

Embedding of Topological Posets in Hyperspaces

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Dedicated to Roger J-B Wets on the occasion of his 85th birthday.

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We study the problem of topologically order-embedding a given topological poset $(X, \preceq)$ in the space of all closed subsets of X which is topologized by the Fell topology and ordered by set inclusion. We show that this can be achieved whenever $(X, \preceq)$ is a topological semilattice (resp. lattice) or a topological po-group, and X is locally compact and order-connected (resp. connected). We give limiting examples to show that these results are tight, and provide several applications of them. In particular, a locally compact version of the Urysohn-Carruth metrization theorem is obtained, a new fixed point theorem of Tarski-Kantorovich type is proved, and it is found that every locally compact and connected Hausdorff topological lattice is a completely regular ordered space.

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1. Introduction

Let $(X, \preceq)$ be a poset, which is to say X is a nonempty set and $\preceq$ is a partial order on X . We say that $(X, \preceq)$ is a *topological poset* if, in addition, X is a topological space and $\preceq$ is a closed subset of $X \times X$.¹ Since the seminal work of Nachbin [33], such structures, in which topology and order are intimately linked, have been studied extensively. In addition, they play major roles in a diverse set of applied fields such as vector optimization [20], set optimization [22], general relativity [29, 31], rational decision-making [9, 15], and domain theory [17]. Special cases of topological posets also arise in functional analysis, often in the garb of an ordered topological vector space [1, 35].

The prototypical example of a poset is $(2^X, \subseteq)$ where 2^X is the power set of X and $\subseteq$ is the set inclusion ordering on 2^X . This example is, in fact, universal in the sense

¹The literature is not unified in this terminology. Some authors refer to what we call a topological poset here as a *po-space* or an *ordered topological space*. In addition, a partial order $\preceq$ on a space X that is closed in $X \times X$ is sometimes called *closed*, sometimes *continuous*, and sometimes *closed-continuous*.

that it contains a copy of $(X, \preceq)$ for any partial order $\preceq$ on X . Indeed, there is a natural way of order-embedding $(X, \preceq)$ into $(2^X, \subseteq)$ by associating to each $x \in X$ its *principal ideal* $x^\downarrow := \{z \in X : z \preceq x\}$. Clearly, the partial order axioms dictate that the mapping $x \mapsto x^\downarrow$ from X into 2^X – this is called the *canonical order-embedding* – is injective and preserves order structures in the sense that $x \preceq y$ iff $x^\downarrow \subseteq y^\downarrow$. When $(X, \preceq)$ is a topological poset, the canonical order-embedding maps points in X to nonempty closed subsets of X . The natural codomain for this map is thus $(\mathcal{C}(X), \subseteq)$ where $\mathcal{C}(X)$ stands for the set of all closed subsets of X . In this case we wish the canonical order-embedding to also preserve the topological structures of the involved posets. This brings up the problem of finding a suitable topology for $\mathcal{C}(X)$ so that $x \mapsto x^\downarrow$ is a topological embedding as well.

While this problem seems to have received surprisingly little attention in the literature, it is certainly not new. A folk theorem of topological order theory, a version of which was put on record first by Lawson [25], says that when $(X, \preceq)$ is a compact Hausdorff topological $\wedge$ -semilattice, and we endow $\mathcal{C}(X)$ with the classical Vietoris topology, then $x \mapsto x^\downarrow$ is indeed a topological embedding. (This result will actually obtain as an immediate corollary of one of our main theorems below.) However, topological posets that arise in applications are often not compact, so this result does not apply. And the available embedding results in the noncompact case are not really satisfactory. Misra [32] extends the above folk theorem to the noncompact case by endowing X with the order topology and taking $\preceq$ to be total, which is rather restrictive. By contrast, Choe and Park [12] work with a topological poset $(X, \preceq)$ which is a C -space.² Every compact topological poset is indeed a C -space, but unfortunately, the latter property is usually difficult to verify. Besides, [12] uses a coarsening of the Vietoris topology on the codomain of the canonical order-embedding that depends on $\preceq$. In other words, the Choe-Park embedding is not universal, in that the topology of the space in which $(X, \preceq)$ is embedded depends on the partial order $\preceq$. This limits the applicability of the embedding.

It is plain that a satisfactory solution to the embedding problem at hand would use a hyperspace topology on $\mathcal{C}(X)$, which is not only independent of $\preceq$, but is also well-behaved. The immediate candidates in this regard are, of course, the Vietoris topology and the Hausdorff metric topology (when X is a metric space), but these topologies become too demanding when the underlying space X is not compact. For example, it is only natural that the sequence of lines $\langle L_n \rangle$ in the plane, where L_n is defined by the equation $y = nx$, converge to the vertical axis, but neither the Vietoris topology nor the Hausdorff metric topology delivers this conclusion.

A weaker, and much better behaved, topology on $\mathcal{C}(X)$ is the *Fell topology*. This topology is widely studied since the seminal contribution of Fell [16], and it is found to have remarkable properties. In Section 3, we formally define the Fell topology and discuss its basic features, but we should note right away that this topology is the same as the Vietoris topology when X is compact and Hausdorff (as well as the Hausdorff metric topology when X is compact and metrizable). While convergence with respect the Fell topology may initially appear somewhat abstract, it is in fact intimately linked to the classical *Kuratowski-Painlevé* (K-P) *convergence* [24]

² This means that $\bigcup_{x \in S} x^\downarrow$ and $\bigcup_{x \in S} \{z \in X : x \preceq z\}$ are closed for every $S \in \mathcal{C}(X)$.

which is often easier to visualize. Most importantly for our purposes, when X is first countable and Hausdorff (in particular, when it is metrizable), the two convergence notions coincide for sequences of closed sets.³ This is important because, since the seminal work of Wijsman [42], it became evident that K-P convergence is fundamental to finite-dimensional convex analysis. As such, it has also led to the investigation of other convergence notions and set topologies in infinite dimensions, including Mosco convergence, Attouch-Wets convergence and the Joly-slice topology. (See, for instance, [3, 5, 28].)

The primary objective of the present paper is to investigate if, and when, the canonical order-embedding from a topological poset $(X, \preceq)$ into $(\mathcal{C}(X), \subseteq)$, where $\mathcal{C}(X)$ is endowed with the Fell topology, is a topological embedding. In Section 4, we observe that $x \mapsto x^\downarrow$ need not be continuous in this sense even when X is a metric continuum (so the folk theorem we mentioned above is not valid for compact topological posets). However, the situation is markedly different in the case of Hausdorff topological $\wedge$ -semilattices. Our first main result shows that $x \mapsto x^\downarrow$ is continuous whenever $(X, \preceq)$ is such a poset. Since the Fell topology reduces to the Vietoris topology when X is a compact Hausdorff space, this result provides a proof for the said folk theorem. However, in the general context of our continuity theorem, the use of the Fell topology is essential. In Section 4, we show that the canonical order-embedding may fail to be continuous if we equip $\mathcal{C}(X)$ with the Vietoris, or the Hausdorff metric, topology, even when $(X, \preceq)$ is a locally compact metric $\wedge$ -semilattice.

Our second main result, also presented in Section 4, shows that the inverse of the canonical order-embedding from $\mathcal{C}_\downarrow(X) := \{x^\downarrow : x \in X\}$ onto X is continuous (with respect to the relative Fell topology), provided that $(X, \preceq)$ is a locally compact topological poset which has connected order-intervals. Putting these results together yields our first embedding theorem: *Every locally compact Hausdorff topological $\wedge$ -semilattice $(X, \preceq)$ with connected order intervals is topologically order-embedded in the topological poset $(\mathcal{C}(X), \subseteq)$ by the canonical map $x \mapsto x^\downarrow$.* But every connected topological lattice has connected order-intervals. This gives our second embedding theorem: *Every locally compact and connected Hausdorff topological lattice $(X, \preceq)$ can be topologically order-embedded in $(\mathcal{C}(X), \subseteq)$.* Put a bit more precisely, $x \mapsto x^\downarrow$ is both a homeomorphism and an order-isomorphism from X onto $\mathcal{C}_\downarrow(X)$ whenever the latter is such a lattice.

Needless to say, the main advantage of universal embedding theorems is that they allow for a unified investigation of a variety of mathematical structures by means of a particular model. Our embedding theorems are useful precisely in this sense. To illustrate, we provide several applications in Section 5. First, we use this theorem to extend the Urysohn-Carruth metrization theorem, which says that every compact metric poset can be remetrized by a radially convex metric, to the context of topological posets that are covered by either of our embedding theorems. Second, we use our positive result on the continuity of the inverse of the canonical order-embedding to give sufficient topological conditions for a continuous $\wedge$ -homomorphism between

³ For a comprehensive account of Kuratowski-Painlevé convergence for nets of closed sets and its relation to the Fell topology, see [5, 23].

$\wedge$ -complete topological $\wedge$ -semilattices to preserve arbitrary infima. Third, we again use that positive result to provide a fixed point theorem in which local compactness and order-connectedness replace the order-homomorphism condition in the classical Kantorovich-Tarski fixed point theorem. Fourth, we revisit Anderson's theorem on local order-convexity of locally compact and connected Hausdorff topological lattices, and provide a very short proof for it. Finally, we utilize our second embedding theorem to prove that every such topological lattice is completely regularly ordered (in the sense of Nachbin).

While our approach in this paper is largely order-theoretic, we should note that analogous topological order-embedding theorems can also be achieved in the present setup by replacing the lattice structure with other order-compatible algebraic structures. In particular, in Section 6, we show that, where $\mathcal{C}(X)$ is endowed with the Fell topology, *every locally compact topological partially ordered group $(X, \cdot, \preceq)$ with connected order intervals is topologically order-embedded in the topological poset $(\mathcal{C}(X), \subseteq)$ by the canonical order-embedding.*

The paper concludes with the specification of a substantial open problem that emanates from the present work.

2. Preliminaries

2.1. Posets⁴

Let X be a nonempty set. By a *partial order* $\preceq$ on X , we mean a reflexive, anti-symmetric and transitive binary relation on X . As usual $x \preceq y$ means $(x, y) \in \preceq$, and $x \preceq y \preceq z$ means both $x \preceq y$ and $y \preceq z$ hold, etc.. For any nonempty $S \subseteq X$, we write $x \preceq S$ to mean that $x \preceq y$ for each $y \in S$ (in which case we say that x is a $\preceq$ -lower bound for S). The expression $S \preceq x$ (which means x is an $\preceq$ -upper bound for S) is similarly interpreted. We let $\bigwedge S$ stand for the greatest $\preceq$ -lower bound for S (if it exists), but as usual, write $x \wedge y$ instead of $\bigwedge\{x, y\}$. In turn, we let $S \wedge T := \{x \wedge y : (x, y) \in S \times T\}$ for any nonempty $S, T \subseteq X$, but write $S \wedge y$ instead of $S \wedge \{y\}$ for any $y \in X$. The expressions $\bigvee S$, $x \vee y$, $S \vee T$, and $S \vee y$ are defined analogously.

As usual, the ordered pair $(X, \preceq)$ is called a *poset* in which case we call X the *carrier* of $\preceq$. As we have already noted in Section 1, a *principal ideal* in this poset is a set of the form $x^\downarrow := \{z \in X : z \preceq x\}$ where $x \in X$. Dual to this notion is that of a *principal filter* which is a set of the form $x^\uparrow := \{z \in X : x \preceq z\}$. By an *order-interval* in $(X, \preceq)$, we mean a set of the form $x^\uparrow \cap y^\downarrow$ for some $x, y \in X$ with $x \preceq y$. For any $S \subseteq X$, we define the $\preceq$ -decreasing closure of S as $S^\downarrow := \{z \in X : z \preceq x \text{ for some } x \in S\}$, and define the $\preceq$ -increasing closure $S^\uparrow$ of S dually. In turn, S is said to be $\preceq$ -decreasing if $S = S^\downarrow$, and $\preceq$ -increasing if $S = S^\uparrow$. On the other hand, S is said to be $\preceq$ -convex if $x^\uparrow \cap y^\downarrow \subseteq S$ for every $x, y \in S$.

By an *order-embedding* from a poset $(X_1, \preceq_1)$ to a poset $(X_2, \preceq_2)$, we mean a map $\phi : X_1 \rightarrow X_2$ such that $x \preceq_1 y$ iff $\phi(x) \preceq_2 \phi(y)$. This map is automatically injective. If, in addition, ϕ is surjective, it is then called an *order-isomorphism*.

⁴ For a broad introduction to the theory of posets and lattices, see [37] or [39].

A poset $(X, \preceq)$ is said to be an *inf-semilattice*, or an $\wedge$ -*semilattice*, if $x \wedge y$ exists for every $x, y \in X$, and a *complete $\wedge$ -semilattice* if $\bigwedge S$ exists for every nonempty $S \subseteq X$. The notions of $\vee$ -*semilattice* and *complete $\vee$ -semilattice* are defined dually. A (*complete*) *lattice* is a poset that is at once a (*complete*) $\wedge$ - and a (*complete*) $\vee$ -semilattice. A *chain* C in $(X, \preceq)$ is a nonempty subset of X such that for any $x, y \in C$, either $x \preceq y$ or $y \preceq x$ holds. By an *antichain* A in $(X, \preceq)$, we mean a set $A \subseteq X$ such that no two distinct members of A are comparable with respect to $\preceq$.

2.2. Topological posets

We denote a topological space (X, τ) simply as X when there is no ambiguity about the topology under consideration. The interior, closure, and boundary of a subset S of X are denoted as $\text{int}(S)$, $\text{cl}(S)$, and $\text{bd}(S)$, respectively. We emphasize that $X \times X$ is always endowed with the product topology throughout the exposition.

A *topological poset* (resp., *metric poset*) is a poset $(X, \preceq)$ whose carrier X is a topological (resp., metric) space, and $\preceq$ is a closed subset of $X \times X$. The following is a basic, but useful, characterization of this closedness axiom [33, Proposition 1].

Proposition 2.1. *Let $(X, \preceq)$ be a poset where X is a topological space. The following conditions are equivalent:*

- (1) $\preceq$ is a closed subset of $X \times X$;
- (2) whenever $x \preceq y$ is false, there exist disjoint neighborhoods V of x and U of y such that V is $\preceq$ -increasing and U is $\preceq$ -decreasing.

It follows readily from this observation that the topology of a topological poset is always Hausdorff, because if x and y are distinct points in X either $x \preceq y$ or $y \preceq x$ fails (by antisymmetry of $\preceq$). It is also worth noting that all principal ideals and filters (hence all order-intervals) in a topological poset are closed. The converse is false. For instance, take any infinite set X and endow it with the cofinite topology. Then, $(X, =)$ is not a topological poset, but all of its principal ideals and filters are closed.

An $\wedge$ -semilattice $(X, \preceq)$ where X is a topological space and $(x, y) \mapsto x \wedge y$ is a continuous map from $X \times X$ into X , is said to be a *topological $\wedge$ -semilattice*. If, in addition, $(X, \preceq)$ is a lattice and $(x, y) \mapsto x \vee y$ is also a continuous map from $X \times X$ into X , we say that $(X, \preceq)$ is a *topological lattice*. By a *Hausdorff topological $\wedge$ -semilattice (lattice)*, we simply mean a topological $\wedge$ -semilattice (lattice) whose carrier is a Hausdorff space. And by a *complete Hausdorff topological $\wedge$ -semilattice*, we mean a Hausdorff topological $\wedge$ -semilattice which happens to be a complete $\wedge$ -semilattice.

Even a topological $\wedge$ -semilattice that satisfies the T_1 -axiom may not have closed principal ideals, let alone be a topological poset. For example, if we order $\mathbb{N}$ as $\cdots \prec 3 \prec 2 \prec 1$, and endow it with the cofinite topology, we end up with a T_1 topological $\wedge$ -semilattice in which no principal ideal is closed. However, as it is well known, this anomaly does not arise in the presence of the Hausdorff axiom. We put this on record for completeness of the exposition.

Proposition 2.2. *A topological $\wedge$ -semilattice $(X, \preceq)$ is a topological poset if and only if X is Hausdorff.*

Proof. We only need to prove the sufficiency. To this end, take any net $\langle (x_\lambda, y_\lambda) \rangle$ in $\preceq$ that converges to some $(x, y) \in X \times X$. Then, $x_\lambda \wedge y_\lambda \rightarrow x \wedge y$ by continuity of $\wedge$. But for each λ , we have $x_\lambda \wedge y_\lambda = x_\lambda$, whence $x_\lambda \wedge y_\lambda \rightarrow x$ as well. As a net can converge to at most one limit in a Hausdorff space, we thus obtain $x \wedge y = x$, that is, $x \preceq y$, as desired. $\square$

Following Gierz et al. [17, Definition VI-5.13], we say that a topological poset $(X, \preceq)$ is *order-connected* if the order interval $x^\uparrow \cap y^\downarrow$ is a connected subset of X for every $x, y \in X$ with $x \preceq y$. This property will play an essential role in what follows. The following observation relates it to the property of connectedness of principal ideals and filters.

Proposition 2.3. *Let $(X, \preceq)$ be a topological poset. If this poset is order-connected, then its principal ideals and filters are connected. If $(X, \preceq)$ is a topological $\wedge$ -semilattice, the converse holds as well.*

Proof. Assume $(X, \preceq)$ is order-connected, and take any $x \in X$. Then we obtain $x^\downarrow = \bigcup \{z^\uparrow \cap x^\downarrow : z \in x^\downarrow\}$, so $x^\downarrow$ is connected, being the union of a collection of connected sets with a point in common, and similarly for $x^\uparrow$. Conversely, if $(X, \preceq)$ is a topological $\wedge$ -semilattice with connected principal filters, it must be order-connected. This is because, in that case, for any $x, y \in X$ with $x \preceq y$, we have $x^\uparrow \cap y^\downarrow = x^\uparrow \wedge y$, whence $x^\uparrow \cap y^\downarrow$ is connected, being the continuous image of the connected set $x^\uparrow$. $\square$

The second assertion of this proposition is false for topological posets in general. For example, let X stand for the boundary of the 2-cell $[-1, 1]^2$, and endow X with the usual topology and the coordinatewise order. This yields a topological poset with connected principal ideals and filters, which is not order-connected (as $x^\uparrow \cap y^\downarrow = \{x, y\}$ where $x = (0, -1)$ and $y = (0, 1)$).

By a *topological order-embedding* from a topological poset $(X_1, \preceq_1)$ into another topological poset $(X_2, \preceq_2)$, we mean an order embedding $\phi : X_1 \rightarrow X_2$ such that ϕ is a topological embedding (that is, it is a homeomorphism from X_1 onto $\phi(X_1)$ where the latter is endowed with the subspace topology).

3. A crash course on the Fell topology

In this section we provide a quick review of a few essentials of hyperspace theory with an emphasis on the Fell topology. On the one hand, this makes the exposition largely self-contained, as most hyperspace concepts that we utilize subsequently are introduced here. On the other hand, this discussion aims to underscore the advantages of our embedding theorems that stem from the remarkable character of the Fell topology. Save for one exception (Proposition 3.1), all of the results mentioned below can be found in the monographs [5] and [23].

Let X be a Hausdorff space. In what follows, we denote the collection of all closed subsets of X by $\mathcal{C}(X)$, and that of all nonempty closed subsets of X by $\mathcal{C}_0(X)$. Formally speaking, by a *hyperspace*, we mean any nonempty subfamily of $\mathcal{C}(X)$ equipped with some topology.

There are a variety of interesting ways in which one may topologize $\mathcal{C}(X)$ in a manner faithful to the topology of X (in the sense of rendering $x \mapsto \{x\}$ an embedding).

The most well-known of these is the *Hausdorff metric topology* which requires X to be a metric space. This topology is induced by the extended real-valued *Hausdorff metric* H on $\mathcal{C}(X)$, which is defined for distinct A and B as

$$H(A, B) := \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}.$$

Here, of course, $d(a, B) := \inf_{b \in B} d(a, b)$ if $B \neq \emptyset$, and $d(a, \emptyset) := \infty$, and similarly for $d(b, A)$, where d is the metric of the space X .

There is a natural way of extending the Hausdorff metric topology to uniform spaces. (See [30] and [43, p. 250].) Let $\mathcal{B}$ be a base for a diagonal uniformity $\mathcal{D}$ on X consisting of symmetric sets, where X is equipped with the (completely regular) topology induced by the uniformity. For any $S \subseteq X$ and $B \in \mathcal{B}$, put

$$B(S) := \{x \in X : (x, s) \in B \text{ for some } s \in S\}.$$

The $\mathcal{D}$ -Hausdorff uniformity on $\mathcal{C}(X)$ has as a base all sets of the form

$$U_B := \{(F_1, F_2) \in \mathcal{C}(X) \times \mathcal{C}(X) : F_2 \subseteq B(F_1) \text{ and } F_1 \subseteq B(F_2)\}$$

where B runs through $\mathcal{B}$. The topology τ_H on $\mathcal{C}(X)$ induced by this separated uniformity (which is independent of the chosen base) is called the *Hausdorff uniform topology* determined by $\mathcal{D}$. If the uniformity of X is metric, this topology becomes the Hausdorff metric topology induced by that metric.

This approach does not extend to the case where X is a space that is not completely regular. The first serious study of a hyperspace topology, where X is any topological space, was conducted in the seminal article of Michael [30]. While Michael called his topology the *finite topology*, the literature refers to it much more commonly as the *Vietoris topology*. To describe this topology, we adopt the notation

$$S^- := \{A \in \mathcal{C}(X) : A \cap S \neq \emptyset\} \quad \text{and} \quad S^+ := \{A \in \mathcal{C}(X) : A \subseteq S\}$$

where S is any subset of X [5]. The *Vietoris topology* τ_V is generated by all sets of the form O^- where O is an open subset of X , and of the form $(X \setminus F)^+$ where $F \in \mathcal{C}(X)$. When X is regular as well as Hausdorff, $\mathcal{C}(X)$ equipped with τ_V is a Hausdorff space. In addition, for both the Hausdorff uniform topology and the Vietoris topology, $\emptyset$ is an isolated point of $\mathcal{C}(X)$.

A weaker topology on $\mathcal{C}(X)$ is the *Fell topology*, which we denote by τ_F . This topology is generated by all sets of the form O^- where O is an open set in X , and of the form $(X \setminus K)^+$ where K is a compact set in X . When X is compact, the Fell topology, the Vietoris topology and the Hausdorff uniform topology all agree. As will be clarified shortly, the Fell topology is also intimately related to the *Kuratowski-Painlevé* (K-P) *convergence* notion for sequences of sets.

We recall that sequence $\langle A_n \rangle$ in $\mathcal{C}(X)$ is said to *K-P converge* to $A \in \mathcal{C}(X)$ – in this case we write $A_n \xrightarrow{\text{K-P}} A$ – if (i) for each $a \in A$ there exists a sequence $\langle a_n \rangle$ such that $a_n \rightarrow a$ and $a_n \in A_n$ for each n ; and (ii) for every strictly increasing sequence $\langle n_k \rangle$ of positive integers and any convergent sequence $\langle a_{n_k} \rangle$ with $a_{n_k} \in A_{n_k}$ for each k , we have $\lim a_{n_k} \in A$ [3, 5, 7, 24].

In the sequel, for any Hausdorff space X , we write $\mathcal{C}^F(X)$ for the topological space $(\mathcal{C}(X), \tau_F)$, and $\mathcal{C}_0^F(X)$ for the subspace $\mathcal{C}_0(X)$ of $\mathcal{C}^F(X)$. (In other words, the topology of $\mathcal{C}_0^F(X)$ is the relative Fell topology.) The following is a list of notable facts about these spaces whose proofs can be found in [3, 5, 23].

- $\mathcal{C}^F(X)$ is compact;
- $\mathcal{C}_0^F(X) = X^-$ is an open subset in $\mathcal{C}^F(X)$;
- $x \mapsto \{x\}$ embeds X in $\mathcal{C}^F(X)$ as a closed subset, from which many important properties that hold in $\mathcal{C}^F(X)$ are forced on the base space;
- When X is first countable, convergence of sequences in $\mathcal{C}^F(X)$ is equivalent to their K-P convergence;
- When X is a uniform space, the Fell topology is coarser than the Hausdorff uniform topology on $\mathcal{C}(X)$;
- $\mathcal{C}^F(X)$ is Hausdorff iff X is locally compact; in this setting, as an open subset of a compact Hausdorff space, $\mathcal{C}_0^F(X)$ is a locally compact Hausdorff space;
- If X is locally compact and has a countable base $\mathcal{V}$ for its topology, then a countable subbase for the Fell topology is

$$\{V^- : V \in \mathcal{V}\} \cup \{(X \setminus \text{cl}(V))^+ : V \in \mathcal{V} \text{ with } V \text{ relatively compact}\}.$$

In this setting, by the Urysohn metrization theorem, $\mathcal{C}^F(X)$ is metrizable.

- If $\langle S_\lambda \rangle$ is a decreasing (resp., increasing) net with respect to $\subseteq$, then $S_\lambda \rightarrow \bigcap S_\lambda$ (resp., $S_\lambda \rightarrow \text{cl}(\bigcup S_\lambda)$) relative to τ_F .

We will always order the members of $\mathcal{C}(X)$ by the set inclusion ordering. Thus, henceforth, whenever we refer to $\mathcal{C}^F(X)$, or $\mathcal{C}_0^F(X)$, as a poset, what we mean is $(\mathcal{C}^F(X), \subseteq)$, or $(\mathcal{C}_0^F(X), \subseteq)$, as the case may be. The next result characterizes exactly when these posets are topological.

Proposition 3.1. *Let X be a Hausdorff space. The following conditions are equivalent:*

- (1) X is locally compact;
- (2) $\subseteq$ is closed in $\mathcal{C}(X) \times \mathcal{C}(X)$ in the Fell topology;
- (3) $\subseteq$ is closed in $\mathcal{C}_0(X) \times \mathcal{C}_0(X)$ in the relative Fell topology.

Proof. (2) implies (3) trivially. On the other hand, if (3) holds, then $\mathcal{C}_0(X)$ is Hausdorff in the relative Fell topology (Proposition 2.1), so X must be locally compact [5, Proposition 5.1.2]. To complete the proof, assume that X is locally compact, and take any two nets $\langle A_\lambda \rangle$ and $\langle B_\lambda \rangle$ in $\mathcal{C}(X)$ with $A_\lambda \subseteq B_\lambda$ for each index λ . Suppose these nets converge to A and B , respectively, relative to the Fell topology. To derive a contradiction, say there is a point a in $A \setminus B$. Since B is closed and X is locally compact, there is a compact neighborhood K of a in X such that $K \cap B = \emptyset$. But this means that $B \in (X \setminus K)^+$, so $\langle B_\lambda \rangle$ must eventually be in $(X \setminus K)^+$. As $\langle A_\lambda \rangle$ lies in $\text{int}(K)^-$ eventually, it follows that $A_\lambda \subseteq B_\lambda$ fails for some λ , a contradiction. Thus, $A \subseteq B$, and we are done. $\square$

The reader is invited to check that $(\mathcal{C}(X), \subseteq)$ equipped with the Vietoris topology is a topological poset, provided that X is regular. On the other hand, when X is a uniform space, endowing $(\mathcal{C}(X), \subseteq)$ with the Hausdorff uniform topology always yields a topological poset.

4. Bicontinuity of the canonical order-embedding

4.1. Continuity of $x \mapsto x^\downarrow$

We begin with giving a simple, if a bit surprising, example that shows that the canonical order-embedding from a topological poset $(X, \preceq)$ into $\mathcal{C}^F(X)$ need not be continuous, even when X is a metric continuum.

Example 4.1. Consider $X := ([-1, 0] \times \{0\}) \cup (\{0\} \times [-1, 0])$ as a metric subspace of the Euclidean plane; X is then a compact and connected metric space. Moreover, $(X, \preceq)$ is a metric poset, where $\preceq$ is the coordinatewise order on X . Now notice that $(-\frac{1}{n}, 0) \rightarrow (0, 0)$ and $(0, -1) \in (0, 0)^\downarrow$, but $(-\frac{1}{n}, 0)^\downarrow$ fails to hit the open ball of radius $\frac{1}{2}$ around $(0, -1)$ in X for each n . It follows that the canonical order-embedding $x \mapsto x^\downarrow$ is not continuous from X into $\mathcal{C}^F(X)$.⁵ $\square$

If we adjoin $(-1, -1)$ to X in this example, we obtain a lattice which is in fact a compact Hausdorff topological $\vee$ -semilattice, and yet continuity of $x \mapsto x^\downarrow$ still fails. However, somewhat surprisingly, this anomaly does not arise if the underlying poset is a Hausdorff topological $\wedge$ -semilattice. This is the content of our next finding.

Theorem 4.2. *Let $(X, \preceq)$ be a Hausdorff topological $\wedge$ -semilattice. Then the map $x \mapsto x^\downarrow$ from X into $\mathcal{C}^F(X)$ is continuous.*

Proof. By Proposition 2.2, X is a topological poset, and hence the range of the order-embedding $x \mapsto x^\downarrow$ is indeed contained in $\mathcal{C}(X)$. Now fix an arbitrary $x \in X$. Take any open set O in X with $x^\downarrow \in O^-$, that is, $x^\downarrow \cap O \neq \emptyset$. We pick any z in this intersection, and note that $x \wedge z = z$ and O is an open neighborhood of z in X . By continuity of $\wedge$, therefore, there exist open subsets U and V of X such that $(x, z) \in U \times V$ and $U \wedge V \subseteq O$. It follows that $y \wedge z \in y^\downarrow \cap O$ for every $y \in U$. Thus, for each $y \in U$, $y^\downarrow \cap O \neq \emptyset$, that is, $y^\downarrow \in O^-$, which means $\{y^\downarrow : y \in U\} \subseteq O^-$.

Next, take any compact subset K of X with $x^\downarrow \in (X \setminus K)^+$, that is, $x^\downarrow \subseteq X \setminus K$. We wish to find an open neighborhood V of x in X such that $\{y^\downarrow : y \in V\} \subseteq (X \setminus K)^+$. To derive a contradiction, suppose there is no such V , and let $\mathcal{V}(x)$ stand for the family of all open neighborhoods of x in X . Then, for every $V \in \mathcal{V}(x)$, there is an $x_V \in V$ such that $x_V^\downarrow$ is not contained within $X \setminus K$, that is, $x_V^\downarrow \cap K \neq \emptyset$. We choose any y_V in $x_V^\downarrow \cap K$ for each $V \in \mathcal{V}(x)$ to form a net $\langle y_V \rangle$ in X with the underlying directed set $(\mathcal{V}(x), \supseteq)$. By compactness of K , this net has a cluster point, say y , in K . It follows that (y, x) belongs to $\text{cl}(\{(y_V, x_V) : V \in \mathcal{V}(x)\})$. As $\preceq$ is closed in $X \times X$, and $y_V \preceq x_V$ for each $V \in \mathcal{V}(x)$, we thus conclude that $y \preceq x$. But then y belongs to $x^\downarrow \cap K$, contradicting $x^\downarrow \subseteq X \setminus K$. $\square$

When X is a compact Hausdorff space, the Vietoris and Fell topologies on $\mathcal{C}(X)$ coincide. Since a continuous injection from a compact space into a Hausdorff space is a topological embedding, the following thus obtains as an immediate consequence of Theorem 4.2. As we have noted in Section 1, this is a folk theorem of topological order theory.

⁵ This does not mean that $(X, \preceq)$ cannot be topologically order-embedded in $\mathcal{C}^F(X)$ in this particular setting. Indeed, $(a, b) \mapsto (a, b)^\downarrow \cup (b, a)^\downarrow$ embeds $(X, \preceq)$ in $\mathcal{C}^F(X)$ topologically as well as order-theoretically.

Corollary 4.3. *The canonical order-embedding from a compact Hausdorff topological $\wedge$ -semilattice $(X, \preceq)$ into $(\mathcal{C}(X), \subseteq)$ is a topological embedding, where $\mathcal{C}(X)$ is endowed with the Vietoris topology.*

It is natural to inquire if one can replace the Fell topology with the Vietoris topology in Theorem 4.2, or at least with the Hausdorff metric topology when $(X, \preceq)$ is a metric $\wedge$ -semilattice. We conclude this section with two examples that show that the answers to these questions are negative. In each case, we use a metric $\wedge$ -semilattice $(X, \preceq)$ where $X \subseteq \mathbb{R}^2$, the metric is the Euclidean metric and $\preceq$ is simply the coordinatewise ordering. In the case of the Vietoris topology, we can even take $X = \mathbb{R}^2$.

Example 4.4. In $\mathbb{R}^2$, we have $(\frac{1}{n}, 0) \rightarrow (0, 0)$, but $\langle (\frac{1}{n}, 0)^\downarrow \rangle$ does not converge to $(0, 0)^\downarrow$ relative to the Vietoris topology. Put $F := \{(\alpha, \beta) \in \mathbb{R}^2 : \beta \leq \ln \alpha\}$, and notice that $(0, 0)^\downarrow \in (X \setminus F)^+$ while $(\frac{1}{n}, 0)^\downarrow$ does not belong to $(X \setminus F)^+$ for any $n \in \mathbb{N}$.⁶ $\square$

The situation with the Hausdorff metric topology is less straightforward. For one thing, in the plane equipped with the Euclidean metric, we have $H(x^\downarrow, y^\downarrow) \leq \|x - y\|_2$ for any $x, y \in \mathbb{R}^2$, so we cannot work with the entire plane to produce the desired counterexample. This is partly responsible for the intricacy of the next construction.

Example 4.5. In $\mathbb{R}^2$, let $A := \{(-1, -k) : k \in \mathbb{N}\}$, let $B := \{(0, -k) : k \in \mathbb{N}\}$ and for each integer $m \geq 2$, put $C_m := \{(-\frac{1}{m}, -k) : k = 1, \dots, m\}$. The carrier of our topological poset is

$$X := A \cup B \cup C_2 \cup C_3 \cup \dots$$

which we view as a metric subspace of $\mathbb{R}^2$ and endow with the coordinatewise order $\preceq$. (See Figure 1.) Clearly, $(X, \preceq)$ is a complete, locally compact and totally disconnected metric space whose only non-isolated points lie in B . Relative to $\preceq$, we have $H((-\frac{1}{n}, -1)^\downarrow, (0, -1)^\downarrow) = 1$ for every $n \geq 2$, so the canonical order-embedding would not be continuous if we endowed $\mathcal{C}(X)$ with the Hausdorff metric. Besides, it is easily seen that $(X, \preceq)$ is a lattice. In what follows, we prove that $(X, \preceq)$ is actually a metric $\wedge$ -semilattice.

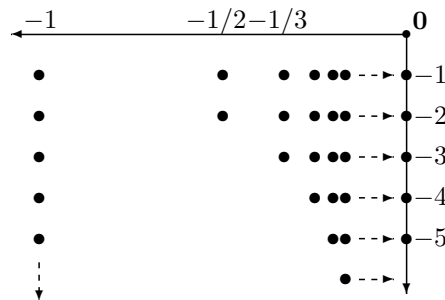


Figure 1

To prove that $\wedge$ is a continuous map from $X \times X$ onto X , it is enough to focus only on the subdomain $X \times B$, because $\wedge$ is a symmetric map and all points in $X \setminus B$ are isolated. We may also reason sequentially, as the carrier space is metric.

⁶ By contrast, $(1, \frac{1}{n})^\downarrow \rightarrow (1, 0)^\downarrow$ in the Fell topology. Indeed, as $\langle (1, \frac{1}{n})^\downarrow \rangle$ is a decreasing sequence in $\mathcal{C}(X)$, we have $(1, \frac{1}{n})^\downarrow \xrightarrow{\text{K-P}} \bigcap_{n \geq 1} (1, \frac{1}{n})^\downarrow = (1, 0)^\downarrow$ in concert with Theorem 4.2.

Take any $x \in X$ and put $y := (0, -k)$ for some $k \in \mathbb{N}$. Let (x_n, y_n) be a sequence in $X \times X$ with $x_n \rightarrow x$ and $y_n \rightarrow y$. In the sequel, we will write $x_n := (\alpha_n, \beta_n)$ and $y_n := (\gamma_n, \mu_n)$ for each $n \in \mathbb{N}$.

First suppose that $x \in B$ as well, that is, $x = (0, -l)$ for some $l \in \mathbb{N}$. This means that $\langle \alpha_n \rangle$ and $\langle \gamma_n \rangle$ converge to 0 and that, eventually, $\beta_n := -l$ and $\mu_n = -k$. In view of the continuity of the minimum functional on $\mathbb{R}^2$, therefore,

$$\lim_{n \rightarrow \infty} x_n \wedge y_n = \lim_{n \rightarrow \infty} (\min\{\alpha_n, \gamma_n\}, \min\{-l, -k\}) = (0, \min\{-l, -k\}) = x \wedge y.$$

Next suppose that $x \in X \setminus B$, and put $x := (\alpha, \beta)$. In this case, x is isolated, so we may assume $x_n = x$ for all n . As $\gamma_n \rightarrow 0 > \alpha$ and $\mu_n \rightarrow -k$, we can also assume that $\gamma_n > \alpha$ and $\mu_n = -k$ for all n . We now claim that with these simplifying assumptions, $\langle x_n \wedge y_n \rangle = \langle x \wedge y_n \rangle$ is a constant sequence. This is verified by considering five separate cases:

$$x \wedge y_n = \begin{cases} x, & \text{if } \alpha = -1 \text{ and } \beta \leq -k, \\ (-1, -k), & \text{if } \alpha = -1 \text{ and } \beta > -k, \\ (-1, -k), & \text{if } -1 < \alpha < -1/k, \\ (\alpha, -k), & \text{if } -1/k \leq \alpha \text{ and } \beta \geq -k, \\ x, & \text{if } -1/k \leq \alpha \text{ and } \beta < -k \end{cases}$$

for every $n \in \mathbb{N}$. It follows $x \wedge y_n = x \wedge y$ for each n in every contingency, and we may thus conclude that $x_n \wedge y_n \rightarrow x \wedge y$ when $x \in X \setminus B$ as well.

4.2. Continuity of the inverse of $x \mapsto x^\downarrow$

We again start with some limiting examples that show that the inverse of the canonical order-embedding need not be continuous without suitable hypotheses.

Example 4.6. Let $X := (-\infty, 0)^2 \cup \{\mathbf{0}\} \cup (0, \infty)^2$ which we view as a subspace of $\mathbb{R}^2$ relative to the usual topology. (Here $\mathbf{0}$ stands for the origin of $\mathbb{R}^2$.) Then, $(X, \preceq)$, where $\preceq$ is the usual coordinatewise order, is a Hausdorff topological lattice. However, the map $x^\downarrow \mapsto x$ is not continuous at $\mathbf{0}^\downarrow$. After all, here we have $(\frac{1}{n}, 1)^\downarrow \xrightarrow{\text{K-P}} \mathbf{0}^\downarrow$, whence $(\frac{1}{n}, 1)^\downarrow \rightarrow \mathbf{0}^\downarrow$ in the Fell topology. $\square$

Example 4.7. Let $\mathbf{0}$ again stand for the 2-vector $(0, 0)$, and put $x_n := (\frac{1}{n}, n)$ for each $n \in \mathbb{N}$. We now consider the poset $(X, \preceq)$ where $X := \{\mathbf{0}, x_1, x_2, \dots\}$ and $\preceq$ is the coordinatewise order. We endow X with the discrete topology (which is the subspace topology X inherits from the Euclidean topology of the plane). Then, $(X, \preceq)$ is a locally compact Hausdorff topological $\wedge$ -semilattice. However, the inverse of the canonical order-embedding in this setting is not continuous. Indeed, we have $x_n^\downarrow = \{\mathbf{0}, x_n\} \xrightarrow{\text{K-P}} \{\mathbf{0}\}$ here, so $x_n^\downarrow \rightarrow \mathbf{0}^\downarrow$ in the Fell topology, but $\langle x_n \rangle$ does not converge to $\mathbf{0}$. $\square$

In the first example above, the difficulty emanates from the lack of local compactness (at $\mathbf{0}$), and in the second one, because order-intervals are not connected. The next result shows that if the underlying topological poset $(X, \preceq)$ is locally compact and possesses connected order-intervals, then the inverse of the associated canonical

order-embedding is indeed continuous on $\mathcal{C}_\downarrow(X) := \{x^\downarrow : x \in X\}$, the set of all principal ideals of $(X, \preceq)$.

Theorem 4.8. *Let $(X, \preceq)$ be a locally compact and order-connected topological poset. Then the map $x^\downarrow \mapsto x$ from $\mathcal{C}_\downarrow(X)$ onto X is continuous (with respect to the relative Fell topology).*

Proof. Let $\langle x_\lambda \rangle$ be a net in X (with the underlying directed set $(\Lambda, \preceq)$), and x a point in X , such that $x_\lambda^\downarrow \rightarrow x^\downarrow$ in the Fell topology. We need to show that $x_\lambda \rightarrow x$.

To derive a contradiction, suppose $\langle x_\lambda \rangle$ does not converge to x in X . As X is locally compact and Hausdorff, there is a local base of compact neighborhoods of x in X [13, p.238]. By replacing Λ by a suitable cofinal subset if necessary, we may thus assume that there exists a compact neighborhood K of x in X such that $x_\lambda \in X \setminus K$ for every $\lambda \in \Lambda$.

We now define an auxiliary net of points in X that converges to x . Let $\mathcal{O}_K(x)$ stand for the family of all open neighborhoods O of x in X such that $O \subseteq \text{int}(K)$. Put

$$\Omega := \left\{ (\lambda, O) : \lambda \in \Lambda, O \in \mathcal{O}_K(x) \text{ and } x_\lambda^\downarrow \cap O \neq \emptyset \right\},$$

and consider the partial order $\blacktriangleleft$ on Ω defined by

$$(\alpha, U) \blacktriangleleft (\beta, V) \quad \text{iff} \quad \alpha \preceq \beta \text{ and } V \subseteq U.$$

Clearly, $(\Omega, \blacktriangleleft)$ is a directed poset. For each $(\lambda, O) \in \Omega$, we pick any $w_{\lambda, O} \in x_\lambda^\downarrow \cap O$. As $x_\lambda^\downarrow \rightarrow x^\downarrow$ in the Fell topology, for every open neighborhood U of x in X , there is a $\lambda_U \in \Lambda$ such that

$$x_{\lambda_U}^\downarrow \in (U \cap \text{int}(K))^- \quad \text{whenever} \quad \lambda_U \preceq \lambda,$$

whence $w_{\lambda, O} \in O \subseteq U$ whenever $(\lambda_U, U \cap \text{int}(K)) \blacktriangleleft (\lambda, O)$. Thus: $w_{\lambda, O} \rightarrow x$.

Note that for any $\lambda \in \Lambda$, the connectedness of $w_{\lambda, O}^\uparrow \cap x_\lambda^\downarrow$ entails that there exists a $z_{\lambda, O} \in \text{bd}(K) \cap w_{\lambda, O}^\uparrow \cap x_\lambda^\downarrow$ for, otherwise, $\text{int}(K)$ and $X \setminus K$ would determine a non-trivial separation of $w_{\lambda, O}^\uparrow \cap x_\lambda^\downarrow$. As $\text{bd}(K)$ is compact, being a closed subset of a compact set, the net $\langle z_{\lambda, O} \rangle$ must have a cluster point z in $\text{bd}(K)$. The proof will be completed by showing that (i) $z \preceq x$ and (ii) $x \preceq z$, hence $z = x$ (which contradicts x belonging to $\text{int}(K)$).

Suppose (i) fails, that is, z lies outside of $x^\downarrow$. Since X is locally compact, z must then have a compact neighborhood C which is disjoint from $x^\downarrow$. As $x_\lambda^\downarrow \rightarrow x^\downarrow$ in the Fell topology, there is a $\lambda_0 \in \Lambda$ such that $x_\lambda^\downarrow \cap C = \emptyset$ for every $\lambda \in \Lambda$ with $\lambda_0 \preceq \lambda$.

On the other hand, since z is a cluster point of $\langle z_{\lambda, O} \rangle$, we have $z_{\lambda, O} \in \text{int}(C)$ for a cofinal set of indices (λ, O) in Ω . As $z_{\lambda, O} \in x_\lambda^\downarrow$ for all λ , this is a contradiction. Conclusion: $z \preceq x$.

Given that $\preceq$ is closed, (ii) follows from the fact that (a) $w_{\lambda, O} \rightarrow x$; (b) z is a cluster point of $\langle z_{\lambda, O} \rangle$; and (c) $w_{\lambda, O} \preceq z_{\lambda, O}$ for every $(\lambda, O) \in \Omega$. To be more precise, take any $V \in \mathcal{O}_K(x)$ and any open neighborhood U of z . Since $x_\lambda^\downarrow \rightarrow x^\downarrow$ in the Fell topology, there exists $\lambda_0 \in \Lambda$ such that $x_\lambda^\downarrow \cap V \neq \emptyset$, that is, $(\lambda, V) \in \Omega$, for

every $\lambda \in \Lambda$ with $\lambda_0 \preceq \lambda$. As z is a cluster point of $\langle z_{\lambda, O} \rangle$, there certainly exists a $(\lambda, W) \in \Omega$ with $\lambda_0 \preceq \lambda$ where both $W \subseteq V$ and $z_{\lambda, W} \in U$ hold. Since $w_{\lambda, W} \in W$ as well, we obtain

$$(w_{\lambda, W}, z_{\lambda, W}) \in W \times U \subseteq V \times U.$$

As $w_{\lambda, W} \preceq z_{\lambda, W}$, this argument proves that (x, z) belongs to the closure of $\preceq$ in $X \times X$. Since $\preceq$ is closed, therefore, $x \preceq z$, and the proof is complete. $\square$

4.3. Embedding theorems

Putting Theorems 4.2 and 4.8 together yields our main embedding result.

Theorem 4.9. *Let $(X, \preceq)$ be a locally compact and order-connected Hausdorff topological $\wedge$ -semilattice. Then, the map $x \mapsto x^\downarrow$ topologically order-embeds $(X, \preceq)$ in $\mathcal{C}^F(X)$.*

For any topological poset $(X, \preceq)$, let $\mathcal{C}_\downarrow^F(X)$ stand for $(\mathcal{C}_\downarrow(X), \tau_F)$, that is, the space of all principal ideals of $(X, \preceq)$ endowed with the relative Fell topology. Theorem 4.9 entails that X can be identified with $\mathcal{C}_\downarrow^F(X)$ topologically, and $(X, \preceq)$ with $(\mathcal{C}_\downarrow^F(X), \subseteq)$ order-theoretically, by means of the same morphism. It follows that, in the context of this theorem, $(\mathcal{C}_\downarrow^F(X), \subseteq)$ is a locally compact and order-connected Hausdorff topological $\wedge$ -semilattice. In particular, even though the $\cap$ operation on $\mathcal{C}^F(X)$ is not continuous (even when X is a metric continuum), it is continuous on $\mathcal{C}_\downarrow^F(X)$. To see this, define $\varphi: X \rightarrow \mathcal{C}_\downarrow^F(X)$ by $\varphi(x) := x^\downarrow$. Then, if $\langle x_\lambda \rangle$ and $\langle y_\lambda \rangle$ are two nets in X with $x_\lambda^\downarrow \rightarrow x^\downarrow$ and $y_\lambda^\downarrow \rightarrow y^\downarrow$ for some $x, y \in X$, we have

$$x_\lambda \wedge y_\lambda = \varphi^{-1}(x_\lambda^\downarrow) \wedge \varphi^{-1}(y_\lambda^\downarrow) \rightarrow \varphi^{-1}(x^\downarrow) \wedge \varphi^{-1}(y^\downarrow) = x \wedge y$$

because φ^{-1} and $\wedge$ are continuous. As $(a \wedge b)^\downarrow = a^\downarrow \cap b^\downarrow$ in the context of any $\wedge$ -semilattice, therefore,

$$x_\lambda^\downarrow \cap y_\lambda^\downarrow = (x_\lambda \wedge y_\lambda)^\downarrow = \varphi(x_\lambda \wedge y_\lambda) \rightarrow \varphi(x \wedge y) = (x \wedge y)^\downarrow = x^\downarrow \cap y^\downarrow$$

in view of the continuity of φ .

If only to highlight the nontrivial nature of this fact, we next present an example that shows that even when $(X, \preceq)$ is a compact, connected, and order-connected topological poset which happens to be an $\wedge$ -semilattice, the intersection operation on $\mathcal{C}_\downarrow^F(X)$ does not have to be (even separately) continuous.

Example 4.10. Let $\mathbf{0}$ stand for the origin of the Hilbert space ℓ_2 , and consider the sequence $a_0 \in \ell_2$ where $a_{0,2k-1} := 0$ and $a_{0,2k} := \frac{1}{k}$ for every $k \in \mathbb{N}$. Next, for each positive integer n , let $a_n \in \ell_2$ be the perturbation of a_0 defined by $a_{n,2n-1} := 1/n$, $a_{n,2n} := 0$, and $a_{n,k} := a_{0,k}$ otherwise. Since $\|a_n - a_0\|_2 = \sqrt{2}/n \rightarrow 0$, the set $\{a_0, a_1, \dots\}$ is compact. We put $C_n := \text{conv}(\{\mathbf{0}, a_n\})$ for each nonnegative integer n , and define $X := C_0 \cup C_1 \cup \dots$ which we consider as a (metric) subspace of ℓ_2 . As X is the continuous image of the compact set $\{a_0, a_1, \dots\} \times [0, 1]$, it is compact. Moreover, being the union of a collection of convex subsets of ℓ_2 with a point in common, X is connected.

Clearly, $(X, \preceq)$ is a topological poset, where $\preceq$ is the coordinatewise order. By construction, for any $x, y \in X$, we have $x \preceq y$ iff $x = \alpha a_n$ and $y = \beta a_n$ for some $n \geq 0$ and $\alpha, \beta \in [0, 1]$ with $\alpha \leq \beta$.

It follows that the order-intervals of $(X, \preceq)$ are convex subsets of ℓ_2 , so $(X, \preceq)$ is order-connected. Finally, while $\preceq$ is total on C_n for any $n \geq 0$, if $x \in C_n$ and $y \in C_m$ for distinct $m, n \geq 0$, we have $x \wedge y = \mathbf{0}$. Thus $(X, \preceq)$ is an $\wedge$ -semilattice as well.

Now, notice that $a_n \rightarrow a_0$ implies $H(a_n^\downarrow, a_0^\downarrow) \rightarrow 0$, where H is the Hausdorff metric. As the Fell topology is coarser than the Hausdorff metric topology on $\mathcal{C}(\ell_2)$, it follows that $a_n^\downarrow \rightarrow a^\downarrow$ in the Fell topology. But $a_n^\downarrow \cap a_0^\downarrow = \{\mathbf{0}\}$ for each $n > 0$ while $a_0^\downarrow \cap a_0^\downarrow = a_0^\downarrow \neq \{\mathbf{0}\}$. Conclusion: $\cap$ is not (separately) continuous on $\mathcal{C}_\downarrow^F(X)$. $\square$

Order enters to the hypotheses of Theorem 4.9 from the channels of the semilattice property and order-connectedness. If we strengthen the former property to being a topological lattice, then we can replace the latter channel with a purely topological one. This is our second embedding theorem.

Theorem 4.11. *Let $(X, \preceq)$ be a locally compact and connected Hausdorff topological lattice. Then, the map $x \mapsto x^\downarrow$ topologically order-embeds $(X, \preceq)$ in $\mathcal{C}^F(X)$.*

Proof. Take any $x, y \in X$ with $x \preceq y$, and consider the map $\eta : X \rightarrow X$ defined by $\eta(z) := (z \vee x) \wedge y$. Then, η is continuous (as $\wedge$ and $\vee$ are continuous) and $x^\uparrow \cap y^\downarrow = \eta(X)$, so $x^\uparrow \cap y^\downarrow$ is connected. It follows that $(X, \preceq)$ is order-connected, and Theorem 4.9 applies. $\square$

5. Applications

In this section we provide five applications of the above theorems to various topics in topological order theory.

5.1. On radially convex metrization

Let $(X, \preceq)$ be a poset. We recall that a metric d on X is *radially convex* (relative to $\preceq$) if

$$x \preceq y \preceq z \quad \text{implies} \quad d(x, z) \geq \max\{d(x, y), d(y, z)\}.$$

This concept builds a tight connection between the order and metric structures that may be imposed on a given set. For example, every order interval of a poset that is endowed with a radially convex metric is bounded with respect to that metric. It is thus of interest when one can remetrize a given metric poset by means of a radially convex metric (without changing the topology). The major result in this regard, due to Carruth [11], is the following:

Theorem 5.1. (The Urysohn-Carruth Metrization Theorem) *Every compact metric poset can be equivalently remetrized by a radially convex metric.*

The classical *Nachbin extension theorem* says that if $(X, \preceq)$ is a compact topological poset and S a nonempty closed subset of X , then every order-preserving and continuous real map on S can be extended to X in an order-preserving and continuous manner.⁷ Carruth [11] uses this result to build a continuous order-embedding from X into the coordinatewise ordered Hilbert cube. This bit can be generalized to the context of locally compact and separable metric posets. But while Carruth's order-embedding is automatically a topological embedding (because X is compact),

⁷ This result obtains by putting together Theorems 4 and 6 of Chapter 2 of Nachbin [33].

it is not clear if this is so in the more general case as well. Indeed, it is presently unknown if the Urysohn-Carruth metrization theorem is valid for locally compact and separable metric posets. However, by using the embedding established in Theorem 4.9, we get a positive result to this query in the case of order-connected metric semilattices.

Before we state the main result of this section, we recall that, given any metric space (X, d) , the *Wijsman topology* is the weak topology on $\mathcal{C}_0(X)$ induced by the family $\{d(x, \cdot) : x \in X\}$ [5, 27]. Thus, a net $\langle S_\lambda \rangle$ in $\mathcal{C}_0(X)$ converges to an $S \in \mathcal{C}_0(X)$ relative to the Wijsman topology iff $d(x, S_\lambda) \rightarrow d(x, S)$ for each $x \in X$.

Theorem 5.2. *Every locally compact, second-countable and order-connected Hausdorff topological $\wedge$ -semilattice $(X, \preceq)$ can be metrized by means of an equivalent radially convex metric.*

Proof. A well-known result that goes back to Vaughan [41] says that every locally compact and second-countable Hausdorff space can be metrized by a boundedly compact metric so that every closed and bounded set in the space is compact. Let d be such a metric on X , and note that X is separable (as it is second-countable). Let $\{x_1, x_2, \dots\}$ be a countable dense set in X and define $\rho_d : \mathcal{C}_0(X) \times \mathcal{C}_0(X) \rightarrow [0, \infty)$:

$$\rho_d(A, B) := \sum_{i \geq 1} 2^{-i} \min\{1, |d(x_i, A) - d(x_i, B)|\}.$$

It is well-known that ρ_d metrizes the Wijsman topology on $\mathcal{C}_0(X)$ [5, p. 37]. Moreover, bounded compactness of d ensures that the Fell topology and the Wijsman topology on $\mathcal{C}_0(X)$ are the same [5, Theorem 5.1.10]. We may thus conclude that the metric topology induced by ρ_d is the Fell topology on $\mathcal{C}_0(X)$. Moreover, $d(x, B) \leq d(x, A)$ for every $x \in X$ and $A, B \in \mathcal{C}_0(X)$ with $A \subseteq B$. Consequently, for every $A, B, C \in \mathcal{C}_0(X)$ with $A \subseteq B \subseteq C$, we have

$$d(x, A) - d(x, C) \geq d(x, B) - d(x, C) \quad \text{for all } x \in X,$$

and it follows that $\rho_d(A, C) \geq \rho_d(B, C)$. As one similarly shows that $\rho_d(A, C) \geq \rho_d(A, B)$, it follows that ρ_d is a radially convex metric on $\mathcal{C}_0(X)$ relative to the containment ordering $\subseteq$.

We now define $D : X \times X \rightarrow [0, \infty)$ by $D(x, y) := \rho_d(x^\perp, y^\perp)$. As the canonical order-embedding from X into $\mathcal{C}_0^F(X)$ is a topological embedding (Theorem 4.9), it is plain that D is a metric on X that induces the topology of X . Moreover, as ρ_d is radially convex relative to $\subseteq$, and $x \preceq y$ iff $x^\perp \subseteq y^\perp$, D is radially convex relative to $\preceq$. $\square$

5.2. On complete $\wedge$ -homomorphisms

Let $(X_1, \preceq_1)$ and $(X_2, \preceq_2)$ be two $\wedge$ -semilattices. We recall that a map $f : X \rightarrow Y$ is *order-preserving* if $x \preceq_1 y$ implies $f(x) \preceq_2 f(y)$, *order-reversing* if $x \preceq_1 y$ implies $f(y) \preceq_2 f(x)$, and that it is an $\wedge$ -homomorphism if $f(x \wedge y) = f(x) \wedge f(y)$ for every $x, y \in X_1$. An $\wedge$ -homomorphism is always order-preserving, but not conversely. If both $(X_1, \preceq_1)$ and $(X_2, \preceq_2)$ are complete $\wedge$ -semilattices, and $f(\bigwedge S) = \bigwedge f(S)$ for every nonempty $S \subseteq X_1$, we say that f is a *complete $\wedge$ -homomorphism*.

Where $(X, \preceq)$ is any poset, we follow Gierz et al. [17] in saying that a nonempty subset S of X is $\preceq$ -filtered if every nonempty finite subset of S has a $\preceq$ -lower bound

in S . For instance, if restricting $\preceq$ to S yields an $\wedge$ -semilattice, then S is sure to be $\preceq$ -filtered. In particular, for every $\preceq$ -decreasing sequence $\langle x_n \rangle$ in X , the set $\{x_1, x_2, \dots\}$ is $\preceq$ -filtered.

In this application, we are interested in understanding when a map between complete Hausdorff topological $\wedge$ -semilattices preserve the infima of filtered sets. This is not a trivial matter in that even when such a map f is a continuous $\wedge$ -homomorphism, and $\langle x_n \rangle$ is a decreasing sequence in the domain of f , the equation $f(x_1 \wedge x_2 \wedge \dots) = f(x_1) \wedge f(x_2) \wedge \dots$ may fail.⁸

Example 5.3. Let $X := \{-1\} \cup (0, 1]$ and $Y := \{-1\} \cup [0, 1]$, and endow these sets with the usual order and topology. This makes both of these sets locally compact and totally bounded metric spaces that are also complete topological lattices. Now define $f : X \rightarrow Y$ by $f(x) := x$, which is of course a $\wedge$ -homomorphism and a topological embedding. And yet $f(1 \wedge \frac{1}{2} \wedge \frac{1}{3} \wedge \dots) = -1$ while $f(1) \wedge f(\frac{1}{2}) \wedge \dots = 0$. $\square$

Our first theorem in this application shows that such counter-examples do not arise so long as the complete topological semilattice in the domain is locally compact *and* order-connected. It may be worth noting that this result imposes virtually no additional conditions on the target complete topological semilattice.

Theorem 5.4. *Let $(X_1, \preceq_1)$ and $(X_2, \preceq_2)$ be two topological posets that happen to be complete $\wedge$ -semilattices. Suppose $(X_1, \preceq_1)$ is locally compact and order-connected, and $f : X_1 \rightarrow X_2$ is an order-preserving and continuous map. Then, $f(\bigwedge S) = \bigwedge f(S)$ for every $\preceq_1$ -filtered subset S of X_1 .*

In the proof of this result we shall utilize the notions of decreasing and essentially decreasing nets in a poset. Let $(X, \preceq)$ be a poset and $\langle x_\lambda \rangle$ a net in X with the underlying directed set $(\Lambda, \trianglelefteq)$. We say that $\langle x_\lambda \rangle$ is $\preceq$ -decreasing if $x_\beta \preceq x_\alpha$ holds for every $\alpha, \beta \in \Lambda$ with $\alpha \trianglelefteq \beta$. More generally, we say that $\langle x_\lambda \rangle$ is *essentially $\preceq$ -decreasing* if for every $\alpha \in \Lambda$ there exists a $\beta \in \Lambda$ such that $x_\lambda \preceq x_\alpha$ holds for every $\lambda \in \Lambda$ with $\beta \trianglelefteq \lambda$.

The following proposition, which highlights another desirable property of the Fell topology, will be used to prove the above theorem.

Proposition 5.5. *Let X be a Hausdorff space and $\langle S_\lambda \rangle$ an essentially $\subseteq$ -decreasing net in $\mathcal{C}(X)$ with the underlying directed set $(\Lambda, \trianglelefteq)$. Then, $S_\lambda \rightarrow \bigcap_{\lambda \in \Lambda} S_\lambda$ in the Fell topology.*

Proof. We put $S := \bigcap_{\lambda \in \Lambda} S_\lambda$. Our goal is to show that $\langle S_\lambda \rangle$ is eventually in any subbasic open neighborhood of S (relative to the Fell topology). To this end, notice first that if $S \in O^-$ for some open set O in X , then $S_\lambda \cap O \supseteq S \cap O \neq \emptyset$, whence $S_\lambda \in O^-$, for each λ . Next, suppose $S \in (X \setminus K)^+$ for some compact $K \subseteq X$. If $K = \emptyset$, we trivially have $S_\lambda \in (X \setminus K)^+$ for each λ , so assume instead K is nonempty. As $K \cap S = \emptyset$, for every $x \in K$ there is a $\lambda(x) \in \Lambda$ such that x does not belong to $S_{\lambda(x)}$. Then, $\{X \setminus S_{\lambda(x)} : x \in K\}$ is an open cover of K , so there is a finite $F \subseteq K$ such that $K \subseteq \bigcup_{x \in F} X \setminus S_{\lambda(x)}$, that is, $K \cap \bigcap_{x \in F} S_{\lambda(x)} = \emptyset$. But, as $\langle S_\lambda \rangle$ is essentially $\subseteq$ -decreasing, there exists a $\beta \in \Lambda$ such that $S_\lambda \subseteq \bigcap_{x \in F} S_{\lambda(x)}$, whence

⁸ For any countable set $\{x_1, x_2, \dots\}$ in an $\wedge$ -semilattice, we write $x_1 \wedge x_2 \wedge \dots$ to mean $\bigwedge \{x_1, x_2, \dots\}$.

$K \cap S_\lambda = \emptyset$, for every $\lambda \in \Lambda$ with $\beta \preceq \lambda$. This proves that $\langle S_\lambda \rangle$ is eventually in $(X \setminus K)^+$, as desired. $\square$

Proof of Theorem 5.4. Let S be a $\preceq_1$ -filtered subset of X_1 , and consider the directed poset $(\mathcal{F}_0(S), \subseteq)$ where $\mathcal{F}_0(S)$ is the family of all nonempty finite subsets of S . Then, for every $A \in \mathcal{F}_0(S)$ there exists a $y_A \in S$ such that $y_A \preceq_1 \bigwedge A$ (but note that $\bigwedge A$ need not lie in S). We note that the net $\langle y_A^\downarrow \rangle$ with the underlying directed set $(\mathcal{F}_0(S), \subseteq)$ is essentially $\subseteq$ -decreasing. Indeed, for any $A \in \mathcal{F}_0(S)$, letting $B := A \cup \{y_A\}$ we find $y_C \preceq_1 \bigwedge C \preceq_1 \bigwedge B \preceq_1 y_A$, whence $y_C^\downarrow \subseteq y_A^\downarrow$, for every $C \in \mathcal{F}_0(S)$ with $B \subseteq C$. Now, it is easily verified that

$$\bigcap_{A \in \mathcal{F}_0(S)} y_A^\downarrow = \left(\bigwedge S \right)^\downarrow.$$

Consequently, by Proposition 5.5, and because $\langle y_A^\downarrow \rangle$ is essentially $\subseteq$ -decreasing, we have $y_A^\downarrow \rightarrow (\bigwedge S)^\downarrow$ relative to the Fell topology. By Theorem 4.8, we have, therefore, $y_A \rightarrow \bigwedge S$. By continuity of f , then, $f(y_A) \rightarrow f(\bigwedge S)$. Since $\bigwedge f(S) \preceq_2 f(y_A)$ for each $A \in \mathcal{F}_0(S)$ and $\preceq_2$ is closed, it follows that $\bigwedge f(S) \preceq_2 f(\bigwedge S)$. On the other hand, $f(\bigwedge S) \preceq_2 \bigwedge f(S)$ simply because $f(\bigwedge S)$ is a $\preceq_2$ -lower bound for $f(S)$ (as f is order-preserving). We conclude that $f(\bigwedge S) = \bigwedge f(S)$. $\square$

The following observation provides a reflection of Theorem 5.4 for essentially decreasing sequences.

Corollary 5.6. *Let $(X_1, \preceq_1)$, $(X_2, \preceq_2)$, and f be as in Theorem 5.4. Then,*

$$f(x_1 \wedge x_2 \wedge \cdots) = \lim f(x_n) = f(x_1) \wedge f(x_2) \wedge \cdots \quad (1)$$

for every essentially $\preceq_1$ -decreasing sequence $\langle x_n \rangle$ in X_1 .

Proof. Let $\langle x_n \rangle$ be an essentially $\preceq_1$ -decreasing sequence in X_1 . Then, $\{x_1, x_2, \dots\}$ is a $\preceq_1$ -filtered subset of X_1 , so Theorem 5.4 entails that the first and third expressions of (1) are equal. Moreover, $\langle x_n^\downarrow \rangle$ is an essentially $\subseteq$ -decreasing sequence in $\mathcal{C}^F(X)$, so by Proposition 5.5, $x_n^\downarrow \rightarrow \bigcap_{i \geq 1} x_i^\downarrow = (x_1 \wedge x_2 \wedge \cdots)^\downarrow$. By Theorem 4.8, therefore, $x_n \rightarrow x_1 \wedge x_2 \wedge \cdots$, so by continuity of f , we get the equality of the first and second expressions of (1). $\square$

Theorem 5.4 does not go so far as to ensure that the subject map f is an $\wedge$ -homomorphism. The following example shows that this is not warranted by the hypotheses of the theorem even in the compact case.

Example 5.7. Let $X_1 := [-1, 1]^2$ and $X_2 := [-2, 2]$. Endowing these sets with the usual order and topology yields order-connected and complete topological lattices whose carriers are metric continua. Consider the map $f : X_1 \rightarrow X_2$ with $f(x, y) := x + y$, which is both order-preserving and continuous. And yet, $f((0, 1) \wedge (1, 0)) = 0$ while $f(0, 1) \wedge f(1, 0) = 1$. $\square$

Having said this, we note that strengthening the order-preservation hypothesis in Theorem 5.4 to being an $\wedge$ -homomorphism furnishes a complete $\wedge$ -homomorphism.

Theorem 5.8. *Let $(X_1, \preceq_1)$ and $(X_2, \preceq_2)$ be as in Theorem 5.4. Then, every continuous $\wedge$ -homomorphism $f: X_1 \rightarrow X_2$ is complete.*

Proof. Let S be any nonempty subset of X_1 , and again denote by $\mathcal{F}_0(S)$ the collection of all nonempty finite subsets of S . Define

$$T := \left\{ \bigwedge A : A \in \mathcal{F}_0(S) \right\}$$

which is the smallest sub- $\wedge$ -semilattice of $(X_1, \preceq_1)$ that contains S . Note that, for any $A \in \mathcal{F}_0(S)$, we have

$$\bigwedge f(S) \preceq_2 \bigwedge f(A) = f(\bigwedge A),$$

because $A \subseteq S$ and f is an $\wedge$ -homomorphism. Thus, $\bigwedge f(S)$ is a $\preceq_2$ -lower bound for $f(T)$, whence $\bigwedge f(S) \preceq_2 \bigwedge f(T)$. But, obviously, $\bigwedge S = \bigwedge T$, so by Theorem 5.4,

$$f(\bigwedge S) = f(\bigwedge T) = \bigwedge f(T).$$

It follows that $\bigwedge f(S) \preceq_2 f(\bigwedge S)$. As $f(\bigwedge S) \preceq_2 \bigwedge f(S)$ holds simply because f is order-preserving, we are done. $\square$

It is well-known, and is easily proved, that a decreasing net in a compact Hausdorff topological $\wedge$ -semilattice converges to its infimum. (See, for instance, Strauss [40].) Using this fact, one can show easily that every compact Hausdorff topological $\wedge$ -semilattice is a complete $\wedge$ -semilattice. Moreover, again using this result, we can show that a continuous $\wedge$ -homomorphism between compact Hausdorff topological $\wedge$ -semilattices $(X_1, \preceq_1)$ and $(X_2, \preceq_2)$ is a complete $\wedge$ -homomorphism. Theorem 5.8 shows that we can replace the compactness requirements on the domain and codomain in this result with weaker complete $\wedge$ -semilattice requirements, provided that $(X_1, \preceq_1)$ is locally compact and order-connected.

5.3. A fixed point theorem

A poset $(X, \preceq)$ is said to satisfy the *countable-chain condition* if $\bigwedge S$ exists for every nonempty countable chain S in $(X, \preceq)$. This condition plays an important role in a variety of order-theoretic fixed point theorems. One of the most well-known of these is the following:

Theorem 5.9. (The Tarski-Kantorovich Fixed Point Theorem)⁹ *Let $(X, \preceq)$ be a poset that satisfies the countable-chain condition, and $f: X \rightarrow X$ a function such that $f(x_1 \wedge x_2 \wedge \cdots) = f(x_1) \wedge f(x_2) \wedge \cdots$ for every $\preceq$ -decreasing sequence (x_m) in X . If $f(x) \preceq x$ for some $x \in X$, then f has a fixed point.*

The following is a topological variant of this fixed point theorem that devolves from our result about the continuity of the map $x^\downarrow \mapsto x$.

Theorem 5.10. *Let $(X, \preceq)$ be a locally compact and order-connected topological poset that satisfies the countable-chain condition, and $f: X \rightarrow X$ an order-preserving and continuous map. If $f(x) \preceq x$ for some $x \in X$, then f has a fixed point.*

⁹ See [14, p. 15] and [19, p. 630]

Proof. Take any point $x \in X$ with $f(x) \preceq x$, and define $x_1 := x$ and $x_n := f(x_{n-1})$ for each $n = 2, 3, \dots$. As f is order-preserving, $\langle x_n \rangle$ is a $\preceq$ -decreasing sequence in X . In particular, $x_* := x_1 \wedge x_2 \wedge \dots$ exists by the countable-chain condition. Moreover, $\langle x_n^\downarrow \rangle$ is a $\subseteq$ -decreasing sequence in $\mathcal{C}^F(X)$, so Proposition 5.5 entails

$$x_n^\downarrow \rightarrow x_1^\downarrow \cap x_2^\downarrow \cap \dots = x_*^\downarrow.$$

By Theorem 4.8, therefore, $x_n \rightarrow x_*$, whence, by continuity of f , we find that $f(x_n) \rightarrow f(x_*)$. As $\langle f(x_n) \rangle$ is the sequence $\langle x_{n+1} \rangle$, we must conclude $f(x_n) \rightarrow x_*$. Thus: $f(x_*) = x_*$. $\square$

Checking for order-preservation and continuity properties are in most applications easier than checking for the homomorphism property required in the Tarski-Kantorovich theorem. This is the advantage of Theorem 5.10.

5.4. On Anderson's theorem

Let $(X, \preceq)$ be a poset whose carrier is a topological space. Such a poset is said to be *locally convex* if the topology of X has a basis that consists of $\preceq$ -convex sets. A major source of locally convex posets is the following celebrated result.

Theorem 5.11. (Anderson's Theorem) *Every locally compact and connected Hausdorff topological lattice is locally convex.*

Anderson's original proof for this result [2, Theorem 1] is fairly long. By contrast, a straightforward proof for it obtains if we use the second embedding theorem of Section 4. Indeed, Anderson's theorem is an immediate consequence of Theorem 4.11 and the following elementary observation.

Proposition 5.12. *For any Hausdorff space X , both $(\mathcal{C}^F(X), \subseteq)$ and $(\mathcal{C}_\downarrow^F(X), \subseteq)$ are locally convex.*

Proof. By definition, a basis for the Fell topology on $\mathcal{C}(X)$ consists of sets of the form $O_1^- \cap \dots \cap O_k^- \cap (X \setminus K)^+$ where $k \in \mathbb{N}$, $O_1, \dots, O_k$ are open subsets of X and K a compact subset of X . Now suppose A and B are two elements of such a set. Then, $A \subseteq C$ implies $C \in O_i^-$ for each $i = 1, \dots, k$, and $C \subseteq B$ implies $C \in (X \setminus K)^+$. It follows that the standard basis for the Fell topology on $\mathcal{C}(X)$ consists of $\subseteq$ -convex sets, i.e., $(\mathcal{C}^F(X), \subseteq)$ is locally convex. The second assertion follows from the first. $\square$

There is more one can say about locally compact and connected Hausdorff topological lattices. Let $(X, \preceq)$ be a lattice, and $\langle x_\lambda \rangle$ a net in X . For any $x \in X$, recall that $\langle x_\lambda \rangle$ *order-converges* to x if there exist two nets $\langle y_\lambda \rangle$ and $\langle z_\lambda \rangle$ in X such that (i) $\langle y_\lambda \rangle$ is $\preceq$ -increasing and $\langle z_\lambda \rangle$ is $\preceq$ -decreasing; (ii) $y_\lambda \preceq x_\lambda \preceq z_\lambda$ for every λ ; and (iii) $\bigvee y_\lambda = x = \bigwedge z_\lambda$. Now let $\mathcal{B}$ be the collection of all $\preceq$ -convex subsets U of X such that for any net $\langle x_\lambda \rangle$ in X and any $x \in U$ such that $\langle x_\lambda \rangle$ order-converges to x , the net $\langle x_\lambda \rangle$ is eventually in U . It is readily checked that $\mathcal{B}$ is a basis for a topology on X . Lawson [26] refers to the topology generated by this basis as the *convex-order topology* on X .

Proposition 5.13. *Let $(X, \preceq)$ be a locally compact and connected Hausdorff topological lattice. Then, the topology of X is coarser than the convex-order topology.*

Proof. By Anderson's Theorem, the topology of X has a basis that consists of $\preceq$ -convex sets. Let U be any such set, and take any $x \in U$. Suppose $\langle x_\lambda \rangle$ is a net in X that order-converges to x . Then, by definition, there exist two nets $\langle y_\lambda \rangle$ and $\langle z_\lambda \rangle$ in X with the properties (i), (ii) and (iii) stated above. By (i), $\langle z_\lambda^\downarrow \rangle$ is a decreasing net, so $z_\lambda^\downarrow \rightarrow \bigcap z_\lambda^\downarrow$ in the Fell topology. Since $\bigcap z_\lambda^\downarrow = (\bigwedge z_\lambda)^\downarrow$, (iii) entails that $z_\lambda^\downarrow \rightarrow x^\downarrow$ in the Fell topology, whence $z_\lambda \rightarrow x$ by Theorem 4.11. Similarly, $\langle y_\lambda^\uparrow \rangle$ is a decreasing net, so $y_\lambda^\uparrow \rightarrow \bigcap y_\lambda^\uparrow = (\bigvee y_\lambda)^\uparrow = x^\uparrow$ in the Fell topology. Therefore, applying Theorem 4.11 in the context of the dual lattice $(X, \succeq)$, we find $y_\lambda \rightarrow x$. It follows that both $\langle y_\lambda \rangle$ and $\langle z_\lambda \rangle$ are eventually in U . As U is $\preceq$ -convex, (ii) implies that $\langle x_\lambda \rangle$ is eventually in U . In view of the arbitrary choice of x in U , we conclude that U is open in the convex-order topology. $\square$

In passing, we should note that this is only one-half of the story. In general, it may or may not be possible to topologize a given lattice $(X, \preceq)$ to obtain a locally compact and connected Hausdorff topological lattice. But Lawson [26, Theorem 10] shows that if there is such a topology, then it must be finer than the convex-order topology. Combining this with the proposition above thus establishes Lawson's surprising finding: *There is at most one topology on a lattice that would make it a locally compact and connected Hausdorff topological lattice.*

5.5. On Completely Order-Regular Posets

A topological poset $(X, \preceq)$ is said to be a *completely regular ordered space* if

- (1) for every $x, y \in X$ such that $x \preceq y$ is false, there exists a continuous and order-preserving $f : X \rightarrow \mathbb{R}$ with $f(x) > f(y)$; and
- (2) for every $x \in X$ and a neighborhood V of x in X , there are continuous functions $f : X \rightarrow [0, 1]$ and $g : X \rightarrow [0, 1]$ such that f is order-preserving, g is order-reversing, $f(x) = 1 = g(x)$, and $\min\{f(z), g(z)\} = 0$ for all $z \in X \setminus V$.

This concept, which was introduced by Nachbin [33], generalizes the topological notion of Tychonoff space. (For, $(X, =)$ is a completely regular ordered space iff X is completely regular.) Due to its intimate connection to the theory of order-compactifications, completely regular ordered spaces have received quite a bit of attention in the literature. (See, among others, [9, 10, 12, 33, 38].)

A topological poset $(Y, \preceq)$ is said to be an *order-compactification* of $(X, \preceq)$ if Y is compact, and there exists a topological order-embedding from X into Y whose range is dense in Y . A well-known theorem of Nachbin [33] says that $(X, \preceq)$ admits an order-compactification if, and only if, it is a completely regular ordered space. This fact alone motivates finding conditions for a topological poset to qualify as a completely regular ordered space. In this application, we use our embedding theorem to provide such conditions.

Let $(X, \preceq)$ be a locally compact and connected Hausdorff topological lattice. By Theorem 4.11, $x \mapsto x^\downarrow$ is a homeomorphism from X onto $\mathcal{C}_\downarrow^F(X)$, and of course, it is an order-isomorphism between $(X, \preceq)$ and $(\mathcal{C}_\downarrow(X), \subseteq)$. Furthermore, $\text{cl}(\mathcal{C}_\downarrow(X))$,

where the closure is taken with respect to the Fell topology, is a compact subspace of $\mathcal{C}^F(X)$, because $\mathcal{C}^F(X)$ is itself a compact space. Thus, by Proposition 3.1, $(\text{cl}(\mathcal{C}_\downarrow(X)), \subseteq)$ is a compact topological poset, and as such, it is an order-compactification of $(X, \preceq)$. In view of Nachbin's said compactification theorem, therefore:

Theorem 5.14. *Every locally compact and connected Hausdorff topological lattice is a completely regular ordered space.*

This observation is a companion to a related result of Burgess and Fitzpatrick [10]. To see this, we follow Priestley [36] in referring to a topological poset $(X, \preceq)$ as an *I-space* if $O^\uparrow$ and $O^\downarrow$ are open subsets of X for every open $O \subseteq X$. In this terminology, Corollary 4.5 of [10] entails that *every locally compact and locally convex I-space $(X, \preceq)$ is a completely regular ordered space*. Therefore, in view of Anderson's Theorem, Theorem 5.14 replaces the properties of local convexity and being an *I-space* in the noted result of [10] with connectedness in the context of locally compact Hausdorff topological lattices.

6. Canonical order-embedding of a topological po-group

In this section, we show that one can replace the lattice structure in Theorem 4.9 with an alternative algebraic structure that is compatible with the ordering of the ambient poset.

By a *po-group*, we mean an ordered triple $(X, \cdot, \preceq)$ where $(X, \cdot)$ is a group whose law of composition is written multiplicatively, $(X, \preceq)$ is a poset, and $\preceq$ is translation-invariant relative to $\cdot$, that is, $xz \preceq yz$ and $zx \preceq zy$ for every $x, y, z \in X$ with $x \preceq y$.

We denote the identity of $(X, \cdot)$ by $\mathbf{1}$, and adopt the standard notation for *Minkowski product*. That is, $A \cdot B := \{ab : (a, b) \in A \times B\}$ for any nonempty $A, B \subseteq X$, but as usual, we write $x \cdot A$ instead of $\{x\} \cdot A$, and similarly $A \cdot x := A \cdot \{x\}$, for any $x \in X$. In the context of a po-group $(X, \cdot, \preceq)$, we have $x \preceq y$ iff $xy^{-1} \in \mathbf{1}^\downarrow$ iff $y^{-1}x \in \mathbf{1}^\downarrow$, so the partial order $\preceq$ is entirely determined by the principal ideal $\mathbf{1}^\downarrow$. In particular, $x \cdot \mathbf{1}^\downarrow = x^\downarrow = \mathbf{1}^\downarrow \cdot x$ for any $x \in X$.

For any $x, y \in X$, if $z \preceq x$ and $w \preceq y$, then by translation invariance of $\preceq$ we get $zw \preceq zy \preceq xy$, so $zw \in (xy)^\downarrow$. Thus, $x^\downarrow \cdot y^\downarrow \subseteq (xy)^\downarrow$. Conversely, if $z \preceq xy$, then $w := z(xy)^{-1} \preceq \mathbf{1}$, so

$$z = w(xy) = (wx)y \in x^\downarrow \cdot y \subseteq x^\downarrow \cdot y^\downarrow.$$

Conclusion: $x^\downarrow \cdot y^\downarrow = (xy)^\downarrow$. Using this observation, we find readily that $(\mathcal{C}_\downarrow(X), \cdot)$ is a group where the law of composition is the Minkowski product, the identity is $\mathbf{1}^\downarrow$, and $(x^\downarrow)^{-1} = (x^{-1})^\downarrow$ for each $x \in X$. Besides, if $x^\downarrow \subseteq y^\downarrow$, then $x^\downarrow \cdot z^\downarrow = (x \cdot z)^\downarrow \subseteq (y \cdot z)^\downarrow = y^\downarrow \cdot z^\downarrow$ by translation-invariance of $\preceq$, and similarly, $z^\downarrow \cdot x^\downarrow \subseteq z^\downarrow \cdot y^\downarrow$. It follows that $(\mathcal{C}_\downarrow(X), \cdot, \subseteq)$ is a po-group, and obviously, $x \mapsto x^\downarrow$ is a group isomorphism, as well as an order-isomorphism, from $(X, \cdot, \preceq)$ onto this po-group.

A *topological po-group* is a po-group $(X, \cdot, \preceq)$ such that $(X, \cdot)$ is a topological group and $(X, \preceq)$ is a topological poset. The rich structure of such po-groups allows us to sharpen the continuity theorems of Section 4. In particular, the situation for the continuity of the canonical order-embedding improves markedly in this framework.

Proposition 6.1. *Let $(X, \cdot, \preceq)$ be a topological po-group. Then, the map $x \mapsto x^\perp$ from X into $\mathcal{C}^F(X)$ is continuous.*

Proof. Take any $x \in X$ and any open subset O of X with $x^\perp \in O^-$. Then, $x^\perp \cap O \neq \emptyset$ which means there is a $z \in O$ with $zx^{-1} \preceq \mathbf{1}$. We define $U := xz^{-1} \cdot O$ which is an open neighborhood of x . Clearly, if $y \in U$, then $y = (xz^{-1})a$ for some $a \in O$, so $a = (zx^{-1})y \preceq y$, whence $y^\perp$ intersects O . Thus: $\{y^\perp : y \in U\} \subseteq O^-$.

Next, take any compact subset K of X such that $x^\perp \in (X \setminus K)^+$. Then we have $x^\perp \cap K = \emptyset$, so, since $x^\perp$ is a closed and K is compact, a standard result of the theory of topological groups says that there exists an open neighborhood V of $\mathbf{1}$ in X with $(x^\perp \cdot V) \cap K = \emptyset$. But $(x \cdot V)^\perp \subseteq x^\perp \cdot V$, because if $z \preceq xv$ for some $v \in V$, then for $w := z(xv)^{-1}$, we have $w \preceq \mathbf{1}$, hence $wx \preceq x$, and we thus find $z = w(xv) = (wx)v \in x^\perp \cdot V$. Consequently we have $(x \cdot V)^\perp \cap K = \emptyset$, that is, $\{z^\perp : z \in x \cdot V\} \subseteq (X \setminus K)^+$. $\square$

We next combine Proposition 6.1 with Theorem 4.8 to obtain our third embedding theorem.

Theorem 6.2. *Let $(X, \cdot, \preceq)$ be a locally compact and order-connected topological po-group. Then, the map $x \mapsto x^\perp$ topologically order-embeds $(X, \preceq)$ in $\mathcal{C}^F(X)$.*

A *partially ordered topological linear space* X is a Hausdorff topological (real) linear space X which is endowed with a translation-invariant closed partial order $\preceq$ that satisfies $\lambda x \preceq \lambda y$ for every $x, y \in X$ and $\lambda \geq 0$. Such a space is an Abelian topological po-group relative to the addition operation of the space. Moreover, its order-intervals are convex in the geometric sense, and hence connected. Finally, if it is finite-dimensional, it is locally compact. Thus, as an immediate consequence of Theorem 6.2, we get:

Corollary 6.3. *Let X be a finite-dimensional partially ordered topological linear space. Then the canonical order-embedding is a topological embedding from X into $\mathcal{C}^F(X)$.*

The following are two simple applications of these results.

Example 6.4. Let $(X, \cdot, \preceq)$ be a locally compact, second-countable and order-connected topological po-group. Then, there is a radially convex metric on X that metrizes the topology of X . This is proved exactly as we proved Theorem 5.2 but this time using Theorem 6.2 instead of Theorem 4.9.¹⁰ $\square$

Example 6.5. Let $\text{Sym}(n)$ stand for the set of all $n \times n$ symmetric real matrices. We view this set as a linear space relative to the usual matrix addition and scalar multiplication operations. Recall that a matrix $A \in \text{Sym}(n)$ is said to be *positive semidefinite* if $\langle Ax, x \rangle \geq 0$ for every (real) n -vector x . In turn, the *Loewner order* $\succcurlyeq_L$ is the partial order on $\text{Sym}(n)$ defined by $A \succcurlyeq_L B$ iff $A - B$ is positive semidefinite. Then, $(\text{Sym}(n), \succcurlyeq_L)$ is a finite-dimensional partially ordered topological linear space (where the topology is naturally inherited from the usual topology of $\mathbb{R}^{n \times n}$). By

¹⁰ An interesting question at this junction is if one can choose this metric left-invariant as well as radially convex. We do not know the answer to this at present.

Corollary 6.3, therefore, we may embed this space, both topologically and order-theoretically, in $\mathcal{C}^F(\text{Sym}(n))$.

We should note that $(\text{Sym}(n), \succcurlyeq_L)$ is not an $\wedge$ -semilattice. Indeed, a well-known theorem of Kadison [21] says that $(\text{Sym}(n), \succcurlyeq_L)$ is an antilattice, that is, the greatest lower bound of two symmetric $n \times n$ matrices with respect to the Loewner order exists iff these matrices are $\succcurlyeq_L$ -comparable. Thus, this example is not covered by any of our earlier embedding theorems. $\square$

A natural question at this point is if the rich setting of topological po-groups allows us to delete either the local compactness or the order-connectedness hypotheses from the statement of Theorem 6.2. We conclude with two examples that demonstrate that this is not so, even in the Abelian case.

Example 6.6. Let $\mathbf{0}$ stand for the zero sequence, and let $\{e_1, e_2, \dots\}$ be the standard orthonormal base for the Hilbert space ℓ_2 . We endow ℓ_2 with the coordinatewise order $\preceq$ (that is, $x \preceq y$ iff $x_i \leq y_i$ for each $i = 1, 2, \dots$). Relative to this order and its usual norm-topology, ℓ_2 is a Hausdorff topological lattice. (It is, in fact, a Banach lattice.) As such ℓ_2 has convex order-intervals, so it is order-connected, but of course, it is not locally compact.

We claim that $e_n^\downarrow \xrightarrow{\text{K-P}} \mathbf{0}^\downarrow$ holds. To see this, first, take any $x \in \mathbf{0}^\downarrow$, and note that $x = e_n + (x - e_n) \in e_n + \mathbf{0}^\downarrow = e_n^\downarrow$, so the n th term of the constant sequence $\langle x, x, \dots \rangle$ is in $e_n^\downarrow$ for each $n \in \mathbb{N}$, and this sequence obviously converges to x . Second, let $\langle n_k \rangle$ be a strictly increasing sequence of positive integers, and take any convergent sequence $\langle x_{n_k} \rangle$ with $x_{n_k} \in e_{n_k}^\downarrow$ for each k . We need to show that $x := \lim x_{n_k}$ belongs to $\mathbf{0}^\downarrow$, that is, $x_i \leq 0$ for each $i \in \mathbb{N}$. To this end, fix an arbitrary positive integer i . Whenever $k > i$, we have $n_k > i$, so $x_{n_k, i} \leq e_{n_k, i} = 0$. But $\|x_{n_k} - x\|_2 \rightarrow 0$ implies $x_{n_k, j} - x_j \rightarrow 0$ for each $j \in \mathbb{N}$, so it follows that $x_i \leq 0$. In view of the arbitrary choice of i , we thus find $x \in \mathbf{0}^\downarrow$, as desired.

Conclusion: $e_n^\downarrow \xrightarrow{\text{K-P}} \mathbf{0}^\downarrow$, so $e_n^\downarrow \rightarrow \mathbf{0}^\downarrow$ in the Fell topology. But, of course, $\langle e_n \rangle$ does not converge to $\mathbf{0}$. $\square$

We remark that $(\mathcal{C}_\downarrow^F(\ell_2), +)$ is not a topological group: $x^\downarrow \mapsto (-x)^\downarrow$ is not a continuous self-map on $\mathcal{C}_\downarrow^F(\ell_2)$. Indeed, while $e_n^\downarrow \rightarrow \mathbf{0}^\downarrow$ in the Fell topology, the open ball around $\mathbf{0}$ of radius $\frac{1}{2}$ fails to hit $(-e_n)^\downarrow$ for any $n \in \mathbb{N}$, so $(-e_n)^\downarrow$ does not converge to $\mathbf{0}^\downarrow$ in $\mathcal{C}_\downarrow^F(\ell_2)$. On the other hand, if $(X, +, \preceq)$ is a topological po-group that satisfies the conditions of Theorem 6.2, then it is plain that $(\mathcal{C}_\downarrow^F(X), +, \subseteq)$ is such a po-group as well.

Example 6.7. Let X stand for the set of all square-summable $\mathbb{Z}$ -valued sequences. We view this set as a topological subgroup of ℓ_2 , and endow it with the coordinatewise order $\preceq$. It is plain that all but finitely many terms of any element of X are zero. Thus, the topology that X inherits from ℓ_2 is discrete, and hence, locally compact. However, as the only connected subsets of X are the singletons, $(X, \preceq)$ is not order-connected. And sure enough, the inverse of the order-embedding $x \mapsto x^\downarrow$ is not continuous on $\mathcal{C}_\downarrow^F(X)$. Indeed, we have $e_n^\downarrow \xrightarrow{\text{K-P}} \mathbf{0}^\downarrow$ but $\langle e_n \rangle$ does not converge to $\mathbf{0}$. $\square$

7. An open problem

In this paper we looked at when a given topological poset $(X, \preceq)$ can be embedded in the Fell hyperspace $\mathcal{C}^F(X)$ topologically as well as order-theoretically. While our results identify a fairly rich class of topological posets for which this is possible, in our embedding theorems we worked exclusively with the canonical order-embedding $x \mapsto x^\downarrow$. This is certainly a natural starting point, for, as $x \mapsto x^\downarrow$ is always an order-embedding, it reduces the problem to merely studying the bicontinuity of this map. However, one is likely to obtain other embedding results by considering alternative maps from X into $\mathcal{C}(X)$. (Indeed, relative to Example 4.1, we have observed in a following footnote that an embedding can be found even when $x \mapsto x^\downarrow$ fails to be continuous.) Pursuing this direction, and characterizing those topological posets that can be topologically order-embedded in $\mathcal{C}^F(X)$, is left for future research.

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