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Hamilton-Jacobi Equation for State Constrained Bolza Problems with Discontinuous Time Dependence: a Characterization of the Value Function

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Dedicated to Roger J-B Wets on the occasion of his 85th birthday.

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We consider a class of state constrained Bolza problems in which the integral cost is merely continuous w.r.t. the state variable, and the dynamics and the integral cost are allowed to have a discontinuous behaviour w.r.t. the time variable t in the following sense: although they have everywhere one-sided limits in t, they are required to be continuous only for a.e. t. For this class of problems we establish conditions under which the Value Function is characterized as the unique viscosity solution in the class of lower semicontinuous functions to the associated Hamilton-Jacobi equation. We provide some illustrative examples including a 'growth versus consumption' problem in neo-classical macro-economics, one peculiarity of which is the presence of a fractional singularity w.r.t. the state variable.

Keywords: Value function, Hamilton-Jacobi equation, Bolza problem, state constraints.

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1. Introduction

We consider the following state constrained Bolza problem with the initial data $(t, x) \in [S, T] \times \mathbb{R}^n$:

$$(SC_{t,x}) \begin{cases} \text{Minimize } \displaystyle\int_t^T L(s, x(s), \dot{x}(s))\mathrm{d}s + g(x(T)) \\ \text{over arcs } x(\cdot) \in W^{1,1}([t, T], \mathbb{R}^n) \text{ satisfying} \\ \dot{x}(s) \in F(s, x(s)) \quad \text{for a.e. } s \in [t, T], \\ x(s) \in A \quad \text{for all } s \in [t, T],\ x(t) = x, \end{cases}$$

in which $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $L : [S, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are given functions, $F : [S, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a given multivalued function, and $A \subset \mathbb{R}^n$ is a given non-empty closed set. Consider an interval $I \subset [S, T]$, an *F-trajectory* on I is an absolutely continuous arc $x : I \to \mathbb{R}^n$ which satisfies the reference differential inclusion $\dot{x}(s) \in F(s, x(s))$ for a.e. $s \in I$. We say that an F-trajectory $x \in W^{1,1}(I, \mathbb{R}^n)$ is

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feasible on I if $x(s)\in A$ for all $s\in I$. The *value function* $V\colon [S,T]\times\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ is defined by the infimum cost of $(SC_{t,x})$:

$$V(t,x) := \inf(SC_{t,x}),$$

interpreting '$+\infty$' the cost of F-trajectories which are not feasible (that clearly includes the case when $x \notin A$) and the cost of feasible F-trajectories $x(\cdot)$'s such that $g(x(T)) = +\infty$.

The aim of this paper is to characterize $V(\cdot,\cdot)$ as the unique solution in the class of lower semicontinuous (lsc) functions, in a suitable generalized sense that involves *Fréchet* subdifferentials and superdifferentials (also called 'strict' or 'viscosity' sub/super-differentials), to the Hamilton-Jacobi equation related to $(SC_{t,x})$. The characterization of the Value Function as a generalized solution to the Hamilton-Jacobi equation associated with a reference optimal control problem is a long-standing research topic and a lot of work has been done in this context. Our contribution is in the strand which employs constructive methods via nonsmooth analysis and viability theory techniques (cf. widely-known references as [9, 10, 11, 13, 18], and see also the papers [8, 7] for an up-to-date overview), rather than the tools which are typical in the viscosity solutions theory framework (cf. [3, 4]).

In this paper we shall consider 'everywhere in t' characterizations of the Value Function for state constrained optimal control problems in which F and L may have a discontinuous behaviour w.r.t. the time variable t, and the Lagrangian L is merely continuous w.r.t. the state variable x. The most general known class of time-discontinuous problems, which allows to provide an 'everywhere in t' characterization, was introduced in [8] showing that Value Functions for optimal control Mayer problems ($L = 0$) are generalized solutions to the Hamilton-Jacobi equation, expressed in terms of the lower Dini derivatives and of proximal subdifferentials. In this class of problems the time-dependent data are time-discontinuous in the following sense: they have everywhere one-sided limits in t and are continuous on the complement of a zero-measure subset of $[S,T]$.

Extensions of the results in [8] (characterizing the value function as lower Dini derivative type solution and as proximal solution to the Hamilton-Jacobi equation) for Bolza problems with Lagrangians that are just continuous w.r.t. x (for which simple state augmentation techniques would not allow to employ results valid for Mayer problems [8]) have been provided in two recent papers: [6] for the state constraint free case, and [7] for the case with state constraints.

In the state constraint-free case, when the data F, L and g satisfy particular continuity properties (which yield also that the Value Function V is continuous on $[S,T]\times\mathbb{R}^n$), it is well known that, using constructive methods, V can be characterized also (in addition to lower Dini and proximal solutions characterizations) in terms of the Fréchet subdifferentials and superdifferentials, cf. [11, 10, 9]. When the data F, L and g are possibly discontinuous and satisfy the assumptions of interest in this paper (allowing g to be lsc, F and L to have one-sided limits in t, and L to be just continuous in x), and the value function is only lsc an 'equivalent' (lower Dini/proximal/viscosity solutions) characterization has been obtained in [6]. Passing to the state constrained case, a characterization involving Fréchet sub/super-differentials was proved in [12, Theorem 3] in the class of continuous functions (as set of candidate solutions), but still when the data are continuous, F is Lipschitz w.r.t x, and imposing a stronger version of the 'standard inward pointing' conditions (cf. (IPC) below for a 'standard inward pointing' condition). Weakening these continuity assumptions on the data (and imposing a strong version of the inward pointing condition) only partial results are known providing, for instance, just one-side comparison theorems, see [12].

Therefore natural questions are:

(Q1) Is that possible to provide an 'everywhere in t' characterization of the Value Function $V(\cdot,\cdot)$ in terms of the Fréchet subdifferentials and superdifferentials for state constrained optimal control problems in which L has a merely continuous behaviour in x, and F and L may have time-discontinuous behaviour in the sense of [8] (i.e., they have everywhere left and right limits, but are allowed to have time discontinuities on a zero-measure set)?

(Q2) And what happens if we just assume constraint qualifications including ('standard') conditions such as (IPC) below, rather than imposing stronger versions of them?

In this paper, assuming constraint qualifications which comprise the 'standard' ones (such as the inward and outward pointing conditions below (IPC) and (OPC) discussed in Section 1) we provide positive answers to questions (Q1) and (Q2) (see Theorems 2.2, 2.3 and 2.4) for problems involving respectively a locally bounded lsc final cost (coupled with an 'inward/outward pointing' constraint qualification), and a continuous final cost (associated with a mere 'inward pointing' constraint qualification).

We recall that, when the data of the problem are continuous, asymptotic subdifferentials can be removed to characterize the value function as generalized solution to the Hamilton-Jacobi equation (cf. [11, 10, 18]). This significant simplification can be obtained involving a suitable approximation technique (based on the Rockafellar Horizontal Approximation Theorem) of normals to an epigraph set by non-horizontal normals. In [7] it has been shown that this approximation technique is still applicable even in presence of discontinuous time dependent data (in the sense specified above) for the lower Dini and proximal solutions; we show that the idea used in [7] is applicable also in our case and allows us to obtain the same simplification for solution expressed in terms of Fréchet sub/super-subdifferentials.

We provide two examples to show the crucial role, in our results, of the constraint qualifications and of the regularity of the final cost g; moreover, as an illustrative example of our theory we consider an economics model in which the integral cost is merely continuous w.r.t. the state variable x: this is due to an inherent fractional singularity term which is introduced to interpret the production function (cf. [1]).

Terminology and notation: In this paper we write $\mathbb{B}$ the closed unit ball in $\mathbb{R}^n$. We denote the Lebesgue subsets of $[S,T]$ and the Borel subsets of $\mathbb{R}^n$ by $\mathcal{L}$ and $\mathcal{B}^n$ respectively. The (associated) product σ-algebra of sets in $[S,T]\times\mathbb{R}^n$ is written $\mathcal{L}\times\mathcal{B}^n$. For a (nontrivial) subinterval $[t_0,t_1]\subset[S,T]$ we denote by $\mathbb{L}^p([t_0,t_1],\mathbb{R}^n)$ the space of $\mathbb{L}^p$ functions for the Lebesgue measure, that are defined on $[t_0,t_1]$ and take values in $\mathbb{R}^n$, endowed with the usual norm $\|f\|_{\mathbb{L}^p(t_0,t_1)}$.

We write $W^{1,1}([t_0,t_1],\mathbb{R}^n)$ the space of absolutely continuous functions (for the Lebesgue measure) endowed with the norm:

$$\|f\|_{W^{1,1}(t_0,t_1)} := |f(t_0)| + \|\dot{f}\|_{\mathbb{L}^p(t_0,t_1)}, \qquad \text{for all } f \in W^{1,1}([t_0,t_1],\mathbb{R}^n).$$

Let $D \subset \mathbb{R}^m$, we denote by co$D$ the *convex hull* of D; the *indicator function* of D is written $\chi_D : \mathbb{R}^m \to [0,1]$ (this is the function taking value 1 on D and 0 elsewhere). Take a closed set $C \subset \mathbb{R}^m$ and $x \in \mathbb{R}^m$. Then $\mathrm{d}_C(x) := \min_{y\in C}\{|x-y|\}$ is the *distance* of x from the set C. If $f\colon C \subset \mathbb{R}^m \to \mathbb{R}$ is a locally bounded function, we denote its *lower (resp. upper) semicontinous envelope* by:

$$f_*(x) := \liminf_{y \xrightarrow{C} x} f(y) \quad \Big(\text{resp. } f^*(x) := \limsup_{y \xrightarrow{C} x} f(y)\Big), \text{ for every } x \in C.$$

The notation $y \xrightarrow{C} x$ means that we are considering convergent sequences $\{y_i\}_{i\in\mathbb{N}}$ such that $y_i \to x$, and each element y_i belongs to C. An increasing function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a modulus of continuity if $\lim_{s\to 0}\omega(s) = 0$. For arbitrary nonempty closed sets in $\mathbb{R}^m$, C' and C, we denote by $\mathrm{d}_H(C,C')$ the 'Hausdorff distance' between C and C':

$$\mathrm{d}_H(C,C') := \inf\{\beta > 0 \,|\, C' \subset C + \beta\mathbb{B}\} \vee \inf\{\beta > 0 \,|\, C \subset C' + \beta\mathbb{B}\}\,.$$

We shall use several constructs of nonsmooth analysis (detailed dissertations of which can be found in the monographs [2, 10, 9, 16, 18]). Consider a set $D \subset \mathbb{R}^m$, a point $x \in \overline{D}$ and a multifunction $G : D \rightsquigarrow \mathbb{R}^m$. The lim inf and the lim sup of G at x along D (in the Kuratowski sense) are the sets

$$\begin{aligned}
\liminf_{y \xrightarrow{D} x} G(y) &:= \Big\{ v \in \mathbb{R}^m \mid \limsup_{y \xrightarrow{D} x}\, d_{G(y)}(v) = 0 \Big\}, \\
\limsup_{y \xrightarrow{D} x} G(y) &:= \Big\{ v \in \mathbb{R}^m \mid \liminf_{y \xrightarrow{D} x}\, d_{G(y)}(v) = 0 \Big\}.
\end{aligned}$$

The limit of G at x along D (in the Kuratowski sense) exists if the lim inf and the lim sup above are equal. The *Bouligand tangent cone* (alternatively referred to as contingent cone) $T_C(x)$ to a closed set $C \subset \mathbb{R}^m$ at $x \in C$ is defined by:

$$T_C(x) := \left\{ v \in \mathbb{R}^m \mid \liminf_{h\to 0^+} \frac{\mathrm{d}_C(x+hv)}{h} = 0 \right\} = \limsup_{h\to 0^+} \frac{C-x}{h}.$$

The *proximal normal cone* $N_C^P(x)$ and the *strict normal cone* $\hat{N}_C(x)$ to C at x are defined by:

$$N_C^P(x) := \{\xi \in \mathbb{R}^m \,|\, \exists M \geq 0 \quad \text{s.t.} \quad \xi\cdot(y-x) \leq M|y-x|^2,\ \forall y \in C\},$$

and $$\hat{N}_C(x) := \Big\{ \xi \in \mathbb{R}^m \mid \limsup_{y \xrightarrow{C} x} |y-x|^{-1}\, \xi\cdot(y-x) \leq 0 \Big\}.$$

Well known properties are: $\hat{N}_C(x) = \{\xi \in \mathbb{R}^m \,|\, \xi\cdot v \leq 0, \forall v \in T_C(x)\}$ (i.e. $\hat{N}_C(x)$ is the polar cone to $T_C(x)$) and

$$N_C^P(x) \subset \hat{N}_C(x). \tag{1}$$

Consider an extended valued function $\varphi : \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$. We write as usual $\operatorname{dom}(\varphi) := \{x \in \mathbb{R}^m \,|\, \varphi(x) \neq \pm\infty\}$, $\operatorname{epi}\varphi := \{(x,r) \in \mathbb{R}^{m+1} \,|\, r \geq \varphi(x)\}$, and $\operatorname{hyp}\varphi := \{(x,r) \in \mathbb{R}^{m+1} \,|\, r \leq \varphi(x)\}$. Take a lower semicontinuous (lsc) function $\varphi : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ and a point $\bar{x} \in \operatorname{dom}\varphi$. The *proximal subdifferential* of φ at $\bar{x} \in \operatorname{dom}\varphi$ is defined by

$$\partial^P \varphi(\bar{x}) := \{\xi \,|\, (\xi, -1) \in N^P_{\operatorname{epi}\varphi}(\bar{x}, \varphi(\bar{x}))\}.$$

The *Fréchet subdifferential* (also called *strict subdifferential*) of φ at $\bar{x} \in \operatorname{dom}\varphi$ is defined by

$$\partial_- \varphi(\bar{x}) := \{\xi \,|\, (\xi, -1) \in \hat{N}_{\operatorname{epi}\varphi}(\bar{x}, \varphi(\bar{x}))\}.$$

We recall also that, if $\varphi : \mathbb{R}^m \to \mathbb{R} \cup \{-\infty\}$ is an upper semicontinuous function and $\bar{x} \in \operatorname{dom}\varphi$, then the *Fréchet superdifferential* of φ at $\bar{x}$ is defined as $\partial_+\varphi(\bar{x}) := -\partial_-(-\varphi)(\bar{x})$.

2. Characterizations of the Value Function for Bolza problems

2.1. Hypotheses

In this paper we shall invoke the following hypotheses.

(H1): $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is lsc, $F : [S,T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ takes closed, convex, non-empty values, $F(\cdot, x)$ is $\mathcal{L}(S,T)$-measurable for all $x \in \mathbb{R}^n$, the function $L\colon [S,T] \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is $\mathcal{L} \times \mathcal{B}^{n+n}$-measurable and $L(t,x,\cdot)$ is convex for every $t \in [S,T]$ and $x \in \mathbb{R}^n$,

(H2): (i) there exists $c_F \in \mathbb{L}^1(S,T)$ such that
$F(t,x) \subset c_F(t)(1+|x|)\, \mathbb{B}$ for all $x \in \mathbb{R}^n$ and for a.e. $t \in [S,T]$,
(ii) for every $R_0 > 0$, there exists $c_0 > 0$ such that
$F(t,x) \subset c_0\, \mathbb{B}$ for all $(t,x) \in [S,T] \times R_0\mathbb{B}$,

(H3): for every $R_0 > 0$, there exist a modulus of continuity $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ and $k_F \in \mathbb{L}^1(S,T)$ such that
$d_H(F(t,x'), F(t,x)) \leq \omega(|x - x'|)$ for all $x, x' \in R_0\mathbb{B}$ and $t \in [S,T]$, and
$F(t,x') \subset F(t,x) + k_F(t)|x - x'|\, \mathbb{B}$ for all $x, x' \in R_0\mathbb{B}$ and a.e. $t \in [S,T]$,

(H4): for every $R_0 > 0$, $F(\cdot, x)$ has bounded variation uniformly over $x \in R_0\mathbb{B}$, in the sense
$d_H(F(s,x), F(t,x)) \leq \eta(t) - \eta(s)$, for every $[s,t] \subset [S,T]$ and $x \in R_0\mathbb{B}$
for some non-decreasing function of bounded variation $\eta : [S,T] \to [0,\infty)$,

(H5): there exists a set $\mathcal{O} \subset [S,T]$ of full Lebesgue measure such that, for all $(x,v) \in \mathbb{R}^n \times \mathbb{R}^n$, $t \mapsto L(t,x,v)$ has a left (right) limit at all points $t \in (S,T]$ ($t \in [S,T)$, respectively) and $L(t^-,x,v) = L(t^+,x,v) = L(t,x,v)$ at all points $t \in \mathcal{O}$. Furthermore, for each $R_0 > 0$, there exist $c_L \geq c_0$ (c_0 is the positive constant appearing in condition (H2)(ii)), $M_L > 0$ and a modulus of continuity ω_L such that
$|L(t,x,v)| \leq M_L$, for all $(t,x,v) \in [S,T] \times R_0\mathbb{B} \times 2c_L\mathbb{B}$, and
$|L(t,x',v) - L(t,x,v)| \leq \omega_L(|x - x'|)$, for all $x, x' \in R_0\mathbb{B}$, $t \in [S,T]$ and $v \in c_L\mathbb{B}$.

Remark 2.1. It may be deduced from the hypotheses that, for each $s \in [S,T)$ and $t \in (S,T]$, and for all $x \in \mathbb{R}^n$, the following limits (in the sense of Kuratowski) exist and are nonempty

$$\lim_{s'\downarrow s,\, x\to x} F(s',x') = \lim_{s'\downarrow s} F(s',x') \ (=: F(s^+,x))$$

and

$$\lim_{t'\uparrow t,\, x'\to x} F(t',x') = \lim_{t'\uparrow t} F(t',x') \ (=: F(t^-,x)),$$

and for almost every $s \in [S,T)$ and $t \in (S,T]$ we have

$$F(s^+,x) = F(s,x) \ \text{ and } \ F(t^-,x) = F(t,x), \quad \text{for all } x \in \mathbb{R}^n.$$

We observe also that, even if assumptions (H2)(i) and (H4) imply the validity of condition (H2)(ii), we prefer to write it explicitly since it makes the analysis clearer: indeed the positive constant c_0 of (H2)(ii) has a role also in hypothesis (H5). □

We shall establish a characterization of the Value Function as a 'viscosity' solution to the Hamilton Jacobi equation under 'constraint qualification' hypotheses (relative to the reference differential inclusion and the state constraint set), requiring that an arbitrary state trajectory can be approximated by feasible state trajectories which lie in the interior of the state constraint set, except possibly either a right or a left endpoint. We consider two alternative hypotheses of this nature (the 'backward' and 'forward' versions), which depend on whether the approximating state trajectory satisfies a right or left endpoint boundary condition. Since in our paper the Lagrangian L is merely continuous w.r.t. the state variable x, it is necessary to have at hand a $W^{1,1}$ 'distance estimate' which has a continuous behaviour w.r.t. a parameter which quantifies the F-trajectories 'constraint violation'. We shall give also explicit geometric conditions concerning the interaction between the velocity set $F(t,x)$ and the state constraint set under which these hypotheses are satisfied (see (IPC), (OPC) and Theorem 2.4 below). (Notice that limiting attention to the case $L = 0$, i.e. Mayer problems, a simpler $\mathbb{L}^\infty$ distance estimate result would be enough to provide an 'everywhere in t' characterization of Value Function as generalized solution to the Hamilton-Jacobi equation, see [8], and the discussion in [7]).

$(CQ)_{BW}$: For any $r_0 > 0$, there exists a modulus of continuity $\tilde{\theta}(\cdot)$, such that given any interval $[t_0,t_1] \subset [S,T]$, any F-trajectory $\hat{x}(\cdot)$ on $[t_0,t_1]$ with $\hat{x}(t_1) \in A \cap \left(e^{\int_{t_1}^{T} c_F(t)\mathrm{d}t}(r_0+1) - 1\right)\mathbb{B}$, and any $\rho > 0$ such that we have $\rho \geq \max\{d_A(\hat{x}(t)) \,|\, t \in [t_0,t_1]\}$, we can find an F-trajectory $x(\cdot)$ on $[t_0,t_1]$ such that $x(t_1) = \hat{x}(t_1)$, $x(t) \in \operatorname{int} A$, for all $t \in [t_0,t_1)$ and

$$\|\hat{x} - x\|_{W^{1,1}(t_0,t_1)} \leq \tilde{\theta}(\rho). \tag{2}$$

$(CQ)_{FW}$: For any $r_0 > 0$, there exists a modulus of continuity $\theta(\cdot)$, such that given any interval $[t_0,t_1] \subset [S,T]$, any F-trajectory $\hat{x}(\cdot)$ on $[t_0,t_1]$ with $\hat{x}(t_0) \in A \cap \left(e^{\int_{S}^{t_0} c_F(t)\mathrm{d}t}(r_0+1) - 1\right)\mathbb{B}$, and any $\rho > 0$ such that we have $\rho \geq \max\{d_A(\hat{x}(t)) \,|\, t \in [t_0,t_1]\}$, we can find an F-trajectory $x(\cdot)$ on $[t_0,t_1]$ such that $x(t_0) = \hat{x}(t_0)$, $x(t) \in \operatorname{int} A$ for all $t \in (t_0,t_1]$ and

$$\|\hat{x} - x\|_{W^{1,1}(t_0,t_1)} \leq \theta(\rho). \tag{3}$$

2.2. Main results

We are now ready to state our main theorems the proofs of which are provided in Sections 4 and 5.

Theorem 2.2. (Characterization of locally bounded lsc value functions – 'Forward/ Backward Constraint Qualification') *Assume* (H1)–(H5), $(CQ)_{BW}$ *and* $(CQ)_{FW}$. *Suppose, in addition, that* $g_{|A}$ *is locally bounded and satisfies* $((g_{|A})^*)_* = g_{|A}$. *Let* $U\colon [S,T]\times\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ *be an extended valued function. Then assertions* (a) *and* (b) *below are equivalent:*

(a) *U is the value function for $(SC_{t,x})$, i.e. $U = V$.*

(b) *U is lsc on $[S,T]\times\mathbb{R}^n$ and locally bounded on $[S,T]\times A$, satisfies $U(t,x) = +\infty$ whenever $x \notin A$ and*

(i) *for all $(t,x) \in (S,T)\times A$, $(\xi^0,\xi^1) \in \partial_- U(t,x)$*
$$\xi^0 + \inf_{v\in F(t^+,x)} \left[\xi^1\cdot v + L(t^+,x,v)\right] \le 0\,; \tag{4}$$

(ii) *for all $(t,x) \in (S,T)\times \operatorname{int} A$, $(\xi^0,\xi^1) \in \partial_+ U^*(t,x)$*
$$\xi^0 + \inf_{v\in F(t^+,x)} \left[\xi^1\cdot v + L(t^+,x,v)\right] \ge 0\,; \tag{5}$$

(iii) *for all $x\in A$* $\displaystyle\liminf_{\{(t',x')\to(S,x)\ \mid\ t'>S\}} U(t',x') = U(S,x),$
$$(U_{|[S,T]\times A})^*(T,x) = (g_{|A})^*(x) \quad \textit{and} \quad U(T,x) = g(x).$$

Theorem 2.3. (Characterization of continuous Value Functions – 'Forward Constraint Qualification') *Assume* (H1)–(H5) *and* $(CQ)_{FW}$. *Suppose, in addition, that g is continuous on A. Let $U\colon [S,T]\times\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be an extended valued function. Then assertions (a) and (b)′ below are equivalent:*

(a) *U is the value function for $(SC_{t,x})$, i.e. $U = V$.*

(b)′ *U is continuous on $[S,T]\times A$, satisfies $U(t,x) = +\infty$ whenever $x \notin A$ and*

(i) *for all $(t,x) \in (S,T)\times A$, $(\xi^0,\xi^1) \in \partial_- U(t,x)$*
$$\xi^0 + \inf_{v\in F(t^+,x)} \left[\xi^1\cdot v + L(t^+,x,v)\right] \le 0\,; \tag{6}$$

(ii) *for all $(t,x) \in (S,T)\times \operatorname{int} A$, $(\xi^0,\xi^1) \in \partial_+ U(t,x)$*
$$\xi^0 + \inf_{v\in F(t^+,x)} \left[\xi^1\cdot v + L(t^+,x,v)\right] \ge 0\,; \tag{7}$$

(iii) *for all $x\in A$* $\displaystyle\liminf_{\{(t',x')\to(S,x)\ \mid\ t'>S\}} U(t',x') = U(S,x)$, *and* $U(T,x) = g(x)$.

Geometric Conditions for $(CQ)_{BW}$ and $(CQ)_{FW}$

The two characterizations of the value function are valid considering the 'implicit' constraint qualifications $(CQ)_{BW}$ and $(CQ)_{FW}$, which, in general, are not satisfied or could be difficult to verify. We provide here simple geometric conditions on the interaction of the velocity sets $F(t,x)$ with the state constraint A, referred to as 'outward' and 'inward' pointing conditions ((OPC) and (IPC)) respectively, that yield the validity of $(CQ)_{BW}$ and $(CQ)_{FW}$; it is possible to do so however, only when we impose additional hypotheses on the data.

(OPC): for each $s \in [S,T)$, $t \in (S,T]$ and $x \in \partial A$,

$$F(t^-,x) \cap \big(-\operatorname{int} T_A(x)\big) \neq \emptyset \quad \text{and} \quad F(s^+,x) \cap \big(-\operatorname{int} T_A(x)\big) \neq \emptyset\,;$$

(IPC): for each $s \in [S,T)$, $t \in (S,T]$ and $x \in \partial A$,

$$F(t^-,x) \cap \operatorname{int} T_A(x) \neq \emptyset \quad \text{and} \quad F(s^+,x) \cap \operatorname{int} T_A(x) \neq \emptyset\,.$$

Theorem 2.4. *Take a function $U\colon [S,T]\times\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$. Assume that the data for problem (P) satisfy hypotheses* (H1)–(H5). *Assume any one of the supplementary hypotheses are also satisfied:*

(Supp)$_1$: *A is convex, k_F in* (H3) *is constant and the bounded variation function η in* (H4) *can be chosen such that for every $\mu > 0$ and every $[t_0,t_1] \subset [S,T]$ there exists a partition $\{t_0 =: \tilde t_0 < \tilde t_1 < \tilde t_2 < \ldots < \tilde t_M := t_1\}$ such that for each $k = 0,1,\ldots,M-1$ we have*

$$\lim_{\epsilon\downarrow 0} \int_{\tilde t_k+\epsilon}^{\tilde t_{k+1}} \frac{\eta(\tau)-\eta(\tilde t_k^+)}{\tau-\tilde t_k} d\tau \le \mu$$

(this condition is satisfied in particular when $\eta(\cdot)-\eta(\tilde t_k^+)$ has linear growth over $(t_k,t_{k+1}]$ uniformly over index values k of the partition),

(Supp)$_2$: *A has a C^1 boundary,*

(Supp)$_3$: *The running cost L is x-Lipschitz in the sense that the following strengthened form of* (H5) *is satisfied: for every $R_0 > 0$ and $c_L > 0$, there exist a modulus of continuity $\omega_L : \mathbb{R}^+ \to \mathbb{R}^+$, $k_L \in L^1(S,T)$ and a function of bounded variation $\eta_L : [S,T] \to [0,\infty)$ such that*

$$|L(t,x,v) - L(t,x',v)| \le \omega_L(|x-x'|),$$

for all $x,x' \in R_0\mathbb{B}$, $v \in c_L\mathbb{B}$ and $t \in [S,T]$,

$$|L(t,x,v) - L(t,x',v)| \le k_L(t)|x-x'|,$$

for all $x,x' \in R_0\mathbb{B}$, $v \in c_L\mathbb{B}$ and a.e. $t \in [S,T]$, and

$$|L(s,x,v) - L(t,x,v)| \le \eta_L(t) - \eta_L(s),$$

for every $[s,t] \subset [S,T]$, $x \in R_0\mathbb{B}$ and $v \in c_L\mathbb{B}$.

Then

(A): *If* (IPC) *and* (OPC) *are both satisfied, then conditions* (a) *and* (b) *of Theorem* 2.2 *are equivalent*

and

(B): *If* (IPC) *is satisfied and g is continuous, then conditions* (a) *and* (b)$'$ *of Theorem* 2.3 *are equivalent.*

The proof of Theorem 2.4 can be easily deduced following the proof of [7, Theorem 2.1], so we omit it here. We observe that the additional hypothesis (Supp)$_1$ can be taken into account for the characterization of the value function owing to a recent $W^{1,1}$ distance estimate result [7, Theorem 6.1].

Remark 2.5. (1) The notion of generalized solution used in Theorem 2.2 involves (for possibly discontinuous Value Functions) the concepts of lower and upper semicontinuous envelopes. This appears to be a reasonable feature bearing in mind that a well-known approach in viscosity solutions theory suggests, in presence of a locally bounded candidate V to be a solution to an Hamilton-Jacobi equation, to consider its lower and upper semicontinuous envelopes and check whether the properties of being supersolution and subsolution in the viscosity sense are satisfied (cf. [4, 3]). We emphasize the fact that the right coupling between 'regularity of g' and 'constraint qualifications' plays a crucial role in these results and is far to be merely a matter of adding technical assumptions, as highlighted by Examples 3.1 and 3.2 below. If $g_{|A}$ is just lsc, for instance, an incomplete constraint qualification would not yield the desired characterization since the uniqueness cannot be guaranteed; this aspect is clearly illustrated by Example 3.2.

(2) Taking into account the results obtained in [7], under the assumptions of Theorems 2.2, 2.3 and 2.4 the value function is characterized in an equivalent way in terms of lower Dini derivatives, proximal subdifferentials and Fréchet sub/super-differentials.

(3) For the class of optimal control problems considered in our paper the limits $F(t^+, x)$ and $L(t^+, x, v)$ in the value characterization (b) and (b)$'$ of Theorems 2.2 and 2.3 play a key role: for instance they cannot be exchanged with limits $F(t^-, x)$ and $L(t^-, x, v)$ (see the discussion in [6]; similar conclusions can be drawn also for the Dini and proximal solutions, see [8]). □

3. Examples

From Proposition 5.1 below (see Section 5), we know that if conditions (H1)–(H5) and $(CQ)_{FW}$ alone are in force (without $(CQ)_{BW}$ and condition '$((g_{|A})^*)_* = g_{|A}$') then the value function does satisfy property (b) of Theorem 2.2. We give two illustrative examples, which show that the right coupling between the 'regularity of g' and the 'constraint qualification' ($(CQ)_{FW}$ or $(CQ)_{BW}$) plays a crucial role to obtain the characterizations provided by Theorems 2.2 and 2.3 (and Theorem 2.4). In particular, we point out the following important facts: the continuity of g cannot be dropped in Theorem 2.3; the validity of both $(CQ)_{FW}$ and $(CQ)_{BW}$ is required for Theorem 2.2.

Example 3.1. Consider the case in which $n = 1$, $[S, T] = [0, 1]$, $F(t, x) \equiv [0, 1]$, $A = \{x \in \mathbb{R}, x \geq 0\}$, $L = 0$, and

$$g(x) = \begin{cases} -x - 2, & \text{if } x \leq 0 \\ -x, & \text{if } x > 0 \end{cases}$$

Observe that all the assumptions in Theorem 2.2 are satisfied except $(CQ)_{BW}$, because for any $[t_0, t_1] \subset [0, 1]$, any feasible F-trajectory $x(\cdot)$ such that $x(t_1) = 0$ cannot satisfy the condition $x(t) > 0$ for all $t \in [t_0, t_1[$. On the other hand, $(CQ)_{FW}$ is valid since $\partial A = \{0\}$ and $T_A(0) = \mathbb{R}_+$ and, so, (IPC) is satisfied (which guarantees that $(CQ)_{FW}$ is in force, cf. Theorem 2.4). The value function is then, for all $(t, x) \in [0, 1] \times \mathbb{R}$

$$V(t,x) = \begin{cases} t-x-1, & \text{if} \quad x>0, \\ -2, & \text{if} \quad x=0, \\ +\infty, & \text{if} \quad x<0. \end{cases}$$

Clearly the value function V is not continuous, so it cannot satisfy all the conditions appearing in (b)$'$ of Theorem 2.3. But this example tells us that removing the continuity assumption on $g_{|A}$ raises a key issue: even if we consider the value function in the larger set of lsc functions, the uniqueness property would not be satisfied in general. Indeed, V satisfies (b) of Theorem 2.2 but it not the unique: the following function W ($\neq V$) satisfies condition (b) as well

$$W(t,x) := \begin{cases} t-x-1, & \text{if} \quad x>0, \\ -3/2, & \text{if} \quad x=0 \text{ and } t\neq 1, \\ -2, & \text{if} \quad (t,x)=(1,0), \\ +\infty, & \text{if} \quad x<0. \end{cases}$$

The data of Example 3.1 do not satisfy all the hypotheses of Theorem 2.2 in two respects: both hypothesis $(CQ)_{BW}$ and condition '$((g_{|A})^*)_* = g_{|A}$' are not valid. It is well-known that condition '$((g_{|A})^*)_* = g_{|A}$' is crucial for the uniqueness of the solution (in terms of (b)) even for the state constraint-free case (see [6]). The following example shows that, even if '$((g_{|A})^*)_* = g_{|A}$' is satisfied, the absence of one constraint qualification ($(CQ)_{BW}$ in this case) might compromise the characterization of the value function in the sense of (b) of Theorem 2.2.

Example 3.2. Take $n=1$, $[S,T]=[0,1]$, $A=\{x\in\mathbb{R}, x\geq 0\}$, $L=0$,

$$F(t) = \begin{cases} [0,1], & \text{if } t\leq \frac{1}{2} \\ [1/2,1], & \text{if } t>\frac{1}{2} \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 0, & \text{if } x\leq \frac{1}{4} \\ 1, & \text{if } x>\frac{1}{4}. \end{cases}$$

Then for all $(t,x)\in[0,1]\times\mathbb{R}$, the Value Function is

$$V(t,x) = \begin{cases} 0, & \text{if} \quad x=0 \text{ and } t\leq\frac{1}{2}, \\ 1, & \text{if} \quad x>0 \text{ and } t\leq\frac{1}{2}, \\ 0, & \text{if} \quad 0\leq x\leq \frac{t}{2}-\frac{1}{4} \text{ and } t>\frac{1}{2}, \\ 1, & \text{if} \quad x>\frac{t}{2}-\frac{1}{4} \text{ and } t>\frac{1}{2}, \\ +\infty, & \text{if} \quad x<0. \end{cases}$$

We observe that all the assumptions in Theorem 2.2 are satisfied except $(CQ)_{BW}$, and the Value Function V satisfies (b) of Theorem 2.2. However, also the following function satisfies (b)

$$W(t,x) = \begin{cases} \frac{1}{2}, & \text{if} \quad x=0 \text{ and } t<\frac{1}{2}, \\ 1, & \text{if} \quad x>0 \text{ and } t<\frac{1}{2}, \\ 0, & \text{if} \quad x\leq \frac{t}{2}-\frac{1}{4} \text{ and } t\geq\frac{1}{2}, \\ 1, & \text{if} \quad x>\frac{t}{2}-\frac{1}{4} \text{ and } t\geq\frac{1}{2}, \\ +\infty, & \text{if} \quad x<0. \end{cases}$$

Consequently, V is not the unique solution to the Hamilton-Jacobi equation in the sense of (b) in Theorem 2.2.

Economics example

Consider the following 'growth versus consumption' problem coming from the neo-classical macro-economics:

$$\text{(GC)} \qquad \begin{cases} \text{Maximize } \int_0^T (1-u(t))x^\alpha(t)\mathrm{d}t \ \text{ subject to} \\ \dot{x}(t) = -ax(t) + bu(t)x^\alpha(t) \quad \text{for a.e. } t \in [0,T], \\ u(t) \in [0,1] \quad \text{for a.e. } t \in [0,T]\,, \\ x(t) \geq 0 \text{ for all } t \in [0,T]\,, \quad x(0) = x_0\,. \end{cases}$$

in which the data are the constants $a > 0$, $b > 0$, $T > 0$ and $\alpha \in (0,1)$; $x_0 \geq 0$ is a given initial point.

In this model x denotes the (aggregated) economic output; whereas the (normalized) rate of financial return $r(x)$ from economic output x is represented by the production function $r(x) = x^\alpha$. The problem consists in finding the proportion u of rate of return for investment and for expenditure, over a given time horizon $[0,T]$, to maximize total expenditure over the time horizon $[0,T]$. The cost function and underlying dynamic model in (GC) provides a (finite horizon) example of a class of utility and macro economic growth models studied in Solow's classic paper [17], [1]. The dynamic model was revisited in subsequent decades, most notably as the Ramsey Cass Koopmans growth model [5] which incorporates a more precise description of savings behavior.

From a variational analysis point of view, an unusual feature of this optimal control problem is the presence of a fractional singularity introduced by the production function $r(x) = x^\alpha$, with $0 < \alpha < 1$. This is not an artificial construct; a singularity is inherent to such problems, where, typically, the production function is required to satisfy the Inanda conditions [1], which include an 'infinite slope' condition at the origin.

A standard approach in the economics literature to investigate solutions to growth model optimal control problems such as (GC) makes use of the Pontryagin Maximum Principle (cf. [1]), which provides only necessary optimality conditions for (GC). The optimality of the extremals (obtained employing the necessary conditions) can be ensured using 'verification' results. In a recent paper, for the 'soft' state constraint '$x(t) > 0$' (with initial data $x_0 > 0$) Miao and Vinter [15] completely solved problem (GC) applying a non-standard verification technique. When we consider the (full) state constrained problem (GC) (i.e. with '$x(t) \geq 0$'), our theory tells us that this procedure will be successful, if we can show that $\tilde{W} := \tilde{V} \circ (Id, \psi^{-1})$ is a generalized solution to the Hamilton Jacobi equation, expressed in terms of the (Fréchet/viscosity) sub/super-differentials (in the sense of the Proposition 3.3 below). Here, $\tilde{V}$ is the 'extremal cost function' (i.e. the cost of each extremal, parameterized by its initial data) and $\psi \colon [0,\infty) \to [0,\infty)$ is the mapping defined by

$$\psi(x) := x^{1-\alpha} \ \text{ for } x \in [0,\infty)\,. \tag{8}$$

Observe that our theory cannot be directly applied to problem (GC), but it can be shown that making use of the mapping ψ we arrive at a transformed optimal control problem (see the analysis used in the proof of [7, Proposition 3.1]):

$$\text{(GC)'}\qquad \begin{cases} \text{Maximize } \int_0^T (1-u(t))y^{\frac{\alpha}{1-\alpha}}(t)\mathrm{d}t \ \text{ subject to} \\ \dot{y}(t) = -a(1-\alpha)y(t) + (1-\alpha)bu(t) \text{ for a.e. } t \in [0,T],\ y(t) \geq 0\,, \\ u(t) \in [0,1] \quad \text{for a.e. } t. \in [0,T]\,, \quad y(0) = y_0\,, \end{cases}$$

in which $y_0 \geq 0$. The link between the Value Functions of (GC) and (GC)$'$ is given by the relation:

$$V(t,x) = (W \circ (Id, \psi))\,(t,x), \quad \text{for all } (t,x) \in [0,T] \times [0,\infty)\,.$$

Observe now that the 'maximization' problem (GC)$'$ can be easily modified to a 'minimization' problem, the data of which satisfy all the hypotheses of Theorem 2.3. We deduce that the Value Function W is characterized as in the statement of Proposition 3.3 below which summarizes our analysis on example (GC).

Proposition 3.3. *Denote by $V : [0,T] \times [0,\infty) \to \mathbb{R}$ the value function for* (GC) *and let $\psi : [0,\infty) \to [0,\infty)$ be the function defined in (8). Then*

$$V(t,x) = (W \circ (Id, \psi))\,(t,x), \quad \textit{for all } (t,x) \in [0,T] \times [0,\infty)\,,$$

where $W : [0,T] \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is the unique upper semicontinuous function such that W is continuous on $[0,T] \times [0,\infty)$, $W(t,y) = -\infty$ whenever $y < 0$ and

(i) *for all $(t,y) \in (0,T) \times [0,\infty)$, $(\xi^0,\xi^1) \in \partial_+ W(t,y)$*

$$\xi^0 + \sup_{u\in[0,1]} \left(\xi^1 \cdot (-a(1-\alpha)y + (1-\alpha)bu) + (1-u)y^{\frac{\alpha}{1-\alpha}}\right) \geq 0;$$

(ii) *for all $(t,y) \in (0,T) \times (0,\infty)$, $(\xi^0,\xi^1) \in \partial_- W(t,y)$*

$$\xi^0 + \sup_{u\in[0,1]} \left(\xi^1 \cdot (-a(1-\alpha)y + (1-\alpha)bu) + (1-u)y^{\frac{\alpha}{1-\alpha}}\right) \leq 0;$$

(iii) *for all $y \in [0,\infty)$*

$$\limsup_{\{(t',y')\to(0,y),\, t'>0\}} W(t',y') = W(0,y) \quad \textit{and} \quad W(T,y) = 0.$$

4. Preliminary results and hypotheses reduction

Standard reduction techniques (as used, for example, in [8]) permit us to prove Theorems 2.2 and 2.3, without loss of generality, when hypotheses (H2)–(H5) are replaced by stronger hypotheses, labelled (H2)*–(H5)*, in which the constants c_0, c_L and M_L, the continuity moduli ω and ω_L, the integrable function k_F and the function of bounded variation η are all independent of the parameter R_0. Henceforth, we assume (H1) and (H2)*–(H5)*, and we can take $c_L = c_0$.

A key role in our proofs is played by the following Local Weak Invariance result ([8]):

Theorem 4.1. (Local Weak Invariance Theorem) *Take an interval $[S,T]$, a multifunction $\Gamma : \mathbb{R}^k \rightsquigarrow \mathbb{R}^k$, a closed set $D \subset \mathbb{R}^k$, a number $\varepsilon > 0$ and a Γ-trajectory $\bar{x}$ on $[S,T]$. Assume that:*

(i) *the graph of Γ is closed and $\Gamma(x)$ is a non-empty, convex set for each element $x \in \bar{x}([S,T]) + \varepsilon\mathbb{B}$;*

(ii) *there exists a real $c > 0$ such that*

$$\Gamma(x) \subset c(1+|x|)\mathbb{B}, \text{ for all } x \in \bar{x}([S,T]) + \varepsilon\mathbb{B}\,;$$

(iii) *for each $[s,t] \subset [S,T]$, the restriction of $\bar{x}$ to $[s,t]$ is the unique Γ-trajectory on $[s,t]$ with initial condition $\bar{x}(s)$;*

(iv) *$\bar{x}(S) \in D$ and for every $t \in [S,T]$ and $x \in D \cap (\bar{x}(t) + \varepsilon\mathbb{B})$, we have:*

$$\min_{w \in \Gamma(x)} \xi \cdot w \leq 0, \text{ for all } \xi \in N_D^P(x).$$

Then

$$\bar{x}(t) \in D, \text{ for all } t \in [S,T].$$

We shall also invoke a useful Carathéodory's Parametrization Result ([2, Theorems 9.6.2 and 9.7.2]):

Proposition 4.2. *Assume that* (H1), (H2) *and* (H3)*. *Then, there exists a measurable function: $f : [S,T] \times \mathbb{R}^n \times \mathbb{B} \to \mathbb{R}^n$, such that:*

$$\begin{cases} \text{for every } (t,x) \in [S,T] \times \mathbb{R}^n, \ F(t,x) = f(t,x,\mathbb{B})\,; \\ \text{for every } (x,u) \in \mathbb{R}^n \times \mathbb{B}, \ f(\cdot,x,u) \text{ is measurable}; \\ \text{for every } (t,u) \in [S,T] \times \mathbb{B}, \ f(t,\cdot,u) \text{ is } 10nk_F\text{-Lipschitz}; \\ \text{for every } (t,x) \in [S,T] \times \mathbb{R}^n, (u,u') \in \mathbb{B} \times \mathbb{B}, \\ \qquad |f(t,x,u) - f(t,x,u')| \leq 5n \max_{v \in F(t,x)} |v||u-u'|. \end{cases}$$

Moreover, for all $(s,x,u) \in (S,T) \times \mathbb{R}^n \times \mathbb{B}$:

$$\lim_{s' \downarrow s, x' \to x, u' \to u} f(s',x',u') \in F(s^-,x) \text{ and } \lim_{s' \uparrow s, x' \to x, u' \to u} f(s',x',u') \in F(s^+,x). \quad (9)$$

(The inclusions (9) do not appear in [2], but are a consequence of the construction of the parametrization f of F, which is based on the Steiner selection argument used in [2], see [8] for a detailed discussion on this point.)

5. Proof of Theorem 2.2

We proceed to show the implications '(a) $\Rightarrow$ (b)' and '(b) $\Rightarrow$ (a)'.

The first relation '(a) $\Rightarrow$ (b)' is actually valid even if the outward pointing constraint qualification and condition '$((g_{|A})^*)_* = g_{|A}$' are not in force. This is established by the following proposition.

Proposition 5.1. *Assume* (H1)–(H5) *and* $(CQ)_{FW}$. *Suppose, in addition, that $g_{|A}(\cdot)$ is locally bounded. Then the value function V satisfies* (b) *of Theorem* 2.2.

Proof. By means of a similar analysis to that in [18, Chapter 12], we can show that the Value Function V is lsc. We observe that, from the *a priori* boundedness of the F-trajectories, and the local boundedness of $g_{|A}(\cdot)$ and L, it immediately follows that $V_{|[S,T]\times A}$ is locally bounded. Let $(t,x)\in(S,T)\times A$.

Take any $(\xi^0,\xi^1)\in\partial_- V(t,x)$. Then,

$$\xi^0(t'-t)+\xi^1\cdot(x'-x)\ \le\ V(t',x')-V(t,x)+o(|(t'-t,x'-x)|)\qquad\text{for all }(t',x')\quad(10)$$

in which $o(\cdot):\mathbb{R}^+\to\mathbb{R}^+$ is a function satisfying $o(\epsilon)/\epsilon\to 0$ as $\epsilon\downarrow 0$. According to [6, Lemma 3.3], (dealing with Bolza problems with L satisfying (H5)), we can find sequences $h_i\downarrow 0$ and $v_i\to\bar v$, for some $\bar v\in F(t^+,x)$, such that

$$\lim_{i\to+\infty} h_i^{-1}(V(t+h_i,x+h_iv_i)-V(t,x))\ \le\ -L(t^+,x,\bar v).$$

Taking $(t',x')=(t+h_i,x+h_iv_i)$ in (10), and dividing across by h_i, we obtain, for i sufficiently large,

$$\xi^0+\xi^1\cdot v_i\le h_i^{-1}(V(t+h_i,x+h_iv_i)-V(t,x))+h_i^{-1}o(h_i\sqrt{1+|v_i|^2}).$$

Therefore, in the limit as $i\to\infty$, it follows that

$$\xi^0+\inf_{v\in F(t^+,x)}[\xi^1\cdot v+L(t^+,x,v)]\ \le\ \xi^0+\xi^1\cdot\bar v+L(t^+,x,\bar v)\ \le\ 0.$$

Let $(t,x)\in(S,T)\times\operatorname{int}A$ and take $\varepsilon>0$ such that $x+\varepsilon\mathbb{B}\subset\operatorname{int}A$. There exists a sequence $\{(t_i,x_i)\}_{i\in\mathbb{N}}$ in $(S,T)\times\operatorname{int}A\setminus\{(t,x)\}$ that converges to (t,x) such that

$$\lim_{i\to+\infty} V(t_i,x_i)=(V_{|[S,T]\times A})^*(t,x)\ \ (=V^*(t,x)).$$

We claim that, extracting a subsequence if necessary (we do not relabel), we have $t_i\downarrow t$ as $i\to+\infty$. Indeed, if $t_i<t$ for every $i\ge i_0$ for some $i_0\in\mathbb{N}$, then we proceed as follows. We take a strictly decreasing sequence $\{\tau_i\}_{i\ge i_0}$ in $(t,T]$ that converges to t as $i\to+\infty$. We can assume that for all $i\ge i_0$, $x_i\in x+\frac{\varepsilon}{3}\mathbb{B}$ and $\tau_i-t_i\le\frac{\varepsilon}{3c_0}$. Fix any $i\ge i_0$, and take a feasible F-trajectory $\tilde y_i(\cdot)$ on $[t_i,\tau_i]$ such that $\tilde y_i(t_i)=x_i$. Using the principle of optimality, we obtain:

$$V(t_i,x_i)-\int_{t_i}^{\tau_i} L(s,\tilde y_i(s),\dot{\tilde y}_i(s))\mathrm{d}s\le V(\tau_i,\tilde y_i(\tau_i)).$$

Using the local boundedness of L (cf. condition (H5)*), there exists $M_0>0$ such that for every $i\ge i_0$:

$$V(t_i,x_i)-M_0|\tau_i-t_i|\le V(\tau_i,\tilde y_i(\tau_i)).$$

Passing to the limit to both sides and using the upper semicontinuity of $(V_{|[S,T]\times A})^*$, we obtain:

$$\begin{aligned}(V_{|[S,T]\times A})^*(t,x)&=\limsup_{i\to+\infty}V(t_i,x_i)\le\limsup_{i\to+\infty}V(\tau_i,\tilde y_i(\tau_i))+\lim_{i\to+\infty}M_0|t_i-\tau_i|\\&\le\limsup_{i\to+\infty}(V_{|[S,T]\times A})^*(\tau_i,\tilde y_i(\tau_i))\le(V_{|[S,T]\times A})^*(t,x).\end{aligned}$$

Thus $\limsup_{i\to+\infty} V(\tau_i,\tilde{y}_i(\tau_i)) = (V_{|[S,T]\times A})^*(t,x)$ and there exists a subsequence $\{i_k\}_{k\in\mathbb{N}}$ for which

$$V(\tau_{i_k},\tilde{y}_{i_k}(\tau_{i_k})) \xrightarrow[k\to+\infty]{} (V_{|[S,T]\times A})^*(t,x).$$

This confirms our claim.

Now, fix any $\tilde{v}\in F(t^+,x)$. Then, there exists a sequence of vectors $\{v_i\}$ such that $v_i\in F(t_i^+,x_i)$ for all i, and $\lim_{i\to+\infty} v_i=\tilde{v}$. For each i, we consider the arc

$$y_i(s) := x_i + (s-t_i)v_i, \text{ for all } s\in[t_i,T].$$

Using Filippov's Existence Theorem, there exists an F-trajectory $z_i(\cdot)$ that satisfies $z_i(t_i)=x_i$ and such that for every $h\in(0,T-t_i]$

$$\|z_i-y_i\|_{\mathbb{L}^\infty(t_i,t_i+h)} \le K\left(\int_{t_i}^{t_i+h} \mathrm{d}_{F(s,y_i(s))}(v_i)\mathrm{d}s\right), \text{ where } K=\exp\left(\int_S^T k_F(s)\mathrm{d}s\right).$$

From the *a priori* boundedness of F-trajectories, we can pick $R_0>0$ such that, for each i, $|z_i(s)|\le R_0$ for every $s\in[t_i,T]$ and $|\dot{z}_i(s)|\le c_0$, for almost every $s\in[t_i,T]$. Observe that we can find $\bar{h}>0$ such that, for all $i\ge 1$, we have $t_i+\bar{h}<T$ and $z_i(s)\in x+\varepsilon\mathbb{B}\subset \operatorname{int} A$ for every $s\in[t_i,t_i+\bar{h}]$. Moreover, since property $(CQ)_{FW}$ is valid, each $z_i(\cdot)$ can be extended to a feasible F-trajectory on $[t_i,T]$. For every $i\in\mathbb{N}$, define $\delta_i := \max\{|V(t_i,x_i)-V^*(t,x)|,|x_i-x|,|t_i-t|\}$. Extracting a subsequence, we can find a strictly decreasing sequence $\{h_i\}_{i\in\mathbb{N}}$ that converges to 0 such that $h_i\in[\sqrt{\delta_i},\bar{h}\,]$.

For any $i\in\mathbb{N}$ we define $w_i := \frac{1}{h_i}\int_{t_i}^{t_i+h_i}\dot{z}_i(s)\mathrm{d}s$, $e_i := 1-\frac{t-t_i}{h_i}$ and $\tilde{w}_i := w_i-\frac{x-x_i}{h_i}$. One can easily prove that $w_i \xrightarrow[i\to+\infty]{} \tilde{v}$. Observe also that

$$(t+h_ie_i, x+h_i\tilde{w}_i)\in(S,T)\times\operatorname{int}A \quad \text{and} \quad \lim_{i\to+\infty}(e_i,\tilde{w}_i)=(1,\tilde{v}).$$

Therefore recalling also that $(t,x)\in[S,T]\times\operatorname{int}A$, we deduce that

$$\begin{aligned}
&\limsup_{i\to+\infty} h_i^{-1}\left[V^*(t+h_ie_i,x+h_i\tilde{w}_i)-V^*(t,x)\right]\\
&\qquad\qquad = \limsup_{i\to+\infty} h_i^{-1}\left[V^*(t_i+h_i,x_i+h_iw_i)-V^*(t,x)\right]\\
&\qquad\qquad \ge \limsup_{i\to+\infty} h_i^{-1}\left[V(t_i+h_i,x_i+h_iw_i)-V^*(t,x)\right].
\end{aligned}$$

For all $i\in\mathbb{N}$, we have:

$$V(t_i+h_i,x_i+h_iw_i)-V^*(t,x)\ge V(t_i+h_i,x_i+h_iw_i)-V(t_i,x_i)-\delta_i. \tag{11}$$

We have arranged $z_i(\cdot)$ to be feasible for all $i\in\mathbb{N}$, so the principle of optimality yields:

$$V(t_i+h_i,x_i+h_iw_i)-V(t_i,x_i)\ge -\int_{t_i}^{t_i+h_i} L(s,z_i(s),\dot{z}_i(s))\mathrm{d}s, \text{ for all } i\in\mathbb{N}.$$

Hence, dividing across the inequality (11) by h_i, passing to the limit and recalling that $\frac{\delta_i}{h_i} \leq \sqrt{\delta_i}$, we deduce:

$$\begin{aligned}\limsup_{i\to+\infty} h_i^{-1}\left[V^*(t+h_ie_i, x+h_i\tilde{w}_i) - V^*(t,x)\right]\\ \geq -\liminf_{i\to+\infty}\frac{1}{h_i}\int_{t_i}^{t_i+h_i} L(s,z_i(s),\dot{z}_i(s))\mathrm{d}s.\end{aligned} \tag{12}$$

Our hypotheses guarantee that there exists $k_L > 0$ such that $v \mapsto L(s,z,v)$ is k_L-Lipschitz continuous on $c_0\mathbb{B}$ (uniformly w.r.t. s and z), and so, for every i, we have:

$$\begin{aligned}\int_{t_i}^{t_i+h_i} L(s,z_i(s),\dot{z}_i(s))\mathrm{d}s \leq \int_{t_i}^{t_i+h_i} L(s,z_i(s),\tilde{v})\mathrm{d}s + \int_{t_i}^{t_i+h_i} k_L|\dot{z}_i(s)-\tilde{v}|\mathrm{d}s\\ \leq h_i \sup_{\substack{|z-x|\leq c_0h_i+h_i^2\\ 0<s-t\leq h_i+h_i^2}} L(s,z,\tilde{v}) + h_i\, k_L(K\theta_i(h_i)+|v_i-\tilde{v}|),\end{aligned} \tag{13}$$

where θ_i is defined by

$$\theta_i(h) := \begin{cases}\sup_{\substack{0<s-t_i\leq h\\ |x_i-y|\leq c_0h}} \mathrm{d}_H(F(s,y),F(t_i^+,x_i)), & \text{if } h\neq 0,\\ 0, & \text{otherwise,}\end{cases}$$

and satisfies $\theta_i(h_i) \xrightarrow[i\to+\infty]{} 0$ (cf. [6]). Dividing across by h_i in (13) and passing to the limit as $i\to+\infty$, it follows that:

$$\liminf_{i\to+\infty}\frac{1}{h_i}\int_{t_i}^{t+h_i} L(s,z_i(s),\dot{z}_i(s))\mathrm{d}s \leq L(t^+,x,\tilde{v}). \tag{14}$$

Combining (12) with (14) we obtain:

$$\limsup_{i\to+\infty} h_i^{-1}\left[V^*(t+h_ie_i, x+h_i\tilde{w}_i) - V^*(t,x)\right] \geq -L(t^+,x,\tilde{v}).$$

Now, take any $(\xi^0,\xi^1)\in\partial_+V^*(t,x)$. Then,

$$V^*(t',x')-V^*(t,x)-\xi^0(t'-t)-\xi^1\cdot(x'-x) \ \leq\ o(|(t'-t,x'-x)|) \ \text{ for all } (t',x') \tag{15}$$

where $o(\cdot):\mathbb{R}^+\to\mathbb{R}^+$ is a function such that $o(\epsilon)/\epsilon\to 0$ as $\epsilon\downarrow 0$. Setting $(t',x')=(t+h_ie_i,x+h_i\tilde{w}_i)$ in (15), and dividing across by h_i, we have

$$h_i^{-1}(V^*(t+h_ie_i,x+h_i\tilde{w}_i)-V^*(t,x)) - \xi^0e_i - \xi^1\cdot\tilde{w}_i \leq h_i^{-1}o(h_i|(e_i,\tilde{w}_i)|).$$

From these relations, in the limit as $i\to\infty$, it follows that

$$-L(t^+,x,\tilde{v}) - \xi^0 - \xi^1\cdot\tilde{v} \ \leq\ 0.$$

This relation being valid for all $\tilde{v}\in F(t^+,x)$, we obtain (b)(ii).

Concerning condition (b)(iii), the first condition,

$$\liminf_{(t',x')\to(S,x),t'>S} V(t',x') = V(S,x),$$

can be easily deduced from the fact that V satisfies (b)(i).

It remains to prove that $V^*_{|[S,T]\times A}(T,\cdot) = g^*_{|A}(\cdot)$. Since $V_{|[S,T]\times A}(T,\cdot) = g_{|A}(\cdot)$, we immediately deduce that $(V_{|[S,T]\times A})^*(T,x) \geq (g_{|A})^*(x)$ for all $x \in A$. We prove that the converse inequality is also satisfied. Fix any $x \in A$. There exists a sequence $\{(t_i,x_i)\}_{i\in\mathbb{N}}$ in $[S,T]\times A \setminus \{(T,x)\}$ converging to (T,x) such that:

$$\lim_{i\to+\infty} V(t_i,x_i) = \limsup_{\{(t,y)\to(T,x),y\in A\}} V(t,y) = V^*_{|[S,T]\times A}(T,x).$$

Using $(CQ)_{FW}$, we know that, for every i, there exists a feasible F-trajectory $x_i(\cdot)$ on $[t_i,T]$ such that $x_i(t_i) = x_i$. By the principle of optimality:

$$V(t_i,x_i) - \int_{t_i}^{T} L(s,x_i(s),\dot{x}_i(s))\mathrm{d}s \leq V(T,x_i(T)), \text{ for all } i \in \mathbb{N}.$$

Using condition (H5)*, we know that there exists a constant $M_0 > 0$ such that for every $i \in \mathbb{N}$:

$$V(t_i,x_i) - M_0|T-t_i| \leq V(T,x_i(T)) = g(x_i(T)).$$

Using the fact $\lim_{i\to+\infty} x_i(T) = x$, passing to the limit as $i \to +\infty$, we obtain:

$$(V_{|[S,T]\times A})^*(T,x) \leq \limsup_{i\to+\infty} g(x_i(T)) \leq \limsup_{y \xrightarrow{A} x} g(y) = (g_{|A})^*(x).$$

Therefore V satisfies (b)(iii) and the proof of Proposition 5.1 is complete. □

To complete the proof of Theorem 2.2 we must show '(b) ⇒ (a)'. With this objective in mind we assume that (H1), (H2)*-(H5)*, $(CQ)_{FW}$ are in force and consider a lsc function $U : [S,T]\times\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ such that $U(t,x) = +\infty$ whenever $x \notin A$. We prove the following properties:

(A) if U satisfies (b)(i) and $U(T,x) = g(x)$ for all $x \in A$, then $V(t,x) \leq U(t,x)$ for any $(t,x) \in [S,T]\times A$.

(B) if, in addition, $(CQ)_{BW}$ holds, g is locally bounded on A with $((g_{|A})^*)_* = g_{|A}$, and U satisfies the conditions (b)(ii) and (b)(iii), then $V(t,x) \geq U(t,x)$ for any $(t,x) \in [S,T]\times A$.

Take any $(\bar{t},\bar{x}) \in [S,T]\times A$. To establish (A) we need to show that there exists a feasible F trajectory x on $[\bar{t},T]$ such that $x(\bar{t}) = \bar{x}$ and

$$\int_t^T L(s,x(s),\dot{x}(s))\mathrm{d}s + g(x(T)) \leq U(t,x)\,. \tag{16}$$

Define the multifunction: $Q\colon [S,T]\times\mathbb{R}^n \rightsquigarrow \mathbb{R}^n\times\mathbb{R}$

$$Q(\tau,x) := \begin{cases} \Big\{(v,-\eta)\,|\,v\in F(S^+,x), M_0 \geq \eta \geq L(S^+,x,v)\Big\}, \\ \qquad\qquad \text{if} \quad \tau = S, \\ \mathrm{co}\Big\{(v,-\eta)\,|\,v\in \{F(\tau^-,x)\cup F(\tau^+,x)\}\,, M_0 \geq \eta \geq \widetilde{L}(\tau,x,v)\Big\}, \\ \qquad\qquad \text{if} \quad \tau \in\,]S,T[, \\ \Big\{(v,-\eta)\,|\,v\in F(T^-,x), M_0 \geq \eta \geq L(T^-,x,v)\Big\}, \\ \qquad\qquad \text{if} \quad \tau = T\ , \end{cases}$$

where $\widetilde{L}(\tau, x, v) := \min\{L(\tau^+, x, v), L(\tau^-, x, v)\}$. A routine analysis allows to verify that the multifunction Q takes as values non-empty convex sets with elements which are (uniformly) bounded by $c := \sqrt{c_0^2 + M_0^2}$; moreover the graph of Q is closed.

Take any $(\bar{t}, \bar{x}) \in ((S,T) \times A) \cap \mathrm{dom}(U)$. Our aim is to apply the Viability/Weak Invariance Theorem (cf. [2, Theorem 10.1.6], [9, Theorem 12.3], [18, 12.2.1]) to the following differential inclusion:

$$\begin{cases} (\dot{\tau}, \dot{x}, \dot{\ell})(t) & \in & \Gamma(\tau(t), x(t), \ell(t)), \quad \text{for a.e. } t \in [\bar{t}, T], \\ (\tau(t), x(t), \ell(t)) & \in & \text{epi } U, \quad \text{for all } t \in [\bar{t}, T], \\ (\tau(\bar{t}), x(\bar{t}), \ell(\bar{t})) & = & (\bar{t}, \bar{x}, U(\bar{t}, \bar{x})). \end{cases} \tag{17}$$

where $\Gamma : [S,T] \times \mathbb{R}^{n+1} \rightsquigarrow \mathbb{R}^{n+2}$ is defined by

$$\Gamma(\tau, x, \ell) := \begin{cases} \text{co } (\{(0,0,0)\} \cup (\{1\} \times Q(S,x))), & \text{if} \quad \tau = S, \\ \{1\} \times Q(\tau, x), & \text{if} \quad \tau \in (S,T), \\ \text{co } (\{(0,0,0)\} \cup (\{1\} \times Q(T,x))), & \text{if} \quad \tau = T\ . \end{cases}$$

Clearly the multifunction Γ inherits the following properties from Q: the graph of Γ is closed, for all $(\tau, x, \ell) \in [S,T] \times \mathbb{R}^{n+1}$, $\Gamma(\tau, x, \ell)$ is nonempty convex set and $\Gamma(\tau, x, \ell) \subset (c+1)\mathbb{B}$. It remains to check that 'inward pointing condition' of the Weak Invariance Theorem is also satisfied: take any $(\tau, x, \ell) \in$ epi U and any $(\xi^0, \xi^1, -\lambda) \in N^P_{\text{epi } U}(\tau, x, \ell)$, we must show

$$\min_{w \in \Gamma(\tau, x, \ell)} (\xi^0, \xi^1, -\lambda) \cdot w \ \leq \ 0\ . \tag{18}$$

If $\tau = S, T$, then it is immediately verified (taking $w = 0$ that belongs to both $\Gamma(S, x, \ell)$ and $\Gamma(T, x, \ell)$). Suppose then that $S < \tau < T$. Observe that, by the nature of proximal normals to epigraph sets, we know that $\lambda \geq 0$ and we need to check (18) only when $\ell = U(\tau, x)$. We shall show

$$\xi^0 + \min_{v \in F(\tau^-, x) \cup F(\tau^+, x)} \xi^1 \cdot v + \lambda L(\tau^+, x, v) \ \leq \ 0\ . \tag{19}$$

This will confirm the inward pointing condition, indeed it implies

$$\min_{w \in \Gamma(\tau, x, \ell)} (\xi^0, \xi^1, -\lambda) \cdot w \ \leq \ \xi^0 + \min_{v \in \text{co } \{F(\tau^-, x) \cup F(\tau^+, x)\}} \xi^1 \cdot v + \lambda L(\tau^+, x, v) \ \leq \ 0\ .$$

To check (19) we need to consider two cases.

Case 1: $\lambda > 0$. In this case $((1/\lambda)\xi^0, (1/\lambda)\xi^1, -1) \in N^P_{\text{epi } U}((\tau, x), U(\tau, x))$. It follows that

$$((1/\lambda)\xi^0, (1/\lambda)\xi^1) \ \in \ \partial^P U(\tau, x) \subset \partial_- U(t, x).$$

But then, by (c)(i), $(1/\lambda)\left(\xi^0 + \min_{v \in F(\tau^+, x)}[\xi^1 \cdot v + \lambda L(\tau^+, x, v)]\right) \ \leq \ 0$. This implies (19).

Case 2: $\lambda = 0$. In this case, we know from the Rockafellar Horizontal Approximation Theorem ([9, Thm. 11.31], [18, Thm. 4.6.2]) that there exist $(\xi^0_i, \xi^1_i) \to (\xi^0, \xi^1)$, $\lambda_i \downarrow 0$ and $(t_i, x_i) \to (\tau, x)$ such that, for each i, $(\lambda_i^{-1}\xi^0_i, \lambda_i^{-1}\xi_i) \in \partial_- U(t_i, x_i)$.

But then, by condition (c)(i), there exists $v_i \in F(t_i^+, x_i)$ such that

$$\xi_i^0 + \xi_i^1 \cdot v_i + \lambda_i L(\tau^+, x, v_i) \leq 0\,.$$

But $\{v_i\}$ is a bounded sequence. We can therefore arrange, by extracting a subsequence, that $v_i \to \bar{v}$, for some $\bar{v} \in \mathbb{R}^n$. Since $(t,x) \rightsquigarrow F(t^-,x) \cup F(t^+,x)$ is an upper semi-continuous multifunction, it follows that $\bar{v} \in F(\tau^-,x) \cup F(\tau^+,x)$. So, recalling also that L is bounded on bounded sets, in the limit as $i \to \infty$, we obtain

$$0 \geq \xi^0 + \xi^1 \cdot \bar{v} + 0 \times L(\tau^+, x, \bar{v}) \geq \xi^0 + \inf_{v \in F(\tau^-,x) \cup F(\tau^+,x)} \xi^1 \cdot v.$$

We have confirmed (19) in this case, too. Then, the Weak Invariance Theorem is applicable to system (17). Hence there exists $(\tau(\cdot), x(\cdot), \ell(\cdot)) \in W^{1,1}([\bar{t},T], \mathbb{R}\times\mathbb{R}^n\times\mathbb{R})$ satisfying $\tau(t) = t$ and

$$\begin{cases} (\dot{x}(t), \dot{\ell}(t)) \in Q(t, x(t)), \text{for a.e. } t \in [\bar{t}, T] \\ x(\bar{t}) = \bar{x}, \ell(\bar{t}) = U(\bar{t}, \bar{x}) \\ \ell(t) \geq U(t, x(t)) \quad \text{for all} \ \ t \in [\bar{t}, T]\ . \end{cases}$$

Taking into account the definition of the multivalued function Q and the hypotheses on both F and L, we conclude that $x(\cdot)$ is an F-trajectory and that we have $\dot{\ell}(s) \leq -L(s, x(s), \dot{x}(s))$ for a.e. $s \in [\bar{t}, T]$. Hence we have:

$$g(x(T)) = U(T, x(T)) \leq \ell(T) = \ell(\bar{t}) + \int_{\bar{t}}^{T} \dot{\ell}(s)\mathrm{d}s \leq U(\bar{t}, \bar{x}) - \int_{\bar{t}}^{T} L(s, x(s), \dot{x}(s))\mathrm{d}s,$$

which implies: $g(x(T)) + \int_{\bar{t}}^{T} L(s, x(s), \dot{x}(s))\mathrm{d}s \leq U(\bar{t}, \bar{x})$. Thus we obtain:

$$V(\bar{t}, \bar{x}) \leq U(\bar{t}, \bar{x}).$$

It remains to investigate the case in which $\bar{t} = S$: if $(S, \bar{x})$ belongs to $\mathrm{dom}(U)$, then from the first condition in (b)(iii) we can take a sequence $(S_i, x_i) \to (S, \bar{x})$ as $i \to \infty$ such that $S_i \downarrow S$ and $\lim_{i\to\infty} U(S_i, x_i) = U(S, \bar{x})$. For each $i \geq 0$, from the previous argument we know that there exists an F-trajectory $y_i(\cdot)$ such that

$$U(S_i, x_i) \geq g(y_i(T))\ . \tag{20}$$

Extending $y_i(\cdot)$ to all $[S,T]$ by constant extrapolation from the right on $[S, S_i]$, with the help of the Compactness of Trajectories Theorem (cf. [18, Theorem 2.5.3]) we can arrange, by extracting a suitable subsequence, that $y_i(\cdot) \to \bar{y}(\cdot)$ uniformly, for some F-trajectory $\bar{y}(\cdot)$ such that $\bar{y}(\bar{t}) = \bar{x}$. Therefore, since g is lower semicontinuous, in the limit, as $i \to \infty$ from (20) we deduce that

$$U(S, \bar{x}) = \liminf_{i\to\infty} U(S_i, x_i) \geq \liminf_{i\to\infty} g(y_i(T)) \geq g(\bar{y}(T))\ .$$

We have confirmed (16).

We now provide details of the proof of (B).

Take any point $(\bar{t},\bar{x}) \in [S,T] \times A$ and any feasible F-trajectory $x(\cdot)$ on $[\bar{t},T]$ such that $x(\bar{t}) = \bar{x}$. Since $((g_{|A})^*)_*(x(T)) = g_{|A}(x(T))$, there exists a sequence $\{\xi_j\}_{j\in\mathbb{N}}$ in A such that $\lim_{j\to+\infty} \xi_j = x(T)$ and

$$\lim_{j\to+\infty} (g_{|A})^*(\xi_j) = g_{|A}(x(T)). \tag{21}$$

Consider a decreasing sequence $\{\bar{t}_j\}_{j\in\mathbb{N}}$ in $(\bar{t},T]$ that converges to $\bar{t}$. Since we are assuming that $(CQ)_{BW}$ is satisfied, we can combine it with Flippov's Existence Theorem to obtain, for each $j \geq 1$, the existence of a feasible F-trajectory $x_j(\cdot)$ on $[\bar{t}_j,T]$ such that $x_j(T) = \xi_j$, $x_j(t) \in \operatorname{int} A$ for all $t \in [\bar{t}_j,T)$ and:

$$\|x_j(\cdot) - x(\cdot)\|_{W^{1,1}(\bar{t}_j,T)} \leq \tilde{\theta}(\rho_j) + \rho_j,$$

where $\rho_j := \exp\left(\int_S^T k_F(s)\mathrm{d}s\right) |\xi_j - x(T)|$.

Fix any $j \in \mathbb{N}$. We can take an increasing sequence $\{t_{j,i}\}_{i\geq 1}$ in $(\bar{t}_j,T)$, converging to T, and a sequence $\{\omega_{j,i}\}_{i\geq 1}$ in $\operatorname{int} A$, converging to ξ_j such that:

$$\lim_{i\to+\infty} (U_{|[S,T]\times A})^*(t_{j,i},\omega_{j,i}) = \liminf_{\{(t',x')\to(T,\xi_j)\,|\,t'<T, x'\in \operatorname{int} A\}} (U_{|[S,T]\times A})^*(t',x'). \tag{22}$$

Invoking the same argument above, now with the base points $(t_{j,i},\omega_{j,i})$, for each integer $i \geq 1$, we can find a feasible F-trajectory $x_{j,i}(\cdot)$ on $[\bar{t}_j,t_{j,i}]$ such that we have $x_{j,i}(t_{j,i}) = \omega_{j,i}$, $x_{j,i}(t) \in \operatorname{int} A$ for all $t \in [\bar{t}_j,t_{j,i}]$ and:

$$\|x_{j,i}(\cdot)-x_j(\cdot)\|_{W^{1,1}(\bar{t}_j,t_{j,i})} \leq \tilde{\theta}(\rho_{j,i})+\rho_{j,i}, \text{ where } \rho_{j,i} := \exp\left(\int_S^T k_F(s)\mathrm{d}s\right) |\xi_j-\omega_{j,i}|.$$

It is not difficult to see that, using Filippov's Existence Theorem, each $x_{j,i}(\cdot)$ can be extended on $[\bar{t}_j,T]$ obtaining the estimate (we do not relabel)

$$\|x_{j,i}(\cdot) - x_j(\cdot)\|_{W^{1,1}(\bar{t}_j,T)} \leq \tilde{\theta}(\rho_{j,i}) + \rho_{j,i}.$$

Since $x_{j,i}(t) \in \operatorname{int} A$ for all $t \in [\bar{t}_j,t_{j,i}]$, there exist sequences $\{\delta_{j,i}\}_{i\geq 1}$ and $\{\tau_{j,i}\}_{i\geq 1}$ with $\tau_{j,i} \in (t_{j,i},T)$ such that $\delta_{j,i} \downarrow 0$ and $\tau_{j,i} \uparrow T$ as $i \to +\infty$, and

$$x_{j,i}(s) + \delta_{j,i}\mathbb{B} \subset \operatorname{int} A, \text{ for all } s \in [\bar{t}_j,\tau_{j,i}].$$

Let $f : [S,T]\times\mathbb{R}^n\times\mathbb{B} \to \mathbb{R}^n$, be the (Carathéodory) parametrization of F (provided by Proposition 4.2). Using Filippov's Selection Theorem, for each $i \geq 1$, there exists a measurable selection $u_{j,i} : [\bar{t}_j,T] \to \mathbb{B}$ such that $\dot{x}_{j,i}(t) = f(t,x_{j,i}(t),u_{j,i}(t))$ for a.e. $t \in [\bar{t}_j,T]$.

Fix any $i \geq 1$ and any $j \geq 1$. Take $k \in \mathbb{N}$ such that $k \geq \frac{2}{\delta_{j,i}}$. Invoking an argument based on Lusin's Theorem, we can find a continuous control $u^k_{j,i} : [\bar{t}_j,T] \to \mathbb{B}$ and an arc $x^k_{j,i} : [\bar{t}_j,t_{j,i}] \to \mathbb{R}^n$ such that:

$$\begin{cases} \dot{x}^k_{j,i}(t) = f(t,x^k_{j,i}(t),u^k_{j,i}(t)), \text{ for a.e. } t \in [\bar{t}_j,T]\,, \\ \|x_{j,i} - x^k_{j,i}\|_{\mathbb{L}^\infty(\bar{t}_j,T)} \leq \frac{1}{k} \text{ and } x^k_{j,i}(t_{j,i}) = x_{j,i}(t_{j,i}) = \omega_{j,i}\,, \\ \operatorname{meas}\left(\{t \in [\bar{t}_j,T]\,|\,u_{j,i}(t) - u^k_{j,i}(t) \neq 0\}\right) \leq \frac{1}{k}. \end{cases}$$

For every $(t,x) \in [\bar{t}_j,T]\times\mathbb{R}^n$, we define the functions $v^+(t,x) := f(t^+,x,u^k_{j,i}(t))$ and $v^-(t,x) := f(t^-,x,u^k_{j,i}(t))$.

On this basis we then define the multivalued functions $F^k_{j,i}\colon [\bar{t}_j,T]\times\mathbb{R}^n \rightsquigarrow \mathbb{R}^n$, $\Lambda^k_{j,i}\colon [\bar{t}_j,T]\times\mathbb{R}^n \rightsquigarrow \mathbb{R}$ and $\Gamma^k_{j,i}\,:\,[\bar{t}_j,T]\times\mathbb{R}^n\times\mathbb{R} \rightsquigarrow \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}$ by the relations:

$$F^k_{j,i}(t,x) := \mathrm{co}\,\{v^-(t,x), v^+(t,x)\},$$

$$\Lambda^k_{j,i}(t,x) := -\mathrm{co}\,\{L(t^-,x,v^-(t,x)), L(t^+,x,v^+(t,x))\},$$

$$\Gamma^k_{j,i}(t,x,\ell) := \begin{cases} \{1\}\times F^k_{j,i}(t,x)\times\Lambda^k_{j,i}(t,x), & \text{if } t\in[\bar{t}_j,t_{j,i}\,),\\ \mathrm{co}\,\left\{(0,0,0)\cup\{1\}\times F^k_{j,i}(t_{j,i},x)\times\Lambda^k_{j,i}(t_{j,i},x)\right\}, & \text{if } t=t_{j,i}.\end{cases}$$

Consider the arc $(\tau^k_{j,i},x^k_{j,i},\ell^k_{j,i})$ on $[\bar{t}_j,T]$ defined by

$$\begin{aligned} t \;\mapsto\; & (\tau^k_{j,i}(t), x^k_{j,i}(t), \ell^k_{j,i}(t))\\ & := \Big(t, x^k_{j,i}(t), (U_{|[S,T]\times A})^*(\bar{t}_j, x^k_{j,i}(\bar{t}_j)) - \int_{\bar{t}_j}^{t} L(s,x^k_{j,i}(s),\dot{x}^k_{j,i}(s))\mathrm{d}s\Big).\end{aligned}$$

Observe that $[\bar{t}_j,t_{j,i}]\subset(S,T)$ and

$$x^k_{j,i}(t)+\tfrac{\delta_{j,i}}{2}\mathbb{B}\subset x_{j,i}(t)+\delta_{j,i}\mathbb{B}\subset \mathrm{int}\,A,\ \text{ for all } t\in[\bar{t}_j,t_{j,i}].$$

Now, our aim is to apply the Local Weak Invariance (Theorem 4.1) to the constrained differential inclusion:

$$\begin{cases} (\dot{\tau}(t),\dot{y}(t),\dot{\ell}(t))\in\Gamma^k_{j,i}(\tau(t),y(t),\ell(t)),\ \text{ for a.e. } t\in[\bar{t}_j,t_{j,i}],\\ \tau(\bar{t}_j)=\bar{t}_j, y(\bar{t}_j)=x^k_{j,i}(\bar{t}_j), \ell(\bar{t}_j)=(U_{|[S,T]\times A})^*(\bar{t}_j,x^k_{j,i}(\bar{t}_j)),\\ (\tau(t),y(t),\ell(t))\in\mathrm{hyp}\,(U_{|[S,T]\times A})^*,\ \text{ for all } t\in[\bar{t}_j,t_{j,i}],\end{cases} \tag{23}$$

with the reference arc $(\tau^k_{j,i},x^k_{j,i},\ell^k_{j,i})$. We discuss here only the validity of the inward pointing condition (iv) of Theorem 4.1 since it is easy to see that all the assumptions are satisfied. Observe that along a '$\frac{\delta_{j,i}}{2}$-tube' around the arc $(\tau^k_{j,i},x^k_{j,i},\ell^k_{j,i})$ we have $(U_{|[S,T]\times A})^*=U^*$. Therefore the required inward pointing condition to check is: for any $(\tau,x,\ell)\in\mathrm{hyp}\,U^*\cap((\tau^k_{j,i},x^k_{j,i},\ell^k_{j,i})([\bar{t}_j,t_{j,i}])+\frac{\delta_{j,i}}{2}\mathbb{B})$ we have

$$\min_{w\in\Gamma^k_{j,i}(\tau,x,\ell)} w\cdot(-\xi^0,-\xi^1,\lambda)\le 0,\ \text{ for all } (-\xi^0,-\xi^1,\lambda)\in N^P_{\mathrm{hyp}\,U^*}(\tau,x,\ell). \tag{24}$$

Take any $(\tau,x,\ell)\in\mathrm{hyp}\,U^*\cap((\tau^k_{j,i},x^k_{j,i},\ell^k_{j,i})([\bar{t}_j,t_{j,i}])+\frac{\delta_{j,i}}{2}\mathbb{B})$ and any $(-\xi^0,-\xi^1,\lambda)\in N^P_{\mathrm{hyp}\,U^*}(\tau,x,\ell)$ $(\subset\hat{N}_{\mathrm{hyp}\,U^*}(\tau,x,\ell))$. We have $\lambda\ge 0$. Since (24) is clearly verified when $\tau=t_{j,i}$, we assume that $\tau\in[\bar{t}_j,t_{j,i})$. We can also restrict attention to the case $\ell=U^*(\tau,x)$.

Depending on the value of λ we can consider two distinct cases.

If $\lambda>0$, then $\lambda^{-1}(\xi^0,\xi^1)\in\partial_+U^*(\tau,x)$ and so, from condition (d)(ii), we deduce that $\xi^0+\xi^1\cdot v^+(\tau,x)+\lambda L(\tau^+,x,v^+(\tau,x))\ge 0$.

This confirms (24) with $w=(1,v^+(\tau,x),-L(\tau^+,x,v^+(\tau,x)))$.

If $\lambda=0$, then there exist sequences $(\xi^0_m,\xi^1_m)\to(\xi^0,\xi^1)$, $\lambda_m\downarrow 0$ and $(s_m,x_m)\to(\tau,x)$ such that, for each m, $(\lambda_m^{-1}\xi^0_m,\lambda_m^{-1}\xi^1_m)\in\partial_+U^*(s_m,x_m)$.

By (b)(ii), we obtain $\xi^0_m+\xi^1_m\cdot v^+(s_m,x_m)+\lambda_m L((s_m)^+,x_m,v^+(s_m,x_m))\ge 0$.

By extracting suitable subsequences and arguing as in the proof of (A) we deduce that (24) is verified.

Observe that the last condition in (23) means that $\ell^k_{j,i}(t) \le (U_{|[S,T]\times A})^*(\tau^k_{j,i}(t), x^k_{j,i}(t))$, for all $t \in [\bar t_j, t_{j,i}]$. It follows that at $t = t_{j,i}$:

$$(U_{|[S,T]\times A})^*(\bar t_j, x^k_{j,i}(\bar t_j)) - \int_{\bar t_j}^{t_{j,i}} L(s, x^k_{j,i}(s), \dot x^k_{j,i}(s))\mathrm{d}s \le (U_{|[S,T]\times A})^*(t_{j,i}, \omega_{j,i}),$$

for every k, i and j. Applying Lebesgue's dominated convergence, taking a subsequence if necessary and without relabeling, in the limit as $k \to +\infty$ we obtain

$$\begin{aligned} U(\bar t_j, x_{j,i}(\bar t_j)) &\le \liminf_{k\to+\infty} U(\bar t_j, x^k_{j,i}(\bar t_j)) \le \liminf_{k\to+\infty} (U_{|[S,T]\times A})^*(\bar t_j, x^k_{j,i}(\bar t_j)) \\ &\le (U_{|[S,T]\times A})^*(t_{j,i}, \omega_{j,i}) + \int_{\bar t_j}^{t_{j,i}} L(s, x_{j,i}(s), \dot x_{j,i}(s))\mathrm{d}s, \text{ for every } i \text{ and } j. \end{aligned}$$

Then, taking the limit as $i \to +\infty$, we have, for every $j \ge 1$:

$$\begin{aligned} U(\bar t_j, x_j(\bar t_j)) &\le \liminf_{i\to+\infty} U(\bar t_j, x_{j,i}(\bar t_j)) \\ &\le \lim_{i\to+\infty} U^*_{|[S,T]\times A}(t_{j,i}, \omega_{j,i}) + \int_{\bar t_j}^{T} L(s, x_j(s), \dot x_j(s))\mathrm{d}s. \end{aligned} \tag{25}$$

From (22) and the upper semicontinuity of $(U_{|[S,T]\times A})^*$ it follows that

$$\begin{aligned} \liminf_{\{(t',x')\to(T,\xi_j)\,|\,t'<T, x'\in \mathrm{int}\,A\}} (U_{|[S,T]\times A})^*(t',x') &\le \limsup_{\{(t',x')\to(T,\xi_j), x'\in A\}} (U_{|[S,T]\times A})^*(t',x') \\ &\le (U_{|[S,T]\times A})^*(T,\xi_j). \end{aligned}$$

Using these relations and the fact that $(U_{|[S,T]\times A})^*(T,\cdot) = (g_{|A})^*(\cdot)$ (from (25)), we deduce that

$$U(\bar t_j, x_j(\bar t_j)) \le (g_{|A})^*(\xi_j) + \int_{\bar t_j}^{T} L(s, x_j(s), \dot x_j(s))\mathrm{d}s, \text{ for every } j \ge 1.$$

Recalling that the sequence $\{\xi_j\}_{j\in\mathbb{N}}$ satisfies (21) and that U is lsc, passing to the limit as $j \to +\infty$ we obtain:

$$U(\bar t, x(\bar t)) \le \liminf_{j\to+\infty} U(\bar t_j, x_j(\bar t_j)) \le g_{|A}(x(T)) + \int_{\bar t}^{T} L(s, x(s), \dot x(s))\mathrm{d}s,$$

which concludes the proof since $x(\cdot)$ was an arbitrary feasible F-trajectory such that $x(\bar t) = \bar t$. $\square$

6. Proof of Theorem 2.3

We observe that, if the inward pointing constraint qualification is in force and g is continuous on A, we obtain that $V_{|[S,T]\times A}$ is continuous too, and so, in view of Theorem 2.2 we obtain that (a)$\Rightarrow$(b)$'$ ((b)$'$ is the simplified version of (b) which appears in the statement of Theorem 2.3). As a consequence, we can restrict attention to the implication (b)$' \Rightarrow$ (a). To this aim, we assume (H1)–(H5) and take an extended valued function $U\colon [S,T]\times\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ such that $U_{|[S,T]\times A}$ is continuous and $U(t,x) = +\infty$ when $x \notin A$, and employing a consolidated approach we divide the proof of the desired property into two steps:

(A)′ if U satisfies (b)′(i) and $U(T,x) = g(x)$ for all $x \in A$, then $V(t,x) \le U(t,x)$ for any $(t,x) \in [S,T] \times A$.

(B)′ if, in addition, $(CQ)_{BW}$ holds, and U satisfies (b)′(ii) and (b)′(iii), then $V(t,x) \ge U(t,x)$ for any $(t,x) \in [S,T] \times A$.

Since (A)′ can be easily deduced as in step (A) of the proof of Theorem 2.2, consider here only (B)′.

Take $(\bar{t},\bar{x}) \in [S,T] \times A$, and a feasible F-trajectory $x(\cdot)$ on $[\bar{t},T]$ such that $x(\bar{t}) = \bar{x}$. With the help of property $(CQ)_{FW}$, we can find a sequence of feasible F-trajectories $\{x_j\}_{j\ge1}$ on $[\bar{t},T]$ such that, for every $j \ge 1$, $x_j(\bar{t}) = \bar{x}$, $x_j(t) \in \operatorname{int} A$ for all $t \in (\bar{t},T]$, and:

$$\|x - x_j\|_{W^{1,1}(\bar{t},T)} \le \frac{1}{j} \,. \tag{26}$$

Take a sequence $\{t_j\}_{j\ge1}$ in $(\bar{t},T)$ such that $t_j \downarrow \bar{t}$. Invoking Proposition 4.2 (on the Carathéodory parametrization), Filippov's Selection Theorem and Lusin's Theorem as in the proof of Theorem 2.2, for each j, we can find a sequence of F-trajectories $\{x_j^i\}_{i\in\mathbb{N}}$ on $[t_j,T]$ such that

$$\|x_j - x_j^i\|_{W^{1,1}(t_j,T)} \le \frac{1}{i} \text{ and } x_j^i(t_j) = x_j(t_j), \tag{27}$$

and the arc $(\tilde{\tau},\tilde{x},\tilde{\ell})$ on $[t_j,T]$ defined by

$$t \mapsto (\tilde{\tau},\tilde{x},\tilde{\ell})(t) := \Big(t, x_j^i(t), U(t_j,x_j^i(t_j)) - \int_{t_j}^{t} L(s,x_j^i(s),\dot{x}_j^i(s))\mathrm{d}s\Big),$$

is such that

$$\begin{cases} \tau(t_j) = t_j, y(t_j) = x_j^i(t_j), \ell(t_j) = U(t_j,x_j^i(t_j)), \\ (\tau(t),y(t),\ell(t)) \in \operatorname{hyp} U, \ \text{for all } t \in [t_j,T]. \end{cases} \tag{28}$$

The last condition in (28) yields $\tilde{\ell}(t) \le U(t,x_j^i(t))$, for all $t \in [t_j,T]$. It follows that for $t = T$:

$$U(t_j,x_j^i(t_j)) - \int_{t_j}^{T} L(s,x_j^i(s),\dot{x}_j^i(s))\mathrm{d}s \le U(T,x_j^i(T)). \tag{29}$$

Bearing in mind (27), and passing to the limit in (29) as $i \to +\infty$, we obtain

$$U(t_j,x_j(t_j)) \le U(T,x_j(T)) + \int_{t_j}^{T} L(s,x_j(s),\dot{x}_j(s))\mathrm{d}s, \ \text{for every } j \ge 1.$$

Now invoking (26), the feasibility of $x_j(\cdot)$, the continuity of U on $[S,T] \times A$, and condition (b)′(iii), we take the limit as $j \to +\infty$ in the previous inequality deducing:

$$U(\bar{t},\bar{x}) \le \lim_{j\to+\infty} U(t_j,x_j(t_j)) \le g(x(T)) + \int_{\bar{t}}^{T} L(s,x(s),\dot{x}(s))\mathrm{d}s.$$

This shows that $U(\bar{t},\bar{x}) \le V(\bar{t},\bar{x})$. □

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