Journal of Convex Analysis
Volume 30 (2023), No. 2, 481–498

Existence and Statistical Estimation of Equilibria in Stochastic Electoral Competitions

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Dedicated to Roger J-B Wets on the occasion of his 85th birthday.

Received: May 29, 2022
Accepted: December 20, 2022

We establish the existence of a Nash equilibrium in a class of stochastic games representing electoral competitions between two candidates facing a continuum of voters. The underlying uncertainty in such games is due to the fact that candidates are uncertain about the fraction of the voters that will actually vote for them on election day. The payoff functions of the candidates are neither continuous nor concave. Therefore, we transform a game in this class into a new game with nicer geometric and continuity properties. We establish the existence of an equilibrium for the new game using a standard fixed point argument, and we finally show that an equilibrium of the new games is also an equilibrium of the original game. We further show the equilibria we found can be approximated statistically using non-parametric maximum likelihood estimators of the distribution of the random variable representing the uncertainty in the game.

Keywords: Stochastic voting models, Nash Equilibrium, log-concave random variables, non-parametric maximum likelihood estimators.

2010 Mathematics Subject Classification: 91A80, 90C15, 62F12.

1. Introduction

In this paper, we investigate the existence of equilibria in voting models with two candidates facing a continuum of voters. Uncertainty is introduced to these models by postulating that each candidate is uncertain about the fraction of voters that will vote for him. More specifically, once the candidates announce their policies, each candidate can accurately compute the fraction of voters that prefer one policy over another. However, the candidates are uncertain about how many of these voters will actually vote for them on election day [7, 9, 14, 15]. The most common reasons for such uncertainty are (i) factors that are unobservable by the candidates such as voters' non-policy preferences (e.g. preferences over personal attributes of the candidates that may include race, gender, age, political party), (ii) random events that impact the actual behavior of the voters on election day. This includes events that affect voter turnout such as election day weather [17, 20], mistakes by voters during voting, and the uncertain impact of newly implemented voter registration regulations. The result of polls may also impact the turnout on election day; some voters might become complacent about the victory of their candidate, and therefore decide not to vote [16, 28].

ISSN 0944-6532 / \$ 2.50

In mathematical terms, we are interested in simultaneously solving a pair of stochastic optimization problems. We have two decision makers (two candidates) with interdependent payoff functions. For $i \in \{1,2\}$, decision maker i has payoff

$$U_i(t_1,t_2) = \int_{\mathbb{R}} g_i(t_1,t_2,\xi_i)h_i(\xi_i)d\xi_i$$

where T is compact interval on the real line, and ξ_i is a random variable with a pdf h_i on the real line. For every $t_2 \in T$, the first decision maker constructs a set-valued map $BR_1 : T \rightrightarrows T$ representing his best response to a move t_2 by the second decision maker

$$BR_1(t_2) = \operatorname{argmax}_{t_1 \in T} U_1(t_1,t_2).$$

Computing $BR_1(t_2)$ amounts to solving a family *anticipative stochastic optimization problems* that are parameterized by t_2 [30]. Similarly, for every $t_1 \in T$, the second decision maker constructs her best response set-valued map

$$BR_2(t_1) = \operatorname{argmax}_{t_2 \in T} U_2(t_1,t_2).$$

Let $G = (U_1, U_2, T \times T)$ denote the game played by the two decision makers. We are interested in the existence and the statistical estimation of the (pure strategy) Nash equilibria of the game $G = (U_1, U_2, T \times T)$. In other words, we are interested in the existence and the approximation of a pair $(t_1^*, t_2^*) \in T \times T$ such that $t_1^* \in BR_1(t_2^*)$ and $t_2^* \in BR_2(t_1^*)$. In the application we have in mind, the payoff function U_i is discontinuous (in fact, it is neither upper semi-continuous nor lower semi-continuous), and the set-valued map BR_i is not convex-valued. Therefore, the standard method establishing the existence of an equilibrium using fixed point theorems cannot be used. To circumvent this problem, we transform the problem into a simpler game, show that the new game has an equilibrium, and then show that the equilibrium of the new game is actually an equilibrium for the original game.

Very often, we only have limited information regarding the distribution of the random variable impacting voter turn out. Therefore, we show that our existence result is stable with respect to certain types of approximations, and then we follow techniques developed in [13, 19] to approximate the equilibria in our model by sampling the underlying random variable.

2. A model of electoral competition

2.1. The policy space

We assume each candidate announces a position t in policy space $T = [0,1]$. Such announcement can be thought of as the general position of the candidate on the left-right political spectrum. More generally, we can assume that the candidates are evaluated by voters on a single issue [23].

2.2. The voters

We have a continuum of voters where every voter is identified with a point in $T = [0,1]$ corresponding to his ideal policy. The preferences of a voter $\hat{t}$ over policies in T are described by a utility function of the type $u(t) = v(|t - \hat{t}|)$ where v is a strictly decreasing concave function. Hence, each voter prefers policies that are closer to his ideal policy. Furthermore, for $i \in \{1, 2\}$, candidate i believes that the distribution of ideal policies of the voters on $[0, 1]$ is given by a pdf f_i and a cdf F_i.

For a pair of announced polices (t_1, t_2), we define $FR_1(t_1, t_2)$ as the fraction of voters that- according to candidate 1- prefer policy t_1 to t_2. Similarly, we define $FR_2(t_1, t_2)$ as the fraction of voters that -according to candidate 2- prefer policy t_2 to t_1. Clearly, FR_i must reflect the *subjective* belief candidate i has about the preferences of the voters. Our assumptions on the voters lead to

Assumption A: Candidate 1 believes that, for any (t_1, t_2), the fraction of voters that prefer t_1 over t_2 is given by

$$FR_1(t_1, t_2) = \begin{cases} F_1(\frac{t_1+t_2}{2}) & \text{if } t_1 < t_2 \\ \frac{1}{2} & \text{if } t_1 = t_2 \\ 1 - F_1(\frac{t_1+t_2)}{2}) & \text{if } t_2 < t_1. \end{cases} \tag{2.1}$$

Similarly, candidate 2 believes that the fraction of voters that prefer t_2 over t_1 is given by

$$FR_2(t_1, t_2) = \begin{cases} 1 - F_2(\frac{t_1+t_2}{2}) & \text{if } t_1 < t_2 \\ \frac{1}{2} & \text{if } t_1 = t_2 \\ F_2(\frac{t_1+t_2)}{2}) & \text{if } t_2 < t_1. \end{cases} \tag{2.2}$$

Moreover, we assume that F_1 and F_2 are weakly increasing, continuous, and onto $[0, 1]$.

Assumption B: Assume that $F_1(t^m) = F_2(t^m) = 1/2$ for some $0 < t^m < 1$. Moreover, F_1 is concave on $[\frac{t_m}{2}, 1]$ and F_2 is convex on $[0, \frac{1+t_m}{2}]$.

The most important consequences of Assumptions A and B are:

(1) The functions $FR_1(\cdot, t^m)$ and $FR_2(t^m, \cdot)$ are continuous on $[0, 1]$. Moreover, $FR_1(\cdot, t^m)$ is concave and weakly increasing on $[0, t^m]$ and weakly decreasing on $[t^m, 1]$. On the other hand, $FR_2(t^m, \cdot)$ is concave and weakly decreasing on $[t^m, 1]$ and weakly increasing on $[0, t^m]$.

(2) For any $t_2 > t^m$, $FR_1(\cdot, t_2)$ "jumps down" at

t_2: $\lim_{t \nearrow t_2} FR_1(t, t_2) > FR_1(t_2, t_2) > \lim_{t \searrow t_2} FR_1(t, t_2)$.

Moreover, $FR_1(\cdot, t_2)$ is weakly increasing and concave on $[0, t_2)$ and weakly decreasing on $[t_2, 1]$.

(3) For any $t_1 < t^m$, $FR_2(t_1, \cdot)$ "jumps up" at

t_1: $\lim_{t \nearrow t_1} FR_2(t_1, t) < FR_2(t_1, t_1) < \lim_{t \searrow t_1} FR_2(t_1, t)$.

Moreover, $FR_2(t_1, \cdot)$ is weakly decreasing and concave on $(t_1, 1]$ and weakly increasing on $[0, t_1]$.

As we will see in Section 6, the requirement that both F_1 and F_2 have the same median is not essential. We can also assume a more general "tie-breaking rule" with $FR_i(t, t) = c_i$ for some $0 \leq c_i \leq 1$ where the case of $c_i \neq \frac{1}{2}$ can represent voters' bias toward one of the candidate.

Traditionally, the uncertainty in voting models arises from the assumption that Candidate i believes that the fraction of voters that will actually vote for him on election day is given by

$$\widehat{FR}_i(t_1, t_2, \xi_i) = \phi(FR_i(t_1, t_2), \xi_i) \tag{2.3}$$

where ξ_i is a real-valued random variable representing a "shock" that is independent of t_1 and t_2. Models where the uncertainty is expressed via equation (2.3) are called "error distribution models" [24].[1] Introducing uncertainty via (2.3) seems to be simple and intuitive. However, this approach can lead to serious difficulties when it comes to establishing the existence of equilibria in the model. In particular, using (2.3) requires imposing a "rationality" constraint to insure that value of $\widehat{FR}_i(t_1, t_2, \xi_i)$ remains between 0 and 1 almost surely in ξ_i. The details of the difficulties associated with such a constraint are discussed in the Appendix. To avoid these difficulties, we introduce uncertainty to our model using a slightly different approach. We define Γ_i as the relative weight of voters that will vote for candidate i on election day:

$$\widehat{\Gamma}_i(t_1, t_2, \xi_i) = \frac{\widehat{FR}_i(t_1, t_2, \xi_i)}{1 - \widehat{FR}_i(t_1, t_2, \xi_i)}. \tag{2.4}$$

Assume further that $$\widehat{\Gamma}_i(t_1, t_2, \xi_i) = \psi_i(FR_i(t_1, t_2), \xi_i), \tag{2.5}$$

for some function ψ_i.

Assumption C: A simple majority is needed to win the election. In other words, candidate i believes he will win the election, if

$$\widehat{\Gamma}_i(t_1, t_2, \xi_i) > 1. \tag{2.6}$$

We further assume that ψ_i takes values in $[0, +\infty)$, and the functional form of ψ_i in (2.5) is such that (2.6) holds, if and only if

$$\xi_i < \tau_i(FR_i(t_1, t_2)) \tag{2.7}$$

with a function $\tau_i : [0, 1] \to \mathbb{R}$ that is continuous, increasing, and concave.

The assumptions on τ_i are consistent with the intuition that candidate i believes that his probability of winning increases – at a decreasing rate – as $FR_i(t_1, t_2)$ increases (as his platform becomes more popular than the platform of his opponent). In Section 4, we provide examples where Assumption C holds.

2.3. Distributional assumptions

To make the model tractable, we need to impose log-concavity conditions on the functional forms of H_i, the cdf of ξ_i.

A function $f : \mathbb{R}^n \to [0, +\infty)$ is log-concave on a convex set $D \subseteq \mathbb{R}^n$, if $\log f$ is concave on D (with the convention that $\log 0 = -\infty$). The properties listed in the following lemma can be gleaned from the literature in [3, 6].

[1] It is also possible to introduce uncertainty to a voting model by assuming that candidates are uncertain about how the ideal policies of voters are distributed over the policy space [23, 24].

Lemma 2.1. *Let f and g be two functions from $\mathbb{R}^n$ to $[0, +\infty)$. Let k be a concave function from a convex set $\hat{D} \subseteq \mathbb{R}^m$ to $\mathbb{R}$. Then*

(i) *If f is concave on a convex set D, then it is log-concave on D.*

(ii) *If f and g are both log-concave on a convex set D, then so is their product fg.*

(iii) *Let $Lev_{\geq}^{\alpha,D} f = \{x \in D | f(x) \geq \alpha\}$. Then, f is log-concave on a convex set D implies that, for any α, the set $Lev_{\geq}^{\alpha,D} f$ is convex.*

(iv) *If $f : \mathbb{R} \to \mathbb{R}$ is log-concave on an interval $D \subseteq \mathbb{R}$, then on any subinterval of D, f cannot decrease and then increase.*

(v) *If $f : \mathbb{R} \to \mathbb{R}$ is log-concave and weakly increasing, and the range of k is contained in the domain of f, then the function $f(k(x))$ is log-concave on $\hat{D}$.*

A random variable is log-concave, if its pdf (its density) is log-concave. Log-concave random variables are very common. The class of log-concave densities (pdfs) is very rich and contains most of the commonly used continuous distributions. This includes all non-degenerate normal densities, all Gamma densities with shape parameter ≥ 1, all Weibull densities with exponent ≥ 1, all beta densities with parameter ≥ 1, as well as the logistic and the Gumbel densities. For more examples see [3, 5]. If a random variable is log-concave, then its cdf is log-concave. The converse, however, is not true.

Assumption D: The random variable ξ_i has a pdf h_i that is continuous of $[0, 1]$, and it has a log-concave cdf H_i that on is log-concave on the range of τ_i.

2.4. Subjective winning probabilities

A candidate's beliefs regarding the preferences of the voters (the function F_i) and regarding the error term impacting voter turn out (the random variable ξ_i) are subjective and possibly private and unknown to the other candidate. Since the continuity of h_i implies a zero probability of the event $\xi_i = \tau_i(FR_i(t_1, t_2))$, each candidate computes his probability of winning the election -given a pair (t_1, t_2)- by computing

$$\pi_i(t_1, t_2) = Pr(\widehat{\Gamma}_i(t_1, t_2, \xi_i) > 1). \tag{2.8}$$

Hence, under assumption C and D, the (subjective) probability of a victory for candidate 1 is given by

$$\pi_1(t_1, t_2) = Pr(\widehat{\Gamma}_1(t_1, t_2, \xi_1) > 1) = H_1(\tau_i(FR_1(t_1, t_2)). \tag{2.9}$$

Similarly, player 2 perceives that her (subjective) probability of winning to be

$$\pi_2(t_1, t_2) = Pr(\widehat{\Gamma}_2(t_1, t_2, \xi_i) > 1) = H_2(\tau_2(FR_2(t_1, t_2)). \tag{2.10}$$

Assumptions C and D imply that π_i inherits the essential features of FR_i that we listed earlier.

Lemma 2.2. *Given assumptions A,B, C, and D, the subjective winning probabilities defined by* (2.9) *and* (2.10) *satisfy the following properties:*

(i) *$\pi_1(\cdot, t^m)$ and $\pi_2(t^m, \cdot)$ are continuous on $[0, 1]$. Moreover, $\pi_1(\cdot, t^m)$ is log-concave and weakly increasing on $[0, t^m]$ and weakly decreasing on $[t^m, 1]$. On the other* *hand, $\pi_2(t^m,\cdot)$ is weakly increasing on $[0,t^m]$, and it is log-concave and weakly decreasing on $[t^m,1]$.*

(ii) *For any $t_2 > t^m$, we have $\lim_{t\nearrow t_2}\pi_1(t,t_2) > \pi_1(t_2,t_2) > \lim_{t\searrow t_2}\pi_1(t,t_2)$.*

Moreover, $\pi_1(\cdot,t_2)$ is weakly increasing and log-concave on $[0,t_2)$, and weakly decreasing on $[t_2,1]$.

(iii) *For any $t_1 < t_m$, $\lim_{t\nearrow t_1}\pi_2(t_1,t) < \pi_2(t_1,t_1) < \lim_{t\searrow t_1}\pi_2(t_1,t)$.*

Moreover, $\pi_2(t_1,\cdot)$ is weakly decreasing and log-concave on $(t_1,1]$, and weakly increasing on $[0,t_1]$.

The proofs of the claims in Lemma 2.2 follow immediately from the definition of FR_i, Assumptions A,B,C, D, the fact H_i and F_i are cdfs and hence weakly increasing, and consequences 1-3 of Assumptions A and B. In particular, the log-concavity claims follow from the log-concavity of H_i (Assumption D), the monotonicity and concavity of τ_i (Assumption C), the concavity of F_1 and $1-F_2$ (Assumption A), together with part (v) of Lemma 2.1.

2.5. Candidates's preferences

The candidates in our model are office and policy motivated. They care about winning the election as well as about the policy implemented by the winner. Suppose t_1 and t_2 are the announced platforms of candidates 1 and 2 respectively. If candidate 1 is elected, policy t_1 will be implemented and the payoff of candidate 1 will be $v_1(t_1) + K_1$ where $v_1(t_1)$ is the utility of implementing t_1, and $K_1 \geq 0$ is additional utility that results from winning the election. On the other hand, if candidate 1 losses the election, then t_2 will be implemented and the payoff of candidate 1 will be simply $v_1(t_2)$. Similarly, if candidate 2 wins the election, policy t_2 will be implemented, and the payoff of candidate 2 will be $v_2(t_2)+K_2$ with $K_2 \geq 0$, whereas if she looses, her payoff will be $v_2(t_1)$. We further assume that $t = 0$ and $t = 1$ are respectively the ideal policies of candidates one and two. More specifically, we posit

Assumption E: Assume $v_1(0) = 0$ and v_1 is decreasing, continuous, and concave on $[0,1]$. Similarly, $v_2(1) = 0$ and v_2 is increasing, continuous, and concave on $[0,1]$.

Assumption E implies that the ideal policies of candidates 1 and 2 are extreme and located at the boundary points of $[0,1]$). As we will see in Section 6, this assumption is not essential to our results.

3. Existence of equilibria

In the model we have introduced so far, Assumptions A through E are not enough to guarantee the existence of an equilibrium [4, Section 4]. Assumptions A and B provide the candidates with an incentive to announce platforms in $[0,1]$ that are close to each other. On the other hand, Assumption E provides the candidates with an incentive to announce distinct and extreme platforms. The relative strength of these incentives determines where the equilibrium – if it exists – will be.

The expected payoff of candidate 1 can be computed as

$$U_1(t_1,t_2) = \int_0^\infty g_1(t_1,t_2,\xi_1)h_1(\xi_1)d\xi_1 \tag{3.1}$$

where

$$g_1(t_1,t_2,\xi_1)=\begin{cases} v_1(t_1)+K_1 & \text{if } \widehat{\Gamma}_1(t_1,t_2,\xi_1)>1\\ \frac{1}{2}[v_1(t_1)+K_1]+\frac{1}{2}v_1(t_2) & \text{if } \widehat{\Gamma}_1(t_1,t_2,\xi_1)=1\\ v_1(t_2) & \text{if } \widehat{\Gamma}_1(t_1,t_2,\xi_1)<1.\end{cases}$$

Therefore, using Assumptions C and D, we obtain

$$\begin{aligned} U_1(t_1,t_2) &= \int_0^{\tau_1(FR_1(t_1,t_2))} g_1(t_1,t_2,\xi_1)h_1(\xi_1)d\xi_1 \;+ \int_{\tau_1(FR_1(t_1,t2))}^{+\infty} g_1(t_1,t_2,\xi_1)h_1(\xi_1)d\xi_1 \\ &= \pi_1(t_1,t_2)l_1(t_1,t_2) \end{aligned} \tag{3.2}$$

where $l_1(t_1,t_2)=[v_1(t_1)-v_1(t_2)+K_1]+v_1(t_2)$. Similarly,

$$U_2(t_1,t_2)=\pi_2(t_1,t_2)l_2(t_1,t_2) \tag{3.3}$$

where $l_2(t_1,t_2)=[v_2(t_2)-v_2(t_1)+K_2]+v_2(t_2)$.

The functions U_1 and U_2 inherit their properties from the properties of the subjective winning probabilities.

Lemma 3.1. *Given Assumptions* A, B, C, D, *and* E, *the functions* U_1 *and* U_2 *defined by* (3.2) *and* (3.3) *satisfy the following:*

(i) $U_1(\cdot,t^m)$ *and* $U_2(t_m,\cdot)$ *are continuous on* $[0,1]$. *Moreover,* $U_1(\cdot,t^m)$ *is log-concave on* $[0,t^m]$ *and weakly decreasing on* $[t^m,1]$ *whereas* $U_2(t^m,\cdot)$ *is log-concave on* $[t^m,1]$ *and weakly increasing on* $[0,t^m]$.

(ii) *for any* $t_2>t^m$, $\lim_{t\nearrow t_2}U_1(t,t_2)<U_1(t_2,t_2)<\lim_{t\searrow t_2}U_1(t,t_2)$.

Furthermore, $U_1(\cdot,t_2)$ *is log-concave on* $[0,t_2)$ *and weakly decreasing over* $[t_2,1]$.

(iii) *for any* $t_1<t^m$, $\lim_{t\nearrow t_1}U_2(t_1,t)>U_2(t_1,t_1)>\lim_{t\searrow t_1}U_2(t_1,t)$.

Furthermore, $U_2(t_2,\cdot)$ *is log-concave on* $(t_1,1]$ *and weakly increasing over* $[0,t_1]$

Proof. Claims (i) through (iii) follow from the equation (3.2), (3.3), the properties of π_i given in Lemma 2.2, and our assumptions on H_i and F_i and v_i. In particular, the log-concavity claims follow from equations (3.2) and (3.3), the log-concavity properties of π_i given by Lemma 2.2, the concavity assumption on v_i, and parts (i) and (ii) of Lemma 2.1. □

The partial derivatives $\frac{\partial U_i}{\partial t_i}$ may fail to exist at a point (t_1,t_2). However, we assume the existence of "directional" partials. For any $\tilde{t}_1$ and $\tilde{t}_2$ in $[0,1]$, let

$$\frac{\partial U_1}{\partial t_1}^{-}(\tilde{t}_1,\tilde{t}_2)=\lim_{t_1\nearrow\tilde{t}_1}\frac{U_1(t_1,\tilde{t}_2)-U_1(\tilde{t}_1,\tilde{t}_2)}{t_1-\tilde{t}_1}$$

and

$$\frac{\partial U_1}{\partial t_1}^{+}(\tilde{t}_1,\tilde{t}_2)=\lim_{t_1\searrow\tilde{t}_1}\frac{U_1(t_1,\tilde{t}_2)-U_1(\tilde{t}_1,\tilde{t}_2)}{t_1-\tilde{t}_1}.$$

We define $\frac{\partial U_2}{\partial t_2}^{-}$ and $\frac{\partial U_2}{\partial t_2}^{+}$ similarly. We also define the constants

$$\alpha_1=\sup_{t_2\in(t^m,1]}\frac{\partial U_1}{\partial t_1}^{+}(t^m,t_2)\quad\text{and}\quad\alpha_2=\inf_{t_1\in[0,t^m)}\frac{\partial U_2}{\partial t_2}^{-}(t_1,t^m).$$

Our next theorem establishes the existence of an equilibrium in our model.

Theorem 3.2. *Consider the functions* U_1 *and* U_2 *defined in* (3.2) *and* (3.3). *Assume that conditions A, B,C,D, and E hold. Assume further that the payoffs of the candidates satisfy any of the following two conditions:*

(a) $\frac{\partial U_1}{\partial t_1}^{-}(t^m, t^m) > 0$ *and* $\frac{\partial U_2}{\partial t_2}^{+}(t^m, t^m) < 0$.

(b) $\alpha_1 < 0$ *and* $\alpha_2 > 0$.

Then, the game $G = (U_1, U_2, [0,1] \times [0,1])$ *has an equilibrium.*

Proof. Assume (a) holds. Then, the assumption $\frac{\partial U_1}{\partial t_1}^{-}(t^m, t^m) > 0$, the log-concavity of U_1 on $[0, t_2)$ -obtained via Lemma 3.1- and (iv) of Lemma 2.1 imply that $U_1(\cdot, t^m)$ is weakly increasing on $[0, t^m]$. Lemma 3.1 also implies that $U_1(\cdot, t^m)$ is weakly decreasing on $[t^m, 1]$. Hence, $t^m \in \operatorname{argmax}_{t_1 \in [0,1]} U_1(t_1, t^m)$.

Similarly, and using $\frac{\partial U_2}{\partial t_2}^{+}(t^m, t^m) < 0$, we can show that $t^m \in \operatorname{argmax}_{t_2 \in [0,1]} U_2(t^m, t_2)$. Therefore, $(t_1^*, t_2^*) = (t^m, t^m)$ is an equilibrium.

Assume (b) holds. Then, $\alpha_1 < 0$ implies that for any $t_2 \geq t^m$ we have

$$\operatorname{argmax}_{t_1 \in [0, t^m]} U_1(t_1, t_2) = \operatorname{argmax}_{t_1 \in [0,1]} U_1(t_1, t_2). \tag{3.4}$$

If $t_2 = t^m$, (3.4) follows from part (i) of Lemma 3.1. When $t_2 > t^m$, $\frac{\partial U_1}{\partial t_1}^{+}(t^m, t_2) < 0$, part (ii) of Lemma 3.2 (i.e. the log-concavity of $U_1(\cdot, t_2)$ on $[0, t_2)$ and the "jump down at t_2), and (iv) of Lemma 2.1 imply that $U_1(\cdot, t_2)$ is weakly decreasing on $[t^m, t_2]$. Furthermore, because of (ii) of Lemma 3.1, $U_1(\cdot, t_2)$ is weakly decreasing on $[t^2, 1]$. Hence, $U_1(\cdot, t_2)$ is weakly deceasing on $[t^m, 1]$, and (3.4) must hold. Using a similar argument, we can show that, if $\alpha_2 > 0$, then for any $t_1 \leq t^m$

$$\operatorname{argmax}_{t_2 \in [t^m, 1]} U_2(t_1, t_2) = \operatorname{argmax}_{t_2 \in [0,1]} U_2(t_1, t_2). \tag{3.5}$$

Let $\hat{G} = (U_1, U_2, [0, t^m] \times [t^m, 1])$ be the gamewhich can be obtained from the game $G = (U_1, U_2, [0,1] \times [0,1])$ by restricting the strategy sets of candidates 1 and 2 to $[0, t^m]$ and $[t^m, 1]$ respectively. The payoff of every candidate is now continuous and log-concave in his own strategy.

Define the set-valued map $BR(x_1, x_2) : [0, t^m] \times [t^m, 1] \rightrightarrows [0, t^m] \times [t^m, 1]$ by

$$BR(t_1, t_2) = BR_1(t_2) \times BR_1(t_2). \tag{3.6}$$

The continuity of U_1 and U_2 on $[0, t^m] \times [t^m, 1]$ (obtained in Lemma 3.1) implies the outer-semi continuity [22, Def. 5.4] of BR. Moreover, the continuity and the log-concavity of U_1 and U_2 on $[0, t^m]$ and $[t^m, 1]$ respectively and (iii) of Lemma 2.1. imply that the set-valued maps BR_1 and BR_2 are closed and convex-valued. Therefore, the set-valued map BR defined in (3.6) is also closed and convex-valued, and it maps the compact set $[0, t^m] \times [t^m, 1]$ into itself. By Kakutani's fixed point theorem, this set-valued map has a fixed point (t_1^*, t_2^*) in $[0, t^m] \times [t^m, 1]$, which gives us an equilibrium for the restricted game $\hat{G}$. Now (3.4) and (3.5) imply that (t_1^*, t_2^*) is also an equilibrium for the original game $G = (U_1, U_2, [0,1] \times [0,1])$. $\square$

Let (t_1^*, t_2^*) be an equilibrium of a game G that satisfies assumptions A through E and assumption (b) of Theorem 3.2. Suppose both are t_1^* and t_2^* in $[t_m, 1]$. Then,

$\alpha_1 < 0$ and the properties of U_1 and U_2 given in Lemma 3.1 imply that candidate 1 can increase his payoff by slightly decreasing t_1, contradicting the fact that (t_1^*, t_2^*) is an equilibrium. Similarly, if both t_1^* and t_2^* are in $[0, t_m]$, then $\alpha_2 > 0$ implies that candidate 2 can increase his payoff by increasing t_2, again contradicting the fact that (t_1^*, t_2^*) is an equilibrium. Moreover, condition (b) and the properties of U_1 and U_2 given in Lemma 3.1 rule out the possibility of having $t_1^* \in [t^m, 1]$ and $t_2^* \in [0, t^m]$. Therefore, we can claim the following partial converse of Theorem 3.2.

Corollary 3.3. *If* G *satisfies assumptions* A *through* E *and assumption* (b) *of Proposition* 3.1*, then* $(t_1^*, t_2^*) \in \mathrm{NE}(G)$ *implies* $t_1^* \in [0, t_m]$ *and* $t_2^* \in [t_m, 1]$.

Furthermore,

$$\operatorname{argmax}_{t_1 \in [0,t^m]} U_1(\cdot, t_2^*) = \operatorname{argmax}_{t_1 \in [0,1]} U_1(\cdot, t_2^*) \tag{3.7}$$

and

$$\operatorname{argmax}_{t_2 \in [t^m,1]} U_2(t_1^*, \cdot) = \operatorname{argmax}_{t_2 \in [0,1]} U_2(t_1^*, \cdot). \tag{3.8}$$

4. Examples and discussion of assumptions

Our assumption on the policy space and candidate preferences are standard in the literature on voting games. However, unlike our assumptions A and B, most current models of electoral competitions assume that the candidates have identical beliefs regarding voters' policy preferences (i.e. $F_1 = F_2$). Pre-election debates between candidates often reflect a strong disagreement over "what the voters really want". Therefore, assuming $F_1 = F_2$ seems unreasonable. Furthermore, most probabilistic voting models impose assumptions directly on the winning probabilities of the candidates [1, 4, 8, 11, 18, 23, 26]. In particular, these models assume

Agreement Assumption: $\pi_1(t_1, t_2) = 1 - \pi_2(t_1, t_2)$.

Unbiasedness Assumption: $\pi_1(t_1, t_2) = \pi_2(t_2, t_1)$.

These two assumptions imply that the payoffs of the candidates are reciprocally upper semi-continuous, a property that is often needed to establish the existence of equilibria in a discontinuous game [21] and [26]. However, these assumptions will not hold when the candidates disagree about distribution of voter preferences and $F_1 \neq F_2$. These assumptions may also fail when the uncertainty in the model results from random events -such as election day weather or newly implemented voter registration laws – that impact different groups of voters differently. Finally, most of the current existence results impose geometric assumptions (concavity/quasi-concavity/log concavity) directly on the subjective probabilities π_i and over the entire interval $[0, 1]$. This is inconvenient for two reasons. First, it is difficult to know whether or not these assumptions hold without first computing π_i on all of $[0, 1]$. Second, given the nature of the jump discontinuity π_i has, it is difficult to ascertain that these geometric assumptions hold over all of the policy space $[0, 1]$.[2] Our concavity/log concavity conditions, on the other hand, are imposed on F_i and H_i, respectively the distribution of the voters and the cdf of the error terms, and these conditions hold for a very large class of distributions. The examples in this section illustrate how Theorem 3.2 can be used to establish the existence of equilibria without imposing direct assumptions on π_i.

[2] In fact, some authors have assumed that π_i is concave on the entire interval $[0, 1]$, which is clearly impossible due to the discontinuity of π_i on the interior of $[0, 1]$. See [4] for details.

The functional form of ψ_i in equation (2.5) is normally left to the modeler. In order to simplify the exposition of our examples, we choose a specific functional from for the function ψ_i in (2.5).

Assumption C': The ratio $\widehat{\Gamma}$ defined in (2.4), is given by

$$\widehat{\Gamma}_i(t_1, t_2, \xi_i) = \psi_i(FR_i, \xi) = \frac{1}{\xi_i} FR_i. \tag{4.1}$$

where ξ_i is random variable distributed on $[0, +\infty)$ with a continuous cdf H_i.

Under Assumption C', we have

$$\widehat{FR}_i(t_1, t_2) = \frac{FR_i(t_1, t_2)}{FR_i(t_1, t_2) + \xi_i}, \tag{4.2}$$

and $$\Gamma_i(t_1, t_2, \xi_i) > 1 \Leftrightarrow \xi_i < FR_i(t_1, t_2),$$

and therefore Assumption C' implies Assumption C.

In all the examples of this section, we assume that Assumption C' holds.

Example 4.1. Assume $\xi_i = \xi_2 = \xi$ are uniformly distributed on $[0, 1]$. Assume further that $f_1 \equiv 1$ on $[0, 1]$ but f_2 is given by the following expression

$$f_2(z) = \begin{cases} \frac{16z}{3} & \text{if } z \leq 1/4 \\ \frac{4}{3} & \text{if } 1/4 \leq z \leq 3/4 \\ -\frac{16z}{3} + \frac{16}{3} & \text{if } 3/4 < z \leq 1. \end{cases} \tag{4.3}$$

Note that Agreement assumption is violated despite the fact that the median for both f_1 and f_2 is $1/2$. Assumption B is satisfied despite the fact that F_2 is not convex on all of $[0, 1]$. For any v_1 and v_2 that satisfy Assumption E, condition (a) of Theorem 3.2 is equivalent to

$$f_1(1/2)K_1 + v_1'(1/2) > 0 \quad \text{and} \quad -f_2(1/2)K_2 + v_2'(1/2) < 0.$$

In other words, condition (a) of Theorem 3.2 holds, if and only if,

$$K_1 \geq \frac{-v_1'(1/2)}{f_1(1/2)} \quad \text{and} \quad K_2 \geq \frac{v_2'(1/2)}{f_2(1/2)}. \tag{4.4}$$

Take for example $v_1 = -t$ and $v_2 = -(1 - t)$. In this case, if $K_1 > 1$ and $K_2 > 3/4$, then $(1/2, 1/2)$ is an equilibrium. If we let $v_1 = -t^2$ and $v_2 = -(1 - t)^2$, then we still have $v_1'(1/2) = -1$ and $v_2'(1/2) = 1$, and our conclusion remains the same; for $K_1 \geq 1$ and $K_2 \geq 3/4$, $(1/2, 1/2)$ is an equilibrium. Intuitively, (4.4) says that if both candidates believe that there is mass of voters around $1/2$ that is large relative to $v_i'(1/2)$, then (t^m, t^m) will be an equilibrium even when K_1 and K_2 are small. Alternatively, if K_1 and K_2 are very large, then (4.4) will hold for a large class of functions f_i and v_i. In fact, the verification of condition (4.4) only requires knowing the values these functions take at the median. □

The next example deals with simple heterogeneity in the beliefs of the candidates regrading the error terms.

Example 4.2. Assume the candidates agree that voters are uniformly distributed over $[0, 1]$. For $i \in \{1, 2\}$, assume h_i is equal to constant β_i on $[0, 1]$ and $0 \leq \beta_i \leq 1$.

Assume $\beta_1 \neq \beta_2$, $v_1(t) = -|t|$, and $v_2(t) = -|1-t|$. Then, for any $K_i \geq 0$ conditions A,B,C,D, and E hold. Since $\pi_1(t,t) = \beta_1 \neq \beta_2 = \pi_2$, this example does not satisfy the Agreement and Unbiasedness assumptions. Moreover,

$$\text{for any } t_2 \geq 1/2\text{:}\quad \frac{\partial U_1}{\partial t_1}^{-}(1/2, t_2) = \alpha_1 = \frac{1}{2}\beta_1[K_1 - 1], \text{ and} \tag{4.5}$$

$$\text{for any } t_1 \leq 1/2,\quad \frac{\partial U_2}{\partial t_2}^{+}(t_1, 1/2) = \alpha_2 = \frac{1}{2}\beta_2[1 - K_2]. \tag{4.6}$$

When $0 \leq K_i \leq 1$ for $i \in \{1,2\}$, (4.5) and (4.6) imply that condition (b) of Theorem 3.2 holds. When $K_i \geq 1$ for $i \in \{1,2\}$, (4.3) and (4.4) imply that condition (a) of Theorem 3.2 holds. Therefore, for any $K_i \geq 0$, if both $K_i \leq 1$ or if both $K_i \geq 1$, then the game has an equilibrium. □

Our last example shows how Theorem 3.2 can be applied when ξ is not constant on $[0,1]$.

Example 4.3. Assume the candidates agree the the voters are uniformly distributed over $[0,1]$. Assume $\xi_1 = \xi_2 = \xi$, where ξ is a random variable with values in $[0,+\infty)$ and a pdf

$$h(\xi) = \frac{1}{(\xi+1)^2}.$$

Then, $H(\xi) = \frac{\xi}{\xi+1}$. Using Assumption C', we obtain

$$\pi_1(t_1,t_2) = \begin{cases} \frac{t_1+t_2}{2+t_1+t_2} & \text{if } t_1 < t_2 \\ \frac{1}{3} & \text{if } t_1 = t_2 \\ \frac{2-(t_1+t_2)}{4-(t_1+t_2)} & \text{if } t_2 < t_1. \end{cases};\ \text{ similarly, }\ \pi_2(t_1,t_2) = \begin{cases} \frac{2-(t_1+t_2)}{4-(t_1+t_2)} & \text{if } t_1 < t_2 \\ \frac{1}{3} & \text{if } t_1 = t_2 \\ \frac{t_1+t_2}{2+t_1+t_2} & \text{if } t_2 < t_1. \end{cases}$$

Our particular choice of h implies that both candidates believe that the voters exhibit some base against them in the following sense; given identical platforms, each candidate believes that his chance of winning the election is only $\frac{1}{3}$ and that $\pi_1(t,t) \neq 1 - \pi_2(t,t)$.

Assume $v_1(t) = -t$, $v_2(t) = -(1-t)$. We apply Theorem 3.2 to obtain the following results:

If $K_1 = K_2 = 0$, then $(t_1^*, t_2^*) = (0,1)$ is the only pure strategy equilibrium; note first that if (t_1^*, t_2^*) is an equilibrium, then $t_1^* < t_2^*$. Furthermore, if $0 < t_1 < t_2 \leq 1$, then t_1 cannot be a best response of candidate 1 to t_2. Finally $t_1 = 0$ is a best response of candidate 1 to $t_2 = 1$ and $t_2 = 1$ is a best response of candidate 2 to $t_1 = 0$ because $U_1(\cdot, 1)$ is decreasing and $U_2(0, \cdot)$ is increasing on $[0,1]$.

If $0 < K_i \leq 3/2$ for $i \in \{1,2\}$, then condition (b) of Theorem 3.2 holds. In particular, routine calculations show that $\alpha_1 \leq 0$, if for $1/2 < t_2 \leq 1$, we have

$$2K_1 < 9/4 + t_2 + t_2^2.$$

The above inequality holds for any $1/2 \leq t_2 \leq 1$ as long as $K_1 \leq \frac{3}{2}$. Hence, $K_1 \leq \frac{3}{2}$ implies that $\alpha_1 \leq 0$. Similarly, we can show that $K_2 \leq \frac{3}{2}$ also implies that $\alpha_2 \geq 0$.

Finally, if $K_i > 3/2$ for $i \in \{1,2\}$, then condition (a) of Proposition 3.2 holds, and $(1/2, 1/2)$ is a pure strategy equilibrium for the game. In fact, if $K_i \geq 4$, then for any $t_2 > 1/2$, $\frac{\partial U_1}{\partial t_1}(\cdot, t_2) \geq 0$ on $[0, 1/2]$, and for any $t_1 < 1/2$, $\frac{\partial U_2}{\partial t_2}(t_1, \cdot) \leq 0$ on $[t^m, 1]$. Therefore, $(1/2, 1/2)$ is the only pure strategy equilibrium of this game, and we recover the classic "median voter theorem": when the outcome of the election is deterministic and the candidates care mostly about winning, they will choose the same platform corresponding to the position of the median voter. □

5. Statistical approximation of equilibria

When only limited information is available about the random error term ξ_i, the equilibria of the $G = (U_i, U_2, T)$ cannot be computed directly and must be inferred from historical data or collected samples. In a seminal paper, Dupacŏvá and Wets [13] studied the statistical consistency of the solutions of stochastic optimization problems. In this section, we follow the approach of [13] by approximating the equilibria of the game G through a process of sampling. One of the main insights of the literature on statistical approximation of stochastic optimization problems is that the usefulness of a statistical estimator in a specific application cannot be determined exclusively by its consistency properties [10]. Therefore, we focus on a specific non-parametric maximum likelihood estimator (NPMLE). This estimator is based on an optimization problem with soft (i.e. shape) constraints [10, 25]. Moreover, it has nice consistency properties, well-known convergence rates, and well-developed numerical tools to compute it. Furthermore, since this estimator is not based on kernel estimation techniques, it does not require tuning parameters (order of the kernel and the bandwidth) ([12, 29]. Most importantly, when the estimated density is log-concave, the NPMLE approximating densities will also be log-concave. This fact plays a crucial role when we are approximating the equilibria of a game G using the equilibria of a sequence of approximating games G^n. The goal of our approximation scheme is to show that any cluster point of the equilibria of G^n is actually an equilibrium of G. Therefore, it is important to be able to claim that the games in the (tail of) sequence G^n do in fact have equilibria. Otherwise, we end up with a vacuously true statement. Hence, we need to know that the payoff functions in G^n have the geometric properties required by our existence result, which can only happen if our statistical estimators are log-concave.

Henceforth, we assume that ξ_i is log-concave for $i \in \{1, 2\}$. Let h_i^n and H_i^n be respectively the empirical pdf and cdf associated with sample $\{\xi_i^1, \cdots, \xi_i^n\}$ of n iid draws from ξ_i. Note that H_i^n and h_i^n are random since they depend on the sample. Following [27] and [12], the log-concave NPMLE estimator of h_i^n is obtained as follows: Start with the functional

$$\Psi_i^n(\phi_i) = \int \phi dF_i^n - \int exp\, \phi(x)dx. \tag{5.1}$$

Solve
$$\hat{\phi}_i^n = \text{argmax}_{\phi\text{concave}}\, \Psi_i^n(\phi). \tag{5.2}$$

Set
$$\hat{h}_i^n = exp\, \hat{\phi}_i^n, \tag{5.3}$$

$$\hat{H}_i^n(\xi) = \int_{-\infty}^{\xi} \hat{h}_i^n(r)dr. \tag{5.4}$$

The following lemma lists some important facts about the log-concave NPMLE:

Lemma 5.1. *The log-concave NPMLE estimator of a log-concave pdf h_i has the following properties*

(i) *The function $\hat{\phi}_i^n$ defined in* (5.1) *exists and is unique.*

(ii) $E(\hat{H}_i^n) = E(H_i^n)$ *and* $Var(\hat{H}_i^n) \leq Var(H_i^n)$.

(iii) $||\hat{H}_i^n - H_i||_\infty \to 0$ *in probability and* $||h_1^n - h_i||_1 \to 0$ *in probability.*

The proofs of the above claims can be found in [12, 29]. In particular, (iii) above is Corollary 4.2 in [12].

Constructing G^n: We use sampling and (5.1) through (5.4) to construct sequences $\hat{H}_i^n$ and $\hat{h}_i^n$.

We then define the approximate payoff of player i as

$$U_i^n(t_1, t_2) = \hat{H}_i^n(FR_i(t_1, t_2))l_i(t_1, t_2) \tag{5.5}$$

and we denote the set of Nash equilibria of the game $G^n = (U_1^n, U_2^n, [0,1] \times [0,1])$ by $\mathrm{NE}(G^n)$.

Let $$\alpha_1^n = \sup_{t_2 \in (t^m, 1]} \frac{\partial U_1^n}{\partial t_1}^+ (t^m, t_2) \quad \text{and} \quad \alpha_2^n = \inf_{t_1 \in [0, t^m)} \frac{\partial U_2^n}{\partial t_2}^- (t_1, t^m).$$

All the terms $\hat{H}_i^n$, $\hat{h}_i^n$, U_i^n, α_i^n, and G_i^n are random because of their dependence on the sample drawn. Moreover, as a result of (iii) of Lemma 5.1, and through the iterative extraction of subsequences if needed, we will assume − without loss of generality − that $\hat{H}_i^n$ uniformly converges to H almost surely, and that $\hat{h}_i^n$ point-wise converges to h_i almost surely. Hence, all our subsequent claims in Lemma 5.2 and Theorem 5.3 of this section are to be interpreted as holding almost surely.

Lemma 5.2. *If the game G satisfies assumptions A through E and condition* (b) *of Theorem* 3.2*, then for large enough n,* $\mathrm{NE}(G^n)$ *is not empty, and for every* $(t_1^n, t_2^n) \in \mathrm{NE}(G^n)$ *we have*

$$t_1^n \in \mathrm{argmax}_{t \in [0, t^m]}\, U_1^n(\cdot, t_2^n) \text{ and } t_2^n \in \mathrm{argmax}_{t \in [t^m, 1]}\, U_1^n(t_2^n, \cdot).$$

Proof. The uniform convergence of $\hat{H}_i^n$, the point-wise convergence of h_i^n, and the fact that $\alpha_1 < 0$ and $\alpha_2 > 0$ imply that for large enough n, $\alpha_1^n < 0$ and $\alpha_2^n > 0$. Together with the log-concavity of $\hat{H}_i^n$ this allows us to apply Theorem 3.2 to G^n, and thus $\mathrm{NE}(G^n) \neq \emptyset$ for all large enough values of n. Moreover, Corollary 3.3 implies that, for any $(t_1^n, t_2^n) \in \mathrm{NE}(G^n)$, we have $t_1^n \in [0, t^m]$ and $t_2^n \in [t^m, 1]$. Therefore, $t_1^n \in \mathrm{argmax}_{t \in [0, t^m]}\, U_1^n(\cdot, t_2^n)$ and $t_2^n \in \mathrm{argmax}_{t \in [t^m, 1]}\, U_1^n(t_1^n, \cdot)$. □

Since we want to establish that any cluster point of equilibria of G^n is an equilibrium of the original game G, we recall some basic definitions necessary for our analysis [22, Chap. 4 and 7]. Let S_n be a sequence of sets in $\mathbb{R}^n$. The set $\mathrm{Ls}\, S_n$ is defined by

$$\mathrm{Ls}\, S_n = \{x \in \mathbb{R}^n | \exists \text{a (sub)sequence} x_{n_k} \in S_{n_k} \text{such that} x_{n_k} \to x\}.$$

The indicator function of a set $S \subset \mathbb{R}^n$ is defined as

$$\delta_S(x) = \left\{ \begin{array}{c} 0, \text{ if } x \in S \\ -\infty, \text{ otherwise.} \end{array} \right\}$$

A sequence of functions $f_n : \mathbb{R}^n \to \overline{\mathbb{R}}$ hypo converge to $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, and we write $f_n \xrightarrow{h} f_0$, if the followings conditions hold

(i) for any $x_n \to x$, we have $\limsup f_n(x_n) \leq f(x)$.

(ii) for any x, there exists $x_n \to x$ such that $\liminf f_n(x_n) \geq f(x)$.

Moreover, hypo convergence of f_n to f implies $\operatorname{Ls} \operatorname{argmax} f_n \subseteq \operatorname{argmax} f$.

Theorem 5.3. *Suppose the game* $G = (U_1, U_2, [0,1] \times [0,1])$ *satisfies assumptions* A *through* E *and condition* (b) *of Theorem* 3.2.
Then, $\operatorname{Ls} \mathrm{NE}(G^n) \neq \emptyset$ *and* $\operatorname{Ls} \mathrm{NE}(G^n) \subseteq \mathrm{NE}(G)$.

Proof. The fact $\operatorname{Ls} \mathrm{NE}(G^n) \neq \emptyset$ follows from Lemma 2.2 and the compactness of $[0,1]$. Let $(t_1^*, t_2^*) \in \operatorname{Ls} \mathrm{NE}(G^n)$. This implies the existence of a (sub) sequence $(t_1^n, t_2^n) \in \mathrm{NE}(G^n)$ with $(t_1^n, t_2^n) \to (t_1^*, t_2^*)$. Define the following functions on $[0, t^m]$

$$\tilde{U}_1^n(\cdot, t_2^n) = U_1^n(\cdot, t_2^n) + \delta_{[0,t^m]}(\cdot) \text{ and } \tilde{U}_1(\cdot, t_2^*) = U_1(\cdot, t_2) + \delta_{[0,t^m]}(\cdot).$$

For the second player, define the following functions on $[t^m, 1]$

$$\tilde{U}_2^n(t_1^n, \cdot) = U_2^n(t_1^n, \cdot) + \delta_{[t^m,1]}(\cdot) \text{ and } \tilde{U}_2(t_1, \cdot) = U_2(t_1, \cdot) + \delta_{[t^m,1]}(\cdot).$$

Almost sure convergence assumptions on $\hat{H}_i^n$ and $\hat{h}_i^n$ and the continuity of l_i imply that $\tilde{U}_i^n \xrightarrow{h} \tilde{U}_i$. Hence, $\operatorname{Ls} \operatorname{argmax}_{t_1 \in [0,t^m]} U_1^n(\cdot, t_2^n) \subseteq \operatorname{argmax}_{t_1 \in [0,t^m]} U_1(\cdot, t_2)$. Now Lemma 5.2 and Corollary 3.3 imply

$$\operatorname{Ls} \operatorname{argmax}_{[0,1]} U_1^n(\cdot, t_2^n) \subseteq \operatorname{argmax}_{t_1 \in [0,1]} U_1(\cdot, t_2).$$

Similarly, we can show that

$$\operatorname{Ls} \operatorname{argmax}_{[0,1]} U_2^n(t_1^n, \cdot) \subseteq \operatorname{argmax}_{t_2 \in [0,1]} U_2(t_1, \cdot).$$

Therefore, $(t_1^*, t_2^*) \in \mathrm{NE}(G)$. □

6. Extensions

Assuming that the candidates are located at the extreme points of the interval $[0,1]$ is not essential to our results. Consider a game that satisfies all our previous assumptions except now assume $v_1(z) = -|a - z|$ and $v_2 = -|b - z|$ such that $0 < a < t^m < b < 1$. In other words, the ideal policies of candidates 1 and 2 are a and b respectively. Consider the game played between the two candidates over the interval $[a, b]$. Theorem 3.2 implies that the game on $[a, b]$ has an equilibrium (t_1^*, t_2^*) with $a \leq t_1^* \leq t^m \leq t_2^* \leq 1$, which is also an equilibrium for the game on $[0,1] \times [0,1]$

Theorem 3.2 can also be easily modified to allow for the beliefs of the candidates cdfs F_1 and F_2 to have different medians.

Assumption A': Candidate 1 believes that, for any (t_1, t_2), the fraction of voters that prefer t_1 over t_2 is given by

$$FR_1(t_1, t_2) = \begin{cases} F_1(\frac{t_1+t_2}{2}) & \text{if } t_1 < t_2 \\ c_1 & \text{if } t_1 = t_2 \\ 1 - F_1(\frac{t_1+t_2)}{2}) & \text{if } t_2 < t_1 \end{cases}$$

where $0 \leq c_1 \leq 1$.

Similarly, candidate 2 believes that the fraction of voters that prefer t_2 over t_1 is given by

$$FR_2(t_1,t_2) = \begin{cases} 1 - F_2(\frac{t_1+t_2}{2}) & \text{if } t_1 < t_2 \\ c_2 & \text{if } t_1 = t_2 \\ F_2(\frac{t_1+t_2)}{2}) & \text{if } t_2 < t_1 \end{cases}$$

where $0 \leq c_1 \leq 1$. Moreover, we assume are F_1 and F_2 weakly increasing, continuous, and onto $[0,1]$.

Assumption B': There exist two points $0 < t_1^m < t_2^m < 1$ and two constants $0 \leq c_1 \leq c2$ such that $F_1(t_1^m) = c_1$ and $F_2(t_2^m) = c_2$. Moreover, F_1 is concave on $[\frac{t_1^m}{2}+1, 1]$ and F_2 is convex on $[0, \frac{1+t_2^m}{2}]$.

Define the constants

$$\tilde{\alpha}_1 = \sup_{t_2 \in (t_1^m, 1]} \frac{\partial U_1}{\partial t_1}^+ (t_1^m, t_2) \text{ and } \tilde{\alpha}_2 = \inf_{t_1 \in [0, t_2^m)} \frac{\partial U_2}{\partial t_2}^- (t_1, t_2^m).$$

Our assumptions now imply

(i) the functions $U_1(\cdot, t_1^m)$ and $U_2(t_2^m, \cdot)$ are continuous on $[0,1]$. Moreover, $U_1(\cdot, t_1^m)$ is log-concave on $[0, t_1^m]$ and weakly decreasing on $[t_1^m, 1]$. $U_2(t_1^m, \cdot)$ is log-concave on $[t_2^m, 1]$ and it is weakly increasing on $[0, t_2^m]$.

(ii) for any $t_2 > t_1^m$, $\lim_{t \nearrow t_2} U_1(t,t_2) < U_1(t_2,t_2) < \lim_{t \searrow t_2} U_1(t,t_2)$.
Furthermore, $U_1(\cdot, t_2)$ is log-concave on $[0, t_2)$ and weakly decreasing over $[t_2, 1]$.

(iii) for any $t_1 < t_2^m$, $\lim_{t \nearrow t_1} U_2(t_1,t) > U_2(t_1,t_1) > \lim_{t \searrow t_1} U_2(t_1,t)$.

Furthermore, $U_2(t_1, \cdot)$ is log-concave on $(t_1, 1]$ and weakly increasing over $[0, t_1]$.

Following the steps in the proof of Theorem 3.2, we can prove the following generalization.

Theorem 6.1. *Let the game G satisfies assumptions* A', B', C, D, E. *Assume further that $\tilde{\alpha}_1 < 0$ and $\tilde{\alpha}_2 > 0$. Then, the game G has an equilibrium.*

7. Appendix

A common approach to error distribution models is to assume

$$\widehat{FR}_i(t_1,t_2) = FR_i(t_1,t_2) + \xi \tag{7.1}$$

with ξ a continuous random variable taking values in some interval $[-\beta, \beta]$ with $\beta \leq 1$ and $H_i(0) = \gamma$. In fact, ξ_i is often assumed to be uniformly distributed on some interval $[-\beta, \beta]$ [26, 24]. The expected payoff of candidate 1 can be computed using (3.1), except now we use

$$g_1(t_1,t_2,\xi) = \begin{cases} v_1(t_1) + K_1 & \text{if } \widehat{FR}_1(t_1,t_2,\xi) > 1/2 \\ \frac{1}{2}[v_1(t_1) + K_1] + \frac{1}{2}v_1(t_2) & \text{if } \widehat{FR}_1(t_1,t_2,\xi) = 1/2 \\ v_1(t_2) & \text{if } \widehat{FR}_1(t_1,t_2,\xi) < 1/2. \end{cases}$$

Assuming that i_i is continuous, the expected payoff of candidate i is now given by

$$U_i(t_1,t_2) = \pi_i(t_1,t_2)l_i(t_1,t_2)$$

where
$$\pi_i(t_1,t_2) = 1 - H_i(\frac{1}{2} - FR_i(t_1,t_2)), \tag{7.2}$$

and as before
$$l_1(t_1,t_2) = [v_1(t_1) - v_1(t_2) + K_1] + v_1(t_2)$$

and
$$l_2(t_1,t_2) = [v_2(t_2) - v_2(t_1) + K_2] + v_2(t_2).$$

The expected payoff of candidate 2 can be similarly computed [26]. The fact that the error term in (7.1) is both additive and independent of the announced policies is very convenient for computing U_1 and U_2. However, it introduces a rationality/internal consistency constraint to the decision problem of each agent: given the definition of $\widehat{FR_i}$ as a fraction of voters, we must have, for $i \in 1, 2$,

$$Pr\{0 \le \widehat{FR_i}(t_1,t_2,\xi) \le 1\} = 1. \tag{7.3}$$

When $\widehat{FR_i}$ is given by (7.1), (7.3) becomes

$$1 - H_i(1 - FR_i(t_1,t_2)) + H_i(-FR_i(t_1,t_2)) = 0. \tag{7.4}$$

Assumptions A and B (as well as similar assumptions used in the literature on error distribution models) imply that for some t_1 and t_2, $FR_i(t_1,t_2)$ can be arbitrarily close to 0 or to 1. For such values of t_1 and t_2, (7.3) and (7.4) will be violated no matter what β is.[3] We can define set-valued maps corresponding to the constraint (7.3)

$$S_1(t_2) = \{t_1 \in [0,1] \text{ such condition 7.3 holds.}\}$$

$$S_2(t_1) = \{t_2 \in [0,1] \text{ such condition 7.3 holds,}\}$$

with the understanding that these set-valued map can be empty-valued for some t_1 and t_2 due condition (7.4). We modify the definition of an equilibrium (t_1^*, t_2^*) by requiring

$$t_1^* \in \text{argmax}_{t_1 \in S_1(t_2^*)} U_1(t_1,t_2^*) \text{ and } t_2^* \in \text{argmax}_{t_1 \in S_2(t_2^*} U_1(t_1^*,t_2). \tag{7.5}$$

Establishing the existence of such an equilibrium is difficult. One possible approach is to use the notion of variational continuity of constrained games [2, Def. 3]. Unfortunately, applying the results in [2] requires imposing strong conditions directly on $\pi_i(t_1,t_2)$. This, in turn, will considerably limit the type of distributions F_i and H_i that we can consider in our model.

Note that if we modify (7.1) by limiting (truncating) the support of ξ_i to insure that $0 \le FR_i(t_1,t_2) + \xi_i \le 1$, then in effect, we violate the independence of ξ_i and the policy choice vector (t_1,t_2).

Introducing uncertainty via Γ_i and (2.4) does induce a rationality constraint

$$Pr(0 \le \widehat{\Gamma}_i(t_1,t_2,\xi_i)) = 1. \tag{7.6}$$

However, this simple constraint holds for all t_1 and t_2 in $[0,1]$ as long as ψ has a range in $[0,\infty)$.

[3] This fact was not considered in the results of [26] and the results of Chapter 3 in [24].

Finally, introducing uncertainty as a multiplicative term

$$\widehat{FR}_i(t_1, t_2, \xi_i) = \xi_i \times FR_i(t_1, t_2)), \tag{7.7}$$

where ξ_i is a random variable with values in $[0, 1]$, does not require a rationality constraint. However, this model does not allow (unobservable) factors that can make $\widehat{FR}_i(t_1, t_2)$ larger than $FR_i(t_1, t_2)$. Moreover, when $t_1 = t_2 = t$ and ξ_i is continuously distributed, $\Pr(\xi \times FR_i(t, t) > \frac{1}{2}) = Pr(\xi \times \frac{1}{2} > \frac{1}{2}) = 0$. If a simple majority is needed to win, then this means that candidate 1 believes any tie in platforms will cause him to lose the election. This in turn would rule out any convergence results that we normally expect in a Downsian setting.[4]

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