequence,

$$\lambda(\gamma(0)+t\gamma'(0))=a_\lambda+t\lambda(\gamma'(0))\leq a_\lambda$$

for all $\lambda\in\Lambda_0$ and $t\geq 0$.

For all $\lambda \in \Lambda \setminus \Lambda_0$, we have $\lambda(\gamma(0)) < a_\lambda$. As $\Lambda \setminus \Lambda_0$ is a finite set, we find $\varepsilon > 0$ such that $\lambda(\gamma(0)) + t\lambda(\gamma'(0)) \leq a_\lambda$ for all $t \in [0, \varepsilon]$ and all $\lambda \in \Lambda \setminus \Lambda_0$. For all $t \in [0, \varepsilon]$, we then have $\lambda(\gamma(0) + t\gamma'(0)) \leq a_\lambda$ for all $\lambda \in \Lambda$. Thus $\gamma(0) + t\gamma'(0) \in M$, by (5). $\square$

Lemma 5.2. *Let E and F be finite-dimensional real vector spaces, $M \subseteq E$ and $N \subseteq F$ be convex polytopes with non-empty interior and $\phi\colon U \to V$ be a C^1-map between open subsets $U \subseteq M$ and $V \subseteq N$. For $x \in M$, let $M(x)$ be the face of M generated by x, and $E(x) := \mathrm{span}_{\mathbb{R}}(M(x) - x)$. For $y \in N$, let $N(y)$ be the face of N generated by y, and $F(y) := \mathrm{span}_{\mathbb{R}}(N(y) - y)$. For $x \in U$, we have:*

(a) $\phi'(x)(E(x)) \subseteq F(\phi(x))$;

(b) *If $\phi'(x)\colon E \to F$ is injective, then* $\dim E(x) \leq \dim F(\phi(x))$;

(c) *If $\phi\colon U \to V$ is a C^1-diffeomorphism, then $\phi'(x)(E(x)) = F(\phi(x))$.*

Proof. (a) Let $w \in E(x)$. Since $x \in \operatorname{algint} M(x)$, there exists $r > 0$ such that $x \pm tw \in M(x) \subseteq M$ for all $t \in [0, r]$. After shrinking r, we may assume that $x \pm tw \in U$ for all $t \in [0, r]$ and thus $\phi(x \pm tw) \in N$. Since $\pm\phi'(x)(w) = \frac{d}{dt}\big|_{t=0}\phi(x \pm tw)$, Lemma 5.1 provides $\varepsilon > 0$ such that

$$\phi(x) \pm t\phi'(x)(w) \in N \quad \text{for all } t \in [0, \varepsilon].$$

Notably, $v_+ := \phi(x) + \varepsilon\phi'(x)(w) \in N$ and $v_- := \phi(x) - \varepsilon\phi'(x)(w) \in N$. Since $(1/2)v_+ + (1/2)v_- = \phi(x) \in N(\phi(x))$ and $N(\phi(x))$ is a face of N, we deduce that $v_+, v_- \in N(\phi(x))$. Thus $\varepsilon\phi'(x)(w) = v_+ - \phi(x) \in N(\phi(x)) - \phi(x) \subseteq F(\phi(x))$, whence also $\phi'(x)(w) \in F(\phi(x))$.

(b) is immediate from (a).

(c) Set $y := \phi(x)$. By (a), we have $\phi'(x)(E(x)) \subseteq F(y)$ and $(\phi^{-1})'(y)(F(y)) \subseteq E(\phi^{-1}(y)) = E(x)$. Since $(\phi^{-1})'(y) = (\phi'(x))^{-1}$, applying $\phi'(x)$ to the latter inclusion we get $F(y) \subseteq \phi'(x)(E(x))$, whence $F(y) = \phi'(x)(E(x))$. $\square$

Lemma 5.2 (c) is analogous to the well-known boundary invariance for manifolds with corners (as in [18, Theorem 1.2.12 (a)]).

Definition 5.3. Let E be a finite-dimensional real vector space of dimension n and $M \subseteq E$ be a convex polytope with non-empty interior. The *index* of $x \in M$ is defined as

$$\mathrm{ind}_M(x) := \dim(E/E(x)) = n - \dim E(x) = n - \dim M(x),$$

where $M(x)$ is the face of M generated by x and $E(x) := \mathrm{span}_{\mathbb{R}}(M(x) - x)$. For $i \in \{0, 1, \ldots, n\}$, we define

$$\partial_i(M) := \{x \in M \colon \mathrm{ind}_M(x) = i\}.$$

In the situation of Definition 5.3, the following holds.

Lemma 5.4. *For each $i \in \{0, 1, \ldots, n\}$, the set $\partial_i(M)$ is an $(n-i)$-dimensional smooth submanifold of E. If $\mathcal{F}_{n-i}(M)$ is the finite set of all faces of M of dimension $n - i$, then the topology induced by M on $\partial_i(M)$ makes the latter the topological sum of the algebraic interiors* $\mathrm{algint}(F)$ *for $F \in \mathcal{F}_{n-i}(M)$. The connected components of $\partial_i(M)$ are the sets* $\mathrm{algint}(F)$ *for $F \in \mathcal{F}_{n-i}(M)$; they are open and closed in $\partial_i(M)$.*

Proof. If $F, G \in \mathcal{F}_{n-i}(M)$ such that $F \neq G$ and $F \cap G \neq \emptyset$, then $F \cap G$ is a face of M of dimension $< n-i$ and $F \cap G \subseteq F \setminus \operatorname{algint}(F)$ as well as $F \cap G \subseteq G \setminus \operatorname{algint}(G)$, entailing that $\operatorname{algint}(F) \cap \operatorname{algint}(G) = \emptyset$. Thus

$$\operatorname{algint}(F) \cap \operatorname{algint}(G) = \emptyset \quad \text{for all } F, G \in \mathcal{F}_{n-i}(M) \text{ such that } F \neq G. \tag{6}$$

Let $F \in \mathcal{F}_{n-i}(M)$. For each $G \in \mathcal{F}_{n-i}(M)$, we have

$$\operatorname{algint}(F) \cap G \subseteq F \cap G \subseteq F \setminus \operatorname{algint}(F)$$

by the preceding and thus $\operatorname{algint}(F) \cap G = \emptyset$. Thus

$$K := \bigcup_{G \in \mathcal{F}_{n-i}(M) \setminus \{F\}} G$$

is a compact subset of E such that $\operatorname{algint}(F) \cap K = \emptyset$. Note that $F = H \cap M$ for some hyperplane H in E (faces of polyhedra being exposed), whence

$$F = M \cap \operatorname{aff}(F).$$

Now $\operatorname{algint}(F) = U \cap \operatorname{aff}(F)$ for some open subset $U \subseteq E$. After replacing U by its intersection with the open set $E \setminus K$ which contains $\operatorname{algint}(F)$, we may assume that $U \cap \operatorname{algint}(G) = \emptyset$ for all $G \in \mathcal{F}_{n-i}(M) \setminus \{F\}$ and hence

$$\partial_i(M) \cap U = U \cap \operatorname{algint}(F) = \operatorname{algint}(F),$$

showing that $\operatorname{algint}(F)$ is open in $\partial_i(M)$. The topology induced by E therefore makes $\partial_i(M)$ the topological sum of the $\operatorname{algint}(F)$ with $F \in \mathcal{F}_{n-i}(M)$. If we choose $x \in \operatorname{algint}(F)$, then $E(x) = \operatorname{aff}(F) - x$ and

$$\phi\colon U \to U - x, \quad y \mapsto y - x$$

is a C^∞-diffeomorphism between open subsets of E such that

$$\phi(U \cap \partial_i(M)) = \phi(\operatorname{algint}(F)) = \phi(U \cap \operatorname{aff}(F)) = (U - x) \cap E(x).$$

Thus $\partial_i(M)$ is a submanifold of E of dimension $\dim(E(x)) = n - i$. $\square$

Lemma 5.5. *Let E be a finite-dimensional real vector space, $M \subseteq E$ be a convex polytope with non-empty interior, and $k \in \mathbb{N} \cup \{\infty\}$. Then $\operatorname{Diff}^k_{\mathrm{fr}}(M)$ is a normal subgroup of finite index in $\operatorname{Diff}^k(M)$.*

Proof. We write $\mathcal{F}(M)$ for the set of all faces of M. Assume $n := \dim(E)$ and let $\phi \in \operatorname{Diff}^k(M)$. By Lemma 5.2 (c), we have $\operatorname{ind}_M(\phi(x)) = \operatorname{ind}_M(x)$ for each $x \in M$, whence $\phi(\partial_i(M)) \subseteq \partial_i(M)$ for each $i \in \{0, 1, \ldots, n\}$ and thus $\phi(\partial_i(M)) = \partial_i(M)$, as M is the disjoint union of $\partial_0(M), \ldots, \partial_n(M)$ and ϕ is surjective. Since $\partial_i(M)$ is a submanifold of E, the inclusion map $\partial_i(M) \to E$ is smooth, and thus also the inclusion map $j_i\colon \partial_i M \to M$. Thus $\phi|_{\partial_i(M)} = \phi \circ j_i$ is C^k. As this map has image in $\partial_i(M)$, which is a submanifold of E, we deduce that

$$\phi_i := \phi|_{\partial_i(M)}\colon \partial_i(M) \to \partial_i(M)$$

is C^k. Note that $(\phi^{-1})_i$ is inverse to ϕ_i, whence ϕ_i is a C^k-diffeomorphism and hence a homeomorphism. For each $F \in \mathcal{F}_{n-i}(M)$, the algebraic interior $\operatorname{algint}(F)$ is open and closed in the topological sum $\partial_i(M)$. Hence, $\phi(\mathrm{algint}(F)) = \phi_i(\mathrm{algint}(F))$ is both open and closed in $\partial_i(M)$. Being also non-empty and connected, $\phi(\mathrm{algint}(F))$ is a connected component of $\partial_i(M)$ and thus $\phi(\mathrm{algint}(F)) = \mathrm{algint}(G)$ for some $G \in \mathcal{F}_{n-i}(M)$. As a consequence,

$$\phi(F) = \phi(\overline{\mathrm{algint}(F)}) = \overline{\phi(\mathrm{algint}(F))} = \overline{\mathrm{algint}(G)} = G.$$

Thus $\phi_*(F) := \phi(F) \in \mathcal{F}(M)$ for each $F \in \mathcal{F}(M)$. Since $(\mathrm{id}_M)_*$ is the identity map on $\mathcal{F}(M)$ and $(\phi \circ \psi)_* = \phi_* \circ \psi_*$ for all $\phi, \psi \in \mathrm{Diff}^k(M)$, we get a group homomorphism

$$\pi\colon\ \mathrm{Diff}^k(M) \to \mathrm{Sym}(\mathcal{F}(M)), \quad \phi \mapsto \phi_*$$

to the group of all permutations of the finite set $\mathcal{F}(M)$. Thus $\mathrm{Diff}^k_{\mathrm{fr}}(M) = \ker(\pi)$ is a normal subgroup of $\mathrm{Diff}^k(M)$ of finite index. $\square$

Proof of Theorem 1.1. In view of the local description of Lie group structures (analogous to Proposition 18 in [4, Chapter III, §1, no. 9]), we need only show that the group homomorphism

$$I_\psi\colon\ \mathrm{Diff}^\infty_{\mathrm{fr}}(M) \to \mathrm{Diff}^\infty_{\mathrm{fr}}(M), \quad \phi \mapsto \psi \circ \phi \circ \psi^{-1}$$

is smooth for each $\psi \in \mathrm{Diff}(M)$. Since $\mathrm{Diff}^\infty_{\mathrm{fr}}(M)$ is a smooth submanifold of $C^\infty(M, E)$, we need only show that I_ψ is smooth as a map to $C^\infty(M, E)$, which will hold if we can show that the map

$$I_\psi^\wedge\colon\ \mathrm{Diff}^\infty_{\mathrm{fr}}(M) \times M \to E, \quad (\phi, x) \mapsto I_\psi(\phi)(x) = \psi(\phi(\psi^{-1}(x)))$$

is smooth (see [2, Theorem 3.25 and Lemma 3.22]). We know that the evaluation map $\varepsilon\colon C^\infty(M, E) \times M \to E$, $(f, x) \mapsto f(x)$ is smooth (see [2, Proposition 3.20 and Lemma 3.22]). Since $\mathrm{Diff}^\infty_{\mathrm{fr}}(M)$ is a submanifold of $C^\infty(M, E)$, also the restriction of ε to a map

$$\mathrm{Diff}^\infty_{\mathrm{fr}}(M) \times M \to E$$

is smooth. As this map takes its values in M, also its corestriction

$$\theta\colon\ \mathrm{Diff}^\infty_{\mathrm{fr}}(M) \times M \to M, \quad (\phi, x) \mapsto \phi(x)$$

is smooth. The formula

$$I_\psi^\wedge(\phi, x) = \psi(\theta(\phi, \psi^{-1}(x)))$$

now shows that $I_\psi^\wedge$ is a smooth function of (ϕ, x), as required. $\square$

Remark 5.6. If $E = \mathbb{R}^n$ and $M \subseteq E$ is a polytope with non-empty interior, then

$$\mathrm{Diff}_{\partial M}(M) := \{\phi \in \mathrm{Diff}(M)\colon \phi(x) = x \text{ for all } x \in \partial M\}$$

is a subgroup of $\mathrm{Diff}^\infty_{\mathrm{fr}}(M)$. Since

$$C^\infty_{\partial M}(M, E) := \{f \in C^\infty(M, E)\colon f(x) = 0 \text{ for all } x \in \partial M\}$$

is a closed vector subspace of $C^\infty_{\mathrm{str}}(M, E)$ and the chart

$$\Phi_\infty\colon\ \mathrm{Diff}^\infty_{\mathrm{fr}}(M) \to \Omega_\infty \subseteq C^\infty_{\mathrm{str}}(M, E)$$

satisfies $\Phi_\infty(\mathrm{Diff}_{\partial M}(M)) = \Omega_\infty \cap C^\infty_{\partial M}(M, E)$, we see that $\mathrm{Diff}_{\partial M}(M)$ is a submanifold (and hence a Lie subgroup) of $\mathrm{Diff}^\infty_{\mathrm{fr}}(M)$ modeled on $C^\infty_{\partial M}(M, E)$, and thus also of $\mathrm{Diff}(M)$ (in the sense of [15, Definition 3.1.10]).

Remark 5.7. Let E be a finite-dimensional real vector space, $n := \dim(E)$, $M \subseteq E$ be a convex polytope with non-empty interior, and $k \in \mathbb{N}_0 \cup \{\infty\}$. Using Lemma 5.4, we obtain the following more intuitive characterization: A function $f \in C^k(M, E)$ is stratified (and thus in $C^k_{\mathrm{str}}(M, E)$) if and only if $(\mathrm{id}_M, f)(\partial_i(M)) \subseteq T(\partial_i(M))$ for each $i \in \{0, \ldots, n\}$, identifying $T(\partial_i(M))$ with a subset of $TM = M \times \mathbb{R}^n$ as usual. That is, $f|_{\partial_i(M)}$ can be considered as a C^k-vector field of the $(n-i)$-dimensional C^∞-manifold $\partial_i(M)$ for all $i \in \{0, \ldots, n\}$.

A. Lipschitz perturbations of automorphisms

We record a known fact concerning Lipschitz perturbations of invertible linear operators (compare, e.g., Theorem 5.3 in the unpublished work [11]).

Lemma A.1. *Let $(E, \|\cdot\|)$ be a Banach space with $E \neq \{0\}$. Let $A\colon E \to E$ be an invertible, continuous linear map. Let $x \in E$, $r > 0$ and $f\colon \overline{B}^E_r(x) \to E$ be a Lipschitz continuous map such that $\mathrm{Lip}(f) < \frac{1}{\|A^{-1}\|_{\mathrm{op}}}$. Then the image of $\phi\colon \overline{B}^E_r(x) \to E$, $y \mapsto Ay + f(y)$ satisfies*

$$\phi(\overline{B}^E_r(x)) \supseteq \overline{B}^E_{ar}(\phi(x))$$

with $a := \frac{1}{\|A^{-1}\|_{\mathrm{op}}} - \mathrm{Lip}(f) > 0.$

Proof. Given $c \in \overline{B}^E_{ar}(\phi(x))$, consider the map

$$h\colon \overline{B}^E_r(x) \to E, \quad v \mapsto v + A^{-1}(c - \phi(v)).$$

Then $\mathrm{Lip}(h) \leq \|A^{-1}\|_{\mathrm{op}} \mathrm{Lip}(f) < 1$. In fact, for all $v, w \in \overline{B}^E_r(x)$, we have

$$\begin{aligned}
\|h(w) - h(v)\| &= \|w - v + A^{-1}(\phi(v) - \phi(w))\| \\
&\leq \|A^{-1}\|_{\mathrm{op}} \|\phi(v) - Av - (\phi(w) - Aw)\| \\
&= \|A^{-1}\|_{\mathrm{op}} \|f(v) - f(w)\| \leq \|A^{-1}\|_{\mathrm{op}} \mathrm{Lip}(f) \|w - v\|.
\end{aligned}$$

If we can show that h maps $\overline{B}^E_r(x)$ into itself, then h has a fixed point w by the contraction mapping principle, and $w = h(w) = w + A^{-1}(c - \phi(w))$ implies that $\phi(w) = c$. Note that

$$\|A^{-1}\|_{\mathrm{op}}(\mathrm{Lip}(f) + a) = 1$$

and $z := c - \phi(x) \in \overline{B}^E_{ar}(0)$. Thus

$$\begin{aligned}
\|h(v) - x\| &= \|A^{-1}(\phi(x) - Ax - (\phi(v) - Av) + z)\| \\
&\leq \|A^{-1}\|_{\mathrm{op}}(\|f(x) - f(v)\| + \|z\|) \leq \|A^{-1}\|_{\mathrm{op}}(\mathrm{Lip}(f) + a) r = r
\end{aligned}$$

for each $v \in \overline{B}^E_r(x)$, whence $h(v) \in \overline{B}^E_r(x)$. $\square$

B. Non-existence of Michor-type local additions

Let us consider smooth manifolds with corners as in [19], which are assumed finite-dimensional and paracompact. If M is a finite-dimensional smooth manifold with corners of dimension n, one can define its tangent bundle TM with fibre $\mathbb{R}^n$ and obtains the subset ${}^iTM$ of all tangent vectors v of the form

$$v = \dot{\gamma}(0) \in T_{\gamma(0)}M$$

for some smooth curve $\gamma\colon [0,1[\to M$, the so-called *inner* tangent vectors (see [19, p. 20]). If $M = [0,\infty[^k \times \mathbb{R}^{n-k}$ with $k \in \{0,\ldots,n\}$, we identify TM with $M \times \mathbb{R}^n$ as usual; then

$${}^iTM = \{(x,y) \in M \times \mathbb{R}^n \colon (\forall j \in \{1,\ldots,k\})\; x_j = 0 \;\Rightarrow\; y_j \geq 0\},$$

writing $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n)$ in components. Hence ${}^iTM$ is a convex subset of $\mathbb{R}^n \times \mathbb{R}^n$ and its interior $M^0 \times \mathbb{R}^n$ is dense in ${}^iTM$. Notably, ${}^iTM$ is a locally convex subset of $\mathbb{R}^{2n}$ with dense interior. Since every n-dimensional smooth manifold M with corners locally looks like $[0,\infty[^n$, we see that ${}^iTM$ locally looks like ${}^iT([0,\infty[^n)$, whence ${}^iTM$ is a smooth manifold with rough boundary in the sense of [15, Chapter 3]. It therefore makes sense to speak about smooth functions on ${}^iTM$. (A smaller class of modeling sets and corresponding generalized manifolds is proposed in [19, p. 20, Remark]). Consider $\pi_{TM}\colon TM \to M$, $T_xM \ni v \mapsto x$. According to [19, §10.2, p. 90], local additions on a smooth manifold M with corners are defined as follows:

B.1. A local addition τ on M is a smooth mapping $\tau\colon {}^iTM \to M$ satisfying

(A1) $(\tau, \pi_{TM})\colon {}^iTM \to M \times M$ is a diffeomorphism onto an open neighbourhood of the diagonal in $M \times M$;

(A2) $\tau(0_x) = x$ for all $x \in M$.

Here $0_x \in T_xM$ is the 0-vector in the tangent space T_xM for $x \in M$. Michor claims that every smooth manifold with corners admits a local addition (see [19, p. 90, Lemma]). However:

Proposition B.2. *If M is a smooth manifold with corners such that $\partial M \neq \emptyset$, then M does not admit a smooth local addition in Michor's sense.*

Proof. Suppose we could find a smooth local addition $\tau\colon {}^iTM \to M$ in Michor's sense. We derive a contradiction. Let $n \in \mathbb{N}$ be the dimension of M. For $p \in \partial M$, there exist $k \in \{1,\ldots,n\}$, an open p-neighbourhood $U \subseteq M$ and a C^∞-diffeomorphism $\phi\colon U \to [0,\infty[^k \times \mathbb{R}^{n-k} =: V$ such that $\phi(p) = 0$. Let

$$F := {}^iTV \subseteq TV = V \times \mathbb{R}^n$$

be the set of all $(x,y) \in V \times \mathbb{R}^n$ with $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n)$ such that $y_j \geq 0$ for all $j \in \{1,\ldots,k\}$ such that $x_j = 0$. Then F is a convex subset of $\mathbb{R}^n \times \mathbb{R}^n$ with dense interior. Moreover, $\tau^{-1}(U)$ is an open 0_p-neighbourhood in ${}^iTM$ and

$$Q := T\phi((TU) \cap \tau^{-1}(U))$$

is an open $(0,0)$-neighbourhood in $F \subseteq V \times \mathbb{R}^n$, whence Q is a locally convex subset of $\mathbb{R}^n \times \mathbb{R}^n$ with dense interior. We obtain a smooth map

$$\sigma := \phi \circ \tau \circ T\phi^{-1}|_Q \colon Q \to V \subseteq \mathbb{R}^n$$

such that

$$\psi := (\sigma, \mathrm{pr}_1) \colon Q \to V \times V, \quad (x,y) \mapsto (\sigma(x,y), x)$$

is a C^∞-diffeomorphism onto an open neighbourhood of the diagonal in $V \times V$. For some $\rho > 0$, we have $P := \,]{-\rho}, \rho[^{2n} \cap F \subseteq Q$. Write $\sigma|_P = (\sigma_1, \ldots, \sigma_n)$ in terms of the components $\sigma_1, \ldots, \sigma_n \colon P \to \mathbb{R}$. Using the partial maps

$$\sigma_0 := \sigma(0, \cdot) \colon [0, \infty[^k \times \mathbb{R}^{n-k} \to \mathbb{R}^n \quad \text{and} \quad \sigma^0 := \sigma(\cdot, 0) \colon [0, \rho[^k \times]{-\rho}, \rho[^{n-k} \to \mathbb{R}^n$$

of $\sigma|_P$, the Jacobian of ψ at $(0,0) \in P \subseteq \mathbb{R}^n \times \mathbb{R}^n$ can be regarded as the block matrix

$$J_\psi(0,0) = \begin{pmatrix} J_{\sigma^0}(0) & J_{\sigma_0}(0) \\ \mathbf{1} & \mathbf{0} \end{pmatrix}.$$

The invertibility of the Jacobian implies that the final n columns must be linearly independent. Notably, we must have

$$\frac{\partial \sigma_1}{\partial y_j}(0,0) \neq 0$$

for some $j \in \{1, \ldots, n\}$. Let $e_1, \ldots, e_n \subseteq \mathbb{R}^n$ be the standard basis vectors. For small $t \geq 0$, we have $(0, te_j) \in Q$, whence $\sigma(0, te_j) \in V$, entailing that $\sigma_1(0, te_j) \geq 0$ and thus

$$\frac{\partial \sigma_1}{\partial y_j}(0,0) = \lim_{t \to 0_+} \frac{\sigma_1(0, te_j)}{t} \geq 0,$$

exploiting that $\sigma_1(0,0) = 0$. There exists $\theta > 0$ such that

$$\varepsilon := \theta \, \frac{\partial \sigma_1}{\partial y_j}(0,0) - 1 > 0.$$

There exists $\delta > 0$ such that

$$t(1, \ldots, 1, -\theta e_j) \in Q$$

for all $t \in [0, \delta]$ (with 1 in the first n slots). Then

$$h \colon [0, \delta] \to \mathbb{R}, \quad t \mapsto \sigma_1(t(1, \ldots, 1, -\theta e_j))$$

is a smooth function such that

$$h'(0) = \sum_{i=1}^n \underbrace{\frac{\partial \sigma_1}{\partial x_i}(0,0)}_{=\delta_{1,i}} - \theta \, \frac{\partial \sigma_1}{\partial y_j}(0,0) = 1 - \theta \frac{\partial \sigma_1}{\partial y_j}(0,0) = -\varepsilon < 0,$$

where $\delta_{1,i}$ denotes Kronecker's delta. Since $h(0) = 0$, after shrinking δ, we can achieve that $h(t) < 0$ for all $t \in \,]0, \delta]$. Thus $\sigma_1(\delta(1, \ldots, 1, -\theta e_j)) = h(\delta) < 0$, contradicting the fact $\sigma(Q) \subseteq V$ and thus $\sigma_1(x,y) \geq 0$ for all $(x,y) \in Q$. $\square$

It would not help to assume that, instead, the local addition τ is only defined on an open neighbourhood Ω of the 0-section in ${}^iTM$, and replace (A1) with the requirement that $(\tau, \pi_{TM}) \colon \Omega \to M \times M$ be a C^∞-diffeomorphism onto an open subset of $M \times M$. In fact, we can replace ${}^iTM$ with Ω in the preceding proof.

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