

On the Continuity of the Continuous Steiner Symmetrization

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Dedicated to Roger J-B Wets on the occasion of his 85th birthday.

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Starting from the Brock's construction of Continuous Steiner Symmetrization of sets, the problem of modifying continuously a given domain up to obtain a ball, preserving its measure and with decreasing first eigenvalue of the Laplace operator, is considered. For a large class of cases it is shown this is possible, while the general question remains still open.

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1. Introduction

The problem of *rounding* more and more a given set $\Omega \subset \mathbb{R}^d$, keeping fixed its measure and asymptotically reaching a ball of the same measure, enters in a number of problems and has been widely considered in the literature. More precisely, given a bounded open set $\Omega \subset \mathbb{R}^d$, the goal is to construct a family of domains (Ω_t) , with $t \in [0, 1]$, such that $\Omega_0 = \Omega$, $\Omega_1 = \Omega^*$ where Ω^* is a ball of the same measure as Ω , and $|\Omega_t| = |\Omega|$ for all $t \in [0, 1]$, where by $|\cdot|$ we denote the Lebesgue measure.

In addition, we require that the mapping $t \mapsto \Omega_t$ be *continuous* with respect to some suitable topology, and that the family (Ω_t) satisfy some monotonicity property that will be specified later.

We notice that, without the last monotonicity requirement, a very simple construction would provide a solution. Take indeed a set Ω and a point x_0 far enough from Ω ; denoting by $B(x_0, r)$ the ball of center x_0 and radius r and by ω_d the Lebesgue measure of the unit ball in $\mathbb{R}^d$, the family

$$\Omega_t = (1 - t)^{1/d} \Omega \cup B(x_0, r_t) \quad \text{with } r_t = \left(\frac{t|\Omega|}{\omega_d} \right)^{1/d}$$

satisfies the measure constraint $|\Omega_t| = |\Omega|$, is such that $\Omega_0 = \Omega$ and $\Omega_1 = \Omega^*$, and is continuous in several useful topologies. An example of such a family (Ω_t) is illustrated in Figure 1.1.

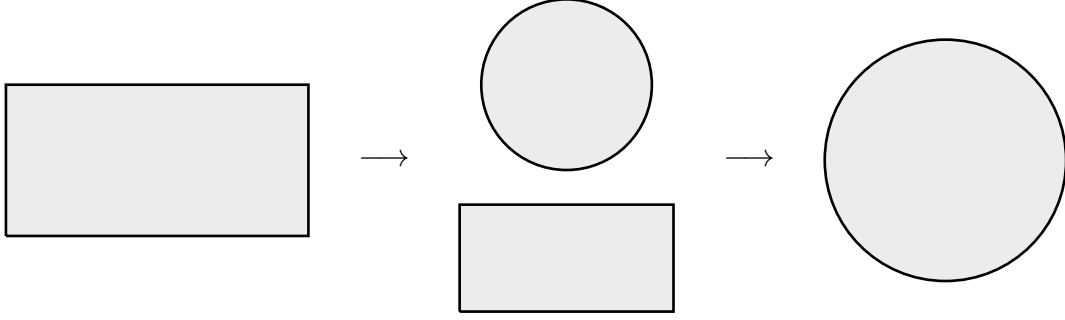


Figure 1.1: The sets Ω_0 , $\Omega_{1/2}$, Ω_1 when Ω is the rectangle $]0, 2[\times]0, 1[$.

The additional monotonicity conditions that we impose consists in the requirement that a suitable shape functional F is monotone. For instance we could consider:

- $F(\Omega) = P(\Omega)$, the *perimeter* in the sense of De Giorgi, and we require $P(\Omega)$ is nonincreasing;
- $F(\Omega) = \mathcal{H}^{d-1}(\Omega)$, the *Hausdorff $d - 1$ dimensional measure*, and we require $\mathcal{H}^{d-1}(\Omega)$ is nonincreasing;
- $F(\Omega) = T(\Omega)$, the *torsional rigidity* defined below, and we require $T(\Omega)$ is nondecreasing;
- $F(\Omega) = \lambda(\Omega)$, the *first eigenvalue* of the Dirichlet Laplacian defined below, and we require $\lambda(\Omega)$ is nonincreasing;
- $F(\Omega) = h(\Omega)$, the *Cheeger constant*, and we require $h(\Omega)$ is nonincreasing.

In this paper we focus the attention mostly on the first eigenvalue $\lambda(\Omega)$ and on the torsional rigidity $T(\Omega)$.

More precisely, $\lambda(\Omega)$ is the first eigenvalue of the Laplace operator $-\Delta$ with Dirichlet conditions on $\partial\Omega$, that is the minimal value λ such that the PDE

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases}$$

has a nonzero solution. Equivalently, by the min-max principle (see for instance [11]) $\lambda(\Omega)$ can be defined through the minimization of the Rayleigh quotient, as

$$\lambda(\Omega) = \min \left\{ \left[\int_{\Omega} |\nabla u|^2 dx \right] \left[\int_{\Omega} u^2 dx \right]^{-1} : u \in H_0^1(\Omega), u \neq 0 \right\}.$$

An important bound for $\lambda(\Omega)$ is the *Faber-Krahn inequality* (see for instance [11], [12])

$$\lambda(\Omega^*) \leq \lambda(\Omega),$$

which can be stated in a scaling free form as

$$|\Omega|^{2/d} \lambda(\Omega) \geq |B|^{2/d} \lambda(B),$$

where B is any ball in $\mathbb{R}^d$.

The torsional rigidity $T(\Omega)$ is defined as $\int_{\Omega} u_{\Omega} dx$, where u_{Ω} is the unique solution of the PDE

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases}$$

or equivalently through the maximization problem

$$T(\Omega) = \max \left\{ \left[\int_{\Omega} u dx \right]^2 \left[\int_{\Omega} |\nabla u|^2 dx \right]^{-1} : u \in H_0^1(\Omega), u \neq 0 \right\},$$

where the maximum is reached by u_{Ω} itself. Also for $T(\Omega)$ an important inequality holds, the *Saint-Venant inequality*

$$T(\Omega) \leq T(\Omega^*),$$

which can be stated in a scaling free form as

$$|\Omega|^{-(d+2)/d} T(\Omega) \leq |B|^{-(d+2)/d} T(B)$$

where B is any ball in $\mathbb{R}^d$.

The monotonicity properties we require to the family (Ω_t) are then:

- the mapping $t \mapsto \lambda(\Omega_t)$ is nonincreasing;
- the mapping $t \mapsto T(\Omega_t)$ is nondecreasing.

Concerning the continuity of the map $t \mapsto \Omega_t$ our requirement is that the solutions u_t of the PDEs

$$\begin{cases} -\Delta u_t = f & \text{in } \Omega_t, \\ u_t \in H_0^1(\Omega_t), \end{cases}$$

vary continuously in t with respect to the strong $H^1(\mathbb{R}^d)$ convergence, for every right-hand side $f \in L^2(\mathbb{R}^d)$. This is the γ -convergence, that we describe more precisely in Section 2.

When instead of a continuous family (Ω_t) we consider the discrete case of a sequence (Ω_n) such that

- (i) $\Omega_0 = \Omega$, $|\Omega_n| = |\Omega|$ for every n , $\Omega_n \rightarrow \Omega^*$ in the γ -convergence,
- (ii) $\lambda(\Omega_{n+1}) \leq \lambda(\Omega_n)$ and $T(\Omega_{n+1}) \geq T(\Omega_n)$ for every n ,

we have the problem that was first considered by Steiner, who proposed to use successive symmetrizations through different hyperplanes. More precisely, given a domain $\Omega \subset \mathbb{R}^d$ and a direction ν , the *Steiner symmetrization* of Ω with respect to ν is defined as

$$\Omega_{\nu}^* = \left\{ x \in \mathbb{R}^d : |x \cdot \nu| < \frac{\varphi(\pi(x))}{2} \right\}.$$

Here $\pi(x) = x - \nu(x \cdot \nu)$ is the projection of a point $x \in \mathbb{R}^d$ on the hyperplane orthogonal to ν and, for each y in this hyperplane,

$$\varphi(y) = \mathcal{H}^1(\Omega \cap \pi^{-1}(y))$$

is the length of the y -section of Ω , where by $\mathcal{H}^1$ we denote the 1-dimensional Hausdorff measure.

Note that the set Ω_ν^* has the same volume of Ω and is symmetric with respect to the hyperplane orthogonal to ν . In addition, it is well-known (see for instance [1]) that the Steiner symmetrization decreases the first eigenvalue and increases the torsional rigidity, that is

$$\lambda(\Omega_\nu^*) \leq \lambda(\Omega) \quad \text{and} \quad T(\Omega_\nu^*) \geq T(\Omega).$$

By repeating this symmetrization procedure for a dense sequence of directions ν , one obtains a sequence Ω_n of sets, all with the same measure, which γ -converge as $n \rightarrow \infty$ to the ball Ω^* .

The question is now to pass from the discrete Steiner symmetrization to a continuous one. Since successive Steiner symmetrizations allow to pass from a generic set to a ball, it is enough to construct a continuous family Ω_t of sets which transforms a set Ω into its Steiner symmetrization Ω_ν^* for a fixed direction ν . An explicit construction of a family Ω_t was proposed by Brock in [4] (see also [5]) and was called *Continuous Steiner Symmetrization*. We shortly recall the Brock's construction in Section 3.

Unfortunately, the Brock's construction provides the γ -continuity of the family Ω_t only in very particular situations, as for instance when the initial domain Ω is convex, while in general discontinuities may occur, due to irregularities of the domains Ω_t . On the other hand, the γ -continuity would be very useful in several situations, as for instance in the study of some Blaschke-Santaló diagrams, as illustrated in [8].

In the present paper we show that a modification of Brock's construction could be enough to provide the required γ -continuity of the family Ω_t , at least for a larger class of domains Ω . In [8] a similar construction was made for polyhedral domains Ω . Even if the arguments are not complete, we believe it could help to better understand the difficulties behind the Continuous Steiner Symmetrization.

In the last section we consider a possible alternative approach based on the De Giorgi theory of minimizing movements.

2. The γ -convergence

In this section we recall the definition and the main properties of γ -convergence; for all details, proofs, and generalization to the class of capacitary measures, we refer the interested reader to [6]. For simplicity, we make the assumption that all the domains we consider are included in a given bounded open subset D of $\mathbb{R}^d$, which is satisfied for the domains we consider later. In the following, for every domain Ω , a function in $H_0^1(\Omega)$ is considered extended by zero on $\mathbb{R}^d \setminus \Omega$.

Definition 2.1. A sequence (Ω_n) of domains is said to γ -converge to a domain Ω if for every $f \in L^2(\mathbb{R}^d)$ the solutions $u_{n,f}$ of the PDEs

$$\begin{cases} -\Delta u = f & \text{in } \Omega_n \\ u \in H_0^1(\Omega_n) \end{cases}$$

converge weakly in $H^1(\mathbb{R}^d)$ to the solution u_f of the PDE

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u \in H_0^1(\Omega). \end{cases}$$

We summarize here below the main properties of the γ -convergence. We refer to [6] for all the details, properties, and proofs.

- The weak $H^1(\mathbb{R}^d)$ convergence of $u_{n,f}$ to u_f is equivalent to the strong $H^1(\mathbb{R}^d)$ convergence. Indeed, integrating by parts we obtain

$$\int |\nabla u_{n,f}|^2 dx = \int u_{n,f} f dx \rightarrow \int u_f f dx = \int |\nabla u_f|^2 dx.$$

- In the definition above it is not difficult to show that it is equivalent to require the weak $H^1(\mathbb{R}^d)$ convergence of $u_{n,f}$ to u_f for every $f \in L^2(\mathbb{R}^d)$ or for every $f \in H^{-1}(\mathbb{R}^d)$. Indeed, if $f \in H^{-1}(\mathbb{R}^d)$ it is enough to approximate f by a sequence $f_k \in L^2(\mathbb{R}^d)$, in the H^{-1} norm, to obtain for every test function ϕ

$$\begin{aligned} \left| \int \nabla u_{n,f} \nabla \phi dx - \int \nabla u_f \nabla \phi dx \right| &= \left| \langle f, \phi \rangle_{H_0^1(\Omega_n)} - \langle f, \phi \rangle_{H_0^1(\Omega)} \right| \\ &\leq \left| \langle f_k, \phi \rangle_{H_0^1(\Omega_n)} - \langle f_k, \phi \rangle_{H_0^1(\Omega)} \right| + \varepsilon_k \|\phi\| \\ &= \left| \int \nabla u_{n,f_k} \nabla \phi dx - \int \nabla u_{f_k} \nabla \phi dx \right| + \varepsilon_k \|\phi\|. \end{aligned}$$

where $\varepsilon_k \rightarrow 0$. Passing to the limit first as $n \rightarrow \infty$ and then as $k \rightarrow \infty$ gives what claimed.

- The γ -convergence can be defined in a similar way for *quasi-open* sets $\Omega \subset D$ or more generally for *capacitary measures* μ confined into D (that is $\mu = +\infty$ outside D). Quasi-open sets are sets of positivity $\{u > 0\}$ of functions $u \in H^1(\mathbb{R}^d)$, while capacitary measures are regular nonnegative Borel measures μ on D , possibly $+\infty$ valued, such that $\mu(E) = 0$ for every Borel set $E \subset D$ with $\text{cp}(E) = 0$. For all details on quasi-open sets and capacitary measures we refer the interested reader to the book [6]. Here we only recall that for a capacitary measure μ the corresponding PDE is formally written as

$$\begin{cases} -\Delta u + \mu u = f & \text{in } D \\ u \in H_0^1(D) \cap L_\mu^2(D) \end{cases}$$

and has to be intended in the weak sense, that is, $u \in H_0^1(D) \cap L_\mu^2(D)$ and

$$\int_D \nabla u \nabla \phi dx + \int_D u \phi d\mu = \langle f, \phi \rangle$$

for all $\phi \in H_0^1(D) \cap L_\mu^2(D)$. We notice that open sets or more generally quasi-open sets can be seen as capacitary measures: for a given domain Ω the capacitary measure representing it is the measure ∞_{Ω^c} defined as

$$\infty_{\Omega^c}(E) = \begin{cases} 0 & \text{if } \text{cp}(E \cap \Omega) = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

- In Definition 2.1 it is possible to show (see Remark 4.3.10 of [6]) that requiring the convergence of the solutions u_n to u for every right-hand side f is equivalent to require the convergence $u_n \rightarrow u$ only for $f \equiv 1$ and in the $L^2(D)$ sense. In particular, calling u_μ the unique solution of the PDE $-\Delta u + \mu u = 1$ in $H_0^1(D) \cap L_\mu^2(D)$, the quantity

$$d_\gamma(\mu_1, \mu_2) = \|u_{\mu_1} - u_{\mu_2}\|_{L^2(D)} \quad (1)$$

is a distance on the space $\mathbb{M}$ of capacitary measures, which is equivalent to γ -convergence, and so $\mathbb{M}$ endowed with the distance d_γ above is a compact metric space. Since the solutions u_μ are all equi-bounded (for instance they are all below by the solution w of the Dirichlet problem $-\Delta w = 1$ on $H_0^1(D)$, which is a bounded function) the L^2 norm in (1) can be replaced by any L^p norm, with $1 \leq p < +\infty$. In particular, if $p = 1$ and $\Omega_1 \subset \Omega_2$ we have

$$\|u_{\Omega_1} - u_{\Omega_2}\|_{L^1} = \int u_{\Omega_2} dx - \int u_{\Omega_1} dx = T(\Omega_2) - T(\Omega_1),$$

and the γ -convergence is then reduced to the convergence of the corresponding torsional rigidities.

- The first eigenvalue $\lambda(\Omega)$ (as well as all the other eigenvalues $\lambda_k(\Omega)$) and the torsional rigidity $T(\Omega)$ are continuous with respect to the γ -convergence.
- The Lebesgue measure $|\Omega|$, or more generally integral functionals as $\int_\Omega f(x) dx$ with $f \geq 0$ and measurable, are lower semicontinuous with respect to the γ -convergence on the domains Ω .
- As stated above, the space $\mathbb{M}$ of capacitary measures, endowed with the γ -convergence, is a compact metric space. On the contrary, the family of open sets (or also quasi-open sets) is not compact in $\mathbb{M}$; it is actually a dense subset of $\mathbb{M}$. The first example of a sequence of open sets Ω_n which γ -converges to a capacitary measure which is not a domain (actually to the Lebesgue measure) was given in [9].
- Several subclasses of $\mathbb{M}$ are dense with respect to the γ -convergence (see Proposition 4.3.7 and Remark 4.3.8 of [6]). For instance:
 - the class of measures $a(x) dx$ with $a \geq 0$ and smooth;
 - the class of smooth domains $\Omega \subset D$;
 - the class of polyhedral domains $\Omega \subset D$;
 - the class of measures of the form $a(x) d\mathcal{H}^{d-1}$ with $a \geq 0$ and smooth, where $\mathcal{H}^{d-1}$ is the $d - 1$ dimensional Hausdorff measure;
 - the class of measures of the form $\mathcal{H}^{d-1} \llcorner S$, where $S \subset D$ is a smooth $d - 1$ manifold.

3. The Brock's construction

We summarize rapidly here the construction by Brock (see [4], [5]) of the continuous Steiner symmetrization, together with the properties important for our purpose. The first construction is for the unidimensional case; here taking the variable t in $[0, +\infty]$ or in $[0, 1]$ does not make any real difference.

- If I is the interval $]a, b[$, then the continuous Steiner symmetrization I^t is the interval $]a^t, b^t[$, where

$$a^t = (a - b + e^{-t}(a + b))/2, \quad b^t = (b - a + e^{-t}(a + b))/2.$$

- If A is an open subset of $\mathbb{R}$ we consider the properties:
 - (i) $A(0) = A$;
 - (ii) if I is an interval with $I \subset A(s)$, then $I^t \subset A(s + t)$ for every $t \geq 0$.

We define then the continuous Steiner symmetrization A^t as

$$A^t = \bigcap \{A(t) : A(t) \text{ satisfies (i) and (ii)}\}.$$

In [4] Brock proves that if A is open then A^t are open sets; in addition the following monotonicity property hold:

$$A \subset B \implies A^t \subset B^t \text{ for every } t.$$

- Finally, if $A \subset \mathbb{R}$ is only measurable, we have

$$A = \bigcap_n A_n \setminus N$$

with A_n open sets and N Lebesgue negligible. We then define the continuous Steiner symmetrization A^t of A as

$$A^t = \bigcap_n A_n^t.$$

This definition is unique up to a nullset, and we still call continuous Steiner symmetrization a family A^t such that $|A^t \Delta (\bigcap_n A_n^t)| = 0$.

We can now pass to define the continuous Steiner symmetrization for subsets of $\mathbb{R}^d$, with respect to a hyperplane that, with no loss of generality, we can suppose to be $\mathbb{R}^{d-1}$. For a general set A we define the projection of A on $\mathbb{R}^{d-1}$ as

$$A' = \{x' \in \mathbb{R}^{d-1} : (x', y) \in A \text{ for some } y \in \mathbb{R}\},$$

and for $x' \in A'$ the intersection of A with $(x', \mathbb{R})$ as

$$A(x') = \{y \in \mathbb{R} : (x', y) \in A\}.$$

Note that $A(x')$ is a one-dimensional set. When A is an open subset of $\mathbb{R}^d$ we define its continuous Steiner symmetrization A^t by

$$A^t = \{x = (x', y) : x' \in A', y \in (A(x'))^t\}. \quad (2)$$

If $A \subset \mathbb{R}^d$ is only measurable, we define its continuous Steiner symmetrization by the same formula as (2), but up to Lebesgue negligible sets.

We stress that, for a bounded quasi-open set A , the previous construction only provides a measurable set defined up to a set of zero Lebesgue measure. In order to obtain that the symmetrized sets be still quasi-open and defined quasi-everywhere,

it is convenient, for a bounded quasi-open set A , to define (by an abuse of notation) the symmetrized set A^t in the following way: consider a decreasing sequence of bounded open sets (A_n) with $\text{cp}(A_n \setminus A) \rightarrow 0$ and $A \subset A_n$. For any $t \in [0, 1]$ the set A_n^t is well defined, and by monotonicity we may define $A_n^t \supset A_{n+1}^t$. Then (A_n^t) is γ -convergent and we define

$$A^t = \gamma - \lim_{n \rightarrow \infty} A_n^t.$$

In this way, the set A^t is quasi-open. More details on this issue can be found in [6]; in particular, the proofs that the construction above is independent of the sequence A_n and that the Lebesgue measure is preserved, are still missing.

The continuous Steiner symmetrization can be defined for any positive measurable function u by symmetrizing its level sets:

$$\forall s > 0 \quad \{u^t > s\} := \{u > s\}^t.$$

The main properties of the Brock's construction are summarized here below, where $\lambda_k(\Omega)$ denotes the k -th eigenvalue of the Dirichlet Laplacian in Ω .

Proposition 3.1. *For every bounded quasi-open set $\Omega \subset \mathbb{R}^d$ and every positive integer k the mapping $t \mapsto \lambda_k(\Omega^t)$, is lower semicontinuous on the left and upper semicontinuous on the right.*

When the starting set Ω is convex, or more generally when the one-dimensional sections $\Omega(x')$ above are intervals, the γ -continuity actually occurs. However, this is not always the case, as the example of Figure 3.1 shows. Up to the moment when the internal fracture appears the γ -continuity is verified; on the other hand, the Brock's construction removes the fracture instantaneously, and the γ -continuity is lost.

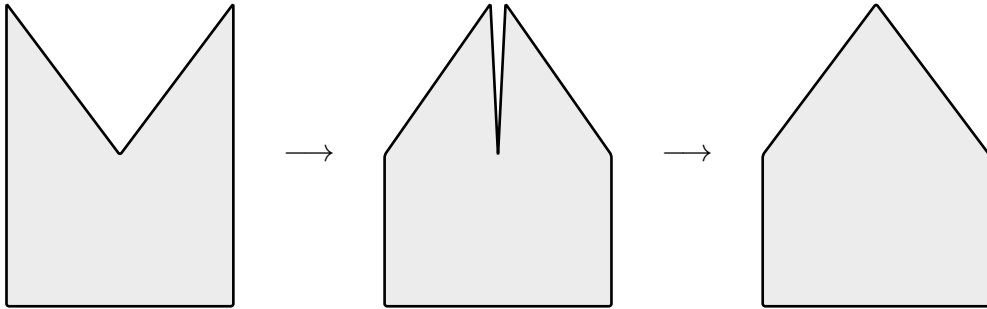


Figure 3.1: A set Ω such that $t \mapsto \lambda(\Omega_t)$ is discontinuous.

Since the torsional rigidity $T(\Omega_t)$ is increasing along the family (Ω_t) , it has only countably many discontinuity points. Let t_0 be one of these points and assume that at t_0 we have two domains Ω^-, Ω^+ such that $\Omega^- \subset \Omega^+$ and

$$\begin{cases} T(\Omega_t) \rightarrow T(\Omega^-) & \text{as } t \rightarrow t_0 \text{ from the left} \\ T(\Omega_t) \rightarrow T(\Omega^+) & \text{as } t \rightarrow t_0 \text{ from the right} \end{cases} \quad (3)$$

In other words Ω^- is the domain with fractures, while Ω^+ is the domain where the fractures have been removed.

Remark 3.2. In the one-dimensional case the existence of a γ -continuous family (Ω_t) cannot be obtained in general, since starting by Ω_0 made of two segments and ending by Ω_1 made of a single segment will necessarily produce a discontinuity of $T(\Omega_t)$ at some point t_0 , independently of the construction of the family (Ω_t) . $\square$

In the case $d \geq 2$ on the contrary, we can fill the discontinuity between Ω^- and Ω^+ by constructing a γ -continuous family (Ω_t) , with Ω_t increasing with respect to the set inclusion, and $\Omega_0 = \Omega^-$, $\Omega_1 = \Omega^+$.

Theorem 3.3. *Let $d \geq 2$ and let $\Omega_0 \subset \Omega_1$ be two bounded open sets. Then there exists a γ -continuous family Ω_t of open sets ($t \in [0, 1]$) such that*

$$\Omega_s \subset \Omega_t \quad \text{for every } s < t. \quad (4)$$

Proof. We denote by C a large cube containing Ω_1 and by $\Gamma(t)$ a Peano curve from $[0, 1]$ onto C , that is a continuous mapping $\Gamma : [0, 1] \rightarrow \mathbb{R}^d$ such that $\Gamma([0, 1]) = C$; we also choose $\Gamma(0) \in \Omega_0$. We define

$$\Omega_t = (\Omega_1 \setminus \Gamma([0, 1 - t])) \cup \Omega_0 \quad \text{for every } t \in [0, 1].$$

Note that Ω_t are open subsets of $\mathbb{R}^d$ and that for $t = 0$ we obtain Ω_0 , while for $t = 1$ we obtain Ω_1 . The family Ω_t above clearly satisfies the monotonicity property (4).

In order to show that the family Ω_t is γ -continuous, it is enough to prove that

$$\text{cp}(\Omega_{t_n} \triangle \Omega_t) \rightarrow 0 \quad \text{whenever } t_n \rightarrow t.$$

This comes from the fact that the mapping $\Gamma(t)$ is uniformly continuous, so that

$$|\Gamma(t) - \Gamma(t_n)| \leq \omega(|t - t_n|)$$

for a suitable modulus of continuity ω . Therefore Ω_t and Ω_{t_n} differ by a set which has a diameter less than $2\omega(|t - t_n|)$, hence of capacity which vanishes as $t_n \rightarrow t$. $\square$

Remark 3.4. Since the proof of Theorem 3.3 is only based on capacitary arguments, the same statement is valid in the more general case when Ω_0 and Ω_1 are quasi-open sets. $\square$

Remark 3.5. When working with polyhedral domains (i.e. whose boundary is made of a finite number of subsets of hyperplanes) we are in the situation above. In fact, if Ω is a polyhedral domain, the Brock's construction provides a family Ω_t made of polyhedral domains, and we have a finite number of discontinuity points $t_1, t_2, \dots, t_N$. In addition, for every discontinuity point t_k , the fracture S is a $d - 1$ dimensional polyhedral set, $\Omega^- = \Omega_{t_k}$ while $\Omega^+ = \Omega_{t_k} \setminus S$, and then Theorem 3.3 applies. $\square$

In several situations (see for instance [8]), thanks to the γ -density of polyhedral domains in the class of all domains, Remark 3.5 is sufficient to achieve the required goals. However, the question of existence of γ -continuous paths (Ω_t) , with monotone $\lambda(\Omega_t)$ and $T(\Omega_t)$, between a general domain Ω_0 and the ball B with the same Lebesgue measure, remains.

Similar questions arise if, instead of the quantities $\lambda(\Omega_t)$ and $T(\Omega_t)$, one considers for instance the perimeter $P(\Omega_t)$, requiring the continuity of the map $t \mapsto P(\Omega_t)$ and its decreasing monotonicity.

The procedure of *removing fractures* mentioned after (3) needs to be more rigorous. This can be made through the following result.

Proposition 3.6. *Let Ω_0 be a given quasi open set and let $m \geq |\Omega_0|$. Then there exists a quasi open set $\hat{\Omega}$ solving the shape optimization problem*

$$\min \{ \lambda(\Omega) : \Omega_0 \subset \Omega, |\Omega| \leq m \}.$$

Proof. The proof can be obtained directly by applying the existence result contained in [7]. $\square$

In an analogous way we can obtain a solution for the shape optimization problem

$$\max \{ T(\Omega) : \Omega_0 \subset \Omega, |\Omega| \leq m \}.$$

In particular, the case $m = |\Omega_0|$ is interesting; this allows to obtain, for every given Ω_0 , an optimal domain $\hat{\Omega}$ containing Ω_0 and with the same measure as Ω_0 , which solves simultaneously the two shape optimization problems

$$\begin{cases} \min \{ \lambda(\Omega) : \Omega_0 \subset \Omega, |\Omega| = |\Omega_0| \}, \\ \max \{ T(\Omega) : \Omega_0 \subset \Omega, |\Omega| = |\Omega_0| \}. \end{cases}$$

Indeed, if Ω_1 is an optimal domain for the eigenvalue optimization problem and Ω_2 an optimal domain for the torsion optimization problem, it is enough to take $\hat{\Omega} = \Omega_1 \cup \Omega_2$.

In other words, if Ω_0 is a Lipschitz domain, we have $\hat{\Omega} = \Omega_0$ while, in the case the set Ω_0 presents some internal fractures, the set $\hat{\Omega}$ removes them.

4. The minimizing movement approach

An alternative approach to the Brock's construction of the family Ω_t through the Continuous Steiner Symmetrization could be the use of the De Giorgi minimizing movement theory, introduced in [10] (see for instance [2], [3] for a detailed presentation and further developments).

In our framework of shape functionals, the metric space X could be the one of all measurable subsets Ω of the Euclidean space $\mathbb{R}^d$ with a prescribed Lebesgue measure, say $|\Omega| = 1$, endowed with the L^1 distance

$$d(\Omega_1, \Omega_2) = |\Omega_1 \Delta \Omega_2|.$$

Given a shape functional F defined on X one can consider the so-called *implicit Euler scheme* of time step ε and initial condition Ω_0 , which provides a discrete family $\Omega_{n,\varepsilon}$ constructed recursively in the following way:

$$\Omega_{0,\varepsilon} = \Omega_0, \quad \Omega_{n+1,\varepsilon} \in \operatorname{argmin}_{\Omega \in X} \left\{ F(\Omega) + \frac{1}{2\varepsilon} |\Omega \Delta \Omega_{n,\varepsilon}|^2 \right\}.$$

We may then set $\Omega_{t,\varepsilon} = \Omega_{[t/\varepsilon],\varepsilon}$, where $[\cdot]$ stands for the integer part function, and say that Ω_t is a family of sets constructed by the minimizing movement procedure associated to the shape functional F if for every $t \in [0, T]$ we have

$$|\Omega_t \triangle \Omega_{[t/\varepsilon],\varepsilon}| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

If the limit above occurs only for a sequence (ε_n) (independent of t), we say that Ω_t is a generalized minimizing movement.

It is easy to see that the discrete sequence $\Omega_{n,\varepsilon}$ is such that $F(\Omega_{n,\varepsilon})$ decreases. It would be interesting to show, at least in the particular cases when the shape functional $F(\Omega)$ is the first eigenvalue $\lambda(\Omega)$, the opposite $-T(\Omega)$ of the torsional rigidity, or the perimeter $P(\Omega)$, or some convex combination of them, that the map $t \mapsto F(\Omega_t)$ is continuous and decreasing.

We do not know if the map $t \mapsto F(\Omega_t)$ above is continuous and decreasing, and the cases in which, as $t \rightarrow \infty$, the limit domain is a ball. Some results in this direction, in the case $F(\Omega) = P(\Omega)$ can be found in [14], while some partial results in the case of spectral functionals can be found in [13].

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