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Various Perturbations and Relaxations of the Sweeping Process

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Dedicated to Roger J-B Wets on the occasion of his 85th birthday.

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The paper is concerned with diverse types of perturbations of dynamical sweeping processes along with their relaxation via Young measures. Skorohod-like problems with Volterra integro-differential perturbations are explored. Sweeping processes coupled with rough differential equations as well as with fractional differential equations are studied. Periodicity and asymptotic properties are also provided.

Keywords: Sweeping process, perturbation, prox-regular set, normal cone, relaxation, Skorohod problem, fractional derivative, fractional integral, rough signal, Young integral, Young measure, periodicity, asymptotic property.

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1. Introduction

Given an interval I with initial point $T_0 \in I$ and a multimapping $C\colon I \rightrightarrows E$ from I into nonempty closed sets of a Hilbert space E, the sweeping process as introduced and studied by Moreau in the 1970s (see [42, 43, 45]) corresponds to the differential inclusion

$$\begin{cases} u(T_0) = a,\ u(t) \in C(t)\ \forall t \in I \\ -\frac{du}{dt}(t) \in N_{C(t)}u(t), \end{cases}$$

where $a \in C(T_0)$ is the initial state and $N_{C(t)}(\cdot)$ is a normal cone to the set $C(t)$. Mechanical motivations and interpretations can be found in the above Moreau's works dealing with convex sets $C(t)$. The present paper is devoted to diverse new properties of the perturbed dynamical form

$$\begin{cases} u(T_0) = a,\ u(t) \in C(t)\ \forall t \in I \\ -\frac{du}{dt}(t) \in N_{C(t)}u(t) + \Gamma(t, u(t)), \end{cases}$$

where Γ is a mapping or multimapping from $I \times E$ into E. Such dynamical systems received in the literature many applications in Economics, electrical circuits, crowd motion, etc. Most of our results are related to prox-regular sets $C(t)$.

The paper is organized as follows. In Section 2 we recall the main concepts of Variational Analysis involved in the text, mainly the notion of prox-regular sets

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and certain fundamental properties of such sets. After stating in Section 3 some basic features related to dynamical sweeping processes associated to a Carathéodory mapping $f\colon I \times E \to E$ with $f(t,x)$ Lipschitz in x, that is, differential inclusions in the form

$$\begin{cases} u(T_0) = a,\ u(t) \in C(t)\ \forall t \in I \\ -\frac{du}{dt}(t) \in N_{C(t)}u(t) + f(t,u(t)), \end{cases}$$

we establish in Theorem 3.6 a new result for the Skorohod problem driven by the sweeping process in the line of [19, 21, 40, 53]. The principal novelty of the Skorohod type problem in Theorem 3.6 is that a Volterra type integral perturbation is considered there. Other Volterra type integral perturbations are contained in Section 4 whereas Section 5 is concerned with integro-differential sweeping processes coupled with fractional differential equations. Section 6 is devoted to perturbations involving Young integral. Optimization problems subject to constraints defined by sweeping processes coupled with rough differential equations are the object of Section 7 as well as optimization problems under sweeping processes with Volterra integral type perturbations. Section 8 develops a relaxation study for the set of solutions associated to sweeping processes perturbed by BV selections of certain multimappings as well as for the set of solutions of sweeping processes with Volterra integral perturbations. Relaxations with Young measures are considered in the same Section 8 along with Bolza problems whose objective functions involve the Young integral. Mixed semicontinuous perturbations of sweeping processes are analyzed in Section 9 and perturbations of second order sweeping processes are studied in Section 10. Periodicity of solutions of the aforementioned dynamical perturbed sweeping processes are largely studied in Section 11 while the final Section 12 provides diverse asymptotic properties.

2. Preliminaries

Throughout the paper $\mathbb{R}_+ := [0,+\infty[$ and $\mathbb{N}$ is the set of positive integers. The space of continuous mappings from a topological space X into another one Y will be denoted $\mathcal{C}(X,Y)$ or $\mathcal{C}_Y(X)$. Given a nonempty subset S of a Hilbert space E its *support function* $\delta^*(\cdot,S)$ is defined as the Legendre-Fenchel conjugate of its indicator function $\delta(\cdot,S)$ with $\delta(x,S) = 0$ if $x \in S$ and $\delta(x,S) = +\infty$ if $x \notin S$, that is,

$$\delta^*(\xi,S) = \sup\{\langle \xi, x\rangle : x \in S\} \quad \text{for all } \xi \in E.$$

So, one has
$$y \in \overline{\mathrm{co}}\, S \quad \text{if and only} \quad \langle \cdot, y\rangle \le \delta^*(\cdot,S), \tag{2.1}$$

where $\overline{\mathrm{co}}$ stands for the closed convex hull. The *distance function* from/to the set S will be denoted d_S or $d(\cdot,S)$ depending on the context, that is,

$$d_S(x) = d(x,S) := \inf_{y\in S} \|x-y\| \quad \text{for all } x \in E.$$

By convention, we will write $|S| := \{\|y\| : y \in S\}$. As usual $\mathcal{B}(E)$ is the *Borel σ-field* of E and $\mathcal{L}(J)$ is the *Lebesgue σ-field* of any interval J of $\mathbb{R}$. For any subset P of an interval J we use the standard notation $\mathbf{1}_P$ for the function given by $\mathbf{1}_P(t) = 1$ if $t \in P$ and $\mathbf{1}_P(t) = 0$ otherwise. The next parts of this section are devoted to recall some concepts which will be needed in the development of the paper.

We start with the concept of absolute continuity of multimappings with time variable. Given an interval $I = [T_0, T]$ (or an unbounded interval I, or a non-closed interval I), one of the best way to translate the absolute continuity of a mapping $\zeta\colon I \to E$ is to require that ζ is derivable Lebesgue almost everywhere on I, a.e. on I for short, with its derivative $\dot{\zeta} := \frac{d\zeta}{dt}$ Lebesgue integrable on I and that

$$\zeta(t) - \zeta(s) = \int_s^t \dot{\zeta}(\tau)\, d\tau \quad \text{for all } s, t \in I.$$

When ζ defined on an unbounded interval I (or a non-closed interval I) is derivable a.e. on I and the latter equality still holds for all $s, t \in I$ with $\dot{\zeta}$ locally Lebesgue integrable on I, one says that ζ is locally absolutely continuous on I. Clearly, ζ is locally absolutely continuous on I if and only if it is absolutely continuous on every compact interval included in I. When I is a compact interval in $\mathbb{R}$ the space of absolutely continuous mappings from I into E is usually denoted $W^{1,1}(I, E)$.

Taking I as any interval of $\mathbb{R}$, a multimapping $C\colon I \rightrightarrows E$ is known to be *absolutely continuous* (resp. *locally absolutely continuous*) on I if there is an absolutely continuous (resp. locally absolutely continuous) function $v\colon I \to \mathbb{R}$ such that

$$d(x, C(t)) \le d(x, C(s)) + |v(t) - v(s)| \quad \text{for all } x \in E \text{ and } s, t \in I; \tag{2.2}$$

such an absolutely continuous (resp. locally absolutely continuous) function $v(\cdot)$ will be called a *control function of the absolute continuity (resp. local absolute continuity) of the multimapping* $C(\cdot)$. It is worth noticing that the graph

$$\operatorname{gph} C := \{(t, x) \in I \times E : x \in C(t)\}$$

of the multimapping $C\colon I \rightrightarrows E$ is closed in $I \times E$ whenever $C(\cdot)$ is (locally) absolutely continuous with $C(t)$ closed for all $t \in I$. Indeed, consider any sequence $(t_n, x_n)_n$ in $\operatorname{gph} C$ converging to (t, x) in $I \times E$. For the control function $v(\cdot)$ of (local) absolute continuity, we have

$$d(x_n, C(t)) \le d(x_n, C(t_n)) + |v(t_n) - v(t)| = |v(t_n) - v(t)|,$$

so as $n \to \infty$ the continuity of $v(\cdot)$ and of $d(\cdot, C(t))$ gives $d(x, C(t)) = 0$, hence $x \in C(t)$ since $C(t)$ is closed. Then $(t, x) \in \operatorname{gph} C$, which justifies that $\operatorname{gph} C$ is closed in $I \times C$. (In fact, we see that $\operatorname{gph} C$ is closed in $I \times E$ whenever the above function $v(\cdot)$ is merely continuous.)

In certain situations it can be useful to look at the absolute continuity property of $C(\cdot)$ with the control function $v(\cdot)$ in the form

$$\operatorname{haus}\big(C(s), C(t)\big) \le |v(t) - v(s)| \quad \text{for all } s, t \in I,$$

where haus denotes the *Hausdorff-Pompeiu distance* given for subsets S_1, S_2 of the space E by

$$\operatorname{haus}(S_1, S_2) = \max\{\operatorname{exc}(S_1, S_2), \operatorname{exc}(S_2, S_1)\},$$

and $\operatorname{exc}(S_1, S_2) = \sup_{x \in S_1} d(x, S_2)$ with the usual convention that this supremum is 0 if $S_1 = \emptyset$ (resp. $+\infty$ if $S_1 \neq \emptyset$ and $S_2 = \emptyset$). One calls $\operatorname{exc}(S_1, S_2)$ the excess of S_1 over S_2 (see, e.g., [54, 57]). Very often, it can be convenient to use the abbreviation "AC" for "*absolutely continuous*", in particular when we will have to consider that property for single valued mappings. Similarly, we will often write, for short, "BVC" for "*continuous with bounded variation*" mappings, and "BVRC" for mappings which are "*right continuous with bounded variation*"; we refer, e.g., to [44] for such mappings.

Let $\varphi\colon E \to \mathbb{R} \cup \{+\infty\}$ be a function on E. For any $x \in E$ we denote by $N_S x$ the *Clarke normal cone* of S at x and by $\partial\varphi(x)$ the *Clarke subdifferential* of φ at x; we refer to [24] for the definitions. One has $N_S x = \emptyset$ for $x \notin S$ and $\partial\varphi(x) \subset \gamma\mathbb{B}_E$ whenever the function φ is finite on an open neighborhood U of x and γ-Lipschitz on U (see [24]); further $\partial\varphi(y) \neq \emptyset$ for any $y \in U$. Above $\mathbb{B}_E$ is the closed unit ball of E centered at zero. The Clarke normal cone is linked with the Clarke subdifferential of the distance function in the sense that for $x \in S$ one has the equality

$$N_S x = \mathrm{cl}_w\big(\mathbb{R}_+ \partial d_S(x)\big), \tag{2.3}$$

where cl_w denotes the weak closure in E. One also has for $x \in S$

$$\tau\partial d_S(x) \subset \tau'\partial d_S(x) \quad \text{for all reals } 0 \le \tau \le \tau'.$$

A remarkable property of the Clarke subdifferential of the function φ is that the multimapping

$$x \mapsto \partial\varphi(x) \text{ is norm-to-weak upper semicontinuous} \tag{2.4}$$

on E whenever φ is real-valued and locally Lipschitz on E; this is known to be equivalent to the norm$\times$weak closedness of the graph

$$\mathrm{gph}\,\partial\varphi := \{(x,\xi) \in E \times E : \xi \in \partial\varphi(x)\}$$

of $\partial\varphi$ since each point in E has a neighborhood whose range is contained in some weakly compact set in E.

The Clarke normal cone of a prox-regular set possesses diverse subregularity properties of great importance in the analysis of sweeping processes. The prox-regularity of sets is related to proximal normals. A vector $\xi \in E$ is called a *proximal normal vector* to the set S at x if there are $y \in E$ and a real $\tau > 0$ such that $\xi = \tau(y - x)$ with $x \in \mathrm{Proj}_S y$, where

$$\mathrm{Proj}_S y := \{p \in S : \|y - p\| = d_S(y)\},$$

that is, the set of nearest points of y in S. The set of such vectors ξ is denoted by $N_S^P x$ and called the *proximal normal cone* of S at x, so clearly $N_S^P x = \emptyset$ whenever $x \notin S$. It is also clear that a vector $\xi \in \mathbb{B}_E$ is a proximal normal of S at $x \in S$ if and only if for some real $\tau > 0$

$$x \in \mathrm{Proj}_S(x + t\xi) \quad \text{for all } t \in]0, \tau].$$

The prox-regularity of S amounts to requiring the uniformity of such a real $\tau > 0$. Given an extended real $r \in]0, +\infty]$, a nonempty closed set S of the Hilbert space E is defined to be *r-prox-regular* provided that for any $x \in S$ and $\xi \in \mathbb{B}_E \cap N_S^P x$ one has

$$x \in \mathrm{Proj}_S(x + t\xi) \quad \text{for every positive real } t \le r.$$

The next theorem recalls some basic properties of prox-regular sets; see, e.g., [49, 57] for (a)–(c), and [56, Proposition 10.4] for (d).

Theorem 2.1. *Let S be a nonempty closed set of a Hilbert space E which is r-prox-regular for some $r \in]0,+\infty]$. Let $C\colon [T_0,T] \rightrightarrows E$ be a multimapping with nonempty r-prox-regular sets which is absolutely continuous on $[T_0,T]$. The following hold.*

(a) *For $x \in S$ one has $\zeta \in N_S x$ if and only if*

$$\langle \zeta, y-x\rangle \le \frac{\|\zeta\|}{2r}\|y-x\|^2 \quad \textit{for all}\, y \in S;$$

further, $N_S(\cdot)\cap \gamma\mathbb{B}_E$ is (γ/r)-hypomonotone in the sense that

$$\langle \zeta-\xi, x-y\rangle \ge -\frac{\gamma}{r}\|x-y\|^2$$

for all $x,y \in S$, $\zeta \in N_S(x)\cap\gamma\mathbb{B}_E$ and $\xi \in N_S(y)\cap\gamma\mathbb{B}_E$.

(b) *For any $x \in S$ one has $N_S^P x = N_S x$.*

(c) *The metric projection Proj_S is single valued and continuous on the open r-enlargement $U_r(S) := \{u \in E : d_S(u) < r\}$ of S, and the square distance function d_S^2 is Fréchet differentiable on $U_r(S)$ with*

$$\nabla(\frac{1}{2}d_S^2)(u) = u - \mathrm{Proj}_S u \quad \textit{for all}\, u \in U_r(S).$$

(d) *For any net $(t_j)_{j\in J}$ in $[T_0,T]$ converging to $t \in [T_0,T]$, any net $(x_j)_{j\in J}$ in E converging to $x \in C(t)$, and any net $(\zeta_j)_{j\in J}$ in E converging weakly to ζ with $\zeta_j \in \partial d_{C(t_j)}(x_j)$ for every $j \in J$, one has $\zeta \in \partial d_{C(t)}(x)$.*

Given a mapping z on an interval $[T_0,T]$ of $\mathbb{R}$ with values in $\mathbb{R}^d$, for s,t in this interval with $s<t$ we denote (as in [30]) its (first) *variation on* $[s,t]$ by $|z|_{1-var;[s,t]}$, that is,

$$|z|_{1-var;[s,t]} := \sup\left(\sum_{i=0}^{p-1}\|z_{t_{i+1}} - z_{t_i}\|_{\mathbb{R}^d}\right),$$

where the supremum is taken over all finite sequences $(t_0,t_1,\cdots,t_p)$, $p \in \mathbb{N}$, with $s = t_0 < t_1 < \cdots < t_p = t$, and where we put $z_t := z(t)$ by convenience. By convention, $|z|_{1-var;[s,t]} = 0$ when $s=t$, and it is known that

$$|z|_{1-var;[t_1,t_3]} = |z|_{1-var;[t_1,t_2]} + |z|_{1-var;[t_2,t_3]} \quad \text{for } t_1 \le t_2 \le t_3 \text{ in } I.$$

When $|z|_{1-var;[s,t]} < +\infty$ one says that the mapping z is of bounded (or finite) variation on $[s,t]$. We denote by $\mathcal{C}^{1-var}([T_0,T],\mathbb{R}^d)$ the space of continuous mappings $z\colon [T_0,T] \to \mathbb{R}^d$ of bounded variation on $[T_0,T]$ with values in $\mathbb{R}^d$. A remarkable property of any mapping $z \in \mathcal{C}^{1-var}([T_0,T],\mathbb{R}^d)$ is that (see [30, Proposition 1.12])

$$\text{the function } t \mapsto |z|_{1-var;[T_0,t]} \text{ is continuous on } [T_0,T]. \tag{2.5}$$

As usual, we also denote by $\mathcal{L}(\mathbb{R}^d,\mathbb{R}^e)$ the vector space of linear mappings Λ from $\mathbb{R}^d$ into $\mathbb{R}^e$ endowed with the operator norm

$$|\Lambda| := \sup_{x\in\mathbb{R}^d, \|x\|_{\mathbb{R}^d=1}} \|\Lambda(x)\|_{\mathbb{R}^e}.$$

Given a mapping $z \in \mathcal{C}^{1-var}([T_0, T], \mathbb{R}^d)$, for any closed subinterval J of $[T_0, T]$ and any continuous mapping $\Lambda\colon [T_0, T] \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ we have for its *Riemann-Stieljes integral* $\int_J \Lambda(\tau)\, dz_\tau$ that (see, e.g., Proposition 2.2 in Friz and Victoir [30])

$$\left| \int_J \Lambda(\tau)\, dz_\tau \right| \le |\Lambda(\cdot)|_{\infty;J} |z|_{1-var;J}, \tag{2.6}$$

where $|\Lambda(\cdot)|_{\infty;J} := \sup_{\tau \in J} |\Lambda(\tau)|$. If a sequence of continuous mappings $(\Lambda_n(\cdot))$ from $[T_0, T]$ into $\mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ converges uniformly to $\Lambda(\cdot)$, Proposition 2.7 in Friz and Victoir [30] says that

$$\int_J \Lambda_n(\tau)\, dz_\tau \longrightarrow \int_J \Lambda(\tau)\, dz_\tau \quad \text{as } n \to \infty. \tag{2.7}$$

When we write, as above and in all the remaining of the paper, $\mathbb{R}^d$, $\mathbb{R}^e$ etc, it is implicit that d, e are positive integers.

We recall now the following Gronwall lemma.

Lemma 2.2. *Let I be a nonempty interval of $\mathbb{R}$ and $\phi\colon I \to \mathbb{R}_+$ be a non-negative function which is locally absolutely continuous on I. Let $\alpha, \beta\colon I \to \mathbb{R}$ be locally Lebesgue integrable on I. Assume that*

$$\frac{d\phi}{dt}(t) \le \alpha(t) + \beta(t)\phi(t) \ \text{a.e. } t \in I.$$

Then for any $\theta, t \in I$ with $\theta \le t$ one has

$$\phi(t) \le \int_\theta^t \left(\alpha(s) \exp\left[\int_s^t \beta(\tau)\, d\tau \right] \right) ds + \phi(\theta) \exp\left[\int_\theta^t \beta(\tau)\, d\tau \right].$$

Another more simple lemma is the following.

Lemma 2.3. *Let $\gamma\colon I \to \mathbb{R}_+$ be a non-negative function on an interval I of $\mathbb{R}$ and $\rho\colon I \to \mathbb{R}_+$ be a non-negative locally absolutely continuous function on the interval I such that*

$$\frac{d}{dt}\left(\frac{1}{2}\rho(t)^2\right) \le \gamma(t)\rho(t) \ \text{a.e. } t \in I.$$

Then one has

$$\frac{d\rho}{dt}(t) \le \gamma(t) \ \text{a.e. } t \in I.$$

Proof. Let $P \subset I$ be with $I \setminus P$ Lebesgue null and such that at each $t \in P$ the function ρ is derivable and the inequality in the assumption

$$\frac{d}{dt}\left(\frac{1}{2}\rho(t)^2\right) \le \gamma(t)\rho(t)$$

holds true. At any $t \in P$ the usual chain rule gives $\frac{d}{dt}\left(\frac{1}{2}\rho(t)^2\right) = \rho(t)\frac{d\rho}{dt}(t)$, so if $\rho(t) > 0$ we deduce that $\frac{d\rho}{dt}(t) \le \gamma(t)$. On the other hand, at any $t \in P$ with $\rho(t) = 0$ the inequality $\rho(\tau) - \rho(t) \ge 0$ for all $\tau \in I$ (since $\rho(\tau) \ge 0$), ensures that $\frac{d\rho}{dt}(t) = 0$. We can conclude that $\frac{d\rho}{dt}(t) \le \gamma(t)$ for all $t \in P$ (since $\gamma(\cdot) \ge 0$). $\square$

3. A Skorohod problem with new perturbations of sweeping process

We begin with the following version of a classical lemma for which we provide a proof for completeness. Given an extended real $p \in [1, \infty]$, a compact interval I of $\mathbb{R}$ and a separable Hilbert space E, we will often denote, for short, $L^p_E(I)$ in place of $L^p_E(I, dt)$.

Lemma 3.1. *Let E, G be separable Hilbert spaces, $I = [T_0, T]$ and S be a nonempty subset of G. Let $\Gamma\colon I \times S \rightrightarrows E$ be a multimapping from $I \times S$ into the set of nonempty weakly compact convex sets of E. Let $(\xi_n)_n$ be a sequence in $L^1_E(I)$ converging to ξ in $L^1_E(I)$ with respect to the $\sigma(L^1_E(I), L^\infty_E(I))$ topology. Let u and u_n be mappings from I into S for $n \in \mathbb{N}$ and let β be a function from I into $\mathbb{R}_+$.*

(a) *If $\xi_n(t) \in \beta(t)\Gamma(t, u(t))$ for almost every $t \in I$, then one has*

$$\xi(t) \in \beta(t)\Gamma(t, u(t)) \quad a.e.\ t \in I.$$

(b) *If $(u_n)_n$ converges almost everywhere to u with $\xi_n(t) \in \beta(t)\Gamma(t, u_n(t))$ for almost every $t \in I$ and if, for almost every $t \in I$, the multimapping $\Gamma(t, \cdot)$ is upper semicontinuous for E endowed with the weak topology, then*

$$\xi(t) \in \beta(t)\Gamma(t, u(t)) \quad a.e.\ t \in I.$$

(c) *Assume that Γ is upper semicontinuous for E endowed with the weak topology. Assume also that $(u_n)_n$ converges almost everywhere to u and that there is a Lebesgue negligible subset N of I for which for each $t \in I \setminus N$ there is a sequence $(t_n)_n$ in I converging to t such that $\xi_n(t) \in \beta(t)\Gamma(t_n, u_n(t))$ for every $n \in \mathbb{N}$. Then one has*

$$\xi(t) \in \beta(t)\Gamma(t, u(t)) \quad a.e.\ t \in I.$$

Proof. We prove (c) first. By Mazur's lemma there exists a sequence $(\widehat{\xi}_n)_n$ converging strongly to ξ in $L^1_E(I)$ with

$$\widehat{\xi}_n \in \mathrm{co}\,\{\xi_k : k \geq n\} \quad \text{for all } n \in \mathbb{N}.$$

Extracting a subsequence we may suppose that $(\widehat{\xi}_n)_n$ converges almost everywhere on I to ξ, so there is a Lebesgue negligible set $P \subset I$ such that for each $t \in I \setminus P$ we have $\widehat{\xi}_n(t) \in \mathrm{co}\,\{\xi_k(t) : k \geq n\}$ for all $n \in \mathbb{N}$ as well as

$$\widehat{\xi}_n(t) \to \xi(t) \quad \text{and} \quad u_n(t) \to u(t) \quad \text{as } n \to \infty.$$

Fix any $t \in I \setminus (N \cup P)$. Then we have

$$\xi(t) \in \bigcap_{n \in \mathbb{N}} \overline{\mathrm{co}}\,\{\xi_k(t) : k \geq n\},$$

hence for each fixed $y \in E$

$$\langle y, \xi(t)\rangle \leq \inf_{n \in \mathbb{N}} \sup_{k \geq n} \langle y, \xi_k(t)\rangle = \limsup_{n \to \infty} \langle y, \xi_n(t)\rangle,$$

which in turn gives

$$\langle y, \xi(t)\rangle \leq \limsup_{n \to \infty} \delta^*(y, \beta(t)\Gamma(t_n, u_n(t))) = \beta(t) \limsup_{n \to \infty} \delta^*(y, \Gamma(t_n, u_n(t))).$$

Since the support function $(\tau, x) \mapsto \delta^*(y, \Gamma(\tau, x))$ is upper semicontinuous relative to $I \times S$ (see, e.g., Theorem II-20 in [23]), we deduce that

$$\langle y, \xi(t)\rangle \leq \beta(t)\delta^*(y, \Gamma(t, u(t))) = \delta^*(y, \beta(t)\Gamma(t, u(t))).$$

This being true for every $y \in E$, it follows from (2.1) that $\xi(t) \in \beta(t)\Gamma(t, u(t))$, which confirms the property (b).

Regarding (a) and (b) it suffices to apply the above arguments with appropriate changes. □

We continue by recalling the result in [55, Theorem 4.4] in a form for which the proof in [55] is still valid.

Theorem 3.2. *Let E be a finite-dimensional Euclidean space, let $I := [T_0, T]$, and let $C\colon I \rightrightarrows E$ be a multimapping with nonempty closed values which is absolutely continuous with $v(\cdot)$ as a control function of absolute continuity. Let $\Gamma\colon I \times E \rightrightarrows E$ be a multimapping with nonempty compact convex values which is $\mathcal{L}(I) \otimes \mathcal{B}(E)$ measurable with respect to (t, x) and upper semicontinuous with respect to x. Assume that there exists a Lebesgue integrable function $m\colon I \to \mathbb{R}_+$ such that for all $t \in I$ and all $x \in C(t)$*

$$\max\{\|y\| : y \in \Gamma(t, x)\} \leq m(t). \tag{3.1}$$

Then for any $a \in C(T_0)$ the sweeping process

$$\begin{cases} u(T_0) = a,\ u(t) \in C(t),\ \forall t \in I; \\ -\frac{du}{dt}(t) \in N_{C(t)}u(t) + \Gamma(t, u(t)), \quad a.e.\ t \in I \end{cases}$$

admits at least one absolutely continuous solution $u(\cdot)$ satisfying

$$-\frac{du}{dt}(t) \in \big(|\dot{v}(t)| + m(t)\big)\partial d_{C(t)}(u(t)) + \Gamma(t, u(t)) \quad a.e.\ t \in I,$$

so $\|\dot{u}(t)\| \leq |\dot{v}(t)| + 2m(t)$ a.e. $t \in I$.

A similar theorem also holds true in infinite dimensions under the prox-regularity of the sets $C(t)$. It can be seen as consequence of [26, 55].

Theorem 3.3. *Let E be a separable Hilbert space, let I be either $[T_0, T]$ or $[T_0, +\infty[$, and let $r \in]0, +\infty]$. Let $C\colon I \rightrightarrows E$ be a multimapping with nonempty closed values such that $C(t)$ is r-prox-regular for all $t \in I$, and let $a \in C(T_0)$. Let $f\colon I \times E \to E$ be a mapping with $f(\cdot, x)$ Lebesgue integrable on I and let $\beta, \lambda\colon I \to \mathbb{R}_+$ be functions such that*

$$\|f(t, x) - f(t, y)\| \leq \lambda(t)\|x - y\| \text{ for all } t \in I,\ x, y \in E$$

and

$$\|f(t, x)\| \leq \beta(t)(1 + \|x\|) \quad \text{for all } t \in I,\ x \in E.$$

(I) *If both $\beta(\cdot)$ and $\lambda(\cdot)$ are in $L^1_{\mathbb{R}_+}(I)$ and if $C(\cdot)$ is absolutely continuous on I with $v(\cdot)$ as a control function of absolute continuity, then the sweeping process*

$$\begin{cases} u(T_0) = a,\ u(t) \in C(t),\ \forall t \in I; \\ -\frac{du}{dt}(t) \in N_{C(t)}u(t) + f(t, u(t)), \quad a.e.\ t \in I \end{cases}$$

admits one and only one absolutely continuous solution $u(\cdot)$ on I, and this solution satisfies

$$-\frac{du}{dt}(t) \in \big(|\dot v(t)| + \|f(t,u(t))\|\big)\partial d_{C(t)}(u(t)) + f(t,u(t)) \quad a.e.\ t \in I,$$

so for almost every $t \in I$

$$\|\dot u(t) + f(t,u(t))\| \le |\dot v(t)| + \|f(t,u(t))\| \quad and \quad \|\dot u(t)\| \le |\dot v(t)| + 2\|f(t,u(t))\|.$$

Further, with

$$\kappa := \|a\| + \left(\exp\left\{2\int_I \beta(s)\,ds\right\}\right)\int_I \big[2\beta(s)(1+\|a\|) + |\dot v(s)|\big]\,ds,$$

one has $\|u(\cdot)\|_\infty \le \kappa$ and for a.e. $t \in I$

$$\|f(t,u(t))\| \le (1+\kappa)\beta(t),\ \|\dot u(t) + f(t,u(t))\| \le (1+\kappa)\beta(t) + |\dot v(t)|.$$

In particular, for any $h \in L^1_E(I)$ the differential inclusion

$$\begin{cases} u(T_0) = a,\ u(t) \in C(t),\ \forall t \in I; \\ -\frac{du}{dt}(t) \in N_{C(t)}u(t) + h(t), \quad a.e.\ t \in I \end{cases}$$

has one and only one absolutely continuous solution $u(\cdot)$ on I, and this solution has the additional properties that

$$-\frac{du}{dt}(t) \in \big(|\dot v(t)| + \|h(t)\|\big)\partial d_{C(t)}(u(t)) + h(t) \quad a.e.\ t \in I,$$

and $\|\dot u(t)\| \le |\dot v(t)| + 2\|h(t)\| \quad a.e.\ t \in I.$

(II) *If both $\beta(\cdot)$ and $\lambda(\cdot)$ are locally Lebesgue integrable on $I = [T_0, +\infty[$ and if $C(\cdot)$ is locally absolutely continuous on $[T_0, +\infty[$ with $v(\cdot)$ as a control function of local absolute continuity, then the sweeping process*

$$\begin{cases} u(T_0) = a,\ u(t) \in C(t),\ \forall t \in I; \\ -\frac{du}{dt}(t) \in N_{C(t)}u(t) + f(t,u(t)), \quad a.e.\ t \in I \end{cases}$$

admits one and only one locally absolutely continuous solution $u(\cdot)$ on $I = [T_0, +\infty[$, and this solution satisfies

$$-\frac{du}{dt}(t) \in \big(|\dot v(t)| + \|f(t,u(t))\|\big)\partial d_{C(t)}(u(t)) + f(t,u(t)) \quad a.e.\ t \in I,$$

so $\|\dot u(t)\| \le |\dot v(t)| + 2\|f(t,u(t))\| \quad a.e.\ t \in I.$

Using the latter theorem we establish two lemmas useful for the sequel. Recall first that a subset S of E is *ball-compact* provided that $S \cap \tau\mathbb{B}_E$ is compact in E for every real $\tau > 0$.

Lemma 3.4. *Let E be a separable Hilbert space, let $I := [T_0, T]$, and let $r \in]0, +\infty]$. Let $C\colon I \rightrightarrows E$ be a multimapping with nonempty closed values which is absolutely continuous with $v(\cdot)$ as a control function of absolute continuity and such that $C(t)$ is r-prox-regular for all $t \in I$. Let $a \in C(T_0)$. Let $(h_n)_n$ be a sequence in $L^1_E(I)$ for which there is some function $\varrho \in L^1_{\mathbb{R}_+}(I)$ such that for every $n \in \mathbb{N}$*

$$\|h_n(t)\| \le \varrho(t) \quad a.e.\ t \in I.$$

For each $n \in \mathbb{N}$ let $u_n \colon I \to E$ be the unique absolutely continuous solution of

$$\begin{cases} u_n(T_0) = a, \ u_n(t) \in C(t) \ \forall\, t \in I \\ -\frac{du_n}{dt}(t) \in N_{C(t)} u_n(t) + h_n(t). \end{cases}$$

(a) *If the sequence $(u_n)_n$ converges pointwise to a mapping u on I, then for any $\sigma(L^1_E(I), L^\infty_E(I))$ limit h of a subsequence of $(h_n)_n$, the mapping u is absolutely continuous and solution of the differential inclusion*

$$\begin{cases} u(T_0) = a, \ u(t) \in C(t) \ \forall\, t \in I \\ -\frac{du}{dt}(t) \in N_{C(t)} u(t) + h(t). \end{cases} \tag{3.2}$$

(b) *If $C(t)$ is ball-compact for every $t \in I$, then the sequence $(u_n, h_n)_n$ admits a subsequence converging to (u, h) in the space $\mathcal{C}_E(I) \times L^1_E(I)$ with respect to the $\| \cdot \|_\infty \times \sigma(L^1_E(I), L^\infty_E(I))$ topology and for any such limit (u, h) the mapping u is absolutely continuous and solution of the differential inclusion*

$$\begin{cases} u(T_0) = a, \ u(t) \in C(t) \ \forall\, t \in I \\ -\frac{du}{dt}(t) \in N_{C(t)} u(t) + h(t). \end{cases} \tag{3.3}$$

Proof. Putting $\beta(t) := |\dot{v}(t)| + \varrho(t)$ for all $t \in I$, Theorem 3.3 says that u_n is the absolutely continuous solution of

$$\begin{cases} u_n(T_0) = a, \ u_n(t) \in C(t) \ \forall\, t \in I \\ -\frac{du_n}{dt}(t) \in \beta(t) \partial d_{C(t)}(u_n(t)) + h_n(t). \end{cases} \tag{3.4}$$

(a) By the pointwise convergence of $(u_n)_n$ to u, we note that $u(T_0) = a$ and $u(t) \in C(t)$ since $C(t)$ is closed for every $t \in I$. We also note that there is some Lebesgue null set $N \subset I$ such that for every $n \in \mathbb{N}$ one has $\|h_n(t)\| \le \varrho(t)$ for all $t \in I \backslash N$, hence (keeping in mind that $\varrho \in L^1_{\mathbb{R}_+}(I)$) the sequence $(h_n)_n$ admits at least a $\sigma(L^1_E(I), L^\infty_E(I))$ convergent subsequence. Let h be any $\sigma(L^1_E(I), L^\infty_E(I))$ limit of a subsequence of $(h_n)_n$ that we do not relabel. By (3.4) we have $\|\dot{u}_n(t)\| \le 2\beta(t)$ for all $n \in \mathbb{N}$ and $t \in I \setminus N$, so according to the integrability of $\beta(\cdot)$ some subsequence of $(\dot{u}_n)_n$ (that we do not relabel) $\sigma(L^1_E(I), L^\infty_E(I))$ converges to some mapping ξ in $L^1_E(I)$. For each $t \in I$ and each $y \in E$ we can write for every $n \in \mathbb{N}$

$$\langle y, u_n(t) - a \rangle = \int_I \langle \mathbf{1}_{[T_0,t]}(\tau) y, \frac{du_n}{d\tau}(\tau) \rangle \, d\tau,$$

hence passing to the limit as $n \to \infty$ we obtain for every $y \in E$

$$\langle y, u(t) - a \rangle = \int_I \langle \mathbf{1}_{[T_0,t]}(\tau) y, \xi(\tau) \rangle \, d\tau = \langle y, \int_{T_0}^t \xi(\tau) \, d\tau \rangle.$$

This being true for every $y \in E$ it ensues that $u(t) - a = \int_{T_0}^t \xi(\tau) \, d\tau$ for every $t \in I$, hence u is absolutely continuous on I with $\dot{u}(\tau) = \xi(\tau)$ a.e. $\tau \in I$. This tells us that $(\dot{u}_n + h_n)_n$ converges $\sigma(L^1_E(I), L^\infty_E(I))$ to $\dot{u} + h$, which, combined with Lemma

3.1 and the $\|\cdot\|$-to-weak upper semicontinuity of $x \mapsto \partial d_{C(t)}(x)$ (see (2.4)) gives by (3.4) the inclusion

$$-\dot{u}(t) - h(t) \in \beta(t)\partial d_{C(t)}(u(t)) \subset N_{C(t)}u(t) \text{ a.e. } t \in I.$$

It results that u is solution of the differential inclusion (3.2), which translates the assertion (a).

(b) By (3.4) we have $\|\dot{u}_n(t)\| \le 2\beta(t)$ a.e. $t \in I$, so the sequence $(u_n)_n$ is equicontinuous. Further, putting $\eta := \|a\| + 2\int_I \beta(\tau)\,d\tau$, for each $t \in I$ we also have $u_n(t) \in C(t) \cap \eta\mathbb{B}_E$, and the latter set is compact in E by assumption in (b). By the Arzelà-Ascoli theorem $(u_n)_n$ admits at least a subsequence converging uniformly on I to some mapping $u\colon I \to E$. Then the assertion (b) directly follows from (a) above. □

The second lemma is concerned with a compactness property of the set of solutions of the differential inclusion (3.2) when the mapping h therein varies in a suitable subset of $L^1_E(I)$.

Lemma 3.5. *Let E be a separable Hilbert space, let $I := [T_0, T]$, and let $r \in]0, +\infty]$. Let $C\colon I \rightrightarrows E$ be a multimapping with nonempty closed values which is absolutely continuous with $v(\cdot)$ as a control function of absolute continuity and such that $C(t)$ is r-prox-regular for all $t \in I$. Let $a \in C(T_0)$. Let $\varrho \in L^1_{\mathbb{R}_+}(I)$ (resp. $\varrho \in L^\infty_{\mathbb{R}_+}(I)$) and*

$$\mathcal{H} := \{h \in L^1_E(I) : \|h(t)\| \le \varrho(t) \text{ a.e. } t \in I\},$$

so $\mathcal{H}$ is convex and $\sigma(L^1_E(I), L^\infty_E(I))$ (resp. $\sigma(L^\infty_E(I), L^1_E(I))$) compact.
Assume in addition that $C(t)$ is ball-compact for all $t \in I$ (resp. $C(t)$ is ball compact for all $t \in I$ and $\dot{v} \in L^\infty_{\mathbb{R}}(I)$). For each $h \in \mathcal{H}$ denote by u_h the unique solution of the differential inclusion

$$\begin{cases} u(T_0) = a, \ u(t) \in C(t), \ \forall t \in I; \\ -\frac{du}{dt}(t) \in N_{C(t)}u(t) + h(t), \quad a.e.\ t \in I. \end{cases}$$

Then the following hold.
(a) *The solution set $\mathcal{X}_{\mathcal{H}} := \{u_h : h \in \mathcal{H}\}$ is compact in $\mathcal{C}_E(I)$ with respect to the topology of uniform convergence.*
(b) *The mapping $\mathcal{H} \ni h \mapsto u_h$ from $\mathcal{H}$ into $\mathcal{C}_E(I)$ is sequentially continuous for $\mathcal{C}_E(I)$ endowed with the topology of uniform convergence on I and $\mathcal{H}$ with the induced $\sigma(L^1_E(I), L^\infty_E(I))$ topology (resp. $\sigma(L^\infty_E(I), L^1_E(I))$ topology).*

Proof. The convexity and compactness of $\mathcal{H}$ are clear.
(a) Assume in addition that each set $C(t)$ is ball-compact and define $\beta\colon I \to \mathbb{R}_+$ by $\beta(t) := |\dot{v}(t)| + \varrho(t)$ for all $t \in I$. Then $\|\dot{u}_h(t)\| \le 2\beta(t)$ for a.e. $t \in I$ according to Theorem 3.3 above, so the collection $(u_h)_{h\in\mathcal{H}}$ is equicontinuous. Further, for $\gamma(t) := \|a\| + \int_{T_0}^t 2\beta(s)\,ds$ we have for each $t \in I$

$$u_h(t) \in C(t) \cap \gamma(t)\mathbb{B}_E \quad \text{for all } h \in \mathcal{H}.$$

Since $C(t) \cap \gamma(t)\mathbb{B}_E$ is compact in E by assumption and since it can be seen by Lemma 3.4(a) that $\mathcal{X}_{\mathcal{H}}$ is closed in $\mathcal{C}_E(I)$, the Arzelà-Ascoli theorem ensures that $\mathcal{X}_{\mathcal{H}}$ is compact in $\mathcal{C}_E(I)$, proving (a).

(b) Regarding (b) take any $h \in \mathcal{H}$ and any sequence $(h_n)_n$ in $\mathcal{H}$ converging to h with respect to the $\sigma(L^1_E(I), L^\infty_E(I))$ topology. By (b) in Lemma 3.4 the sequence $(u_{h_n})_n$ admits a subsequence converging uniformly on I to an absolutely continuous solution u_∞ of the differential inclusion

$$\begin{cases} u(T_0) = a,\ u(t) \in C(t)\ \forall t \in I \\ -\frac{du}{dt}(t) \in N_{C(t)}u(t) + h(t)\ \text{a.e. } t \in I. \end{cases}$$

By the uniqueness of the solution of the latter differential inclusion (due to Theorem 3.3) we have $u_\infty = u_h$. We have then obtained that, for any sequence $(h_n)_n$ in $\mathcal{H}$ converging to $h \in \mathcal{H}$ with respect to the $\sigma(L^1_E(I), L^\infty_E(I))$ topology, the sequence $(u_{h_n})_n$ contains a subsequence converging uniformly on I to u_h. This ensures that the mapping $\mathcal{H} \ni h \mapsto u_h$ from $\mathcal{H}$ into $\mathcal{C}_E(I)$ is sequentially continuous for $\mathcal{C}_E(I)$ endowed with the topology of uniform convergence on I and $\mathcal{H}$ with the induced $\sigma(L^1_E(I), L^\infty_E(I))$ topology.

The arguments for the sequential $\sigma(L^\infty_E(I), L^1_E(I))$ continuity are similar, and hence omitted. □

The next theorem presents a new result for the Skorohod problem driven by the sweeping process in the line of Castaing, Monteiro Marques and Raynaud de Fitte [21], Castaing, Marie and Raynaud de Fitte [19], Rascanu [53], Maticiuc, Rascanu, Slominski and Topolewski [40].

Theorem 3.6. *Let $I := [0,1]$ and $C\colon I \rightrightarrows \mathbb{R}^e$ be an absolutely continuous closed valued multimapping with $v(\cdot)$ as a control function of absolute continuity. Let $z \in C^{1-var}(I, \mathbb{R}^d)$ and let $b\colon I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ be a continuous integrand operator for which there is a real constant $M_b > 0$ such that*

(a) $|b(t,x)| \le M_b$ *for all* $(t,x) \in I \times \mathbb{R}^e$*;*

(b) $|b(t,x) - b(t,y)| \le M_b\|x-y\|_{\mathbb{R}^e}$ *for all* $(t,x,y) \in I \times \mathbb{R}^e \times \mathbb{R}^e$*.*

Let also $g\colon I \times I \times \mathbb{R}^e \to \mathbb{R}^e$ be a continuous mapping for which there is a real constant $M_g > 0$ such that

(c) $\|g(t,s,x)\|_{\mathbb{R}^e} \le M_g$ *for all* $(t,s,x) \in I \times I \times \mathbb{R}^e$*;*

(d) $\|g(t,s,x) - g(t,s,y)\|_{\mathbb{R}^e} \le M_g\|x-y\|_{\mathbb{R}^e}$ *for all* $(t,s) \in I \times I,\ x,y \in \mathbb{R}^e$*.*

Given $a \in C(0)$ there exist BVC mappings $x\colon I \to \mathbb{R}^e$ and $h\colon I \to \mathbb{R}^e$, and absolutely continuous mappings $k\colon I \to \mathbb{R}^e$ and $u\colon I \to \mathbb{R}^e$ satisfying

$$\begin{cases} x(0) = u(0) = a, \quad x(t) = h(t) + k(t) + u(t),\ \forall t \in I \\ h(t) = \int_0^t b(\tau, x(\tau))\, dz_\tau,\ \forall t \in I \\ k(t) = \int_0^t g(t,s,x(s))\, ds,\ \forall t \in I \\ u(t) \in C(t),\ \forall t \in I, \quad -\frac{du}{dt}(t) \in N_{C(t)}u(t) + k(t),\ a.e.\ t \in I, \end{cases}$$

where $h(t) = \int_0^t b(\tau, x(\tau))\, dz_\tau$ is the Riemann-Stieljes integral and where the mapping k with $k(t) = \int_0^t g(t,s,x(s))\, ds$ is the Volterra type perturbation;
further, for $M := \max\{M_b, M_g\}$ the mapping u satisfies

$$-\frac{du}{dt}(t) \in (|\dot{v}(t)| + M)\partial d_{C(t)}(u(t)) + k(t),\ a.e.\ t \in I.$$

Proof. Define $M := \max\{M_b, M_g\}$ and $\beta(\cdot) := |\dot{v}(\cdot)| + M$. For each $t \in I$ set

$$x^0(t) = a,\ h^1(t) = \int_0^t b(\tau, a)\, dz_\tau$$

and note by (2.6) that

$$\left| \int_0^t b(\tau, a)\, dz_\tau \right| \le |b(\cdot, a)|_{\infty;[0,1]} |z|_{1-var;[0,t]}. \tag{3.5}$$

Moreover, writing for $s, t \in I$

$$\int_0^t b(\tau, a)\, dz_\tau - \int_0^s b(\tau, a)\, dz_\tau = \int_s^t b(\tau, a)\, dz_\tau,$$

we see by condition (a) that

$$\|h^1(t) - h^1(s)\| \le M|z|_{1-var;[s,t]} \tag{3.6}$$

for all $0 \le s \le t \le 1$, hence in particular for every $t \in I$

$$\|h^1(t)\| \le M|z|_{1-var;[0,t]} \le M|z|_{1-var;[0,1]}.$$

For each $t \in I$ set $k^1(t) = \displaystyle\int_0^t g(t, s, x^0(s))\, ds$, so k^1 is continuous on I.

Using conditions (c) and (d) we also have the estimates

$$\|k^1(t)\| \le M\ \forall t \in I, \text{ and } \|k^1(t) - k^1(\tau)\| \le M|t - \tau|\ \forall\, \tau, t \in I.$$

By Theorem 3.2 above there is an AC mapping $u^1 \colon I \to \mathbb{R}^e$ solution of the differential inclusion

$$\begin{cases} u^1(0) = a,\ u^1(t) \in C(t),\ \forall t \in I; \\ -\frac{du^1}{dt}(t) \in N_{C(t)} u^1(t) + k^1(t),\ \text{a.e. } t \in I \end{cases}$$

with
$$u^1(t) = a + \int_0^t \frac{du^1}{ds}(s)\, ds \quad \text{for all } t \in I$$

and such that for the above function $\beta(\cdot)$ with $\beta(t) = |\dot{v}(t)| + M$

$$-\frac{du^1}{dt}(t) \in \beta(t) \partial d_{C(t)}(u^1(t)) + k^1(t) \text{ and } \left\| \frac{du^1}{dt}(t) \right\| \le 2\beta(t) \text{ a.e. } t \in I. \tag{3.7}$$

Put for each $t \in I$

$$x^1(t) = h^1(t) + k^1(t) + u^1(t) = \int_0^t b(\tau, x^0(\tau))\, dz_\tau + \int_0^t g(t, s, x^0(s))\, ds + u^1(t).$$

Then $x^1(\cdot)$ is BVC with $x^1(0) = a$. Continuing in this way we are going to construct by induction sequences of mappings $(h^n(\cdot))$, $(k^n(\cdot))$, $(u^n(\cdot))$ and $(x^n(\cdot))$ such that

$$h^n(t) = \int_0^t b(\tau, x^{n-1}(\tau))\, dz_\tau,\ \ k^n(t) = \int_0^t g(t, s, x^{n-1}(s))\, ds,$$

for all $t, \tau \in I$: $\quad \|k^n(t)\| \le M \quad$ and $\quad \|k^n(t) - k^n(\tau)\| \le M|t - \tau|$, for all $0 \le s \le t \le 1$:

$$\|h^n(t) - h^n(s)\| \le M|z|_{1-var;[s,t]}, \ \|h^n(t)\| \le M|z|_{1-var;[0,t]} \le M|z|_{1-var;[0,1]},$$

$u^n(\cdot)$ is an AC solution of the differential inclusion

$$\begin{cases} u^n(0) = a, \ u^n(t) \in C(t), \ \forall t \in I; \\ -\frac{du^n}{dt}(t) \in \beta(t)\partial d_{C(t)}(u^n(t)) + k^n(t), \ \text{a.e. } t \in I, \end{cases}$$

and for all $t \in I$: $$x^n(t) = h^n(t) + k^n(t) + u^n(t).$$

We already defined $x^0(\cdot) \equiv a$ and we already constructed $h^1(\cdot)$, $k^1(\cdot)$, $u^1(\cdot)$ and $x^1(\cdot)$ fulfilling the required properties. Suppose that the desired mappings are constructed for $j = 2, \cdots, n-1$ with the required properties. For each $t \in I$ put

$$h^n(t) = \int_0^t b(\tau, x^{n-1}(\tau))\, dz_\tau \quad \text{and} \quad k^n(t) = \int_0^t g(t, s, x^{n-1}(s))\, ds.$$

Clearly, for all $t, \tau \in I$ we have

$$\|k^n(t)\| \le M \quad \text{and} \quad \|k^n(t) - k^n(\tau)\| \le M|t - \tau|.$$

It ensues by (2.6) that $\|h^n(t) - h^n(s)\| \le M|z|_{1-var;[s,t]}$ for all $0 \le s \le t \le 1$, and hence in particular

$$\|h^n(t)\| \le M|z|_{1-var;[0,t]} \le M|z|_{1-var;[0,1]} \quad \text{for all } t \in I.$$

By Theorem 3.2 again there is an AC mapping $u^n \colon I \to \mathbb{R}^e$ solution of the differential inclusion

$$\begin{cases} u^n(0) = a, \ u^n(t) \in C(t), \ \forall t \in I; \\ -\frac{du^n}{dt}(t) \in N_{C(t)}u^n(t) + k^n(t), \ \text{a.e. } t \in I \end{cases}$$

with $$u^n(t) = a + \int_0^t \frac{du^n}{ds}(s)\, ds \quad \text{for all } t \in I$$

and such that for the above function $\beta(\cdot) := |\dot{v}(\cdot)| + M$

$$-\frac{du^n}{dt}(t) \in \beta(t)\partial d_{C(t)}(u^n(t)) + k^n(t) \quad \text{and} \quad \left\|\frac{du^n}{dt}(t)\right\| \le 2\beta(t) \text{ a.e. } t \in I.$$

Put for each $t \in I$

$$x^n(t) = h^n(t) + k^n(t) + u^n(t) = \int_0^t b(\tau, x^{n-1}(\tau))\, dz_\tau + \int_0^t g(t, s, x^{n-1}(s))\, ds + u^n(t).$$

Then $x^n(\cdot)$ is BVC with $x^n(0) = a$ and

$$-\frac{du^n}{dt}(t) \in \beta(t)\partial d_{C(t)}(u^n(t)) + k^n(t) \ \text{ and } \ \left\|\frac{du^n}{dt}(t)\right\| \le 2\beta(t) \text{ a.e. } t \in I. \tag{3.8}$$

The induction is then justified.

Our next step is a limit process. Since $\left\|\frac{du^n}{dt}(t)\right\| \leq 2\beta(t)$ for a.e. $t \in I$, we see that $(u^n(\cdot))_n$ is equi-absolutely continuous. According to this and the boundedness of $(u_n(t))_n$ for each $t \in I$, we may suppose by the Arzelà-Ascoli theorem that the sequence $(u_n(\cdot))_n$ converges uniformly on I to a mapping $u\colon I \to E$. Further, $u(0) = a$ and for every $t \in I$ we have $u(t) \in C(t)$ since $u^n(t) \in C(t)$ for any integer $n \geq 1$. By the inequalities $\|k^n(t) - k^n(s)\| \leq M|t-s|$ and $\|k_n(t)\| \leq M$ the Arzelà-Ascoli theorem allows us to suppose also that the sequence $(k^n(\cdot))_n$ converges uniformly to a Lipschitz mapping k satisfying $\|k(t) - k(s)\| \leq M|t-s|$. Keeping in mind that u_n is solution of the differential inclusion

$$\begin{cases} u^n(0) = a, \ u^n(t) \in C(t), \ \forall t \in I; \\ -\frac{du^n}{dt}(t) \in N_{C(t)}u^n(t) + k^n(t), \ \text{a.e. } t \in I, \end{cases}$$

the uniform convergence of $(u_n)_n$ and $(k_n)_n$ on I to u and k respectively entails by (a) in Lemma 3.4 that u is the absolutely continuous solution of the differential inclusion

$$\begin{cases} u(0) = a, \ u(t) \in C(t) \ \forall \ t \in I \\ -\frac{du}{dt}(t) \in N_{C(t)}u(t) + k(t). \end{cases}$$

On the other hand, recalling that $t \mapsto |z|_{1-var;[0,t]}$ is continuous by (2.5) and that

$$\|h^n(t) - h^n(s)\| \leq M|z|_{1-var;[s,t]} \quad \text{for all } 0 \leq s \leq t \leq 1,$$

we see that the sequence $(h^n(\cdot))_n$ is equicontinuous on I with $\|h^n(t)\| \leq M|z|_{1-var;I}$, hence by the Arzelà-Ascoli theorem again we may suppose that the sequence $(h^n(\cdot))$ converges uniformly on I to a continuous mapping $h(\cdot)$. This convergence of $(h_n)_n$ to h and the latter inequality involving $h_n(t)$ and $h_n(s)$ imply that

$$\|h(t) - h(s)\| \leq M|z|_{1-var;[s,t]} \quad \text{for all } 0 \leq s \leq t \leq 1,$$

which in turn combined with the continuity of h guarantees that h is a BVC mapping from I into E.

The above convergence properties also ensure that the sequence $x^n(\cdot) = h^n(\cdot) + k^n(\cdot) + u^n(\cdot)$ converges uniformly on I to $x(\cdot) := h(\cdot) + k(\cdot) + u(\cdot)$, and the sequence $\big(b(\cdot, x^{n-1}(\cdot))\big)$ converges also uniformly on I to $b(\cdot, x(\cdot))$ as $n \to \infty$. Then it follows from (2.7) that the sequence $\int_0^t b(\tau, x^{n-1}(\tau))\, dz_\tau$ converges as $n \to \infty$ to $\int_0^t b(\tau, x(\tau))\, dz_\tau$ uniformly with respect to $t \in I$. For each $t \in I$ by condition (c) the sequence $\big(g(t, \cdot, x^{n-1}(\cdot))\big)$ converges pointwise to $g(t, \cdot, x(\cdot))$, hence by the Lebesgue dominated convergence theorem

$$\int_0^t g(t, s, x^{n-1}(s))\, ds \to \int_0^t g(t, s, x(s))\, ds \quad \text{as } n \to \infty,$$

which means that $k(t) = \int_0^t g(t, s, x(s))\, ds$ for all $t \in I$. Further, by identifying the limits we can write for each $t \in I$

$$x(t) = \lim_{n\to\infty} x^n(t) = \lim_{n\to\infty} \int_0^t b(\tau, x^{n-1}(\tau))\, dz_\tau + \lim_{n\to\infty} \int_0^t g(t, s, x^{n-1}(s))\, ds + \lim_{n\to\infty} u^n(t).$$

All the desired properties of the theorem are therefore established, so that the proof is complete. $\square$

4. Towards an integral Volterra type perturbation of sweeping process

In this section we establish an existence result for sweeping process with Volterra type integral perturbation $\int_0^t g(t,s,u(s))\,ds$ with $g\colon I\times I\times E\to E$. The result complements similar ones in [8, 9] for such perturbations; see also [25] for an integral perturbation in the less general form $\int_0^t g_0(s,u(s))\,ds$ with $g_0\colon I\times E\to E$. The authors in [8] employed an appropriate extended catching-up algorithm along with a new Gronwall-like inequality. Here we use a fixed point argument. A new version with only the continuity of $g(t,s,\cdot)$ is then obtained when the sets $C(t)$ are required to be ball-compact.

Theorem 4.1. *Let E be a separable Hilbert space, $I:=[0,1]$, and let $r\in]0,+\infty]$. Let $C\colon I\rightrightarrows E$ be a multimapping with nonempty values which is absolutely continuous with $v(\cdot)$ as a control function of absolute continuity and such that $C(t)$ is r-prox-regular and ball-compact for all $t\in I$. Let $a\in C(0)$ and let $g\colon I\times I\times E\to E$ be a mapping satisfying*

($\mathcal{G}_1$) *$x\mapsto g(t,s,x)$ is continuous on E for every $(t,s)\in I\times I$;*

($\mathcal{G}_2$) *$(t,s)\mapsto g(t,s,x)$ is $\mathcal{L}(I)\otimes\mathcal{L}(I)$-measurable for every $x\in E$;*

($\mathcal{G}_3$) *there is $\rho(\cdot)\in L^1_{\mathbb{R}_+}(I)$ such that*

$$\|g(t,s,x)\|\le\rho(t)\quad\textit{for every } (t,s,x)\in I\times I\times E.$$

Then the integro-differential inclusion

$$\begin{cases} u(0)=a,\ u(t)\in C(t),\ \forall t\in I;\\ -\frac{du}{dt}(t)\in N_{C(t)}u(t)+\int_0^t g(t,s,u(s))\,ds\ \text{a.e. } t\in I\end{cases}$$

admits at least an absolutely continuous solution. Moreover, the solution set is compact in $\mathcal{C}_E(I)$.

If instead of ($\mathcal{G}_1$) there exists $L(\cdot)\in L^1_{\mathbb{R}_+}(I)$ such that the condition

($\mathcal{G}_1'$) *$\|g(t,s,x)-g(t,s,y)\|\le L(t)\|x-y\|$ for all $t,s\in I$ and $x,y\in E$*

is fulfilled, then the solution u of the above integro-differential inclusion is unique and

$$-\frac{du}{dt}(t)\in(|\dot v(t)|+\rho(t))\partial d_{C(t)}(u(t))+\int_0^t g(t,s,u(s))\,ds\ \text{a.e. } t\in I; \tag{4.1}$$

further, this solution is Lipschitz on I whenever the multimapping C is Lipschitz on I and $\rho(\cdot)$ is a constant function.

Proof. Let $\mathcal{H}:=\{h\in L^1_E(I):\|h(t)\|\le\rho(t),\ a.e.\ t\in I\}$.
The set $\mathcal{H}$ is convex and $\sigma(L^1_E(I),L^\infty_E(I))$ compact. For each $h\in\mathcal{H}$, according to Theorem 3.3 denote by u_h the unique absolutely continuous solution of the differential inclusion

$$\begin{cases} u(0)=a,\ u(t)\in C(t),\ \forall t\in I;\\ -\frac{du}{dt}(t)\in N_{C(t)}u(t)+h(t)\ \text{a.e. } t\in I.\end{cases}$$

The assertion (a) in Lemma 3.5 tells us that the solution set is compact in $\mathcal{C}_E(I)$.

For each $h\in\mathcal{H}$ put $\Phi(h)(t):=\displaystyle\int_0^t g(t,s,u_h(s))\,ds\quad$ for all $t\in I$.

First, we notice that $t \mapsto \int_0^t g(t,s,u_h(s))\,ds$ is $\mathcal{L}(I)$-measurable by Fubini-Lebesgue measurability according to $(\mathcal{G}_1) - (\mathcal{G}_2) - (\mathcal{G}_3)$, and second, $\|\Phi(h)(t)\| \leq \rho(t)$ for all $t \in I$. It ensues that $h \mapsto \Phi(h)$ is a mapping from $\mathcal{H}$ into $\mathcal{H}$. The existence of solution of the differential inclusion in the proposition will follow by applying a fixed point theorem to $\Phi\colon \mathcal{H} \to \mathcal{H}$. Since $\mathcal{H}$ is convex and $\sigma(L^1_E(I), L^\infty_E(I))$ compact, it is enough to verify that Φ is sequentially $\sigma(L^1_E(I), L^\infty_E(I))$ continuous. Let $(h_n)_n$ be a sequence in $\mathcal{H}$ converging to $h \in \mathcal{H}$ with respect the $\sigma(L^1_E(I), L^\infty_E(I))$ topology. By (b) in Lemma 3.5 the sequence $(u_{h_n})_n$ converges uniformly on I to u_h. This gives that for each $t \in I$ one has as $n \to \infty$

$$\Phi(h_n)(t) = \int_0^t g(t,s,u_{h_n}(s))\,ds \to \int_0^t g(t,s,u_h(s))\,ds = \Phi(h)(t).$$

Since $\|\Phi(h_n)(t)\| \leq \rho(t)$ for all $n \in \mathbb{N}$ and $t \in I$ and since $(\Phi(t_n))_n$ converges pointwise on I to $\Phi(h)$, this sequence $(\Phi(h_n))_n$ converges $\sigma(L^1_E(I), L^\infty_E(I))$ to $\Phi(h)$, which translates the required continuity of the mapping $\Phi\colon \mathcal{H} \to \mathcal{H}$. Consequently, Φ has a fixed point $h \in \mathcal{H}$, and this ensures that

$$\begin{cases} u_h(0) = a,\ u_h(t) \in C(t),\ \forall t \in I; \\ -\frac{du_h}{dt}(t) \in N_{C(t)} u_h(t) + h(t) \ a.e.\ t \in I \end{cases}$$

with $h(t) = \int_0^t g(t,s,u(s))\,ds$ for all $t \in I$. The existence of solution of the differential inclusion in the proposition is then proved.

Regarding the compactness in $\mathcal{C}_E(I)$ of the solution set, it is obtained by repeating the compactness arguments in Lemma 3.5.

Finally, the uniqueness property under $(\mathcal{G}'_1)$ can be seen by using Gronwall lemma (see Lemma 2.2) and the hypomonotonicity of truncations of the normal cone $N_{C(t)}$ (see, e.g., [26, 8]); and the inclusion (4.1) in the statement of the proposition follows from Theorem 3.3. Further, if the multimapping C is Lipschitz on I, we easily see that the solution $u(\cdot)$ is Lipschitz on I since its derivative $\frac{du}{dt}$ is bounded on I according to the above uniqueness property along with the inclusion (4.1) and the boundedness assumption $(\mathcal{G}_3)$. □

5. Integro-differential sweeping processes coupled with fractional differential equations

We are interested in the following fractional order boundary problem involving an evolution differential inclusion governed by the sweeping process with Volterra integral perturbation:

$$(^*) \qquad D^\alpha h(t) + \lambda D^{\alpha-1} h(t) = u(t),\ t \in [0,1], \tag{5.1}$$

$$(^{**}) \qquad I^\beta_{0^+} h(t)\,|_{t=0} := \lim_{t\to 0} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h(s)\,ds = 0, \tag{5.2}$$

$$(^{***}) \qquad h(1) = I^\gamma_{0^+} h(1) = \int_0^1 \frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)} h(s)\,ds, \tag{5.3}$$

$$(^{****}) \qquad -\frac{du}{dt}(t) \in N_{C(t)} u(t) + \int_0^t f(t,s,u(s),h(s))\,ds \ \text{ a.e. } t \in [0,1]; \tag{5.4}$$

where $\alpha \in]1,2]$, $\beta \in [0, 2-\alpha]$, $\lambda \geq 0$, $\gamma > 0$ are given constants, D^α is the standard *Riemann-Liouville fractional derivative*, Γ is the standard Gamma function and $f\colon [0,1] \times [0,1] \times E \times E \to E$ is a single-valued mapping.

Definition 5.1. (Fractional Bochner integral) Let E be a separable Hilbert space and f be a mapping from $[0,1]$ into E. The *fractional Bochner-integral* of order $\alpha > 0$ of the mapping f is defined by

$$I^\alpha_{\theta^+} f(t) := \int_\theta^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds,\ t > \theta.$$

We refer to [38], [41], [48], for the general theory of Fractional Calculus and Fractional Differential Equations.

We denote by $W^{\alpha,1}_{B,E}([0,1])$ the space of all continuous mappings from I into the separable Hilbert space E such that their Riemann-Liouville fractional derivatives of order $\alpha - 1$ are continuous and their Riemann-Liouville fractional derivatives of order α are Bochner integrable.

For the proof of our theorems we will need some elementary features from reference [15] recalled in Lemma 5.2 and Theorem 5.3 below.

The Green function and its properties

Let $\alpha \in]1,2]$, $\beta \in [0, 2-\alpha]$, $\lambda \geq 0$, $\gamma > 0$ and $G\colon [0,1] \times [0,1] \to \mathbb{R}$ be the function defined by

$$G(t,s) = \varphi(s) I^{\alpha-1}_{0^+}(\exp(-\lambda t)) + \begin{cases} \exp(\lambda s) I^{\alpha-1}_{s^+}(\exp(-\lambda t)), & 0 \leq s \leq t \leq 1, \\ 0, & 0 \leq t \leq s \leq 1, \end{cases}$$

where $$\varphi(s) = \frac{\exp(\lambda s)}{\mu_0} \left[\left(I^{\alpha-1+\gamma}_{s^+}(\exp(-\lambda t)) \right)(1) - \left(I^{\alpha-1}_{s^+}(\exp(-\lambda t)) \right)(1) \right]$$

with $$\mu_0 = \left(I^{\alpha-1}_{0^+}(\exp(-\lambda t)) \right)(1) - \left(I^{\alpha-1+\gamma}_{0^+}(\exp(-\lambda t)) \right)(1).$$

Lemma 5.2. *Let G be the Green function defined above and let E be a separable Hilbert space. Then the following hold.*

(a) *$G(\cdot,\cdot)$ satisfies the following estimate*

$$|G(t,s)| \leq \frac{1}{\Gamma(\alpha)} \left(\frac{1+\Gamma(\gamma+1)}{|\mu_0| \Gamma(\alpha) \Gamma(\gamma+1)} + 1 \right) =: M_G.$$

(b) *If $u \in W^{\alpha,1}_{B,E}([0,1])$ satisfies boundary conditions (*), (**), (***), then*

$$u(t) = \int_0^1 G(t,s) \left(D^\alpha u(s) + \lambda D^{\alpha-1} u(s) \right) ds \quad \text{for every } t \in [0,1].$$

(c) *For any $f \in L^1_E([0,1])$ and for $u_f\colon [0,1] \to H$ defined by*

$$u_f(t) := \int_0^1 G(t,s) f(s) ds \quad \text{for} \quad t \in [0,1],$$

one has $$I^\beta_{0^+} u_f(t)\,|_{t=0} = 0 \quad \text{and} \quad u_f(1) = \left(I^\gamma_{0^+} u_f \right)(1).$$

Moreover, one also has $u_f \in W^{\alpha,1}_{B,E}([0,1])$ *and*

$$(D^{\alpha}u_f)(t) + \lambda\left(D^{\alpha-1}u_f\right)(t) = f(t) \quad \textit{for all} \quad t \in [0,1].$$

From now on, given an interval I of $\mathbb{R}$ and a multimapping $X\colon I \rightrightarrows E$ from I into a separable Hilbert space E which is *Lebesgue measurable* (see [23] for the definition) with nonempty values, we will denote (as usual) by S^1_X the set of all *Lebesgue integrable selections* $\zeta\colon I \to E$ of X, i.e., ζ is a Lebesgue integrable mapping on I which is a *selection* of X in the sense that $\zeta(t) \in X(t)$ for all $t \in I$. When the Lebesgue measurable multimapping X is *integrably bounded*, i.e., there is a non-negative Lebesgue integrable function $\eta\colon I \to \mathbb{R}_+$ such that $X(t) \subset \eta(t)\mathbb{B}_E$ for all $t \in I$, it is clear that S^1_E coincides with the set of all measurable selections of X.

The following theorem characterizes the topological structure of the solution set.

Theorem 5.3. *Let E be a separable Hilbert space, let $X\colon [0,1] \rightrightarrows E$ be a measurable multimapping with nonempty compact convex values such that $X(t) \subset \delta\,\mathbb{B}_E$ for all $t \in [0,1]$, where δ is a positive constant, and let S^1_X be (as above) the set of all measurable selections of X. Then the $W^{\alpha,1}_{B,E}([0,1])$-solutions set of problem*

$$\begin{cases} D^{\alpha}u(t) + \lambda D^{\alpha-1}u(t) = f(t), f \in S^1_X, \ \text{a.e. } t \in [0,1] \\ I^{\beta}_{0^+}u(t)\,|_{t=0} = 0, \quad u(1) = I^{\gamma}_{0^+}u(1) \end{cases} \tag{5.5}$$

is a compact convex subset in $\mathcal{C}_E([0,1])$.

We are now in a position to establish our first theorem in this section. In its statement we keep fixed as above the real constants $\alpha \in]1,2]$, $\beta \in]0, 2-\alpha]$, $\lambda \geq 0$ and $\gamma > 0$.

Theorem 5.4. *Let $I = [0,1]$ and let E be a separable Hilbert space. Let $r \in]0,+\infty]$ and $C\colon I \rightrightarrows E$ be an absolutely continuous multimapping whose values are r- prox regular and ball-compact. Let $a \in C(0)$ and let $f\colon I \times I \times E \times E \to E$ be a mapping such that*

(i) *$f(\cdot,\cdot,x,y)$ is $\mathcal{L}(I) \otimes \mathcal{L}(I)$ measurable for all $(x,y) \in E \times E$;*

(ii) *$f(t,s,\cdot,\cdot)$ is continuous on $E \times E$ for all $(t,s) \in I \times I$;*

(iii) *for some real constant $\mu > 0$*

$$\|f(t,s,x,y)\| \leq \mu \quad \textit{for all}\,(t,s,x,y) \in I \times I \times E \times E;$$

(iv) *for some real constant $M > 0$*

$$\|f(t,s,x_1,y) - f(t,s,x_2,y)\| \leq M\|x_1 - x_2\| \ \textit{ for all}\, t,s \in I \textit{ and } x_1,x_2,y \in E.$$

Then there is a $W^{\alpha,1}_{B,E}([0,1])$ mapping $x\colon I \to E$ and an absolutely continuous mapping $u\colon I \to E$ satisfying

$$\begin{cases} D^{\alpha}x(t) + \lambda D^{\alpha-1}x(t) = u(t), \ t \in I \\ I^{\beta}_{0^+}x(t)\,|_{t=0} = 0, \quad x(1) = I^{\gamma}_{0^+}x(1), \quad u(t) \in C(t), \, t \in I \\ -\frac{du}{dt}(t) \in N_{C(t)}u(t) + \int_0^t f(t,s,u(s),x(s))\,ds \quad \text{a.e. } t \in I \\ u(0) = a. \end{cases}$$

Proof. Without loss of generality we may suppose that $M \geq \mu$. For any continuous mapping $h\colon I \to E$, the mapping $f_h\colon I \times I \times E \to E$ defined by

$$f_h(t,s,x) := f(t,s,x,h(t)) \quad \text{for all } (t,s,x) \in I \times I \times E$$

is $\mathcal{L}(I) \otimes \mathcal{L}(I)$ measurable for every $x \in E$, continuous on E for every $(t,s) \in I \times I$, and satisfies

$$|f_h(t,s,x)\| \leq M \quad \text{for all } (t,s,x) \in I \times I \times E, \text{ and}$$

$$\|f_h(t,s,x) - f_h(t,s,y)\| \leq M\|x-y\| \quad \text{for all } (t,s,x,y) \in I \times I \times E \times E.$$

Then, given $a \in C(0)$, by Theorem 4.1 there is a *unique* AC solution u_h to the differential inclusion

$$\begin{cases} u_h(0) = a, \quad u_h(t) \in C(t), \ \forall t \in I \\ -\frac{du_h}{dt}(t) \in N_{C(t)}u_h(t) + \int_0^t f(t,s,u_h(s),h(s))\,ds \ \text{a.e. } t \in I. \end{cases}$$

We claim that $(u_h)_h$ is uniformly bounded and equi-absolutely continuous. Indeed, denoting $v(\cdot)$ a control function of absolute continuity of $C(\cdot)$ on I, we know by Theorem 4.1 that $\|\dot{u}_h(t)\| \leq \rho(t)$ a.e. $t \in I$ for the non-negative integrable function $\rho(\cdot) := |\dot{v}(\cdot)| + 2M$. This justifies the equi-absolute continuity property of $(u_h)_h$. Putting $L := \|a\| + \int_0^1 (|\dot{v}(t)| + 2M)\,dt$, we see that $\|u_h(t)\| \leq L$ for all $t \in I$, which confirms the uniform boundedness of $(u_h)_h$ and finishes the justification of the claim.

Now let us consider the set $\mathcal{X}$ defined by

$$\mathcal{X} := \{\xi_\ell : I \to E : \ \ell \in S^1_{L\mathbb{B}_E}\},$$

each mapping ξ_ℓ being given for every $t \in I$ by

$$\xi_\ell(t) = \int_0^1 G(t,s)\ell(s)\,ds,$$

where G is the foregoing Green function. We note that $\mathcal{X}$ is closed in $\mathcal{C}_E(I)$, convex and equi-Lipschitz (see Theorem 5.3 and [17, Theorem 4.1]). For each $h \in \mathcal{X}$ let us set (again with the above Green function G)

$$\Phi(h)(t) = \int_0^1 G(t,s)u_h(s)\,ds, \quad \text{for all } t \in I.$$

Then it is clear that $\Phi(h) \in \mathcal{X}$. Further, we have $\|u_h(t)\| \leq L$ for all $t \in I$ so that the set $\Gamma(t) := \{x \in C(t) : \|x\| \leq L\}$ is compact in E and non empty because $u_h(t) \in \Gamma(t)$. As consequence, for any $h \in \mathcal{X}$ and for any $t \in I$, the following inclusion holds

$$\Phi(h) \in \mathcal{Y} := \{\xi_g\colon I \to E : g \in S^1_{\overline{\mathrm{co}}(\Gamma)}\},$$

where as above $\xi_g(t) = \int_0^1 G(t,s)g(t)\,ds$ for all $t \in I$. By [17, Theorem 4.1] again $\mathcal{Y}$ is convex, compact in $\mathcal{C}_E(I)$ and equi-Lipschitz. Hence $\Phi(\mathcal{X})$ is equicontinuous and relatively compact in the Banach space $\mathcal{C}_E(I)$ because $\Phi(\mathcal{X}) \subset \mathcal{Y}$. Now we check that Φ is continuous relative to $\mathcal{X}$. It is enough to show that, if $(h_n)_n$ converges uniformly to h in $\mathcal{X}$, then the sequence $(u_{h_n})_n$, where each u_{h_n} is the unique absolutely continuous solution of the differential inclusion

$$\begin{cases} u_{h_n}(0) = a, \quad u_{h_n}(t) \in C(t), \ \forall t \in I \\ -\dot{u}_{h_n}(t) \in N_{C(t)} u_{h_n}(t) + \int_0^t f(t, s, u_{h_n}(s), h_n(s))\, ds \quad \text{a.e. } t \in I, \end{cases}$$

uniformly converges to the unique absolutely continuous solution u_h of the differential inclusion

$$\begin{cases} u_h(0) = a, \quad u_h(t) \in C(t), \ \forall t \in I \\ -\dot{u}_h(t) \in N_{C(t)} u_h(t) + \int_0^t f(t, s, u_h(s), h(s))\, ds \quad \text{a.e. } t \in I. \end{cases}$$

This needs a careful look. We note that $(u_{h_n})_n$ is equicontinuous by what precedes (since for every $n \in \mathbb{N}$ one has $\|\dot{u}_{h_n}(t)\| \le \rho(t)$ for almost all $t \in I$). Further, $\{u_{h_n}(t) : n \in \mathbb{N}\}$ is included in the compact set $\Gamma(t)$ for every $t \in I$. The Arzelà-Ascoli theorem tells us that $\{u_{h_n} : n \in \mathbb{N}\}$ is relatively compact in $\mathcal{C}_E(I)$. So by extracting a subsequence, we may suppose that $(u_{h_n})_n$ converges uniformly on I to some mapping $\zeta \colon I \to E$ with

$$\zeta(t) = \zeta(0) + \int_0^t \dot{\zeta}(s)\, ds \quad \text{for all } t \in I,$$

along with $(\dot{u}_{h_n})_n$ converging weakly in $L^1_E(I)$ to $\dot{\zeta}$ with $\|\dot{\zeta}(t)\| \le \rho(t)$ for a.e. $t \in I$.

Note that $f(t, s, u_{h_n}(s), h_n(s)) \to f(t, s, \zeta(s), h(s))$ for all $(t, s) \in I \times I$.

For simplicity, define

$$q_n(t) := \int_0^t f(t, s, u_{h_n}(s), h_n(s))\, ds, \quad \text{for all } t \in I,$$

$$q(t) := \int_0^t f(t, s, \zeta(s), h(s))\, ds, \quad \text{for all } t \in I.$$

First, we notice that these mappings are Lebesgue measurable by the Fubini-Bochner property and the separability of the space E. Second, by condition (iii) and the uniform boundedness of $(u_{h_n})_n$ and u_h, we also notice that $(q_n)_n$ and q are uniformly bounded, since

$$\|q_n(t)\| \le M \quad \text{and} \quad \|q(t)\| \le M, \quad \text{for all } n \in \mathbb{N} \text{ and } t \in I.$$

Consequently, $(q_n)_n$ is a sequence of measurable and uniformly bounded mappings which converges pointwise to the measurable mapping q. Therefore, the sequence $(\dot{u}_{h_n} + q_n)_n$ converges weakly to $\dot{\zeta} + q$ in $L^1_E(I)$. This combined with Lemma 3.4 and Theorem 3.3 gives

$$\begin{cases} -\frac{d\zeta}{dt}(t) \in N_{C(t)} \zeta(t) + q(t) \ \text{ a.e. } t \in I \\ \zeta(0) = a, \ \zeta(t) \in C(t) \ \forall t \in I, \end{cases}$$

so using the uniqueness of solution of the latter differential inclusion (due to Theorem 4.1) we obtain that $\zeta = u_h$.

Now let us write by Lemma 5.2(a)

$$\|\Phi(h_n)(t)-\Phi(h)(t)\| = \left\|\int_0^1 G(t,s)u_{h_n}(s)\,ds - \int_0^1 G(t,s)u_h(s)\,ds\right\|$$
$$= \left\|\int_0^1 G(t,s)[u_{h_n}(s)-u_h(s)]\,ds\right\| \le \int_0^1 M_G\,\|u_{h_n}(s)-u_h(s)\|\,ds.$$

Since $\|u_{h_n}(\cdot)-u_h(\cdot)\|\to 0$ uniformly on I as $n\to\infty$, we deduce that

$$\sup_{t\in I}\|\Phi(h_n)(t)-\Phi(h)(t)\| \le \int_0^1 M_G\|u_{h_n}(s)-u_h(s)\|\,ds \to 0,$$

which entails that $\Phi(h_n)\to\Phi(h)$ uniformly on I, as desired. Then $\Phi\colon \mathcal{X}\to\mathcal{X}$ is continuous with $\Phi(\mathcal{X})$ relatively compact in $\mathcal{C}_E(I)$, hence by the Schauder theorem Φ has a fixed point, say $h=\Phi(h)\in\mathcal{X}$. This means that for every $t\in I$

$$h(t)=\Phi(h)(t)=\int_0^1 G(t,s)u_h(s)\,ds,$$

with
$$\begin{cases} u_h(0)=a, \quad u_h(t)\in C(t),\ \forall t\in I\\ -\dot u_h(t)\in N_{C(t)}u_h(t)+\int_0^t f(t,s,u_h(s),h(s))\,ds \quad \text{a.e. } t\in I.\end{cases}$$

According to Lemma 5.2, this translates that we have just shown that there exists a mapping $h\in W^{\alpha,1}_{B,E}(I)$ satisfying

$$\begin{cases} D^\alpha h(t)+\lambda D^{\alpha-1}h(t)=u_h(t),\\ I^\beta_{0+}h(t)\,|_{t=0}=0, \quad h(1)=I^\gamma_{0+}h(1)\\ u_h(0)=a, \quad u_h(t)\in C(t)),\forall t\in[0,1]\\ -\dot u_h(t)\in N_{C(t)}u_h(t)+\int_0^t f(t,\tau,h(\tau),u_h(\tau))d\tau \quad \text{a.e. } t\in I.\end{cases}$$

The proof of the theorem is then complete. □

Several variants are available by considering either time dependent m-accretive operators or other types of fractional equations, e.g., Caputo-type fractional equations with Caputo fractional derivatives. We study below an example of a Caputo fractional differential inclusion governed by the sweeping process. For the sake of completeness, we recall the definition of Caputo derivative and some basic needed properties for the corresponding fractional calculus. Throughout the rest of this section, we assume $\alpha\in\,]1,2]$ and we take E as a separable Hilbert space.

Definition 5.5. The *Caputo fractional derivative* of order $\gamma>0$ of a mapping $h\colon[0,T]\to E$, ${}^cD^\gamma h\colon[0,T]\to E$, is defined by

$${}^cD^\gamma h(t)=\frac{1}{\Gamma(n-\gamma)}\int_0^t \frac{h^{(n)}(s)}{(t-s)^{1-n+\gamma}}ds.$$

Here $n=[\gamma]+1$ and $[\gamma]$ denotes the integer part of γ.

Consider the spaces $\mathcal{C}^1_E([0,T])=\{u\in\mathcal{C}_E([0,T]):\dfrac{du}{dt}\in\mathcal{C}_E([0,T])\}$, where $\frac{du}{dt}$ is the usual derivative of u as in the previous sections, and

$${}^cW^{\alpha,\infty}_{B,E}([0,T]):=\{u\in C^1_E([0,T]):{}^cD^{\alpha-1}u\in\mathcal{C}_E([0,T]);\ {}^cD^\alpha u\in L^\infty_E([0,T])\},$$

where ${}^cD^{\alpha-1}u$ and ${}^cD^{\alpha}u$ are the fractional Caputo derivatives of order $\alpha-1$ and α of u, respectively.

The next two lemmas recall and summarize some properties of a Green function given in [13]. This concept of Green function will be involved in Theorem 5.8.

Lemma 5.6. *Let E be a separable Hilbert space and let $G\colon [0,T]\times[0,T]\to\mathbb{R}$ be the function defined by*

$$G(t,s)=\begin{cases} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}-\dfrac{1+t}{T+2}\Big[\dfrac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big], & \textit{if } \ 0\le s<t,\\ -\dfrac{1+t}{T+2}\Big[\dfrac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big] & \textit{if } \ t\le s<T.\end{cases}$$

The following assertions hold:

(i) *Given $f\in L^\infty_E([0,T])$ and $u_f\colon [0,T]\to E$ defined by*

$$u_f(t)=\int_0^T G(t,s)f(s)ds,\ \forall t\in[0,T],$$

then one has

$$u_f(0)-\frac{du_f}{dt}(0)=0,\quad u_f(T)+\frac{du_f}{dt}(T)=0,$$

$${}^cD^{\alpha-1}u_f(t)=\int_0^t f(s)ds-\frac{I^\alpha f(T)+I^{\alpha-1}f(T)}{(T+2)\Gamma(3-\alpha)}t^{2-\alpha},\ \forall t\in[0,T],$$

$${}^cD^{\alpha}u_f(t)=f(t),\ \forall t\in[0,T].$$

(ii) *Assume that u is a ${}^cW^{\alpha,\infty}_{B,E}([0,T])$-solution to*

$$\begin{cases} {}^cD^\alpha u(t)=\sigma(t),\ t\in[0,T]\\ u(0)-\frac{du}{dt}(0)=0,\quad u(T)+\frac{du}{dt}(T)=0\end{cases}$$

where $\sigma\in L^\infty_E([0,T])$, then $u(t)=\int_0^T G(t,s)\sigma(s)ds$ for all $t\in[0,T]$.

The second crucial lemma is also taken from [13].

Lemma 5.7. *Let E be a separable Hilbert space and let $X\colon [0,T]\rightrightarrows E$ be a convex weakly compact valued measurable multimapping for which there is some real-valued function $\gamma\in L^1([0,T])$ such that $\sup\{\|y\| : y\in X(t)\}\le\gamma(t)$ for all $t\in[0,T]$. Then the ${}^cW^{\alpha,\infty}_{B,E}([0,T])$-solutions set $\mathcal{X}$ to*

$$\begin{cases} {}^cD^\alpha u(t)\in X(t),\ t\in[0,T]\\ u(0)-\frac{du}{dt}(0)=0,\quad u(T)+\frac{du}{dt}(T)=0,\end{cases}$$

is bounded, closed and equicontinuous. If $X(t)$ is compact convex for all $t\in[0,T]$, then $\mathcal{X}$ is compact convex in $\mathcal{C}_E([0,T])$.

We are now ready to state and prove our second theorem of this section. It is concerned with an existence result for a Caputo fractional differential inclusion driven by the sweeping process with Volterra integral type perturbation.

Theorem 5.8. *Let $I := [0, T]$ and let E be a separable Hilbert space. Let $r \in]0, +\infty]$ and $C\colon I \rightrightarrows E$ be an absolutely continuous multimapping whose values are r- prox regular and ball-compact. Let $a \in C(0)$ and let $f\colon I \times I \times E \times E \to E$ be a mapping such that*

(i) *$f(\cdot, \cdot, x, y)$ is $\mathcal{L}(I) \otimes \mathcal{L}(I)$ measurable for all $(x, y) \in E \times E$;*

(ii) *$f(t, s, \cdot, \cdot)$ is continuous on $E \times E$ for all $(t, s) \in I \times I$;*

(iii) *for some real constant $\mu > 0$*

$$\|f(t, s, x, y)\| \leq \mu \quad \textit{for all } (t, s, x, y) \in I \times I \times E \times E;$$

(iv) *for some real constant $M > 0$*

$$\|f(t, s, x_1, y) - f(t, s, x_2, y)\| \leq M\|x_1 - x_2\| \quad \textit{for all } t, s \in I \textit{ and } x_1, x_2, y \in E.$$

Then there is a ${}^cW^{\alpha,\infty}_{B,E}(I)$ mapping $x\colon I \to E$ and an absolutely continuous mapping $u\colon I \to E$ satisfying

$$\begin{cases} {}^cD^\alpha x(t) = u(t), \ t \in I \\ x(0) - \frac{dx}{dt}(0) = 0, \quad x(T) + \frac{dx}{dt}(T) = 0 \\ u(0) = a \textit{ and } u(t) \in C(t), \ \forall t \in I \\ -\frac{du}{dt}(t) \in N_{C(t)}u(t) + \int_0^t f(t, s, u(s), x(s))\, ds \quad \text{a.e. } t \in I. \end{cases}$$

Proof. We repeat the arguments of the proof of Theorem 5.4. Nevertheless, this needs to be careful. Suppose without loss of generality $M \geq \mu$. For any continuous mapping $h\colon I \to E$, the mapping $f_h\colon I \times I \times E \to E$ defined by

$$f_h(t, s, x) := f(t, s, x, h(t)) \quad \text{for all } (t, s, x) \in I \times I \times E$$

is clearly $\mathcal{L}(I) \otimes \mathcal{L}(I)$ measurable on I for every $x \in E$, continuous on E for every $(t, s) \in I \times I$, and satisfies the estimate

$$\|f_h(t, s, x)\| \leq M \quad \text{for all } (t, s, x) \in I \times I \times E$$

and the Lipschitz property

$$\|f_h(t, s, x) - f_h(t, s, y)\| \leq M\|x - y\| \quad \text{for all } (t, s, x, y) \in I \times I \times E \times E.$$

Then, given $a \in C(0)$, by Theorem 3.3 there is a unique AC solution u_h to the differential inclusion

$$\begin{cases} u_h(0) = a, \quad u_h(t) \in C(t), \ \forall t \in I \\ -\frac{du_h}{dt}(t) \in N_{C(t)}u_h(t) + \int_0^t f(t, s, u_h(s), h(s))\, ds \ \text{ a.e. } t \in I \end{cases}$$

with $(u_h)_h$ uniformly bounded and equi-absolutely continuous: indeed, considering a control function of absolute continuity $v(\cdot)$ of $C(\cdot)$ and the non-negative integrable function $\rho(\cdot)$ with $\rho(t) := |\dot{v}(t)| + 2M$, Theorem 4.1 gives $\|\dot{u_h}(t)\| \leq \rho(t)$ for a.e. $t \in I$, which confirms the equi-absolute continuity of $(u_h)_h$; on the other hand, the inequality $\|\dot{u}(\cdot)\| \leq \rho(\cdot)$ entails, with

$$L = \|a\| + \int_0^1 (|\dot{v}(t)| + 2M)\, dt,$$

that $\|u_h(t)\| \leq L$ for all $t \in I$, implying the equi-boundedness of $(u_h)_h$.

Now consider the set $\mathcal{X}$ defined by $\mathcal{X} := \{\xi_\ell \colon I \to E : \ \ell \in S^1_{L\mathbb{B}_E}\}$, where each mapping ξ_ℓ is given by

$$\xi_\ell(t) = \int_0^1 G(t,s)\ell(s)\,ds \quad \text{for all } t \in I,$$

the function G being the Green function used in Lemma 5.6. We observe that the set $\mathcal{X}$ is closed in $\mathcal{C}_E(I)$, convex and equicontinuous by Lemma 5.7.

For each $h \in \mathcal{X}$ let us put (again with the above Green function G)

$$\Phi(h)(t) = \int_0^1 G(t,s)u_h(s)\,ds, \quad \text{for all } t \in I.$$

Then it is evident that $\Phi(h) \in \mathcal{X}$. Further, we have $\|u_h(t)\| \le L$ for all $t \in I$ so that the set $\Gamma(t) := \{x \in C(t) : \|x\| \le L\}$ is compact in E and non empty because $u_h(t) \in \Gamma(t)$. Therefore, for any $h \in \mathcal{X}$ and for any $t \in I$, the following inclusion holds

$$\Phi(h) \in \mathcal{Y} := \{\xi_g \colon I \to E : g \in S^1_{\overline{\text{co}}(\Gamma)}\},$$

where as above $\xi_g(t) = \int_0^1 G(t,s)g(t)\,ds$ for all $t \in I$. By Lemma 5.7 again $\mathcal{Y}$ is convex, compact in $\mathcal{C}_E(I)$ and equicontinuous. Thus, $\Phi(\mathcal{X})$ is equicontinuous and relatively compact in the Banach space $\mathcal{C}_E(I)$ since $\Phi(\mathcal{X}) \subset \mathcal{Y}$. Let us verify the continuity of Φ relative to $\mathcal{X}$. For this, it is sufficient to show that, if $(h_n)_n$ converges uniformly to h in $\mathcal{X}$, then the sequence $(u_{h_n})_n$, where each u_{h_n} is the unique absolutely continuous solution of the differential inclusion

$$\begin{cases} u_{h_n}(0) = a, \quad u_{h_n}(t) \in C(t), \ \forall t \in I \\ -\dot{u}_{h_n}(t) \in N_{C(t)}u_{h_n}(t) + \int_0^t f(t,s,u_{h_n}(s),h_n(s))\,ds \quad \text{a.e. } t \in I, \end{cases}$$

converges uniformly to the unique absolutely continuous solution u_h of the differential inclusion

$$\begin{cases} u_h(0) = a, \quad u_h(t) \in C(t), \ \forall t \in I \\ -\dot{u}_h(t) \in N_{C(t)}u_h(t) + \int_0^t f(t,s,u_h(s),h(s))\,ds, \quad \text{a.e. } t \in I. \end{cases}$$

We note that $(u_{h_n})_n$ is equicontinuous since for every $n \in \mathbb{N}$ one has $\|\dot{u}_{h_n}(t)\| \le \rho(t)$ for almost every $t \in I$. Further, $\{u_{h_n}(t) : n \in \mathbb{N}\}$ is included in the compact set $\Gamma(t)$ for every $t \in I$. The Arzelà-Ascoli theorem assures us that $\{u_{h_n} : n \in \mathbb{N}\}$ is relatively compact in $\mathcal{C}_E(I)$. Then, by extracting a subsequence, we may suppose that $(u_{h_n})_n$ converges uniformly on I to some mapping $\zeta \colon I \to E$ with

$$\zeta(t) = \zeta(0) + \int_0^t \dot{\zeta}(s)\,ds, \quad \text{for all } t \in I,$$

along with $(\dot{u}_{h_n})_n$ converging weakly in $L^1_E(I)$ to $\dot{\zeta}$ with $\|\dot{\zeta}(t)\| \le \rho(t)$ for almost every $t \in I$. Note that

$$f(t,s,u_{h_n}(s),h_n(s)) \to f(t,s,\zeta(s),h(s)), \quad \text{for all } t,s \in I.$$

For simplicity, define

$$q_n(t) := \int_0^t f(t,s,u_{h_n}(s),h_n(s))\,ds, \quad \text{for all } t \in I,$$

and

$$q(t) := \int_0^t f(t,s,\zeta(s),h(s))\,ds, \quad \text{for all } t \in I.$$

First, we observe that these mappings are Lebesgue measurable by the Fubini-Bochner property and the separability of the space E. Second, by condition (iii) and the uniform boundedness of $(u_{h_n})_n$ and ζ, we also notice that $(q_n)_n$ and q are uniformly bounded, since

$$\|q_n(t)\| \le M \quad \text{and} \quad \|q(t)\| \le M, \quad \text{for all } n \in \mathbb{N} \text{ and } t \in I.$$

Consequently, $(q_n)_n$ is a sequence of measurable and uniformly bounded mappings which converges pointwise to the measurable mapping q. Therefore, the sequence $(\dot{u}_{h_n} + q_n)_n$ converges weakly to $\dot{\zeta} + q$ in $L^1_E(I)$. This combined with Lemma 3.4 and Theorem 3.3 yields

$$\begin{cases} -\frac{d\zeta}{dt}(t) \in N_{C(t)}\zeta(t) + q(t) \text{ a.e. } t \in I \\ \zeta(0) = a,\ \zeta(t) \in C(t)\ \forall t \in I, \end{cases}$$

so using the uniqueness of solution of the latter differential inclusion (due to Theorem 4.1) we obtain that $\zeta = u_h$. Now let us write

$$\|\Phi(h_n)(t) - \Phi(h)(t)\| = \left\| \int_0^1 G(t,s)u_{h_n}(s)\,ds - \int_0^1 G(t,s)u_h(s)\,ds \right\|$$
$$= \left\| \int_0^1 G(t,s)[u_{h_n}(s) - u_h(s)]\,ds \right\| \le \int_0^1 M_G \,\|u_{h_n}(s) - u_h(s)\|\,ds,$$

where M_G is a positive real constant. Using this and the uniform convergence $\|u_{h_n}(\cdot) - u_h(\cdot)\| \to 0$ on I as $n \to \infty$, we obtain that

$$\sup_{t\in I} \|\Phi(h_n)(t) - \Phi(h)(t)\| \le \int_0^1 M_G \|u_{h_n}(s) - u_h(s)\|\,ds \to 0,$$

which entails that $\Phi(h_n) \to \Phi(h)$ uniformly on I, as desired. Then $\Phi\colon \mathcal{X} \to \mathcal{X}$ is continuous with $\Phi(\mathcal{X})$ relatively compact in $\mathcal{C}_E(I)$, hence by the Schauder theorem Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}$. This means that for every $t \in I$

$$h(t) = \Phi(h)(t) = \int_0^1 G(t,s)u_h(s)\,ds,$$

with

$$\begin{cases} u_h(0) = a, \quad u_h(t) \in C(t),\ \forall t \in I \\ -\dot{u}_h(t) \in N_{C(t)}u_h(t) + \int_0^t f(t,s,u_h(s),h(s))\,ds \quad \text{a.e. } t \in I. \end{cases}$$

Coming back to Lemma 5.7 and applying the above notations, it is clear that we have just shown that there exists a mapping $h \in {}^cW^{\alpha,\infty}_{B,E}(I)$ satisfying

$$\begin{cases} {}^cD^\alpha h(t) = u_h(t), \ t \in I \\ h(0) - \frac{dh}{dt}(0) = 0, \quad h(T) + \frac{dh}{dt}(T) = 0 \\ u_h(0) = a \text{ and } u_h(t) \in C(t), \ \forall t \in I \\ -\frac{du_h}{dt}(t) \in N_{C(t)} u_h(t) + \int_0^t f(t,s,u(s),h(s))\, ds \quad \text{a.e. } t \in I. \end{cases}$$

The proof of the theorem is complete. □

6. Perturbed sweeping processes under rough signals

Let us begin by recalling some important features concerning the Young integral.

Young integral.

Let $z \in C^{1-var}([0,T],\mathbb{R}^d)$ the space of continuous mappings with bounded variation defined on $[0,T]$ with values in $\mathbb{R}^d$. We recall the notation in the preliminary Section 2 that $\mathcal{L}(\mathbb{R}^d,\mathbb{R}^e)$ denotes the space of linear mappings Λ from $\mathbb{R}^d$ to $\mathbb{R}^e$ endowed with the operator norm

$$|\Lambda| := \sup_{x\in\mathbb{R}^d, \|x\|_{\mathbb{R}^d}=1} |\Lambda(x)|_{\mathbb{R}^e}.$$

Let $\Delta_T := \{(s,t) : 0 \le s \le t \le T\}$. A funtion $\omega : \Delta_T \to [0,+\infty[$ is a *control function* on $[0,T]$ (see [30, Definition 1.6]) if ω is continuous, superadditive and $\omega(s,s) = 0$ for $0 \le s \le T$. Here the superadditivity of ω means that $\omega(s,\theta) + \omega(\theta,t) \le \omega(s,t)$ for all $0 \le s \le \theta \le t \le T$. Examples of control function are $(s,t) \mapsto |t-s|^\theta$ with $\theta \ge 1$ and $(s,t) \mapsto \int_s^t \rho(\tau)\, d\tau$ where ρ is a non-negative Lebesgue integrable function.

Let us consider a class of continuous integrand operators $b\colon [0,T]\times\mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d,\mathbb{R}^e)$ satisfying the conditions

($\mathcal{B}_1$) $|b(t,x)| \le M$, for all $x \in \mathbb{R}^e$,

($\mathcal{B}_2$) $|b(s,x) - b(t,x)| \le \omega(s,t)$, for all $0 \le s \le t \le T$ and $x \in \mathbb{R}^e$,

($\mathcal{B}_3$) $|b(t,x) - b(t,y)| \le M\|x-y\|$, for all $t \in [0,T]$ and $x \in \mathbb{R}^e$,

where ω is a control function on $[0,T]$ and M is a positive constant. If a sequence of mappings $(u_n)_n$ from $[0,T]$ into $\mathbb{R}^e$ is controlled by a common control function $\alpha\colon \Delta_T \to [0,+\infty[$, say

$$\|u_n(s) - u_n(t)\| \le \alpha(s,t) \quad \text{for all } n \in \mathbb{N} \text{ and all } 0 \le s \le t \le T,$$

then the sequence of maps $(y_n)_n$ from $[0,T]$ into $\mathcal{L}(\mathbb{R}^d,\mathbb{R}^e)$, with $y_n(t) = b(t,u_n(t))$, is uniformly bounded and uniformly bounded in variation. In particular we have $y_n \in C^{1-var}([0,T],\mathcal{L}(\mathbb{R}^d,\mathbb{R}^e))$. Indeed, we have for all $0 \le s \le t \le T$

$$|y_n(s) - y_n(t)| \le |b(s,u_n(s)) - b(t,u_n(s))| + |b(t,u_n(s)) - b(t,u_n(t))|$$

with
$$|b(s,u_n(s)) - b(t,u_n(s))| \le \omega(s,t)$$

and
$$|b(t,u_n(s)) - b(t,u_n(t))| \le M\|u_n(s) - u_n(t)\| \le M\alpha(s,t),$$

so using [30, Proposition 1.11], we have $\sup_n |y_n|_{1-var;[s,t]} < \infty$. Consequently, the Young integral $\int_0^t y_n(s) dz_s$ of y_n along z is well-defined and belongs to the space

$C^{1-var}([0,T],\mathbb{R}^e)$, and according to Friz-Victoir [30, Theorem 6.8] we have the following estimates

$$\begin{aligned}\left\|\int_s^t y_n dz\right\| &\leq \frac{1}{1-2^{1-\theta}}|z|_{1-var;[s,t]}|y_n|_{1-var;[s,t]} + |y_n(s)|\|z(t)-z(s)\|_{\mathbb{R}^d} \\ &\leq \frac{1}{1-2^{1-\theta}}|z|_{1-var;[s,t]}|y_n|_{1-var;[s,t]} + M\|z(t)-z(s)\|_{\mathbb{R}^d} \\ &\leq \frac{1}{1-2^{1-\theta}}|z|_{1-var;[s,t]}\sup_n|y_n|_{1-var;[s,t]} + M\|z(t)-z(s)\|_{\mathbb{R}^d} \qquad (6.1)\end{aligned}$$

for all $0 \leq s \leq t \leq T$ with $\theta = 2$; and the first above inequality also yields

$$\left|\int_0^{\cdot} y_n dz\right|_{1-var;[s,t]} \leq C(1,1)|z|_{1-var;[s,t]}\sup_n(|y_n|_{1-var;[s,t]} + |y_n|_{\infty;[s,t]}) \qquad (6.2)$$

for all $0 \leq s \leq t \leq T$. Putting $g_n(t) := \int_0^t y_n(\tau)dz_\tau$ for all $t \in [0,T]$, we see by (6.1) that $(g_n)_n$ is a sequence of mappings in $C^{1-var}([0,T],\mathbb{R}^e)$ which is uniformly bounded, and by (6.1) again combined with the continuity of $t \mapsto |z|_{1-var;[0,t]}$ (due to (2.5) since $z \in C^{1-var}([0,T],\mathbb{R}^d)$) the sequence $(g_n)_n$ is also equicontinuous; further it is additionally uniformly bounded in variation by (6.2). Altogether, the sequence $(g_n)_n$ is uniformly bounded, equicontinuous and uniformly bounded in variation. The above features hold in the particular case when the sequence $(u_n)_n$ (of mappings from $[0,T]$ into $\mathbb{R}^e$) is uniformly bounded and equi-absolutely continuous in the sense that there is some function $\rho \in L^1_{\mathbb{R}_+}(I)$ such that

$$\|u_n(t) - u_n(s)\| \leq \int_s^t \rho(\tau)\,d\tau \quad \text{for all } n \in \mathbb{N} \text{ and } 0 \leq s \leq t \leq T.$$

This will have some importance for our purpose.

Lemma 6.1. *Let $I := [0,T]$ and $z \in C^{1-var}([0,T],\mathbb{R}^d)$. Let $b\colon I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d,\mathbb{R}^e)$ be (as above) a continuous integrand operator satisfying $(\mathcal{B}_1)$, $(\mathcal{B}_2)$ and $(\mathcal{B}_3)$. Let $(u_n)_n$ be a sequence of equi-absolutely continuous mappings from $[0,T]$ into $\mathbb{R}^e$ converging uniformly on $[0,T]$ to a mapping $u\colon[0,T] \to \mathbb{R}^e$. Then putting for every $t \in I$*

$$\phi_n(t) := \int_0^t b(s,u_n(s))\,dz_s \quad \textit{and} \quad \phi(t) := \int_0^t b(s,u(s))\,dz_s,$$

the sequence of mappings $(\phi_n)_n$ converges uniformly to ϕ on I.

Proof. First, we note that u is absolutely continuous on I. Applying the estimates given the preceding features, the sequence $(\phi_n)_n$ is uniformly bounded, equicontinuous and uniformly bounded in variation. So by the Arzelà-Ascoli theorem, any subsequence of $(\phi_n)_n$ contains a (non relabeled subsequence) converging uniformly to a continuous mapping ϕ_∞ on I. By the Lipschitz condition in $(\mathcal{B}_3)$, the corresponding subsequence $(b(.,u_n(.)))_n$ converges uniformly to $b(.,u(.))$ on I. Since $b(.,u_n(.))$ and $b(.,u(.))$ are bounded and uniformly bounded in variation, by Friz-Victoir [30, Proposition 6.12] the function ϕ_∞ (towards which the corresponding subsequence of the subsequence of $(\phi_n)_n$ converges uniformly) is given by

$$\phi_\infty(t) = \int_0^t b(s, u(s)) dz_s \quad \text{for all } t \in I.$$

This confirms the statement of the lemma. $\square$

Theorem 6.2. *Let $I := [0, T]$ and $C\colon I \rightrightarrows \mathbb{R}^e$ be an absolutely continuous closed convex valued multimapping with $v(\cdot)$ as a control function of absolute continuity. Let $a \in C(0)$, let $z \in C^{1-var}(I, \mathbb{R}^d)$ and let $b\colon I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ be a continuous integrand operator satisfying the conditions*

($\mathcal{B}_1$) $|b(t, x)| \le M$, *for all* $t \in I$ *and* $x \in \mathbb{R}^e$,

($\mathcal{B}_2$) $|b(s, x) - b(t, x)| \le \omega(s, t)$, *for all* $0 \le s \le t \le T$ *and* $x \in \mathbb{R}^e$,

($\mathcal{B}_3$) $|b(t, x) - b(t, y)| \le M\|x - y\|$, *for all* $t \in I$ *and* $x \in \mathbb{R}^e$,

where ω is a control function on $[0, T]$ and M is a positive constant.

Let $f\colon I \times \mathbb{R}^e \times \mathbb{R}^e \to \mathbb{R}^e$ be a Carathéodory mapping for which there is a function $L \in L^1_{\mathbb{R}_+}(I)$ such that

(i) $\|f(t, x, y)\| \le L(t)$ *for all* $t \in I$, $x, y \in \mathbb{R}^e$;

(ii) $\|f(t, x, y_1) - f(t, x, y_2)\| \le L(t)\|y_1 - y_2\|$ *for all* $t \in I$, $x, y_1, y_2 \in \mathbb{R}^e$.

Then the problem

$$\begin{cases} x(t) = \int_0^t b(s, u(s))\, dz_s, \ \forall t \in I \\ -\dot{u}(t) \in N_{C(t)} u(t) + f(t, x(t), u(t)) \ a.e. \ t \in I \\ u(0) = a \end{cases}$$

has a solution $(x, u) \in C^{1-var}(I, \mathbb{R}^e) \times W^{1,1}(I, \mathbb{R}^e)$.

Proof. Let Γ be the compact convex valued multimapping defined by $\Gamma(t) := L(t)\mathbb{B}_{\mathbb{R}^e}$ for all $t \in I$. Let S^1_Γ be the set of all measurable selections of Γ. For each $\ell \in S^1_\Gamma$ denote by ζ_ℓ the unique absolutely continuous solution (see Theorem 3.3) of the differential inclusion

$$\begin{cases} -\dot{\zeta}_\ell(t) \in N_{C(t)} \zeta_\ell(t) + \ell(t) \ \text{ a.e. } t \in I \\ \zeta_\ell(t) \in C(t), \ \forall t \in I, \ \ \zeta_\ell(0) = a. \end{cases}$$

Consider the set $\mathcal{X} := \{\zeta_\ell : \ell \in S^1_\Gamma\}$. It is readily seen that $\mathcal{X}$ is compact in $\mathcal{C}(I, \mathbb{R}^e)$ and equi-absolutely continuous, because by Theorem 3.3

$$\|\dot{\zeta}_\ell(t)\| \le \rho(t) := |\dot{v}(t)| + 2L(t) \quad \text{for all } \ell \in S^1_\Gamma.$$

As a consequence, by Lemma 6.1 the set

$$\mathcal{Y} := \{t \mapsto \int_0^t b(s, x(s))\, dz_s : x \in \mathcal{X}\}$$

is compact in $\mathcal{C}(I, \mathbb{R}^e)$, and so is the set $\overline{\mathrm{co}}\, \mathcal{Y}$.

By Theorem 3.3 again, for each $h \in \overline{\mathrm{co}}\,\mathcal{Y}$ there is a unique absolutely continuous solution of the differential inclusion

$$\begin{cases} -\dot{u}_h(t) \in N_{C(t)}u_h(t) + f(t, h(t), u_h(t)) \text{ a.e. } t \in I \\ u_h(t) \in C(t),\ \forall t \in I, \quad u_h(0) = a, \end{cases}$$

and this solution satisfies $\|\dot{u}_h(t)\| \leq \rho(t) = |\dot{v}(t)| + 2L(t)$ a.e. $t \in I$. Now let us consider the mapping Φ defined on $\overline{\mathrm{co}}\,\mathcal{Y}$ with values in $\mathcal{C}(I, \mathbb{R}^e)$ by putting for every $h \in \overline{\mathrm{co}}\,\mathcal{Y}$

$$\Phi(h)(t) := \int_0^t b(s, u_h(s))\, dz_s \quad \text{for all } t \in I.$$

Since $u_h \in \mathcal{X}$, it is clear that $\Phi(h) \in \mathcal{Y}$, so $\Phi \colon \overline{\mathrm{co}}\,\mathcal{Y} \to \mathcal{Y}$; our aim is to prove the existence result by applying some ideas developed in Castaing et al [17] via a fixed point theorem. For this purpose we first claim that $\Phi \colon \overline{\mathrm{co}}\,\mathcal{Y} \to \mathcal{Y}$ is continuous with respect the topology of uniform convergence on I. To do so, we first show that, if $(h_n)_n$ uniformly converges to h in $\overline{\mathrm{co}}\,\mathcal{Y}$, then the sequence $(u_{h_n})_n$, where u_{h_n} is the absolutely continuous solution of the differential inclusion

$$\begin{cases} u_{h_n}(0) = a, \quad u_{h_n}(t) \in C(t),\ \forall t \in I \\ -\dot{u}_{h_n}(t) \in N_{C(t)}u_{h_n}(t) + f(t, h_n(t), u_{h_n}(t)), \text{ a.e. } t \in I, \end{cases}$$

uniformly converges to the absolutely continuous solution u_h associated with h of the differential inclusion

$$\begin{cases} u_h(0) = a, \quad u_h(t) \in C(t),\ \forall t \in I \\ -\dot{u}_h(t) \in N_{C(t)}u_h(t) + f(t, h(t), u_h(t)), \text{ a.e. } t \in I. \end{cases}$$

Since $(u_{h_n})_n$ is equi-absolutely continuous with for each $n \in \mathbb{N}$ the estimate

$$\|\dot{u}_{h_n}(t)\| \leq \rho(t) \text{ a.e. } t \in I,$$

we may and do suppose that $(u_{h_n})_n$ converges uniformly to a mapping $w \colon I \to \mathbb{R}^e$, so for every $t \in I$ we have

$$f(t, h_n(t), u_{h_n}(t)) \to f(t, h(t), w(t)) \text{ as } n \to \infty.$$

Keeping in mind that $\|f(t, h_n(t), u_{h_n}(t))\| \leq L(t)$ for all $t \in I$ with $L \in L^1_{\mathbb{R}_+}(I)$, we can see that w is absolutely continuous and solution (see Lemma 3.4) of the differential inclusion

$$\begin{cases} w(0) = a, \quad w(t) \in C(t),\ \forall t \in I \\ -\dot{w}(t) \in N_{C(t)}w(t) + f(t, h(t), w(t)) \text{ a.e. } t \in I. \end{cases}$$

By uniqueness we derive that $w = u_h$. Taking into account the uniform convergence of $(u_{h_n})_n$ to u_h on I, we see by applying Lemma 6.1 that $(\Phi(h_n)(\cdot))_n$ converges uniformly to $\Phi(h)(\cdot)$ on I. This confirms the continuity of $\Phi \colon \overline{\mathrm{co}}(\mathcal{Y}) \to \mathcal{Y}$ for the topology of uniform convergence on I.

By the Schauder theorem Φ has a fixed point, say $h = \Phi(h)$, which means

$$h(t) = \Phi(h)(t) = \int_0^t b(s, u_h(s))\, dz_s \quad \text{for all } t \in I$$

along with
$$\begin{cases} u_h(0) = a, \quad u_h(t) \in C(t) \ \forall t \in I \\ -\frac{du_h}{dt}(t) \in N_{C(t)} u_h(t) + f(t, h(t), u_h(t)) \ \text{ a.e. } t \in I. \end{cases}$$

Taking $x(\cdot) = h(\cdot)$ and $u(\cdot) = u_h(\cdot)$, we obtain that $(x, u) \in \mathcal{C}^{1-var}(I, \mathbb{R}^e) \times W^{1,1}(I, \mathbb{R}^e)$ is a solution of the differential inclusion in the statement of the theorem. $\square$

We also have the following variant with integral Volterra type perturbation.

Theorem 6.3. *Let $I := [0, T]$ and $C\colon I \rightrightarrows \mathbb{R}^e$ be an absolutely continuous closed convex valued multimapping wth $v(\cdot)$ as a control function of absolute continuity. Let $a \in C(0)$, let $z \in C^{1-var}(I, \mathbb{R}^d)$ and let $b\colon I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ be a continuous integrand operator satisfying the conditions*

$(\mathcal{B}_1)$ $|b(t, x)| \le M$, *for all $t \in I$ and $x \in \mathbb{R}^e$,*

$(\mathcal{B}_2)$ $|b(s, x) - b(t, x)| \le \omega(s, t)$, *for all $0 \le s \le t \le T$ and $x \in \mathbb{R}^e$,*

$(\mathcal{B}_3)$ $|b(t, x) - b(t, y)| \le M\|x - y\|$, *for all $t \in I$ and $x \in \mathbb{R}^e$,*

where ω is a control function on $[0, T]$ and M is a positive constant.

Let $f\colon I \times I \times \mathbb{R}^e \times \mathbb{R}^e \to \mathbb{R}^e$ be Carathéodory mapping for which there is a function $L \in L^1_{\mathbb{R}_+}(I)$ such that

(i) $\|f(t, s, x, y)\| \le L(t)$ *for all $t \in I$, $x, y \in \mathbb{R}^e$;*

(ii) $\|f(t, s, x, y_1) - f(t, s, x, y_2)\| \le L(t)\|y_1 - y_2\|$ *for all $t, s \in I$, $x, y_1, y_2 \in \mathbb{R}^e$.*

Then the problem

$$\begin{cases} x(t) = \int_0^t b(s, u(s))\, dz_s, \ \forall t \in I \\ -\dot{u}(t) \in N_{C(t)} u(t) + \int_0^t f(t, s, x(s), u(s))\, ds \ a.e. \ t \in I \\ u(0) = a \end{cases}$$

has a solution $(x, u) \in C^{1-var}(I, \mathbb{R}^e) \times W^{1,1}(I, \mathbb{R}^e)$.

Proof. The proof will consist in certain adaptations of the arguments for Theorem 6.2. Let $\Gamma\colon I \rightrightarrows \mathbb{R}^e$ be the compact convex valued multimapping defined by $\Gamma(t) := L(t)\mathbb{B}_{\mathbb{R}^e}$ for all $t \in I$. Let S^1_Γ be the set of all measurable selections of Γ. For each $\ell \in S^1_\Gamma$ denote by ζ_ℓ the unique absolutely continuous solution (see Theorem 3.3) of the usual perturbed sweeping process

$$\begin{cases} -\dot{\zeta}_\ell(t) \in N_{C(t)} \zeta_\ell(t) + \ell(t) \ \text{ a.e. } t \in I \\ \zeta_\ell(t) \in C(t), \ \forall t \in I, \quad \zeta_\ell(0) = a. \end{cases}$$

Theorem 3.3 says that $\|\dot{\zeta}_\ell(t)\| \le \rho(t) := |\dot{v}(t)| + 2L(t)$ for all $\ell \in S^1_\Gamma$, hence the set $\mathcal{X} := \{\zeta_\ell : \ell \in S^1_\Gamma\}$ is readily seen to be compact in $\mathcal{C}(I, \mathbb{R}^e)$ and equi-absolutely continuous.

By Lemma 6.1 the set

$$\mathcal{Y} := \{t \mapsto \int_0^t b(s, x(s))\, dz_s : x \in \mathcal{X}\}$$

is then compact in $\mathcal{C}(I, \mathbb{R}^e)$, and so is the set $\overline{\mathrm{co}}\,\mathcal{Y}$. For each $\mathcal{L}(I)$-measurable mapping $h\colon I \to \mathbb{R}^e$, we observe that the mapping $f_h\colon I \times I \times \mathbb{R}^e \to \mathbb{R}^e$ defined by

$$f_h(t, s, x) = f(t, s, h(s), x) \quad \text{for all } (t, s, x) \in I \times I \times \mathbb{R}^e$$

fulfills all the assumptions $(\mathcal{G}_1')$, $\mathcal{G}_2$ and $\mathcal{G}_3$ of the mapping g in Theorem 4.1. Consequently, for each $h \in \overline{\mathrm{co}}\,\mathcal{Y}$ Theorem 4.1 assures us that there is a unique absolutely continuous solution of the differential inclusion

$$\begin{cases} -\dot{u}_h(t) \in N_{C(t)} u_h(t) + \int_0^t f(t, s, h(s), u_h(s))\, ds \text{ a.e. } t \in I \\ u_h(t) \in C(t),\ \forall t \in I, \quad u_h(0) = a, \end{cases}$$

and this solution satisfies by Theorem 3.3 the estimate

$$\|\dot{u}_h(t)\| \le \rho(t) = |\dot{v}(t)| + 2L(t) \text{ a.e. } t \in I.$$

Now let us consider the mapping Φ defined on $\overline{\mathrm{co}}\,\mathcal{Y}$ with values in $\mathcal{C}(I, \mathbb{R}^e)$ by putting for every $h \in \overline{\mathrm{co}}\,\mathcal{Y}$

$$\Phi(h)(t) := \int_0^t b(s, u_h(s))\, dz_s \quad \text{for all } t \in I.$$

Noticing that $\left|\int_0^t f(t, s, h(s) u_h(s))\, ds\right| \le L(t)$, we see that $u_h \in \mathcal{X}$.

Then, for any $h \in \overline{\mathrm{co}}\,\mathcal{Y}$ it is clear that $\Phi(h) \in \mathcal{Y}$, so $\Phi\colon \overline{\mathrm{co}}\,\mathcal{Y} \to \mathcal{Y}$. Our objective is to prove the existence result of the present theorem by applying some ideas developed in Theorem 6.2 via a fixed point theorem. To do so, we first claim that $\Phi\colon \overline{\mathrm{co}}\,\mathcal{Y} \to \mathcal{Y}$ is continuous for the topology of uniform convergence on I. Therefore, we first show that, if $(h_n)_n$ uniformly converges to h in $\overline{\mathrm{co}}\,\mathcal{Y}$, then the sequence $(u_{h_n})_n$, where u_{h_n} is the absolutely continuous solution of the differential inclusion

$$\begin{cases} u_{h_n}(0) = a, \quad u_{h_n}(t) \in C(t),\ \forall t \in I \\ -\dot{u}_{h_n}(t) \in N_{C(t)} u_{h_n}(t) + \int_0^t f(t, s, h_n(s), u_{h_n}(s))\, ds, \text{ a.e. } t \in I, \end{cases}$$

uniformly converges to the absolutely continuous solution u_h associated with h of the differential inclusion

$$\begin{cases} u_h(0) = a, \quad u_h(t) \in C(t),\ \forall t \in I \\ -\dot{u}_h(t) \in N_{C(t)} u_h(t) + \int_0^t f(t, s, h(s), u_h(s))\, ds, \text{ a.e. } t \in I. \end{cases}$$

Since $(u_{h_n})_n$ is equi-absolutely continuous with for each $n \in \mathbb{N}$ the estimate

$$\|\dot{u}_{h_n}(t)\| \le \rho(t) \text{ a.e. } t \in I,$$

we may and do suppose that $(u_{h_n})_n$ converges uniformly to a mapping $w\colon I \to \mathbb{R}^e$, so for every $t \in I$ we have by the Lebesgue dominated convergence theorem

$$\int_0^t f(t, s, h_n(s), u_{h_n}(s))\, ds \to \int_0^t f(t, s, h(t), w(s))\, ds \text{ as } n \to \infty.$$

Keeping in mind that $\left\|\int_0^t f(t,s,h_n(s),u_{h_n}(s))\,ds\right\|\leq L(t)$ for all $t\in I$ with $L\in L^1_{\mathbb{R}_+}(I)$, we can deduce that w is absolutely continuous and solution (see Lemma 3.4) of the differential inclusion

$$\begin{cases} w(0)=a, \quad w(t)\in C(t), \ \forall t\in I \\ -\dot{w}(t)\in N_{C(t)}w(t)+\int_0^t f(t,s,h(s),w(s))\,ds \ \text{ a.e. } t\in I. \end{cases}$$

By uniqueness (see Theorem 4.1 under assumption $(\mathcal{G}_1'))$ we derive that $w=u_h$. Taking into account the uniform convergence of $(u_{h_n})_n$ to u_h on I and applying Lemma 6.1 we see that $(\Phi(h_n)(\cdot))_n$ converges uniformly to $\Phi(h)(\cdot)$ on I. This confirms the continuity of $\Phi\colon \overline{\mathrm{co}}(\mathcal{Y})\to\mathcal{Y}$ with respect to the topology of uniform convergence on I. The Schauder theorem applied to the mapping Φ furnishes a fixed point, say $h=\Phi(h)$, which means

$$h(t)=\Phi(h)(t)=\int_0^t b(s,u_h(s))\,dz_s \quad \text{for all } t\in I$$

along with
$$\begin{cases} u_h(0)=a, \quad u_h(t)\in C(t) \ \ \forall t\in I \\ -\frac{du_h}{dt}(t)\in N_{C(t)}u_h(t)+\int_0^t f(t,s,h(s),u_h(s))\,ds \ \text{ a.e. } t\in I. \end{cases}$$

Taking $x(\cdot)=h(\cdot)$ and $u(\cdot)=u_h(\cdot)$ we obtain that $(x,u)\in\mathcal{C}^{1-var}(I,\mathbb{R}^e)\times W^{1,1}(I,\mathbb{R}^e)$ is a solution of the differential inclusion with integral Volterra type perturbation in the statement of the theorem. □

A deep analysis of Theorem 5.4 and Theorems 6.2 and 6.3 lead us to study the new dynamical system driven by a fractional differential equation along with a sweeping process differential inclusion with rough signal in the form

$$\begin{cases} D^\alpha x(t)+\lambda D^{\alpha-1}x(t)=u(t), t\in I, \\ -\dot{u}(t)\in N_{C(t)}u(t)+f\left(t,u(t),\int_0^t b(\tau,x(\tau))\,dz(\tau)\right), \end{cases}$$

where $\lambda\geq 0$, $\alpha\in\,]1,2]$ and D^α is the standard Riemann-Liouville fractional derivative. By a solution we mean a pair (x,u) where u is an absolutely continuous mapping from I into $\mathbb{R}^e$ and $x\in W^{\alpha,1}_{B,\mathbb{R}^e}(I)$, the space of all continuous mappings in $C(I,\mathbb{R}^e)$ such that their Riemann-Liouville fractional derivatives of order $\alpha-1$ are continuous and their Riemann-Liouville fractional derivatives of order α are Bochner integrable. For such a mapping $x\in W^{\alpha,1}_{B,R^e}(I)$ the Young integral perturbation $t\mapsto\int_0^t b(\tau,x(\tau))\,dz_\tau$ belongs to $C^{1-var}([0,1],\mathbb{R}^e)$. Clearly, the study of these problems is a great novelty. Because of the mixture of the fractional differential inclusion and the perturbed sweeping process differential inclusion with the presence of the rough signal (i.e., Young integral) in the foregoing form, the problem under consideration is complex and more difficult than the traditional problem dealing with either the sole fractional differential inclusion or the mixed fractional differential equation coupled with a perturbed sweeping process involving no rough signal.

We begin with an existence result for the related differential inclusion; another one concerned with minimizers for optimization problems under constraints by such differential inclusions will be examined later in Theorem 7.4. Given $\lambda\geq 0$, $\gamma>0$, $\alpha\in\,]1,2]$, and $\beta\in\,]0,2-\alpha]$, we present our first related result as follows.

Theorem 6.4. *Let $I = [0,1]$, $r \in]0,+\infty]$ and let $C\colon I \rightrightarrows \mathbb{R}^e$ be an absolutely continuous multimapping with $v(\cdot)$ as a control function of absolute continuity and whose values are r-prox regular. Let $a \in C(0)$, $z \in C^{1-var}(I, \mathbb{R}^d)$ and $b\colon I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ be a continuous integrand operator satisfying the conditions*

($\mathcal{B}_1$) $|b(t,x)| \le M,$ *for all* $t \in I$ *and* $x \in \mathbb{R}^e$,

($\mathcal{B}_2$) $|b(s,x) - b(t,x)| \le \omega(s,t),$ *for all* $0 \le s \le t \le T$ *and* $x, y \in \mathbb{R}^e$,

($\mathcal{B}_3$) $|b(t,x) - b(t,y)| \le M\|x-y\|,$ *for all* $t \in I$ *and* $x \in \mathbb{R}^e$,

where ω is a control function on I and M is a positive constant.

Let $f\colon I \times \mathbb{R}^e \times \mathbb{R}^e \to \mathbb{R}^e$ be a map for which there is a real constant $\mu > 0$ such that

(i) $f(\cdot, x, y)$ *is $\mathcal{L}(I)$ measurable for all $(x,y) \in \mathbb{R}^e \times \mathbb{R}^e$;*

(ii) $f(t,\cdot,\cdot)$ *is continuous on $\mathbb{R}^e \times \mathbb{R}^e$ for all $t \in I$;*

(iii) $\|f(t,x,y)\| \le \mu$ *for all $(t,x,y) \in I \times \mathbb{R}^e \times \mathbb{R}^e$;*

(iv) $\|f(t,x_1,y) - f(t,x_2,y)\| \le \mu\|x_1 - x_2\|$, *for all $(t,y) \in I \times \mathbb{R}^e$ and $x_1, x_2 \in \mathbb{R}^e$.*

Then given $\lambda \ge 0$, $\gamma > 0$, $\alpha \in]1,2]$, $\beta \in]0, 2-\alpha]$, there exist a $W^{\alpha,1}_{B,\mathbb{R}^e}(I)$ mapping $x\colon I \to \mathbb{R}^e$ and an absolutely continuous mapping $u\colon I \to \mathbb{R}^e$ satisfying the sweeping process with rough signal coupled with fractional differential equation

$$\begin{cases} D^\alpha x(t) + \lambda D^{\alpha-1} x(t) = u(t), \ t \in I \\ I^\beta_{0^+} x(t)\,|_{t=0} = 0, \quad x(1) = I^\gamma_{0^+} x(1) \\ u(t) \in C(t), \, t \in I, \quad u(0) = a \\ -\frac{du}{dt}(t) \in N_{C(t)} u(t) + f\left(t, u(t), \int_0^t b(s, x(s))\, dz_s\right) \quad \text{a.e. } t \in I. \end{cases}$$

Proof. We may and do suppose $M \ge \mu$. For any continuous mapping $g\colon I \to \mathbb{R}^e$, consider the mapping $f_g\colon I \times \mathbb{R}^e \to \mathbb{R}^e$ defined by

$$f_g(t,x) := f(t,x,g(t)) \quad \text{for all } (t,x) \in I \times \mathbb{R}^e.$$

We see that $f_g(\cdot, x)$ is $\mathcal{L}(I)$-measurable for every $x \in \mathbb{R}^e$, that $f_g(t,\cdot)$ is continuous on $\mathbb{R}^e$ for every $t \in I$, and that

$$\|f_g(t,x)\| \le M \quad \text{for all } (t,x) \in I \times \mathbb{R}^e$$

and $$\|f_g(t,x_1) - f_g(t,x_2)\| \le M\|x_1 - x_2\| \quad \text{for all } (t,x_1,x_2) \in I \times \mathbb{R}^e \times \mathbb{R}^e.$$

Then, given $a \in C(0)$, by Theorem 4.1 there is a *unique* AC solution y_g to the differential inclusion

$$\begin{cases} y_g(0) = a \in C(0), \quad y_g(t) \in C(t), \ \forall t \in I \\ -\frac{dy_g}{dt}(t) \in N_{C(t)} y_g(t) + f(t, y_g(t), g(t)) \ \text{ a.e. } t \in I, \end{cases}$$

and $(y_g)_g$ is uniformly bounded and equi-absolutely continuous. Indeed, for the integrable function $\rho(\cdot) := |\dot v(\cdot)| + 2M$ we know by Theorem 3.3 that $\|\dot y_g(t)\| \le \rho(t)$, then for the real constant $L = \|a\| + \int_0^1 (|\dot v(t)| + 2M)\, dt$ we have $\|y_g(t)\| \le L$ for all $t \in I$. These two latter inequalities justify the uniform boundedness and the equi-absolute continuity of $(y_g)_g$.

Now let us consider the set $\mathcal{X}$ defined by

$$\mathcal{X} := \{\xi_\ell : I \to E : \ \ell \in S^1_{L\mathbb{B}_{\mathbb{R}^e}}\},$$

each mapping ξ_ℓ being given for every $t \in I$ by

$$\xi_\ell(t) = \int_0^1 G(t,s)\ell(s)\, ds,$$

where G is the Green function involved in Lemma 5.2. We note that $\mathcal{X}$ is convex compact and equi-Lipschitz ([17, Theorem 4.1]), more precisely, for some real constant $N > 0$ one has for every $h \in \mathcal{X}$

$$\|h(t) - h(s)\| \le N|t-s|^{\gamma-1} \le N|t-s|.$$

Then for any $h \in \mathcal{X}$, using $(\mathcal{B}_2)$ and $(\mathcal{B}_3)$ we have for all $0 \le s \le t \le 1$

$$\begin{aligned}|b(t,h(t)) - b(s,h(s))| &\le |b(t,h(t)) - b(s,h(t))| + |b(s,h(t)) - b(s,h(s))|\\ &\le \omega(s,t) + N|t-s|,\end{aligned}$$

so that for any $h \in \mathcal{X}$, $b(.,h(.)) \in C^{1-var}([0,1], \mathcal{L}(\mathbb{R}^e,\mathbb{R}^e))$ with $|b(.,h(.))| \le M$ by $(\mathcal{B}_1)$. In particular, for any $t \in I$ the integral $\int_0^t b(s,h(s))\, dz_s$ has a meaning with $b(.,h(.))$ uniformly bounded in variation. In consequence, the set

$$\mathcal{Y} := \{\int_0^{\cdot} b(s,h(s))\, dz_s : h \in \mathcal{X}\}$$

is *compact* in $C([0,1],\mathbb{R}^e)$ by repeating the arguments given in Theorem 6.2. In particular if $(h_n)_n$ in $\mathcal{X}$ converges uniformly to $h \in \mathcal{X}$, then,

$$\textstyle\int_0^{\cdot} b(\tau, h_n(\tau))\, dz_\tau \to \int_0^{\cdot} b(\tau,h(\tau))\, dz_\tau$$

uniformly on I. For each $h \in \mathcal{X}$ let us set (again with the above Green function G involved in Lemma 5.2)

$$\Phi(h)(t) = \int_0^1 G(t,s)u_h(s)\, ds, \quad \text{for all } t \in I.$$

where u_h is the *unique* AC solution to the differential inclusion

$$\begin{cases} u_h(0) = a, \quad u_h(t) \in C(t), \ \forall t \in I\\ -\frac{du_h}{dt}(t) \in N_{C(t)}u_h(t) + f\left(t, u_h(t), \int_0^t b(s,h(s))dz_s\right) \ \text{ a.e. } t \in I\end{cases}$$

Then it is clear that $\Phi(h) \in \mathcal{X}$. Now we verify that Φ is continuous relative to $\mathcal{X}$. It is enough to show that, if $(h_n)_n$ converges uniformly to h in $\mathcal{X}$, then the sequence $(u_{h_n})_n$, where each u_{h_n} is the unique absolutely continuous solution of the differential inclusion

$$\begin{cases} u_{h_n}(0) = a \in C(0), \quad u_{h_n}(t) \in C(t), \ \forall t \in I\\ -\dot{u}_{h_n}(t) \in N_{C(t)}u_{h_n}(t) + f\left(t, u_{h_n}(s), \int_0^t b(s,h_n(s))\, dz_s\right) \quad \text{a.e. } t \in I,\end{cases}$$

uniformly converges to the unique absolutely continuous solution u_h of the differential inclusion

$$\begin{cases} u_h(0) = a \in C(0), \quad u_h(t) \in C(t), \ \forall t \in I \\ -\dot{u}_h(t) \in N_{C(t)} u_h(t) + f\left(t, u_h(s), \int_0^t b(s, h(s))\, dz_s\right) \qquad \text{a.e. } t \in I. \end{cases}$$

We note that $(u_{h_n})_n$ is equicontinuous since for every $n \in \mathbb{N}$ we have $\|\dot{u}_{h_n}(t)\| \le \rho(t)$ for almost all $t \in I$. Further, $\{u_{h_n}(t) : n \in \mathbb{N}\}$ is included in the compact set $L\mathbb{B}_{\mathbb{R}^e}$ for every $t \in I$. The Arzelà-Ascoli theorem tells us that $\{u_{h_n} : n \in \mathbb{N}\}$ is relatively compact in $C(I, \mathbb{R}^e)$. So by extracting a subsequence, we may suppose that $(u_{h_n})_n$ converges uniformly on I to some mapping $\zeta \colon I \to \mathbb{R}^e$ with

$$\zeta(t) = \zeta(0) + \int_0^t \dot{\zeta}(s)\, ds \quad \text{for all } t \in I,$$

along with $(\dot{u}_{h_n})_n$ converging weakly in $L^1_{\mathbb{R}^e}(I)$ to $\dot{\zeta}$ with $\|\dot{\zeta}(t)\| \le \rho(t)$ for a.e. $t \in I$. For simplicity, define

$$q_n(t) := f\left(t, u_{h_n}(t), \int_0^t b(s, h_n(s))\, dz_s\right) \quad \text{for all } t \in I,$$

$$q(t) := f\left(t, \zeta(s), \int_0^t b(s, h(s))\, dz_s\right) \quad \text{for all } t \in I.$$

First, we notice that these mappings are Lebesgue measurable and it is already seen that $\int_0^t b(s, h_n(s))\, dz_s \to \int_0^t b(s, h(s))\, dz_s$ uniformly with respect to $t \in I$. Second, by condition (iii) and the uniform boundedness of $(u_{h_n})_n$ and u_h, we also notice that $(q_n)_n$ and q are uniformly bounded, since

$$\|q_n(t)\| \le M \quad \text{and} \quad \|q(t)\| \le M, \quad \text{for all } n \in \mathbb{N} \text{ and } t \in I.$$

Consequently, $(q_n)_n$ is a sequence of measurable and uniformly bounded mappings which converges pointwise to the measurable mapping q. Therefore, the sequence $(\dot{u}_{h_n} + q_n)_n$ converges weakly to $\dot{\zeta} + q$ in $L^1_E(I)$. This combined with Lemma 3.4 and Theorem 3.3 gives

$$\begin{cases} -\frac{d\zeta}{dt}(t) \in N_{C(t)}\zeta(t) + q(t) \ \text{ a.e. } t \in I \\ \zeta(0) = a, \ \zeta(t) \in C(t) \ \forall t \in I, \end{cases}$$

so using the uniqueness of solution of the latter differential inclusion (due to Theorem 4.1) we obtain that $\zeta = u_h$. Now let us write by Lemma 5.2(a)

$$\Phi(h_n)(t) - \Phi(h)(t) = \int_0^1 G(t,s) u_{h_n}(s)\, ds - \int_0^1 G(t,s) u_h(s)\, ds$$

$$= \int_0^1 G(t,s)[u_{h_n}(s) - u_h(s)]\, ds \le \int_0^1 M_G \|u_{h_n}(s) - u_h(s)\|\, ds.$$

Since $(u_{h_n})_n$ converges uniformly on I to u_h as $n \to \infty$, we deduce that

$$\sup_{t \in I} \|\Phi(h_n)(t) - \Phi(h)(t)\| \le \int_0^1 M_G \|u_{h_n}(s) - u_h(s)\|\, ds \to 0,$$

which entails that $\Phi(h_n) \to \Phi(h)$ uniformly on I, as desired. Then $\Phi\colon \mathcal{X} \to \mathcal{X}$ is continuous, hence by the Schauder theorem Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}$. This means that for every $t \in I$

$$h(t) = \Phi(h)(t) = \int_0^1 G(t,s)u_h(s)\,ds,$$

with
$$\begin{cases} u_h(0) = a, \quad u_h(t) \in C(t), \,\forall t \in I \\ -\dot{u}_h(t) \in N_{C(t)}u_h(t) + f\left(t, u_h(t), \int_0^t b(s,h(s))\,dz_s\right) \quad \text{a.e. } t \in I. \end{cases}$$

According to Lemma 5.2, this means that we have just shown that there exists a mapping $h \in W^{\alpha,1}_{B,\mathbb{R}^e}(I)$ satisfying

$$\begin{cases} D^{\alpha}h(t) + \lambda D^{\alpha-1}h(t) = u_h(t), \\ I^{\beta}_{0+}h(t)\,|_{t=0} = 0, \quad h(1) = I^{\gamma}_{0+}h(1) \\ u_h(0) = a, \quad u_h(t) \in C(t)), \;\forall t \in [0,1] \\ -\dot{u}_h(t) \in N_{C(t)}u_h(t) + f(t, u_h(t), \int_0^t b(s,h(s))dz(s)) \quad \text{a.e. } t \in I. \end{cases}$$

This finishes the proof of the theorem. □

7. Optimization problems governed by rough differential inclusions and sweeping processes with perturbations

Recently, there was an intense activity in optimal control problems involving the sweeping process.

In this spirit we consider the following optimization problem with a dynamical system driven by a rough differential inclusion coupled with a sweeping process:

$$\text{Minimize} \quad \int_0^T L(t, x(t), y(t))\,dt$$

over the pairs $(x,y) \in C^{1-var}([0,T],\mathbb{R}^e) \times W^{1,1}([0,T],\mathbb{R}^e)$ subject to

$$\begin{cases} dx_t = V(x_t)\,dz_t, \; t \in [0,T] \\ x_0 = \psi \in \mathbb{R}^e, \quad y(0) = y_0 \in C(0) \\ -\dot{y}(t) \in N_{C(t)}y(t) + f(t, x(t), y(t)), \quad a.e.\, t \in [0,T]. \end{cases}$$

Here
$$dx_t = V(x_t)\,dz_t, \; x_0 = \psi \tag{7.1}$$

is a rough differential equation where $z \in C^{1-var}([0,T],\mathbb{R}^d)$ and $V : \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ is a bounded continuous mapping. By a solution of (7.1) one means (see [30, Definition 3.3]) a continuous mapping $x\colon [0,T] \to \mathbb{R}^e$ with $x(0) = x_0$ and $x_t = x_0 + \int_0^t V(x_s)\,dz_s$ for all $t \in [0,T]$, or in other words the differential equation (7.1) is understood as the Riemann-Stieltjes integral equation

$$x_t = x_0 + \int_0^t V(x_s)\,dz_s \quad \text{with } x_0 = \psi, \tag{7.2}$$

for any solution $x \in C^{1-var}([0,T],\mathbb{R}^e)$.

The equation is known by [30, Theorem 3.4] to admit at least one solution, and the solution set is equicontinuous by (2.6), which allows us to easily see that the solution set is compact in $\mathcal{C}_{\mathbb{R}^e}([0,T])$ by the Arzéla-Ascoli theorem (note that arguments in [30, Theorem 3.4] yield also to such a compactness result). The differential inclusion

$$-\dot{y}(t) \in N_{C(t)}y(t) + f(t,x(t),y(t)),\ y(0) = y_0 \in C(0) \tag{7.3}$$

corresponds to the sweeping process associated with the moving set $C(t)$ with perturbation f. Here the perturbation $f\colon [0,T]\times\mathbb{R}^e\times\mathbb{R}^e \to \mathbb{R}^e$ is assumed to satisfy the conditions:

($\mathcal{F}_1$) $(x,y)\mapsto f(t,x,y)$ is continuous on $\mathbb{R}^e\times\mathbb{R}^e$ for every fixed $t\in[0,T]$;

($\mathcal{F}_2$) $t\mapsto f(t,x,y)$ is $\mathcal{L}([0,T])$-measurable for all fixed $x,y\in\mathbb{R}^e$;

($\mathcal{F}_3$) f is uniformly bounded, say for some real constant $\mu>0$

$$\|f(t,x,y)\|\le\mu \quad \text{for all } (t,x,y)\in[0,T]\times\mathbb{R}^e\times\mathbb{R}^e;$$

($\mathcal{F}_4$) $f(t,x,\cdot)$ is uniformly Lipschitz in the sense that for some constant $M>0$

$$\|f(t,x,y)-f(t,x,w)\|\le M\|y-w\| \text{ for all } (t,x,y,w)\in[0,T]\times\mathbb{R}^e\times\mathbb{R}^e\times\mathbb{R}^e.$$

A quick glance shows that this problem is of great interest since before investigating the optimal solutions, we are forced to notice that solutions (x,y) of the above couple of equality and inclusion are not classical. Furthermore, it is clear that the optimal problem for such a system is quite general since it contains many results related to classical solutions of similar evolution equations. To the best of our knowledge, the study of optimization problems for a dynamic system driven by a rough differential inclusion coupled with a sweeping process is considered here for the first time.

Proposition 7.1. *Let $I:=[0,T]$ and $C\colon I\rightrightarrows\mathbb{R}^e$ be an absolutely continuous multimapping with $v(\cdot)$ as a control function of absolute continuity. Assume that for some $r\in\,]0,+\infty]$ all the sets $C(t)$ are closed and r-prox regular. Let $V:\mathbb{R}^e\to\mathcal{L}(\mathbb{R}^d,\mathbb{R}^e)$ be a bounded continuous mapping and let $f\colon[0,T]\times\mathbb{R}^e\times\mathbb{R}^e\to\mathbb{R}^e$ be a mapping satisfying the above conditions $(\mathcal{F}_1)-(\mathcal{F}_4)$. Let $z\in C^{1-var}([0,T],\mathbb{R}^d)$ and let $L\colon I\times\mathbb{R}^e\times\mathbb{R}^e\times\mathbb{R}^e\to[0,\infty[$ be a lower semicontinuous integrand such that $L(t,x,y,\cdot)$ is convex on $\mathbb{R}^e$ for every $(t,x,y)\in I\times\mathbb{R}^e\times\mathbb{R}^e$. Then given $\psi\in\mathbb{R}^e$ and $y_0\in C(0)$, the problem of minimizing the cost function $\int_0^T L(t,x(t),y(t),\dot{y}(t))\,dt$ subject to the dynamics*

$$\begin{cases} dx_t = V(x_t)\,dz_t,\ t\in I\\ x_0=\psi\in\mathbb{R}^e, \quad y(0)=y_0\in C(0)\\ -\dot{y}(t)\in N_{C(t)}y(t)+f(t,x(t),y(t)), \quad a.e.\, t\in I. \end{cases}$$

has an optimal solution.

Proof. Let us consider a minimizing sequence $(x_n,y_n)_n$, that is,

$$\lim_{n\to\infty}\int_0^T L(t,x_n(t),y_n(t),\dot{y}_n(t))\,dt = \inf_{(\zeta,\xi)\in\mathcal{X}}\int_0^T L(t,\zeta(t),\xi(t),\dot{\xi}(t))\,dt,$$

where $\mathcal{X}$ is the set of solutions (ζ,ξ) to the above dynamical system. First, according to the foregoing features we can assert that the $C^{1-var}(I,\mathbb{R}^e)$-solution set to (7.1) is nonempty and compact in $C(I,\mathbb{R}^e)$, and so is the $W^{1,1}(I,\mathbb{R}^e)$-solution set to (7.3).

This ensures that for some subsequence

(i) $x_n \to x \in C^{1-var}(I, \mathbb{R}^e)$ uniformly on I, with $x(t) = \psi + \int_0^t V(x(s))\, dz_s$;

(ii) $y_n \to y \in W^{1,1}(I, \mathbb{R}^e)$ uniformly on I, and $\dot{y}_n \to \dot{y}$ weakly in $L^1_{\mathbb{R}^e}(I)$.

According to the lower semicontinuity of the integral functional (see [22, Theorem 8.16]) we obtain

$$\liminf_n \int_0^T L(t, x_n(t), y_n(t), \dot{y}_n(t))\, dt \geq \int_0^T L(t, x(t), y(t), \dot{y}(t))\, dt.$$

On the other hand, $x(0) = \psi$, $y(0) = y_0$, and from the inclusion

$$-\dot{y}_n(t) \in N_{C(t)} y_n(t) + f(t, x_n(t), y_n(t)) \quad \text{a.e. } t \in I$$

we deduce by Lemma 3.4 that we also have

$$-\dot{y}(t) \in N_{C(t)} y(t) + f(t, x(t), y(t)) \, \text{a.e. } t \in I.$$

Altogether, we see that the pair (x, y) is an optimal solution. □

Proposition 7.2. *Let $I := [0,T]$, $f\colon I \times \mathbb{R}^e \to \mathbb{R}^e$ be a Carathéodory mapping such that $\|f(t,x)\| \leq \beta$ for all $(t,x) \in I \times \mathbb{R}^e$. Let $C\colon I \rightrightarrows \mathbb{R}^e$ be an absolutely continuous multimapping with compact convex values. Let $V\colon \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ be a bounded continuous mapping and let $z \in C^{1-var}(I, \mathbb{R}^d)$. Let $A\colon \mathbb{R}^e \to \mathbb{R}^e$ be a continuous coercive symmetric linear operator and let $B\colon \mathbb{R}^e \to \mathbb{R}^e$ be a continuous linear operator. Let $L\colon I \times \mathbb{R}^e \times \mathbb{R}^e \times \mathbb{R}^e \to [0, \infty[$ be a lower semicontinuous integrand such that $L(t,x,y,\cdot)$ is convex on $\mathbb{R}^e$ for every $(t,x,y) \in I \times \mathbb{R}^e \times \mathbb{R}^e$. Then given $\psi \in \mathbb{R}^e$ and $y_0 \in C(0)$, the problem of minimizing the cost function $\int_0^T L(t, x(t), y(t), \dot{y}(t)) dt$ subject to the dynamics*

$$\begin{cases} dx_t = V(x_t) dz_t, \ t \in I \\ x_0 = \psi \in \mathbb{R}^e, \quad y(0) = y_0 \in C(0) \\ f(t, x(t)) + By(t) - A\dot{y}(t) \in N_{C(t)}(\frac{dy}{dt}(t)), \quad a.e. \ t \in I. \end{cases}$$

has an optimal solution.

Proof. As above, consider a minimizing sequence $(x_n, y_n)_n$, so

$$\lim_{n\to\infty} \int_0^T L(t, x_n(t), y_n(t), \dot{y}_n(t)) dt = \inf_{(\zeta,\xi) \in \mathcal{Y}} \int_0^T L(t, \zeta(t), \xi(t), \dot{\xi}(t))\, dt,$$

where $\mathcal{Y}$ is the set of solutions (ζ, ξ) to the above dynamical system. As seen above, the $C^{1-var}(I, \mathbb{R}^e)$-solution set to (7.1) is compact in $C([0,T], \mathbb{R}^e)$, and so is the $W^{1,1}(I, \mathbb{R}^e)$-solution set to

$$\begin{cases} y(0) = y_0 \in C(0) \\ f(t, x(t)) + By(t) - A\dot{y}(t) \in N_{C(t)}(\frac{dy}{dt}(t)), \ \text{a.e. } t \in I. \end{cases}$$

We deduce that for some subsequence

(i) $x_n \to x \in C^{1-var}(I, \mathbb{R}^e)$ uniformly on I, with $x(t) = \psi + \int_0^t V(x(s))\, dz_s$;

(ii) $y_n \to y \in W^{1,1}(I, \mathbb{R}^e)$ uniformly on I, and $\dot{y}_n \to \dot{y}$ weakly in $L^1_{\mathbb{R}^e}(I)$.

This and the lower semicontinuity of the integral functional (see [22, Theorem 8.16]) yield

$$\liminf_n \int_0^T L(t, x_n(t), y_n(t), \dot{y}_n(t))\, dt \geq \int_0^T L(t, x(t), y(t), \dot{y}(t))\, dt.$$

Further, it is clear that $x(0) = \psi$ and $y(0) = y_0$. Then, from the inclusion

$$f(t, x_n(t)) + By_n(t) - A\dot{y}_n(t) \in N_{C(t)}(\frac{dy_n}{dt}(t)) \text{ a.e. } t \in I$$

and from (i) and (ii), we conclude by using the *variational limit* result in Proposition 7.3 below that

$$f(t, x(t)) + By(t) - A\dot{y}(t) \in N_{C(t)}(\frac{dy}{dt}(t)) \text{ a.e. } t \in I. \qquad \square$$

Proposition 7.3. [13] *Let E be a separable Hilbert space and $I := [0, T]$. Let $C\colon I \rightrightarrows E$ be a weakly compact convex valued scalarly measurable multimapping such that $C(t) \subset \kappa(t)\mathbb{B}_E$, for all $t \in I$ for some $\kappa \in L^1_{\mathbb{R}_+}(I)$. Let A be a continuous coercive symmetric linear operator from E into E and B be a continuous compact linear operator from E into E.*

Let $(f_n, f)_{n\in\mathbb{N}}$ be a bounded sequence in $L^\infty_E(I)$ with $\|f_n(t)\| \leq \beta$, $\|f(t)\| \leq \beta$ ($\beta > 0$) for all $n \in \mathbb{N}$ such that $f_n(t)$ converges to $f(t)$ for each $t \in I$.

Let $(v_n, v)_{n\in\mathbb{N}}$ be a bounded sequence in $L^\infty_E(I)$ with $\|v_n(t)\| \leq \gamma$, $\|v(t)\| \leq \gamma$ ($\gamma > 0$) for all $n \in \mathbb{N}$ such that $v_n(t)$ converges weakly to $v(t)$ for each $t \in I$.

Let $(u_n)_n$ be an integrable sequence in $L^1_E(I)$ such that $u_n(t) \in C(t)$ for all $t \in I$ and such that $(u_n)_n$ converges $\sigma(L^1_E(I), L^\infty_E(I))$ in $L^1_E(I)$ to u. Assume that $f_n(t) + Bv_n(t) - Au_n(t) \in N_{C(t)}(u_n(t))$ for all $n \in \mathbb{N}$ and for all $t \in I$. Then $u(t) \in C(t)$ a.e. $t \in I$ and $f(t) + Bv(t) - Au(t) \in N_{C(t)}(u(t))$ a.e. $t \in I$.

Coming back to the context and notation of Theorem 6.4 we also have a result of existence of minimizer for the following minimization problem subject to constraints related to sweeping process differential inclusions with rough signals coupled with fractional differential equations (FDI/sweeping process, for short). Given $\lambda \geq 0$, $\alpha \in]1, 2]$ and $\beta \in]0, 2 - \alpha]$, the result is the following.

Theorem 7.4. *Let $I = [0, 1]$, let $r \in]0, +\infty]$ and let $C\colon I \rightrightarrows \mathbb{R}^e$ be an absolutely continuous multimapping with $v(\cdot)$ as a control function of absolute continuity and such that $C(0)$ is bounded and $C(t)$ is r- prox regular for every $t \in I$. Let $a \in C(0)$, let $z \in C^{1-var}(I, \mathbb{R}^d)$ and let $b\colon I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ be a continuous integrand operator satisfying the conditions*

($\mathcal{B}_1$) $|b(t, x)| \leq M$, *for all $t \in I$ and $x \in \mathbb{R}^e$,*

($\mathcal{B}_2$) $|b(s, x) - b(t, x)| \leq \omega(s, t)$, *for all $0 \leq s \leq t \leq T$ and $x, y \in \mathbb{R}^e$,*

($\mathcal{B}_3$) $|b(t, x) - b(t, y)| \leq M\|x - y\|$, *for all $t \in I$ and $x \in \mathbb{R}^e$,*

where ω is a control function on I and M is a positive constant.

Let $f\colon I \times \mathbb{R}^e \times \mathbb{R}^e \to \mathbb{R}^e$ be a map for which there is a real constant $\mu > 0$ such that

(i) *$f(\cdot, x, y)$ is $\mathcal{L}(I)$ measurable for all $(x, y) \in \mathbb{R}^e \times \mathbb{R}^e$;*

(ii) *$f(t, \cdot, \cdot)$ is continuous on $\mathbb{R}^e \times \mathbb{R}^e$ for all $t, \in I$;*

(iii) *$\|f(t, x, y)\| \leq \mu$ for all $(t, x, y) \in I \times \mathbb{R}^e \times \mathbb{R}^e$;*

(iv) *$\|f(t, x_1, y) - f(t, x_2, y)\| \leq \mu\|x_1 - x_2\|$, for all $(t, y) \in I \times \mathbb{R}^e$ and $x_1, x_2 \in \mathbb{R}^e$.*

Let $\lambda \geq 0$, $\gamma > 0$, $\alpha \in]1,2]$, $\beta \in]0, 2-\alpha]$, and let $L\colon [0,1] \times \mathbb{R}^e \times \mathbb{R}^e \times \mathbb{R}^e \to [0,\infty[$ be a lower semicontinuous integrand such that $L(t,x,y,.)$ is convex on $\mathbb{R}^e$ for every $(t,x,y) \in [0,1] \times \mathbb{R}^e \times \mathbb{R}^e$. Then the problem of minimizing the cost function

$$\int_0^1 L\left(t, \int_0^t b(s,x(s))\,dz_s, u(t), \dot{u}(t)\right) dt$$

over all pairs of mappings (x,u) from I into $\mathbb{R}^e$ with $u(0) \in C(0)$, $x \in W^{\alpha,1}_{B,\mathbb{R}^e}(I)$, $u \in W^{1,1}(I,\mathbb{R}^e)$ and subject to

$$\begin{cases} D^\alpha x(t) + \lambda D^{\alpha-1}x(t) = u(t), \ t \in I \\ I^\beta_{0+}x(t)\,|_{t=0} = 0, \quad x(1) = I^\gamma_{0+}x(1) \\ -\frac{du}{dt}(t) \in N_{C(t)}u(t) + f\left(t, u(t), \int_0^t b(s,x(s))\,dz_s\right) \quad \text{a.e. } t \in I, \end{cases}$$

admits an optimal solution.

Proof. We may and do suppose $M \geq \mu$. Define the constant

$$\sigma := \max_{a \in C(0)} \|a\| + \int_0^1 (|\dot{v}(t)| + 2M)\,dt$$

and the set $\mathcal{X} := \{\xi_\ell\colon I \to E : \ \ell \in S^1_{L\mathbb{B}_{\mathbb{R}^e}}\}$, each mapping ξ_ℓ being given for every $t \in I$ by

$$\xi_\ell(t) = \int_0^1 G(t,s)\ell(s)\,ds,$$

where G is the Green function involved in Lemma 5.2. As in the proof of Theorem 6.4, the set $\mathcal{X}$ is convex and compact in $\mathcal{C}(I,\mathbb{R}^e)$ with respect to the topology of uniform convergence.

Let $(h_n,u_n)_n$ be a mimimizing sequence for the minimization problem in the statement, namely

$$\begin{aligned} &\lim_n \int_0^T L\left(t, \int_0^t b(s,h_n(s))\,dz_s, u_n(t), \dot{u}_n(t)\right) dt \\ &= \inf_{(k,w)} \left[\int_0^T L\left(t, \int_0^t b(s,k(s))\,dz_s, w(t), \dot{w}(t)\right) dt\right], \end{aligned}$$

the infimum being taken over all pairs (k,w) with $k \in W^{\gamma,1}_{B,\mathbb{R}^e}(I)$ and $w \in W^{1,1}(I,\mathbb{R}^e)$ such that

$$\begin{cases} D^\alpha k(t) + \lambda D^{\alpha-1}k(t) = w(t), \\ I^\beta_{0+}k(t)\,|_{t=0} = 0, \quad k(1) = I^\gamma_{0+}k(1) \\ w(0) \in C(0), \quad w(t) \in C(t)), \forall t \in I \\ -\dot{w}(t) \in N_{C(t)}w(t) + f\left(t, w(t), \int_0^t b(s,k(s))\,dz_s\right) \quad \text{a.e. } t \in I. \end{cases}$$

Note that

$$\begin{cases} D^\alpha h_n(t) + \lambda D^{\alpha-1}h_n(t) = u_n(t), \\ I^\beta_{0+}h_n(t)\,|_{t=0} = 0, \quad h_n(1) = I^\gamma_{0+}h_n(1) \\ u_n(0) \in C(0), \quad u_n(t) \in C(t)), \ \forall t \in [0,1] \\ -\dot{u}_n(t) \in N_{C(t)}u_n(t) + f(t, u_n(t), \int_0^t b(s,h_n(s))dz_s) \quad \text{a.e. } t \in I. \end{cases}$$

By compactness of the solution set of the differential inclusion in the statement (as it can be seen through the proof of Theorem 6.4), there is a (non-relabeled) subsequence $(h_n, u_n)_n$ with $(h_n)_n$ converging uniformly on I to $h \in \mathcal{X}$ and with $(u_n)_n$ converging uniformly on I to an AC mapping $u\colon I \to \mathbb{R}^e$ with $\dot{u}_n \to \dot{u}$ weakly in $L^1_{\mathbb{R}^e}(I)$. We can also see through the proof of Theorem 6.4 again that $\int_0^t b(s, h_n(s))\, dz_s \to \int_0^t b(s, h(s))\, dz_s$ uniformly with respect to $t \in I$. So by repeating the arguments in Theorem 6.4, we obtain that u satisfies to the inclusion

$$\begin{cases} u(0) \in C(0), \quad u(t) \in C(t)), \forall t \in [0,1] \\ -\dot{u}(t) \in N_{C(t)}u(t) + f\left(t, u(t), \int_0^t b(s, h(s))\, dz_s\right) \quad \text{a.e. } t \in I. \end{cases}$$

From Lemma 5.2 the set of equalities

$$\begin{cases} D^\alpha h_n(t) + \lambda D^{\alpha-1} h_n(t) = u_n(t), \\ I^\beta_{0^+} h_n(t)\,|_{t=0} = 0, \quad h_n(1) = I^\gamma_{0^+} h_n(1) \end{cases}$$

is equivalent to
$$h_n(t) = \int_0^1 G(t,s) u_n(s)\, ds,$$

with the Green function considered in the same Lemma 5.2. Therefore, by passing to the limit in this equality, we get

$$h(t) = \int_0^1 G(t,s) u(s)\, ds \quad \text{for all } t \in I.$$

Altogether we see by Lemma 5.2 that (h, u) satisfies to the FDI/sweeping process

$$\begin{cases} D^\alpha h(t) + \lambda D^{\alpha-1} h(t) = u(t), \\ I^\beta_{0^+} h(t)\,|_{t=0} = 0, \quad h(1) = I^\gamma_{0^+} h_n(1) \\ u(0) \in C(0), \quad u(t) \in C(t)), \forall t \in [0,1] \\ -\dot{u}(t) \in N_{C(t)}u(t) + f\left(t, u(t), \int_0^t b(s, h(s))\, dz_s\right) \quad \text{a.e. } t \in I. \end{cases}$$

According to the lower semicontinuity of the integral functional (see, e.g., [22, Theorem 8.16]) we obtain

$$\begin{aligned} &\liminf_n \int_0^T L\left(t, \int_0^t b(s, h_n(s)) dz_s, u_n(t), \dot{u}_n(t)\right) dt \\ &\geq \int_0^T L\left(t, \int_0^t b(s, h(s)) dz_s, u(t), \dot{u}(t)\right) dt, \end{aligned}$$

so we see that the pair (h, u) is an optimal solution. □

8. Towards relaxation problems involving sweeping processes

This section analyzes the relaxation of swepping processes subject to perturbations of diverse types.

8.1. Relaxation under perturbation with time-dependent multimapping

In this subsection keeping $I = [0,T]$ and E as a separable Hilbert space, we denote by $L^1_\omega(I,E)$ the space $L^1(I,E)$ equipped with the weak norm $\|\cdot\|_w$ defined by

$$\|h\|_w := \max_{t\in I}\left\|\int_0^t h(s)\,ds\right\| \quad \text{for all } h \in L^1(I,E).$$

If $(h_n)_n$ is an integrably bounded sequence in $L^1(I,E)$ with $\|h_n - h\|_w \to 0$ for a mapping $h \in L^1(I,E)$, then $(h_n)_n$ weakly converges to h in $L^1(I,E)$ (see, e.g., Tolstonogov [59]).

Theorem 8.1. (Relaxation) *Let E be a separable Hilbert space, let $I := [0,1]$, and let $r \in]0,+\infty]$. Let $C\colon I \rightrightarrows E$ be a multimapping with nonempty values which is absolutely continuous with $v(\cdot)$ as a control function of absolute continuity and such that $C(t)$ is r-prox-regular and ball-compact for all $t \in I$. Let $a \in C(0)$ and let $F\colon I \rightrightarrows E$ be a weakly compact valued measurable multimapping for which there is $\alpha \in L^1_{\mathbb{R}_+}(I)$ satisfying*

$$(\mathcal{H}_F) \qquad \max\{\|y\| : y \in F(t)\} \le \alpha(t) \quad \textit{for all}\, t \in I.$$

Let $\mathcal{S}_F$ be the solution set to

$$\begin{cases} u(0) = a,\ u(t) \in C(t),\ \forall t \in I; \\ -\dfrac{du}{dt}(t) \in N_{C(t)}u(t) + F(t) \quad \text{a.e. } t \in I \end{cases}$$

and let $\mathcal{S}_{\overline{\text{co}}F}$ be the solution set to

$$\begin{cases} u(0) = a,\ u(t) \in C(t),\ \forall t \in I; \\ -\frac{du}{dt}(t) \in N_{C(t)}u(t) + \overline{\text{co}}F(t) \quad \text{a.e. } t \in I. \end{cases}$$

Then $\mathcal{S}_F$ is dense in the compact set $\mathcal{S}_{\overline{\text{co}}F}$ of $\mathcal{C}_E(I)$ with respect to the topology of uniform convergence.

Proof. Let $\mathcal{H} := \{h \in L^1_E(I) : \|h(t)\| \le \alpha(t) \text{ a.e. } t \in I\}$. Fix any $\zeta(\cdot) \in \mathcal{S}_{\overline{\text{co}}F}$. This means that there are some $g \in L^1_E(I)$ with

$$g(t) \in \overline{\text{co}}F(t) \quad \text{and} \quad \|g(t)\| \le \alpha(t) \quad \text{a.e.}\, t \in I$$

and an absolutely continuous mapping $u_g\colon I \to E$ coinciding with $\zeta(\cdot)$ such that

$$\begin{cases} -\dot{u}_g(t) \in N_{C(t)}u_g(t) + g(t) \quad \text{a.e.}\, t \in I \\ u_g(0) = a,\ u_g(t) \in C(t),\ \forall t \in I. \end{cases}$$

Similarly, any element in $\mathcal{S}_F$ is in the form u_f, for some $f \in L^1_E(I)$ with

$$f(t) \in F(t) \quad \text{and} \quad \|f(t)\| \le \alpha(t) \quad \text{a.e.}\, t \in I,$$

and with u_f corresponding to the unique absolutely continuous mapping from I into E such that

$$\begin{cases} -\dot{u}_f(t) \in N_{C(t)}u_f(t) + f(t) \quad \text{a.e.}\, t \in I \\ u_f(0) = a,\ u_f(t) \in C(t),\ \forall t \in I. \end{cases}$$

Denote by S^1_F and $S^1_{\overline{\text{co}}F}$ the set of all measurable $L^1_E(I)$-selections of F and $\overline{\text{co}}F$ respectively. Then it is well known [36] that $S^1_{\overline{\text{co}}F} = \overline{\text{co}}S^1_F$ in $L^1_E(I)$. By $\|\cdot\|_{L^1_E(I)}$-to-$\|\cdot\|_\infty$ continuity of $\mathcal{H} \in f \mapsto u_f$ from $\mathcal{H}$ into $\mathcal{C}_E(I)$ (see (b) in Lemma 3.5), for each $\varepsilon > 0$ there is $g_\varepsilon \in \text{co}S^1_F$ such that $\|g - g_\varepsilon\|_{L^1_E(I)} \le \varepsilon$ along with $\|u_g - u_{g_\varepsilon}\|_\infty \le \varepsilon$, where u_{g_ε} is as above the absolutely continuous solution to

$$\begin{cases} -\dot{u}_{g_\varepsilon}(t) \in N_{C(t)}u_{g_\varepsilon}(t) + g_\varepsilon(t) \quad \text{a.e.}\, t \in I \\ u_{g_\varepsilon}(0) = a, \ u_{g_\varepsilon}(t) \in C(t), \ \forall t \in I. \end{cases}$$

There exists a positive integer m, mappings

$$f_i \in S^1_F, \ i = 1, \cdots, m, \tag{8.1}$$

and real numbers $\lambda_i \ge 0$ with $\sum_{i=1}^m \lambda_i = 1$ such that $g_\varepsilon = \sum_{i=1}^m \lambda_i f_i$.

Now consider the compact valued multimapping $\Phi\colon I \rightrightarrows E$ defined by

$$\Phi(t) = \{f_i(t) : \ i = 1, \cdots, m\}, \quad \text{for all}\, t \in I.$$

Then,

$$g_\varepsilon(t) \in \text{co}\,\Phi(t) \text{ a.e. } t \in I. \tag{8.2}$$

By (8.2), we notice that $t \mapsto \Phi(t)$ is compact valued, measurable, and integrably bounded since $|\Phi(t)| \le \varrho(t)$ where $\varrho \in L^1_{\mathbb{R}_+}(I)$ is defined by $\varrho(t) = \sup_{1 \le i \le m} \|f_i(t)\|$, $t \in I$. Then, by Theorem 2.2 in [63], there exists a sequence

$$(h_n)_{n \ge 1} \subset L^1_E(I), \ h_n(t) \in \Phi(t) \subset F(t) \text{ a.e. } t \in I \tag{8.3}$$

such that

$$h_n \to g_\varepsilon \text{ in } L^1_\omega(I, E). \tag{8.4}$$

Thanks to (8.3), (8.4), the integrable boundedness of $t \mapsto \Phi(t)$ and Lemma 2.3 in [59], one deduces that

$$h_n \to g_\varepsilon \text{ weakly in } L^1_E(I). \tag{8.5}$$

Let u_{h_n} be the absolutely continuous solution to

$$\begin{cases} -\dot{u}_{h_n}(t) \in N_{C(t)}u_{h_n}(t) + h_n(t) \ \text{ a.e.}\, t \in I, \\ u_{h_n}(0) = a, \ u_{h_n}(t) \in C(t), \ \forall t \in I. \end{cases}$$

Then by the assertion (b) in Lemma 3.4, $(u_{h_n})_n$ *converges uniformly* to u_{g_ε}. Just we have shown that there is a sequence $(g_n)_n$ in $L^1_E(I)$ and a corresponding sequence $(u_{g_n})_n$ of absolutely continuous mappings from I into E with

$$\begin{cases} -\dot{u}_{g_n}(t) \in N_{C(t)}u_{g_n}(t) + g_n(t) \quad \text{a.e.}\, t \in I \\ g_n(t) \in F(t) \quad \text{a.e.}\, t \in I \\ u_{g_n}(0) = a, \ u_{g_n}(t) \in C(t), \ \forall t \in I \end{cases}$$

such that $g_n \to g$ weakly in $L^1_E(I)$ and $u_{g_n} \to u_g$ uniformly on I. This entails the desired density property. □

8.2. Relaxation for sweeping process with BV perturbation

Let $\alpha \in]0, \infty[$ and E be a separable reflexive Banach space, let

$$\mathrm{exc}(S_1, S_2) := \sup_{x \in S_1} d(x, S_2)$$

be the excess from a subset S_1 of E to a subset S_2 of E as already recalled in Section 2. Let $K\colon [0,1] \rightrightarrows \alpha\mathbb{B}_E$ be a weakly compact convex valued multimapping with bounded right continuous retraction in the sense that there is a bounded and right continuous function $\rho\colon [0,1] \to \mathbb{R}_+$ such that

$$\mathrm{exc}(K(t), K(\tau)) \le \rho(\tau) - \rho(t)$$

for all $t, \tau \in [0, T]$ with $t \le \tau$. Suppose that the graph of K is measurable, that is, $\mathrm{gph}\,(K) \in \mathcal{L}([0,1]) \otimes \mathcal{B}(E)$. Consider the control sets given by

$$S_K^{BVRC} := \{u\colon [0,1] \to E, u \text{ is BVRC},\ u(t) \in K(t),\ \forall t \in [0,1]\}$$

$$S_K^{\infty} := \{u \in L_E^{\infty}([0,1], dt),\ u(t) \in K(t),\ \ \forall t \in [0,1]\}.$$

By Moreau ([44], Prop.5 d, p. 198) and Valadier [64] these sets are non empty and

$$\mathrm{cl}\, S_K^{BVRC} = S_K^{\infty}, \tag{8.6}$$

where cl denotes here the closure with respect to the $\sigma(L_E^{\infty}, L_{E^*}^1)$-topology. Shortly, S_K^{BVRC} is dense in S_K^{∞} with respect to this topology. Then we have the following relaxation result for a control problem governed by sweeping process of the type considered in previous sections.

Theorem 8.2. *Let E be a separable Hilbert space, let $I := [0,1]$, and $r \in]0, +\infty]$. Let $C\colon I \rightrightarrows E$ be a multimapping with nonempty values which is absolutely continuous and such that $C(t)$ is r-prox-regular and ball-compact for all $t \in I$, and let $a \in C(0)$. Let $\alpha \in]0, +\infty[$ and $K\colon I \rightrightarrows \alpha\mathbb{B}_E$ be a nonempty weakly compact convex valued multimapping with measurable graph and with bounded right continuous retraction. Denote $\mathcal{V}_K$ the set of BVRC selections of K and $\mathcal{V}_K^{\infty}$ the set of $L_E^{\infty}(I)$ selections of K, that is, $\mathcal{V}_K := S_K^{BVRC}$ and $\mathcal{V}_K^{\infty} := S_K^{\infty}$. Then the following hold.*

(a) *The AC solution set $\mathcal{S}_{\mathcal{V}_K^{\infty}}$ to the inclusion*

$$\begin{cases} u(0) = a,\ u(t) \in C(t),\ \forall t \in I, & \frac{du}{dt}(t) \in L_E^1(I) \\ -\frac{du}{dt}(t) \in N_{C(t)}u(t) + h(t), & a.e.\ t \in I, h \in \mathcal{V}_K^{\infty} \end{cases}$$

is nonempty and compact in $\mathcal{C}_E([0,1])$.

(b) *The AC solution set $\mathcal{S}_{\mathcal{V}_K}$ to the inclusion*

$$\begin{cases} u(0) = a;\ u(t) \in C(t)),\ \forall t \in I, & \frac{du}{dt}(t) \in L_E^1(I) \\ -\frac{du}{dt}(t) \in N_{C(t)}u(t) + h(t), & a.e.\ t \in I, h \in \mathcal{V}_K \end{cases}$$

is nonempty and is dense in the compact set $\mathcal{S}_{\mathcal{V}_K^{\infty}}$.

Proof. By Theorem 3.3, for each Lebesgue measurable selection h of K, there is a unique AC solution u_h to the inclusion

$$\begin{cases} u_h(0) = a;\ u_h(t) \in C(t),\ \forall t \in I, & \frac{du_h}{dt}(t) \in L^1_E(I) \\ -\frac{du_h}{dt}(t) \in N_{C(t)}u_h(t) + h(t), & a.e.\ t \in I. \end{cases}$$

We know by (a) in Lemma 3.5 that $\mathcal{V}^\infty_K$ is $\sigma(L^\infty_E(I), L^1_E(I))$ compact metrizable and that $\mathcal{S}_{\mathcal{V}^\infty_K}$ is compact in $\mathcal{C}_E([0,1])$. Further, by (b) in the same Lemma 3.5 the mapping $\mathcal{V}^\infty_K \ni h \mapsto u_h$ is continuous from the $\sigma(L^\infty_E(I), L^1_E(I))$ compact metrizable set $\mathcal{V}^\infty_K$ endowed with the $\sigma(L^\infty_E(I), L^1_E(I))$ topology into $\mathcal{C}_E(I)$ endowed with the topology of uniform convergence. Since $\mathcal{V}_K$ is dense in $\mathcal{V}^\infty_K$ with respect to the topology $\sigma(L^\infty_E(I), L^1_E(I))$ (as recalled in the comments preceding the statement of the theorem), the set $\mathcal{S}_{\mathcal{V}_K} = \{u_h : h \in \mathcal{V}_K\}$ is dense in $\mathcal{S}_{\mathcal{V}^\infty_K} = \{u_h : h \in \mathcal{V}^\infty_K\}$ with respect to the topology of uniform convergence on I. □

8.3. Relaxation for sweeping process with Volterra integro-differential perturbation

We provide in this subsection a relaxation result for an integro-differential Volterra sweeping process. The existence of solution for some optimal control problems subject to such dynamics was also shown in Bouach, Haddad and Thibault [9].

Theorem 8.3. *Let E be a separable Hilbert space, let $I := [0,1]$ and let $r \in\,]0,+\infty]$. Let $C\colon [0,1] \rightrightarrows H$ be a multimapping which is absolutely continuous and such that $C(t)$ is r-prox-regular and ball-compact for all $t \in I$, and let $a \in C(0)$.*

Let $\alpha \in]0,+\infty[$ and $K\colon I \rightrightarrows \alpha\mathbb{B}_E$ be a nonempty weakly compact convex valued multimapping with measurable graph and with bounded right continuous retraction.

Let $\varphi\colon I \times I \times E \to \mathbb{R}$ be a function such that $\varphi(\cdot,\cdot,x)$ is $\mathcal{L}(I) \otimes \mathcal{L}(I)$ measurable for all $x \in E$ and for which there is a real constant $M > 0$ such that

(i) $|\varphi(t,s,x)| \le M$ *for all $t,s \in I$ and $x \in I$;*

(ii) $|\varphi(t,s,x) - \varphi(t,s,y)| \le M\|x-y\|$ *for all $t,s \in I$ and $x,y \in E$.*

Then the following hold.

(a) *The AC solution set $\mathcal{S}_{\mathcal{V}^\infty_K}$ to the inclusion*

$$\begin{cases} u(0) = a,\ u(t) \in C(t), \forall t \in I, & \frac{du}{dt}(t) \in L^\infty_E(I) \\ -\frac{du}{dt}(t) \in N_{C(t)}u(t) + \int_0^t \varphi(t,s,u(s))h(s)\,ds,\ a.e.\ t \in I,\ h \in S^\infty_K \end{cases}$$

is non empty and compact in $\mathcal{C}_E([0,1])$ with respect to the topology of uniform convergence.

(b) *The AC solution set $\mathcal{S}_{\mathcal{V}_K}$ to the inclusion*

$$\begin{cases} u(0) = a,\ u(t) \in C(t)),\ \forall t \in I, & \frac{du}{dt}(t) \in L^1_E(I) \\ -\frac{du}{dt}(t) \in N_{C(t)}u(t) + \int_0^t \varphi(t,s,u(s))h(s)\,ds,\ a.e.\ t \in I,\ h \in S^{BVRC}_K \end{cases}$$

is nonempty and dense in the compact set $\mathcal{S}_{\mathcal{V}^\infty_K}$ with respect to the topology of uniform convergence.

Proof. Put $M_0 := \alpha M$. We first note that for each Borel measurable selection h of K, the mapping $f_h(t,s,x) := \varphi(t,s,x)h(s)$ satisfies the conditions

$$\|f_h(t,s,x)\| \leq M_0,$$

$$\|f_h(t,s,x) - f_h(t,s,y)\| \leq M_0\|x-y\|,\ \forall (t,s) \in I \times I,\ \forall (x,y) \in E \times E,$$

and further $(t,s) \mapsto f_h(t,s,x)$ is $\mathcal{L}(I) \otimes \mathcal{L}(I)$ measurable on $I \times I$, in particular if h is a BVRC selection of K and if $\xi \colon I \to E$ is absolutely continuous, then $(t,s) \mapsto \varphi(t,s,\xi(s))h(s)$ is $\mathcal{L}(I) \otimes \mathcal{L}(I)$ measurable and bounded. By Theorem 4.1 for each Borel measurable selection h of K, there is a unique AC solution u_h to the inclusion

$$\begin{cases} u_h(0) = a,\ u_h(t) \in C(t)),\ \forall t \in I,\ \ \frac{du_h}{dt}(t) \in L^1_E(I) \\ -\frac{du_h}{dt}(t) \in N_{C(t)}u_h(t) + \int_0^t \varphi(t,s,u_h(s))h(s)\,ds,\ \text{a.e. } t \in I. \end{cases}$$

So, the sets of solutions are given by

$$\mathcal{S}_{\mathcal{V}_K^\infty} = \{u_h : h \in S_K^\infty\} \quad \text{and} \quad \mathcal{S}_{\mathcal{V}_K} = \{u_h : h \in S_K^{BVRC}\}.$$

Let $(h_n)_n$ in S_K^∞ converging to $h \in S_K^\infty$ with respect to the $\sigma(L^\infty_E(I), L^1_E(I))$ topology. By Lemma 3.4(b) the sequence $(u_{h_n})_n$ admits a subsequence (that we do not relabel) converging uniformly on I to a mapping $u \colon I \to E$. Notice that for each $(t,s) \in I \times I$

$$\varphi(t,s,u_{h_n}(s)) \to \varphi(t,s,u(s)) \quad \text{as } n \to \infty.$$

Define the mappings $\phi_n, \phi \in L^1_E(I)$ by putting for every $t \in I$

$$\phi_n(t) := \int_0^t \varphi(t,s,u_{h_n}(s))h_n(s)\,ds \quad \text{and} \quad \phi(t) := \int_0^t \varphi(t,s,u(s))h(s)\,ds.$$

We assert *the main fact*: $\phi_n \to \phi$ weakly in $L^1_E(I)$. It is clear that ϕ_n and ϕ are Lebesgue measurable by Fubini-Lebesgue property and the separability of the space E. A crucial feature is that $\varphi(t,\cdot,u_{h_n}(\cdot)) - \varphi(t,\cdot,u(\cdot))$ converges to 0, it converges to 0 uniformly on uniformly integrable sets, alias Mackey converges to 0 (cf [11]), hence for $g \in L^\infty_E(I)$

$$\begin{aligned} \lim_n \int_0^1 \langle g(t), \phi_n(t)\rangle\,dt &= \int_0^1 \lim_n \Big[\int_0^t \varphi(t,s,u_{h_n}(s))\langle g(t), h_n(s)\rangle\,ds\Big]\,dt \\ &= \int_0^1 \Big[\int_0^t \varphi(t,s,u(s))\langle g(t), h(s)\rangle\,ds\Big]\,dt = \int_0^1 \langle g(t), \phi(t)\rangle\,dt. \end{aligned}$$

From this weak convergence $\phi_n \to \phi$ in $L^1_E(I)$ and from the fact that u_{h_n} is the absolutely continuous solution of the differential inclusion

$$\begin{cases} u_{h_n}(0) = 0,\ u_{h_n}(t) \in C(t) \\ -\frac{du_{h_n}}{dt}(t) \in N_{C(t)}u_{h_n}(t) + \phi_n(t)\ \text{a.e } t \in I \end{cases}$$

we deduce by (b) in Lemma 3.4 again that the mapping $u \colon I \to E$ is absolutely continuous and solution of the differential inclusion

$$\begin{cases} u(0) = a, \ u(t) \in C(t) \ \forall t \in I \\ -\frac{du}{dt}(t) \in N_{C(t)}u(t) + \phi(t) \text{ a.e } t \in I, \end{cases}$$

or in other words, u is an absolutely continuous solution of

$$\begin{cases} u(0) = a, \ u(t) \in C(t) \ \forall t \in I \\ -\frac{du}{dt}(t) \in N_{C(t)}u(t) + \int_0^t \varphi(t, s, u(s))h(s)\, ds \ \text{ a.e. } t \in I. \end{cases}$$

Then by the uniqueness property in Theorem 4.1 under $(\mathcal{G}_1')$ there, we have $u = u_h$ by definition of u_h. Endowing $\mathcal{C}_E(I)$ with the topology of uniform convergence, it follows that the mapping $\Phi\colon h \mapsto u_h$ from the compact metrizable set $S_K^\infty \subset L_E^\infty(I)$ to $\mathcal{C}_E(I)$ is continuous. Therefore, $\{u_h : h \in S_K^\infty\}$ is compact in $\mathcal{C}_E(I)$. Further, since S_K^{BVRC} is dense in S_K^∞ (see (8.6)), the set $\{u_h : h \in S_K^{BVRC}\}$ is dense in $\{u_h : h \in S_K^\infty\}$. This finishes the proof of the theorem. □

8.4. Relaxation of Bolza problem associated with sweeping process and Riemann-Stieltjes integral perturbation

In some previous sections and subsections we have developed diverse features of the Skorohod problem associated with the sweeping process with Riemann-Stieltjes integral perturbation. In the present subsection we will examine a Bolza problem and a relaxation problem, both associated with a sweeping process with a compact moving set $C(t)$ satisfying for real $\kappa > 0$ the Lipschitz condition

$$\operatorname{haus}(C(t), C(s)) \leq \kappa|t - s|.$$

The relaxation problem is related to Young measures while the Bolza problem is concerned with a sweeping process involving a Young integral perturbation. First, we need some notation and background on Young measures in this special context.

Young measure

We begin with the definition. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Let X be a Polish space (i.e., a complete metrizable space) and let $\mathcal{C}^b(X, \mathbb{R})$ be the space of all bounded continuous real-valued functions defined on X. Let $\mathcal{M}_+^1(X)$ be the set of all Borel probability measures on X equipped with the *narrow topology.* A *Young measure* $\nu\colon \Omega \to \mathcal{M}_+^1(X)$ is, by definition, a *scalarly measurable* mapping from Ω into $\mathcal{M}_+^1(X)$, that is, for every $f \in \mathcal{C}^b(X, \mathbb{R})$, the mapping

$$\omega \mapsto \langle f, \nu_\omega \rangle := \int_X f(x)\, d\nu_\omega(x)$$

is $\mathcal{F}$-measurable. A sequence $(\nu^n)_n$ in the space $\mathcal{Y}(\Omega, \mathcal{F}, P; \mathcal{M}_+^1(X))$ of Young measures *stably converges* to a Young measure $\nu \in \mathcal{Y}(\Omega, \mathcal{F}, P; \mathcal{M}_+^1(X))$ if the following holds:

$$\lim_{n\to\infty} \int_A \left(\int_X f(x)\, d\nu_\omega^n(x) \right) dP(\omega) = \int_A \left(\int_X f(x)\, d\nu_\omega(x) \right) dP(\omega)$$

for every $A \in \mathcal{F}$ and for every $f \in C^b(X, \mathbb{R})$; see [22]. Next, we recall and summarize some results for Young measures while we refer to [22] for more on such measures and on the stable topology.

In the remainder we particularize X as a compact metric space Z, so $\mathcal{M}^1_+(Z)$ will be the space of all Radon probability measures on Z. We will endow $\mathcal{M}^1_+(Z)$ with the *narrow topology* so that $\mathcal{M}^1_+(Z)$ is a compact metrizable space. Let us denote by $\mathcal{Y}(I;\mathcal{M}^1_+(Z))$ the space of all Young measures defined on $I := [0,1]$ endowed with the *stable topology*, so $\mathcal{Y}(I;\mathcal{M}^1_+(Z))$ is a compact metrizable space with respect to this topology. By the Portmanteau Theorem for Young measures [22, Theorem 2.1.3], a sequence (ν^n) in $\mathcal{Y}(I;\mathcal{M}^1_+(Z))$ stably converges to $\nu \in \mathcal{Y}(I;\mathcal{M}^1_+(Z))$ if

$$\lim_{n\to\infty} \int_0^1 \left(\int_Z h_t(z)\, d\nu^n_t(z) \right) dt = \int_0^1 \left(\int_Z h_t(z)\, d\nu_t(z) \right) dt$$

for all $h \in L^1(I, \mathcal{C}(Z))$; here $\mathcal{C}(Z)$ denotes the space of all continuous real valued functions defined on Z endowed with the norm of uniform convergence.

Finally, let Γ be a measurable multimapping defined on I with nonempty compact values in Z and S_Γ be the set of all Lebesgue measurable selections of Γ (alias *original controls*). Let $E := \mathbb{R}^e$ and $f\colon I \times E \times Z \to E$ be a mapping satisfying

(i) for every fixed $t \in I$, $f(t,\cdot,\cdot)$ is continuous on $E \times Z$;

(ii) for every $(x,z) \in E \times Z$, $f(\cdot,x,z)$ is Lebesgue-measurable on I;

(iii) there is a nonnegative real number M such that

$$\|f(t,x,z)\| \le M \;\; \forall (t,x,z) \in I \times H \times Z;$$

(iv) there is a nonnegative real-valued function $\lambda \in L^1(I,\mathbb{R};dt)$ such that

$$\|f(t,x_1,z) - f(t,x_2,z)\| \le \lambda(t)\|x_1 - x_2\| \;\; \forall (t,x_1,z),(t,x_2,z) \in I \times E \times Z.$$

Let $r \in]0,+\infty]$ and $C\colon I \rightrightarrows \mathbb{R}^e$ be a compact valued Lipschitz multimapping with $C(t)$ r-prox-regular for all $t \in I$. Our aim is to present some relaxation problems in the framework of Optimal Control Theory. We consider the evolution inclusion $(\mathcal{PO})$ associated with original controls

$$(\mathcal{PO}) \qquad \begin{cases} -\dot u_\zeta(t) \in N_{C(t)} u_\zeta(t) + f(t, u_\zeta(t), \zeta(t)),\ a.e.\ t \in I, \\ u_\zeta(0) = u_0 \in C(0) \end{cases}$$

where ζ belongs to the set $\mathcal{Z} := S^1_\Gamma$ of all original controls, which means that ζ is a Lebesgue-measurable selection of Γ. The evolution inclusion $(\mathcal{PR})$ associated with relaxed controls corresponds to

$$(\mathcal{PR}) \qquad \begin{cases} -\dot u_\nu(t) \in N_{C(t)} u_\nu(t) + \int_Z f(t, u_\nu(t), z)\, \nu_t(dz), \text{ a.e. } t \in I \\ u_\nu(0) = u_0 \in C(0), \end{cases}$$

where ν belongs to the set $\mathcal{R} := S_\Sigma$ of all relaxed controls, which means that ν is a Lebesgue-measurable selection of the multimapping Σ defined by

$$\Sigma(t) := \left\{ \sigma \in \mathcal{M}^1_+(Z) : \sigma(\Gamma(t)) = 1 \right\}$$

for all $t \in I$, or in other words $\nu_t \in \Sigma(t)$ for all $t \in I$.

Notice also that, for $\nu \in \mathcal{R}$, the mapping

$$h_\nu \colon (t,x) \mapsto \int_Z f(t,x,z)\,\nu_t(dz)$$

inherits the properties:

(1) for every fixed $t \in I$, $h_\nu(t,.)$ is continuous on E;

(2) for every $x \in E$, $h_\nu(.,x)$ is Lebesgue-measurable on I;

(3) there is a constant $M > 0$ such that $\|h_\nu(t,x)\| \le M$ for all (t,x) in $I \times E$;

(4) there is a nonnegative real function $\lambda \in L^1(I,\mathbb{R};dt)$ such that

$$\|h_\nu(t,x_1) - h_\nu(t,x_2)\| \le \lambda(t)\|x_1 - x_2\| \;\; \forall (t,x_1),(t,x_2) \in I \times E.$$

Consequently, for each $\zeta \in \mathcal{Z}$ (resp. $\nu \in \mathcal{R}$), the evolution inclusion $(\mathcal{PO})$ (resp. $(\mathcal{PR})$) has a unique Lipschitz continuous solution (see Theorem 3.3). Moreover, there is an a priori bound $K > 0$ for the Lipschitz ratio of solutions which easily implies that the solution sets $(\mathcal{S}_\mathcal{O})$ and $(\mathcal{S}_\mathcal{R})$ (to $(\mathcal{PO})$ and $(\mathcal{PR})$) are equi-Lipschitz.

Now we state and prove some topological properties of the solution sets $(\mathcal{S}_\mathcal{O})$ and $(\mathcal{S}_\mathcal{R})$, namely we obtain the typical relaxation result that the former is dense in the latter.

Theorem 8.4. *Let $I := [0,1]$ and $r \in]0,+\infty]$. Let $C\colon I \rightrightarrows \mathbb{R}^e$ be an r-prox-regular valued multimapping which is κ-Lipschitz on I. Let Z be a compact metrizable space and $f\colon I \times \mathbb{R}^e \times Z \to \mathbb{R}^e$ be a mapping satisfying the above conditions (i), (ii), (iii) and (iv). Let $\mathcal{Z}$ and $\mathcal{R}$ be the initial controls and relaxed controls respectively, as defined above. Then the following hold.*

(a) *The solution set $(\mathcal{S}_\mathcal{R})$ to*

$$(\mathcal{PR}) \quad \begin{cases} -\dot u_\nu(t) \in N_{C(t)} u_\nu(t) + \displaystyle\int_Z f(t,u_\nu(t),z)\,\nu_t(dz),\ a.e.\, t \in I,\ \nu \in \mathcal{R} \\ u_\nu(0) = u_0 \in C(0) \end{cases}$$

is nonempty and compact in $C(I,\mathbb{R}^e)$.

(b) *The solution set $(\mathcal{S}_\mathcal{O})$ to*

$$(\mathcal{PO}) \begin{cases} -\dot u_\zeta(t) \in N_{C(t)} u_\zeta(t) + f(t,u_\zeta(t),\zeta(t)),\ a.e.\, t \in I,\ \zeta \in \mathcal{Z} \\ u_\zeta(0) = u_0 \in C(0) \end{cases}$$

is dense in $(\mathcal{S}_\mathcal{R})$ with respect to the topology of uniform convergence.

Proof. (a) By Theorem 3.3 for each $\nu \in \mathcal{Z}$ the mapping u_ν is the unique absolutely continuous solution of the differential inclusion

$$\begin{cases} -\dot u_\nu(t) \in (\kappa + M)\partial d_{C(t)}(u_\nu(t)) + \displaystyle\int_Z f(t,u_\nu(t),z)\,\nu_t(dz),\ a.e.\, t \in I \\ u_\nu(0) = u_0 \in C(0), \end{cases}$$

so $\|\dot u_\nu(t)\| \le \kappa + 2M$ and $\|u_\nu(t)\| \le \|u_0\| + \kappa + 2M$. Then $(\mathcal{S}_\mathcal{R})$ is relatively compact in $C(I,\mathbb{R}^e)$, by the Arzelà-Ascoli theorem. Therefore, for any sequence $(\nu^n)_n$ in $\mathcal{R}$, there is for $(u_{\nu^n})_n$ a subsequence still denoted by $(u_{\nu^n})_n$ which converges uniformly on I to a Lipschitz continuous mapping u^∞ with $\|\dot{u}^\infty(t)\| \le \kappa + 2M$ a.e. $t \in I$ and such that $(\dot{u}_{\nu^n})_n$ converges to $\dot{u}^\infty$ with respect to the $\sigma(L^1(I,\mathbb{R}^e;dt), L^\infty(I,\mathbb{R}^e;dt))$ topology. Since $\mathcal{R}$ is compact metrizable for the stable topology, we may also suppose that $(\nu^n)_n$ stably converges to some $\nu^\infty \in \mathcal{R}$. Our objective is to show that $u_{\nu^\infty} = u^\infty$.

From the inclusions

$$-\dot{u}_{\nu^n}(t) \in N_{C(t)} u_{\nu^n}(t) + \int_Z f(t, u_{\nu^n}(t), z)\, \nu_t^n(dz),$$

$$-\dot{u}_{\nu^\infty}(t) \in N_{C(t)} u_{\nu^\infty}(t) + \int_Z f(t, u_{\nu^\infty}(t), z)\, \nu_t^\infty(dz),$$

and from the hypomonotonicity property of $N_{C(t)}$ (see Theorem 2.1) we deduce with $\gamma := (\kappa + 2M)/r$ that with $J_n(t) := \gamma \int_0^t \|u_{\nu_n}(s) - u_{\nu^\infty}(s)\|^2\, ds$

$$\begin{aligned}&\Big\langle \dot{u}_{\nu^n}(t) - \dot{u}_{\nu^\infty}(t), u_{\nu^n}(t) - u_{\nu^\infty}(t) \Big\rangle \\ &\le J_n(t) - \Big\langle h_{\nu^n}(t, u_{\nu^n}(t)) - h_{\nu^\infty}(t, u_{\nu^\infty}(t)), u_{\nu^n}(t) - u_{\nu^\infty}(t) \Big\rangle\end{aligned}$$

where

$$h_{\nu^n}(t, u_{\nu^n}(t)) := \int_Z f(t, u_{\nu^n}(t), z)\, \nu_t^n(dz),$$

$$h_{\nu^\infty}(t, u_{\nu^\infty}(t)) := \int_Z f(t, u_{\nu^\infty}(t), z)\, \nu_t^\infty(dz).$$

Integrating over $[0,t]$ $(t \in I)$ yields

$$\begin{aligned}&\frac{1}{2}\|u_{\nu^n}(t) - u_{\nu^\infty}(t)\|^2 \\ &\le J_n(t) - \int_0^t \Big\langle h_{\nu^n}(s, u_{\nu^n}(s)) - h_{\nu^\infty}(s, u_{\nu^\infty}(s)), u_{\nu^n}(s) - u_{\nu^\infty}(s) \Big\rangle\, ds,\end{aligned}$$

which can be rewritten as

$$\frac{1}{2}\|u_{\nu^n}(t) - u_{\nu^\infty}(t)\|^2 \le L_n(t) + \gamma \int_0^t \|u_{\nu_n}(s) - u_{\nu^\infty}(s)\|^2\, ds,$$

if we set $L_n(t) = \int_0^t \Big\langle u_{\nu^n}(s) - u_{\nu^\infty}(s), -h_{\nu^n}(s, u_{\nu^n}(s)) + h_{\nu^\infty}(s, u_{\nu^\infty}(s)) \Big\rangle\, ds.$

We have $L_n(t) = L_n^1(t) + L_n^2(t) + L_n^3(t)$ where

$$L_n^1(t) = \int_0^t \Big\langle u_{\nu^n}(s) - u_{\nu^\infty}(s), -h_{\nu^n}(s, u_{\nu^n}(s)) + h_{\nu^n}(s, u_{\nu^\infty}(s)) \Big\rangle\, ds,$$

$$L_n^2(t) = \int_0^t \Big\langle u_{\nu^n}(s) - u^\infty(s), -h_{\nu^n}(s, u_{\nu^\infty}(s)) + h_{\nu^\infty}(s, u_{\nu^\infty}(s)) \Big\rangle\, ds,$$

$$L_n^3(t) = \int_0^t \Big\langle u^\infty(s) - u_{\nu^\infty}(s), -h_{\nu^n}(s, u_{\nu^\infty}(s)) + h_{\nu^\infty}(s, u_{\nu^\infty}(s)) \Big\rangle\, ds.$$

As by (iii) $\|h_{\nu^n}(s, u_{\nu^n}(s))\| \le M$ a.e. for all $n \in \mathbb{N} \cup \{\infty\}$ and $u_{\nu^n}(s) \to u^\infty(s)$ for every $s \in I$, we see that $L^2_n(t) \to 0$ as $n \to \infty$ for each $t \in I$. By (i) and (iii) the integrand

$$h(s, z) := \big\langle u^\infty(s) - u_{\nu^\infty}(s), f(s, u_{\nu^\infty}(s), z)\big\rangle$$

is Carathéodory integrable and

$$|h(s, z)| \le M\|u^\infty(s) - u_{\nu^\infty}(s)\|$$

for all $(s, z) \in I \times Z$. Consequently, we have $h \in L^1(I, C(Z))$. Since $(\nu^n)_n$ stably converges to ν^∞, it ensures that for every $t \in I$

$$\lim_{n\to\infty} \int_0^t \left(\int_Z h(s, z)\, \nu^n_s(dz)\right) ds = \int_0^t \left(\int_Z h(s, z)\, \nu^\infty_s(dz)\right) ds.$$

So $\lim_{n\to\infty} L^3_n(t) = 0$, for every $t \in I$. By (iv) we have that

$$|L^1_n(t)| \le \int_0^t \lambda(s)\|u_{\nu^n}(s) - u_{\nu^\infty}(s)\|^2\, ds.$$

Finally, we obtain with $\lambda_0(\cdot) := \gamma + \lambda(\cdot)$

$$\frac{1}{2}\|u_{\nu^n}(t) - u_{\nu^\infty}(t)\|^2 \le L^2_n(t) + L^3_n(t) + \int_0^t \lambda_0(s)\|u_{\nu^n}(s) - u_{\nu^\infty}(s)\|^2\, ds.$$

Keeping in mind that $L^2_n(t) \to 0$ and $L^3_n(t) \to 0$, for all $t \in I$, by Gronwall's lemma (see Lemma 2.2) we have that $u_{\nu^n}(t) \to u_{\nu^\infty}(t)$, for all $t \in I$. This confirms the desired equality $u_{\nu^\infty} = u^\infty$. It results the assertion (a) of the theorem hods true.

(b) The second assertion (b) follows by continuity and density since $\mathcal{Z}$ is known to be dense in $\mathcal{R}$ with respect to the stable topology ([22], Lemma 7.1.1). □

Young integral

Like in Section 6 let $z \in C^{1-var}(I, \mathbb{R}^d)$ with $I := [0, T]$ and let

$$|\Lambda| := \sup_{x\in\mathbb{R}^d, \|x\|_{\mathbb{R}^d}=1} |\Lambda(x)|_{\mathbb{R}^e},$$

for $\Lambda \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$. Let $\Delta_T := \{(s, t) : 0 \le s \le t \le T\}$ and $\omega\colon \Delta_T \to [0, +\infty[$ be a *control function* on $[0, T]$ in the sense of Section 6.

Let $b\colon [0, T] \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ be a continuous integrand operator satisfying the conditions $(\mathcal{B}_1)$, $(\mathcal{B}_2)$ and $(\mathcal{B}_3)$ in Section 6, that is,

$(\mathcal{B}_1)$ $\quad |b(t, x)| \le M, \quad$ for all $x \in \mathbb{R}^e$,

$(\mathcal{B}_2)$ $\quad |b(s, x) - b(t, x)| \le \omega(s, t), \quad$ for all $0 \le s \le t \le T$ and $x \in \mathbb{R}^e$,

$(\mathcal{B}_3)$ $\quad |b(t, x) - b(t, y)| \le M\|x - y\|, \quad$ for all $t \in [0, T]$ and $x \in \mathbb{R}^e$,

where ω is the above control function on $[0, T]$ and M is a positive constant. If a sequence of mappings $(u_n)_n$ from $[0, T]$ into $\mathbb{R}^e$ is controlled by a common control function $\alpha\colon \Delta_T \to [0, +\infty[$, say

$$\|u_n(s) - u_n(t)\| \le \alpha(s, t) \quad \text{for all } n \in \mathbb{N} \text{ and all } 0 \le s \le t \le T,$$

then we have seen in Section 6 that the sequence of mappings $(y_n)_n$ from $[0,T]$ into $\mathcal{L}(\mathbb{R}^d,\mathbb{R}^e)$, with $y_n(t) = b(t,u_n(t))$, is uniformly bounded and uniformly bounded in variation, $\sup_n |y_n|_{1-var;[s,t]} < \infty$. Then, $y_n \in C^{1-var}([0,T],\mathcal{L}(\mathbb{R}^d,\mathbb{R}^e))$. Consequently, the Young integral $\int_0^t y_n(s)dz_s$ of y_n along z is well-defined and belongs to the space $C^{1-var}([0,T],\mathbb{R}^e)$. We have also seen, in the same Section 6, the following estimates

$$\begin{aligned}\left\|\int_s^t y_n\,dz\right\| &\le \frac{1}{1-2^{1-\theta}}|z|_{1-var;[s,t]}|y_n|_{1-var;[s,t]} + |y_n(s)|\|z(t)-z(s)\|_{\mathbb{R}^d}\\ &\le \frac{1}{1-2^{1-\theta}}|z|_{1-var;[s,t]}|y_n|_{1-var;[s,t]} + M\|z(t)-z(s)\|_{\mathbb{R}^d}\\ &\le \frac{1}{1-2^{1-\theta}}|z|_{1-var;[s,t]}\sup_n|y_n|_{1-var;[s,t]} + M\|z(t)-z(s)\|_{\mathbb{R}^d} \qquad (8.7)\end{aligned}$$

for all $0 \le s \le t \le T$ with $\theta = 2$; and the first above inequality also yields

$$\left|\int_0^{\cdot} y_n\,dz\right|_{1-var;[s,t]} \le C(1,1)|z|_{1-var;[s,t]}\sup_n(|y_n|_{1-var;[s,t]} + |y_n|_{\infty;[s,t]}) \qquad (8.8)$$

for all $0 \le s \le t \le T$.

Putting $g_n(t) := \int_0^t y_n(\tau)dz_\tau$ for all $t \in I$, as already noticed in Section 6 by (8.7) we have that $(g_n)_n$ is a sequence of mappings in $C^{1-var}(I,\mathbb{R}^e)$ which is uniformly bounded, and by (8.7) again combined with the continuity of $t \mapsto |z|_{1-var;I}$ (due to (2.5) since $z \in C^{1-var}(I,\mathbb{R}^d)$) the sequence $(g_n)_n$ is also equicontinuous; further it is also uniformly bounded in variation by (8.8). Altogether, the sequence $(g_n)_n$ is uniformly bounded, equicontinuous and uniformly bounded in variation. The above features hold in the particular case when the sequence $(u_n)_n$ is uniformly bounded and equi-Lipschitz.

The next theorem is concerned with a Bolza type problem whose objective function involves Young integral and whose dynamics is a sweeping process with a Volterra type integral perturbation. We keep $z \in C^{1-var}(I,\mathbb{R}^d)$.

Theorem 8.5. *Let $I = [0,T]$ and $b\colon I\times\mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d,\mathbb{R}^e)$ be a continuous integrand operator satisfying $(\mathcal{B}_1)$, $(\mathcal{B}_2)$ and $(\mathcal{B}_3)$ above. Let $\alpha \in]0,+\infty[$ and $K\colon I \rightrightarrows \mathbb{R}^e$ be a weakly compact convex valued multimapping with measurable graph and with bounded right continuous retraction. Let $C\colon I \rightrightarrows \mathbb{R}^e$ be a Lipschitz multimapping whose values are r-prox-regular for a common $r \in]0,+\infty]$, and let $a \in C(0)$. Let $\varphi\colon I\times I\times\mathbb{R}^e \to \mathbb{R}$ be a continuous function for which there is a real $M_\varphi > 0$ such that*

(i) $|\varphi(t,s,x)| \le M_\varphi$ *for all $t,s \in I$ and $x \in \mathbb{R}^e$;*

(ii) $|\varphi(t,s,x)-\varphi(t,s,y)| \le M_\varphi\|x-y\|$ *for all $s,t \in I$ and $x,y \in \mathbb{R}^e$.*

Consider the minimization problem

$$\inf_{(\mathcal{P}_{\mathcal{S}_K^{BVRC}})} \int_0^T \left\langle \int_0^t b(s,u(s))\,dz_s, \zeta(t)\right\rangle dt$$

over all pairs (ζ,u) such that $\zeta \in S_K^{BVRC}$ and u is solution of the dynamical system

$$(\mathcal{P}_{\mathcal{S}_K^{BVRC}}) \qquad \begin{cases} -\dot{u}(t) \in N_{C(t)}u(t) + \int_0^t \varphi(t,s,u(s))\zeta(s)\,ds, \ a.e.\ t \in [0,T]\\ u(0) = a\end{cases}$$

and the similar problem

$$\inf_{(\mathcal{P}_{\mathcal{S}_K^\infty})} \int_0^T \left\langle \int_0^t b(s, u(s))\, dz_s, \zeta(t) \right\rangle dt$$

associated with the dynamic system

$$(\mathcal{P}_{\mathcal{S}_K^\infty}) \qquad \begin{cases} -\dot{u}(t) \in N_{C(t)} u(t) + \int_0^t \varphi(t, s, u(s))\zeta(s)\, ds, \ a.e.\ t \in [0, T] \\ u(0) = a. \end{cases}$$

Then we have $\inf(\mathcal{P}_{\mathcal{S}_K^\infty}) = \inf(\mathcal{P}_{\mathcal{S}_K^{BVRC}})$*, and as consequence,*

$$\inf_{(\mathcal{P}_{\mathcal{S}_K^\infty})} \int_0^T \left\langle \int_0^t b(s, u(s)) dz_s, \zeta(t) \right\rangle dt$$

admits at least a minimizer.

Proof. The inequality $\inf(\mathcal{P}_{\mathcal{S}_K^\infty}) \le \inf(\mathcal{P}_{\mathcal{S}_K^{BVRC}})$ is a simple consequence of the Valadier result mentioned above by observing that a BVRC mapping is Borelian. To show the reverse inequality, put $I := [0, T]$ and fix any $\zeta \in S_K^\infty$. Since S_K^{BVRC} is dense in S_K^∞ with respect to the $\sigma(L_E^\infty, L_E^1)$ topology as already recalled in (8.6), there exists a sequence $(\zeta_n)_n$ in S_K^{BVRC} such that $(\zeta_n)_n$ converges to some ζ with respect to the $\sigma(L_E^\infty(I), L_E^1(I))$ topology. By Theorem 4.1 let u_n be the unique Lipschitz solution to the differential inclusion

$$\begin{cases} -\dot{u}_n(t) \in N_{C(t)} u_n(t) + \int_0^t \varphi(t, s, u_n(s))\zeta_n(s)\, ds, \ \text{a.e. } t \in I \\ u_n(0) = a \end{cases}$$

and let v be the unique Lipschitz solution to

$$\begin{cases} -\dot{v}(t) \in N_{C(t)} v(t) + \int_0^t \varphi(t, s, v(s))\zeta(s)\, ds, \ \text{a.e. } t \in I \\ v(0) = a. \end{cases}$$

According to the first step of the proof of Theorem 8.3, the sequence $(u_n)_n$ admits a (non-relabeled) subsequence converging uniformly on I to v. For simplicity, set $\phi_n(t) = \int_0^t b(s, u_n(s)) dz_s$ for all $t \in I$. Applying Lemma 6.1, the sequence $(\phi_n)_n$ converges uniformly on I to the mapping $\phi\colon I \to \mathbb{R}^e$ defined by

$$\phi(t) = \int_0^t b(s, v(s))\, dz_s \quad \text{for all } t \in I.$$

This yields that

$$\lim_{n\to\infty} \int_0^T \left\langle \int_0^t b(s, u_n(s))\, dz_s, \zeta_n(t)) \right\rangle dt = \int_0^T \left\langle \int_0^t b(s, v(s))\, dz_s, \zeta(t) \right\rangle dt. \tag{8.9}$$

Using the inequality

$$\int_0^T \left\langle \int_0^t b(s, u_n(s)\, dz_s, \zeta_n(t) \right\rangle dt \ge \inf_{(\mathcal{P}_{\mathcal{S}_K^{BVRC}})} \int_0^T \left\langle \int_0^t b(s, u(s))\, dz_s, \zeta(t)) \right\rangle dt$$

for all $n \in \mathbb{N}$, it follows by taking the limit that

$$\int_0^T \left\langle \int_0^t b(s, v(s))\, dz_s, \zeta(t) \right\rangle dt \geq \inf (\mathcal{P}_{\mathcal{S}_K^{BVRC}}).$$

This being true for every $\zeta \in \mathcal{V}_K^\infty$, we obtain

$$\inf (\mathcal{P}_{\mathcal{S}_K^\infty}) \geq \inf (\mathcal{P}_{\mathcal{S}_K^{BVRC}}),$$

hence according to the previous reverse inequality

$$\inf (\mathcal{P}_{\mathcal{S}_K^\infty}) = \inf (\mathcal{P}_{\mathcal{S}_K^{BVRC}}). \tag{8.10}$$

Now, taking for $(\zeta_n)_n$ a minimizing sequence of $\inf (\mathcal{P}_{\mathcal{S}_K^{BVRC}})$ the above reasoning furnishes some $\sigma(L_E^\infty(I), L_E^1(I))$ limit $\zeta \in S_K^\infty$ satisfying (8.9). This and (8.10) allow us to conclude that the latter ζ is a solution for the minimization problem related to $(\mathcal{P}_{\mathcal{S}_K^\infty})$. □

We finish this section with a new type of Bolza problem involving both Young integral and Young measures. Let $I = [0, 1]$ and Z be a compact metric space. Let $h \colon I \times Z \to \mathbb{R}^e$ be a mapping satisfying

(i) for every fixed $t \in I, h(t, \cdot)$ is continuous on Z;

(ii) for every $z \in Z$, $h(\cdot, z)$ is Lebesgue-measurable on I;

(iii) there is a nonnegative real number M such that $\|h(t, z)\| \leq M$ for all $(t, z) \in I \times Z$.

Theorem 8.6. *Let Z and h be as above and, for $I = [0, 1]$, let $z \in C^{1-var}(I, \mathbb{R}^d)$, let $C \colon I \rightrightarrows \mathbb{R}^e$ be a Lipschitz multimapping with r-prox-regular values for a common $r \in]0, +\infty]$ and let $a \in C(0)$. Let $b \colon I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ be a continuous integrand operator satisfying $(\mathcal{B}_1)$, $(\mathcal{B}_2)$ and $(\mathcal{B}_3)$ above. Let us consider the problem*

$$\inf_{(\mathcal{P}_\mathcal{R})} \int_0^1 \left\langle \int_0^t b(s, u(s))\, dz_s, \int_0^t [\int_Z h(s, z)\, \nu_s(dz)]\, ds \right\rangle dt$$

associated, for each Young measure $\nu \in \mathcal{Y}(I; \mathcal{M}_+^1(Z))$, with the dynamic system

$$(\mathcal{P}_\mathcal{R}) \qquad \begin{cases} -\dot{u}(t) \in N_{C(t)} u(t) + \int_0^t [\int_Z h(s, z)\, \nu_s(dz)]\, ds \\ u(0) = a \end{cases}$$

and the problem $\displaystyle \inf_{(\mathcal{P}_\mathcal{O})} \int_0^1 \left\langle \int_0^t b(s, u(s))\, dz_s, \int_0^t h(s, \zeta(s))\, ds \right\rangle dt$

associated, for each measurable mapping $\zeta \colon I \to Z$, with the dynamic system

$$(\mathcal{P}_\mathcal{O}) \qquad \begin{cases} -\dot{u}(t) \in N_{C(t)} u(t) + \int_0^t h(s, \zeta(s))\, ds \\ u(0) = a. \end{cases}$$

Then we have $\inf (\mathcal{P}_\mathcal{R}) = \inf (\mathcal{P}_\mathcal{O})$ *and*

$$\inf_{(\mathcal{P}_\mathcal{R})} \int_0^1 \left\langle \int_0^t b(s, u(s))\, dz_s, \int_0^t [\int_Z h(s, z)\, \nu_s(dz)]\, ds \right\rangle dt$$

admits a minimizer.

Proof. The inequality $\inf(\mathcal{P}_{\mathcal{R}}) \geq \inf(\mathcal{P}_{\mathcal{O}})$ is clear. Let $\nu \in \mathcal{R}$. There is a sequence $(\zeta_n)_n$ in $\mathcal{O}$ which stably converges to ν, so that

$$\lim_n \int_0^t h(s, \zeta_n(s))\, ds = \int_0^t [\int_Z h(s,z)\, \nu_s(dz)]\, ds$$

for all $t \in I$. For each $n \in \mathbb{N}$ let u_n be the unique Lipschitz solution to

$$\begin{cases} -\dot{u}_n(t) \in N_{C(t)} u_n(t) + \int_0^t h(s, \zeta_n(s))\, ds, \text{ a.e. } t \in I \\ u_n(0) = a \end{cases}$$

and let v be the unique Lipschitz solution to

$$\begin{cases} -\dot{v}(t) \in N_{C(t)} v(t) + \int_0^t [\int_Z h(s,z)\, \nu_s(dz)]\, ds, \text{ a.e. } t \in I \\ v(0) = a. \end{cases}$$

We follow arguments for Theorem 8.5. In view of the first step of the proof of Theorem 8.3, the sequence $(u_n)_n$ admits a (non-relabeled) subsequence converging uniformly on I to v. For simplicity set $g_n(t) = \int_0^t b(s, u_n(s))\, dz_s$ for all $t \in I$. Applying Lemma 6.1 the sequence $(g_n)_n$ converges uniformly on I to the mapping $g\colon I \to \mathbb{R}^e$ with

$$g(t) := \int_0^t b(s, v(s))\, dz_s \quad \text{for all } t \in I.$$

Define the mappings $k_n, k\colon I \to \mathbb{R}^e$ by putting for every $t \in I$

$$k_n(t) = \int_0^t h(s, \zeta_n(s))\, ds \quad \text{and} \quad k(t) = \int_0^t \left[\int_Z h(s,z)\, \nu_s(dz)\right] ds,$$

so that $\langle g_n(t), k_n(t)\rangle \to \langle g(t), k(t)\rangle$ as $n \to \infty$. As $(g_n)_n, g$ and $(k_n)_n, k$ are uniformly bounded, we deduce that

$$\begin{aligned} \lim_{n\to\infty} \int_0^1 \left\langle \int_0^t b(s, u_n(s))\, dz_s, \int_0^t h(s, \zeta_n(s))\, ds \right\rangle dt &= \lim_{n\to\infty} \int_0^1 \langle g_n(t), k_n(t)\rangle\, dt \\ = \int_0^1 \langle g(t), k(t)\rangle\, dt &= \left\langle \int_0^t b(s, v(s))\, dz_s, \int_0^t \left[\int_Z h(s,z)\, \nu_s(dz)\right] ds \right\rangle dt. \end{aligned}$$

Since
$$\begin{aligned} &\int_0^1 \left\langle \int_0^t b(s, u_n(s))\, dz_s, \int_0^t h(s, \zeta_n(s))\, ds \right\rangle dt \\ &\geq \inf_{(\mathcal{P}_{\mathcal{O}})} \int_0^1 \left\langle \int_0^t b(s, u(s))\, dz_s, \int_0^t h(s, \zeta(s))\, ds \right\rangle dt \end{aligned}$$

for all $n \in \mathbb{N}$, it follows by taking the limit as $n \to \infty$ that

$$\int_0^1 \left\langle \int_0^t b(s, v(s))\, dz_s, \int_0^t \left[\int_Z h(s,z)\, \nu_s(dz)\right] ds \right\rangle dt \geq \inf(\mathcal{P}_{\mathcal{O}}).$$

This holds for every $\nu \in \mathcal{R}$, hence $\inf(\mathcal{P}_{\mathcal{R}}) \geq \inf(\mathcal{P}_{\mathcal{O}})$, which confirms the desired equality of the theorem according to the reverse inequality above.

Regarding the existence of a minimizer for $(\mathcal{P}_{\mathcal{R}})$ it suffices to take above for $(\zeta_n)_n$ a minimizing sequence of $(\mathcal{P}_{\mathcal{O}})$. $\square$

9. Towards a sweeping process with nonconvex mixed perturbation

In this section we present an existence result of AC solution to a sweeping process with a nonconvex mixed perturbation. Let $I := [0, T]$ and E be a separable Hilbert space. Let $\Gamma\colon I \times E \rightrightarrows E$ be a compact valued multimapping satisfying the following conditions:

(i) Γ is $\mathcal{L}(I) \otimes \mathcal{B}(E)$-measurable;

(ii) for every $t \in I$, at each $x \in E$ such that $\Gamma(t, x)$ is convex, $\Gamma(t, \cdot)$ is upper semicontinuous, and whenever $\Gamma(t, x)$ is not convex, $\Gamma(t, \cdot)$ is lower semicontinuous on some neighborhood of x;

(iii) $\Gamma(t, x) \cap \big(\alpha(t) + \beta(t)\|x\|\big)\mathbb{B}_E \neq \emptyset$ for all $(t, x) \in I \times E$ for some non-negative measurable functions $\alpha, \beta \in L^1_{\mathbb{R}}(I)$.

We recall the following Tolstonogov selection theorem which will be used in the proof of our theorem.

Theorem 9.1. (Tolstonogov [60]) *Let X be a separable Banach space and let $G\colon [0, T] \times X \rightrightarrows X$ be a compact valued multimapping satisfying the following hypotheses:*

(a) *G is $\mathcal{L}([0, T]) \otimes \mathcal{B}(X)$-measurable;*

(b) *for every $t \in [0, T]$, at each $x \in X$ such that $G(t, x)$ is convex, $G(t, \cdot)$ is upper semicontinuous, and whenever $G(t, x)$ is not convex, $G(t, \cdot)$ is lower semi-continuous on some neighborhood of x;*

(c) *there exists a Carathéodory function $g\colon [0, T] \times X \longrightarrow \mathbb{R}_+$ which is integrably bounded and such that for any $x \in X$*

$$G(t, x) \cap B_X[0, g(t, x)] \neq \emptyset \quad a.e.\ t \in [0, T].$$

Then, for any $\varepsilon > 0$ and any compact set $K \subset \mathcal{C}_X([0, T])$, there is a nonempty closed convex valued multimapping $\Psi\colon K \rightrightarrows L^1_X([0, T])$ which has a strongly-weakly sequentially closed graph such that for any $x \in K$ and $h \in \Psi(x)$, one has for a.e. $t \in [0, T]$

$$h(t) \in G(t, x(t)) \quad and \quad \|h(t)\| \leq g(t, x(t)) + \varepsilon.$$

Theorem 9.2. *Let E be a separable Hilbert space, let $I := [0, T]$, and let $C\colon I \rightrightarrows E$ be a multimapping which takes nonempty ball-compact convex values and which is absolutely continuous with $v(\cdot)$ as a control function of absolute continuity. Let $\Gamma\colon I \times E \rightrightarrows E$ be a compact valued multimapping satisfying the above conditions* (i), (ii) *and* (iii), *i.e.:*

(i) *Γ is $\mathcal{L}(I) \otimes \mathcal{B}(E)$-measurable;*

(ii) *for every $t \in I$, at each $x \in E$ such that $\Gamma(t, x)$ is convex, $\Gamma(t, \cdot)$ is upper semicontinuous, and whenever $\Gamma(t, x)$ is not convex, $\Gamma(t, \cdot)$ is lower semicontinuous on some neighborhood of x;*

(iii) $\Gamma(t, x) \cap \big(\alpha(t) + \beta(t)\|x\|\big)\mathbb{B}_E \neq \emptyset$ for all $(t, x) \in I \times E$ *for some non-negative measurable functions $\alpha, \beta \in L^1_{\mathbb{R}}(I)$.*

Then, for each $a \in C(0)$, the differential inclusion

$$\begin{cases} u(0) = a,\ u(t) \in C(t)\ \forall t \in I \\ -\frac{du}{dt}(t) \in N_{C(t)}u(t) + \Gamma(t, u(t)), \quad \text{a.e.}\ t \in I \end{cases}$$

has at least an absolutely continuous solution u on I.

Proof. Step 1. Fix $a \in C(0)$ and $\varepsilon > 0$. Let $\zeta\colon [0,T] \to E$ be the unique absolutely continuous solution to the differential inclusion

$$\begin{cases} \zeta(0) = a,\ \zeta(t) \in C(t)\ \forall t \in I \\ -\dot\zeta(t) \in N_{C(t)}\zeta(t). \end{cases}$$

The existence and uniqueness of such a solution follow from Theorem 3.3. Let $\rho\colon [0,T] \to \mathbb{R}_+$ be the unique absolutely continuous solution to the one-dimensional ordinary differential equation

$$\dot\rho(t) = \alpha(t) + \beta(t)\rho(t) + \varepsilon \quad \text{a.e. } t \in I, \text{ with } \rho(0) = \sup_{t\in[0,T]} \|\zeta(t)\|.$$

For each $h \in M_\varepsilon := \{y \in L^1_E(I) : \|y(t)\| \le \dot\rho(t) \text{ a.e. } t \in I\}$, denote by u_h the unique absolutely continuous solution to the perturbed differential inclusion

$$\begin{cases} -\dot u_h(t) \in N_{C(t)}u_h(t) + h(t) \quad \text{a.e. } t \in I \\ \ u_h(0) = a,\ u(t) \in C(t) \text{ a.e. } t \in I. \end{cases}$$

The existence and uniqueness of solution is due to Theorem 3.3 again. Using the monotonicity of the multimapping $x \mapsto N_{C(t)}x$, one has for a.e. $t \in I$

$$\langle \dot u_h(t) + h(t) - \dot\zeta(t), u_h(t) - \zeta(t)\rangle \le 0,$$

so $\frac{d}{dt}\left(\frac{1}{2}\|u_h(t) - \zeta(t)\|^2\right) \le \|h(t)\|\,\|u_h(t) - \zeta(t)\|$. Then Lemma 2.3 assures us that $\frac{d}{dt}(\|u_h(t) - \zeta(t)\|) \le \|h(t)\|$ for a.e. $t \in I$. This inequality combined with the absolute continuity of $t \mapsto \|u_h(t) - \zeta(t)\|$ and with the equality $\|u_h(0) - \zeta(0)\| = 0$ ensures that for every $t \in I$

$$\|u_h(t) - \zeta(t)\| \le \int_0^t \|h(s)\|\,ds, \quad \text{hence} \quad \|u_h(t)\| \le \rho(0) + \int_0^t \dot\rho(s) = \rho(t).$$

By Lemma 3.5(a) the set $L := \{u_h :\ h \in M_\varepsilon\}$ is compact in $\mathcal{C}_E(I)$ for the topology of uniform convergence.

Step 2. Now, for the given $\varepsilon > 0$, the compact set L in $\mathcal{C}_E(I)$ being constructed we may apply Theorem 9.1 with the mixed semicontinuous compact valued multimapping Γ (in place of G) and with the function g given by $g(t,x) = \alpha(t) + \beta(t)\|x\|$ for all $(t,x) \in I \times E$. Then, there is a nonempty closed convex valued multimapping $\Psi_\Gamma\colon L \rightrightarrows L^1_E(I)$ whose graph is sequentially closed with respect to the topology of uniform convergence in L and the weak topology in $L^1_E(I)$ and such that, for any $x \in L$ and $y \in \Psi_\Gamma(x)$, for a.e. $t \in I$, one has

$$y(t) \in \Gamma(t, x(t)) \text{ and } \|y(t)\| \le \alpha(t) + \beta(t)\|x(t)\| + \varepsilon.$$

This along with the inclusion $u_h \in L$ give for any $y \in \Psi_\Gamma(u_h)$ that for a.e $t \in I$

$$y(t) \in \Gamma(t, u_h(t)) \text{ and } \|y(t)\| \le \alpha(t) + \beta(t)\|u_h(t)\| + \varepsilon \le \alpha(t) + \beta(t)\rho(t) + \varepsilon,$$

where the latter inequality is due to the fact that $\|u_h(t)\| \le \rho(t)$. It ensues that

$$\Psi_\Gamma(u_h) \subset M_\varepsilon = \{y \in L^1_E(I) : \|y(t)\| \le \alpha(t) + \beta(t)\rho(t) + \varepsilon,\ \text{a.e. } t \in I\}.$$

Thus, the multimapping $\Psi_\Gamma \colon L \rightrightarrows L^1_E(I)$ takes nonempty weakly compact convex values and satisfies the following properties (α) and (β):

(α) $\Psi_\Gamma(x) \subset \{h \in M_\varepsilon : \ h(t) \in \Gamma(t, x(t))$ a.e. $t \in I\}$, for each $x \in L$,

(β) Ψ_Γ has closed graph, in the sense that for any $y_n \in \Psi_\Gamma(x_n)$ such that $(x_n)_n$ converges uniformly to $x \in L$ and $(y_n)_n$ $\sigma(L^1_E(I), L^\infty_E(I))$-converges in $L^1_E(I)$ to y, then, $y \in \Psi_\Gamma(x)$, or equivalently the multimapping $\Psi_\Gamma \colon L \rightrightarrows M_\varepsilon$ is upper semicontinuous from $L \subset \mathcal{C}_E(I)$ to M_ε endowed with the induced $\sigma(L^1_E(I), L^\infty_E(I))$-topology.

We also notice that M_ε is convex and $\sigma(L^1_E(I), L^\infty_E(I))$-compact.

Step 3. In this last step we will use the Kakutani-Ky Fan fixed point theorem via the compactness of the solutions set L of the differential inclusion considered in Step 1. For each $h \in M_\varepsilon$, let us set $\Phi(h) = \Psi_\Gamma(u_h)$ where u_h is as above the unique absolutely continuous solution to the differential inclusion

$$\begin{cases} -\dot{u}(t) \in N_{C(t)}u(t) + h(t), \quad \text{a.e. } t \in I \\ u(0) = a, \ u(t) \in C(t) \ \forall t \in I, \end{cases}$$

so we have defined a multimapping $\Phi \colon M_\varepsilon \rightrightarrows M_\varepsilon$. Recall that $L := \{u_h : h \in M_\varepsilon\}$ is compact in $\mathcal{C}_E(I)$. Then, it is clear that Φ is a weakly compact convex valued multimapping from M_ε into M_ε. Once we will justify that Φ has a fixed point h, i.e., $h \in \Phi(h) = \Psi_\Gamma(u_h)$, then we will have the inclusion $h(t) \in \Gamma(t, u_h(t))$ a.e. $t \in I$ as well as the feature that u_h is an absolutely continuous solution on I of the differential inclusion

$$\begin{cases} -\dot{u}_h(t) \in N_{C(t)}u_h(t) + \Gamma(t, u_h(t)), \ \text{a.e. } t \in I \\ u_h(0) = a, \ u_h(t) \in C(t) \ \forall t \in I. \end{cases}$$

For this purpose, using the upper semicontinuity of Ψ_Γ established above and noting by Lemma 3.5(b) that the mapping $M_\varepsilon \ni h \mapsto u_h$ is continuous from M_ε endowed with the induced $\sigma(L^1_E(I), L^\infty_E(I))$ topology into L endowed with the topology of uniform convergence, we see that the multimapping $\Phi \colon M_\varepsilon \rightrightarrows M_\varepsilon$ is upper semicontinous for the induced $\sigma(L^1_E(I), L^\infty_E(I))$ topology. Then we can apply the Kakutani-Ky Fan fixed point theorem, and the proof of the theorem is finished. □

Remark 9.3. The proof is mutatis mutandis the same as in Castaing et al [14] dealing with the inclusion

$$\begin{cases} -\dot{u}(t) \in A(t)u(t) + F(t, u(t)), \ \text{a.e. } t \in I \\ u(0) = u_0 \in D(A(0)), \end{cases}$$

where $A(t) \colon D(A(t)) \subset E \rightrightarrows E$ is a time dependent maximal monotone operator "absolutely continuous in variation" in a separable Hilbert space E with $D(A(t))$ ball-compact for every $t \in I$. The above theorem should be valid in separable Hilbert space with $C(t)$ ball-compact and r-prox-regular by suitable modification. There are several variants (Cf [5, 34, 35, 62]).

10. Towards second order problems involving the sweeping process

Diverse results are known in the literature for second order sweeping processes with derivative inside the moving set, see, e.g., [1, 18] and the references therein. This section studies some other variants.

10.1. AC solutions for second order dynamical systems with sweeping processes

We start with the case of a Carathéodory perturbation.

Theorem 10.1. *Let E be a separable Hilbert space, $I := [0,T]$, and let $r \in]0,+\infty]$. Let $C\colon I \rightrightarrows E$ be a multimapping which is absolutely continuous with $v(\cdot)$ as a control function of absolute continuity and such that $C(t)$ is ball-compact and r-prox-regular for all $t \in I$. Let $f\colon I \times E \times E \to E$ be a Carathéodory mapping for which there is a function $M \in L^1_{\mathbb{R}_+}(I)$ such that*

(i) $\|f(t,x,y)\| \le M(t)$ *for all* $t \in I$, $x, y \in E$

(ii) $\|f(t,x,y) - f(t,x,z)\| \le M(t)\|y - z\|$ *for all* $t \in I$, $x, y, z \in E$.

Then, for any $(x_0, u_0) \in E \times C(0)$, the problem

$$\begin{cases} x(t) = x_0 + \int_0^t u(s)ds, \ \forall t \in I \\ -\dot{u}(t) \in N_{C(t)}u(t) + f(t, x(t), u(t)) \, a.e. \, t \in I \\ x(0) = x_0, u(0) = u_0 \end{cases}$$

has an AC solution $(x,u)\colon I \to E \times E$.

Proof. Let us define $\beta, \rho \in L^1_{\mathbb{R}_+}(I)$ by putting for each $t \in I$

$$\beta(t) := M(t) + |\dot{v}(t)| \quad \text{and} \quad \rho(t) := \max\{2\beta(t), \|u_0\| + 2\int_0^t \beta(s)\, ds\}.$$

Let us consider the closed convex subset $\mathcal{X}$ of the Banach space $\mathcal{C}_E(I)$ (depending on $\rho(\cdot)$ and) defined by

$$\mathcal{X} := \{\zeta_\ell \in W^{1,1}(I,E) : \zeta_\ell(t) = x_0 + \int_0^t \ell(s)\, ds, \ \ell \in L^1_E(I), \ \|\ell(s)\| \le \rho(s) \text{ a.e. } s \in I\}.$$

Then $\mathcal{X}$ is bounded in $\mathcal{C}_E(I)$ and equi-absolutely continuous. By Theorem 3.3, for each $h \in \mathcal{X}$, there is a *unique* AC $= W^{1,1}(I,E)$ solution to the inclusion

$$\begin{cases} -\dot{u}_h(t) \in N_{C(t)}u_h(t) + f(t, h(t), u_h(t)) \quad \text{a.e. } t \in I \\ u_h(t) \in C(t), \forall t \in I \\ u_h(0) = u_0 \in C(0), \end{cases}$$

and this solution satisfies $\|\dot{u}_h(t)\| \le 2\beta(t)$ a.e. $t \in I$.

Now let us consider the mapping Φ defined on $\mathcal{X}$ with values in $\mathcal{C}_E(I)$ by putting for every $h \in \mathcal{X}$

$$\Phi(h)(t) := x_0 + \int_0^t u_h(s)\, ds \quad \text{for all } t \in I.$$

Since $\|u_h(t)\| \le \rho(t)$ for all $t \in I$, it is clear that $\Phi(h) \in \mathcal{X}$. Our aim is to prove the existence theorem by applying some ideas developed in Castaing et al. [17] via a fixed point theorem. Nevertheless, this needs a careful look using the estimation of the absolutely continuous solutions given above.

For this purpose we first claim that $\Phi\colon \mathcal{X} \to \mathcal{X}$ is continuous and for any $h \in \mathcal{X}$ and for any $t \in I$ the inclusion holds

$$\Phi(h)(t) \in u_0 + \int_0^t \overline{\mathrm{co}}\, C_0(s)\, ds,$$

where $C_0(s) := C(s) \cap \rho(s)\mathbb{B}_E$ for all $s \in I$. We note that $\mathrm{gph}\, C$ is closed in $I \times E$ as seen in Section 2, so $\mathrm{gph}\, C_0$ is $\mathcal{L}(I) \otimes \mathcal{B}(E)$-measurable. We deduce that the multimapping $\overline{\mathrm{co}}\, C_0(\cdot)$ is scalarly $\mathcal{L}(I)$-measurable, hence it is $\mathcal{L}(I)$-measurable (see [23, Theorem II-37]). Since $t \mapsto \overline{\mathrm{co}}\, C_0(t)$ is a compact convex valued, $\mathcal{L}(I)$-measurable, and integrably bounded multimapping, the set $\int_0^t \overline{\mathrm{co}}\, C_0(s)\, ds$ is compact in E for every $t \in I$ (see [10]). Since in addition $\Phi(\mathcal{X})$ is equicontinuous, it is relatively compact in the Banach space $\mathcal{C}_E(I)$ by the Arzelà-Ascoli theorem. Now we check that Φ is continuous. It is sufficient to show that, if (h_n) uniformly converges to h in $\mathcal{X}$, then the sequence $(u_{h_n})_n$, where u_{h_n} is the absolutely continuous solution u_{h_n} associated with h_n of the differential inclusion

$$\begin{cases} u_{h_n}(0) = u_0 \in C(0), \quad u_{h_n}(t) \in C(t), \forall t \in I \\ -\dot{u}_{h_n}(t) \in N_{C(t)} u_{h_n}(t) + f(t, h_n(t), u_{h_n}(t)), \quad \text{a.e. } t \in I, \end{cases}$$

uniformly converges to the absolutely continuous solution u_h associated with h of the differential inclusion

$$\begin{cases} u_h(0) = u_0, \quad u_h(t) \in C(t), \forall t \in I \\ -\dot{u}_h(t) \in N_{C(t)} u_h(t) + f(t, h(t), u_h(t)) \quad \text{a.e. } t \in I. \end{cases}$$

Since $(u_{h_n})_n$ is equi-absolutely continuous with for each $n \in \mathbb{N}$ the estimate

$$\|\dot{u}_{h_n}(t)\| \le \rho(t) \text{ a.e. } t \in I,$$

we may suppose that $(u_{h_n})_n$ converges uniformly to a mapping $w\colon I \to E$, so for every $t \in I$ we have

$$f(t, h_n(t), u_{h_n}(t)) \to f(t, h(t), w(t)) \quad \text{as } n \to \infty.$$

Keeping in mind that $\|f(t, h_n(t), u_{h_n}(t))\| \le M(t)$ for all $t \in I$ with $M \in L^1_{\mathbb{R}_+}(I)$, we can apply Lemma 3.4 to see that w is absolutely continuous and solution of the differential inclusion

$$\begin{cases} w(0) = u_0, \quad w(t) \in C(t), \ \forall t \in I \\ -\dot{w}(t) \in N_{C(t)} w(t) + f(t, h(t), w(t)) \quad \text{a.e. } t \in I. \end{cases}$$

By uniqueness (see Theorem 3.3) we derive that $w = u_h$. Further, writing

$$\|\Phi(h_n)(t) - \Phi(h)(t)\| = \left\| \int_0^t u_{h_n}(s)\, ds - \int_0^t u_h(s)\, ds \right\| \le \int_0^T \|u_{h_n}(s) - u_h(s)\|\, ds,$$

and taking into account the uniform convergence of $(u_{h_n})_n$ to $w = u_h$ on I, it follows

$$\sup_{t \in [0,T]} \|\Phi(h_n)(t) - \Phi(h)(t)\| \le \int_0^T \|u_{h_n}(s) - u_h(s)\|\, ds \to 0,$$

so that $(\Phi(h_n))_n$ converges uniformly on I to $\Phi(h)$ in $\mathcal{C}_E(I)$. This confirms the continuity of $\Phi\colon \mathcal{X} \to \mathcal{X}$. Thus according to the relative compactness in $\mathcal{C}_E(I)$ of $\Phi(\mathcal{X})$, by the Schauder theorem Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}$, which means

$$h(t) = \Phi(h)(t) = x_0 + \int_0^t u_h(s)\, ds,\ t \in I,$$

$$\begin{cases} u_h(0) = u_0, \quad u_h(t) \in C(t),\ \forall t \in I \\ -\frac{du_h}{dt}(t) \in N_{C(t)} u_h(t) + f(t, h(t), u_h(t)) \text{ a.e. } t \in I. \end{cases}$$

Then, taking $x(\cdot) = h(\cdot)$ and $u(\cdot) = u_h(\cdot)$, we obtain that (x, u) is an absolutely continuous solution of the differential inclusion in the statement of the theorem. □

10.2. Second-order differential equations with many points boundary conditions coupled with sweeping processes

To finish this section we present the study of a second-order differential equation with m-points boundary conditions coupled with the sweeping process in a separable Hilbert space E.

For the sake of completeness, we recall and summarize some results developed in [12]. By $W^{2,1}_E([0,1])$ we denote the set of all continuous mappings in $\mathcal{C}_E([0,1])$ such that their first derivatives are continuous and their second derivatives belong to $L^1_E([0,1])$.

Lemma 10.2. *Assume that E is a separable Hilbert space. Let m be an integer with $m > 3$ and let $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$, $\gamma > 0$, and $\alpha_i \in \mathbb{R}$ $(i = 1, \cdots, m-2)$ satisfying the condition*

$$\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma\eta_i) \neq 0. \tag{10.1}$$

Let $G\colon [0,1] \times [0,1] \to \mathbb{R}$ be the function defined by

$$G(t,s) = \begin{cases} \frac{1}{\gamma}\left(1 - \exp(-\gamma(t-s))\right), & 0 \le s \le t \le 1 \\ 0, & t < s \le 1 \end{cases} + \frac{A}{\gamma}\left(1 - \exp(-\gamma t)\right)\phi(s),$$

where

$$\phi(s) = \begin{cases} 1 - \exp(-\gamma(1-s)) - \sum_{i=1}^{m-2} \alpha_i \left(1 - \exp(-\gamma(\eta_i - s))\right), & 0 \le s < \eta_1, \\ 1 - \exp(-\gamma(1-s)) - \sum_{i=2}^{m-2} \alpha_i \left(1 - \exp(-\gamma(\eta_i - s))\right), & \eta_1 \le s \le \eta_2, \\ \ldots\ldots. & \\ 1 - \exp(-\gamma(1-s)), & \eta_{m-2} \le s \le 1, \end{cases}$$

and
$$A = \left(\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma\eta_i)\right)^{-1}. \tag{10.2}$$

Then the following properties hold.

(i) *For every fixed $s \in [0,1]$, the function $G(\cdot,s)$ is right derivable on $[0,1[$ and left derivable on $]0,1]$. Its derivative is given by*

$$\left(\frac{\partial G}{\partial t}\right)_+(t,s) = \begin{cases} \exp(-\gamma(t-s)), & 0 \le s \le t < 1 \\ 0, & 0 \le t < s < 1 \end{cases} + A\exp(-\gamma t)\phi(s),$$

$$\left(\frac{\partial G}{\partial t}\right)_-(t,s) = \begin{cases} \exp(-\gamma(t-s)), & 0 \le s < t \le 1 \\ 0, & 0 < t \le s \le 1 \end{cases} + A\exp(-\gamma t)\phi(s).$$

(ii) *$G(\cdot,\cdot)$ and $\frac{\partial G}{\partial t}(\cdot,\cdot)$ satisfies*

$$|G(t,s)| \le M_G \quad \text{and} \quad \left|\frac{\partial G}{\partial t}(t,s)\right| \le M_G \quad \forall (t,s) \in [0,1]\times[0,1],$$

where $M_G = \max\{\gamma^{-1},1\}\left[1+|A|\left(1+\sum_{i=1}^{m-2}|\alpha_i|\right)\right].$

(iii) *If $u \in W_H^{2,1}([0,1])$ with $u(0) = x$ and $u(1) = \sum_{i=1}^{m-2}\alpha_i u(\eta_i)$, then*

$$u(t) = e_x(t) + \int_0^1 G(t,s)(\ddot u(s) + \gamma \dot u(s))\,ds, \quad \forall t \in [0,1], \tag{10.3}$$

where $e_x(t) = x + A(1-\sum_{i=1}^{m-2}\alpha_i)(1-\exp(-\gamma t))x.$

(iv) *Let $f \in L_E^1([0,1])$ and let $u_f\colon [0,1] \to E$ be the function defined by*

$$u_f(t) = e_x(t) + \int_0^1 G(t,s)f(s)\,ds \quad \forall t \in [0,1]. \tag{10.4}$$

Then we have $u_f(0) = x$ *and* $u_f(1) = \sum_{i=1}^{m-2}\alpha_i u_f(\eta_i)$.
Further, the function u_f is derivable on $[0,1]$ and its derivative $\dot u_f$ is defined by

$$\dot u_f(t) = \lim_{h\to 0}\frac{u_f(t+h)-u_f(t)}{h} = \dot e_x(t) + \int_\tau^1 \frac{\partial G}{\partial t}(t,s)f(s)\,ds,$$

with as in (iii): $\dot e_x(t) = \gamma A(1-\sum_{i=1}^{m-2}\alpha_i)\exp(-\gamma t)x.$

(v) *If $f \in L_E^1([0,1])$, the function $\dot u_f$ is scalarly derivable, and its weak derivative $\ddot u_f$ satisfies*

$$\ddot u_f(t) + \gamma \dot u_f(t) = f(t) \quad \text{a.e. } t \in [0,1].$$

The following proposition is a direct consequence of Lemma 10.2.

Proposition 10.3. *Let $f \in L_E^1([0,1])$. The m-points boundary problem*

$$\begin{cases} \ddot u(t) + \gamma \dot u(t) = f(t),\, t \in [0,1] \\ u(0) = x, u(1) = \sum_{i=1}^{m-2}\alpha_i u(\eta_i), \end{cases} \tag{10.5}$$

has a unique $W_E^{2,1}([0,1])$*-solution* u_f*, with integral representation formulas*

$$\begin{cases} u_f(t) = e_x(t) + \int_0^1 G(t,s) f(s)\, ds,\ t \in [0,1] \\ \dot u_f(t) = \dot e_x(t) + \int_0^1 \frac{\partial G}{\partial t}(t,s) f(s)\, ds,\ t \in [0,1], \end{cases}$$

where
$$\begin{cases} e_x(t) & = x + A(1 - \sum_{i=1}^{m-2} \alpha_i)(1 - \exp(-\gamma t))x \\ \dot e_x(t) & = \gamma A \left(1 - \sum_{i=1}^{m-2} \alpha_i\right) \exp(-\gamma t) x \\ A & = \left(\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma(\eta_i))\right)^{-1}. \end{cases}$$

The following result is crucial for our purpose. For shortness, we omit the proof, one can find the details in [12, Theorem 5.1].

Proposition 10.4. *With the hypotheses and notation of Proposition* 10.4*, let* E *be a separable Hilbert space and let* $X\colon [0,1] \rightrightarrows E$ *be a* $(\mathcal{L}(I), \mathcal{B}(E))$*-measurable compact convex valued and integrably bounded multimapping. Then the solution set of* $W_E^{2,1}([0,1])$*-solutions to*

$$\begin{cases} \ddot u_f(t) + \gamma \dot u_f(t) = f(t),\ t \in [0,1],\ f \in S_X^1 \\ u_f(0) = x, \quad u_f(1) = \sum_{i=1}^{m-2} \alpha_i u_f(\eta_i), \end{cases}$$

is bounded, convex, equicontinuous and compact in $\mathcal{C}_E([0,1])$*.*

Now comes the existence result with a second-order differential inclusion with m-points boundary condition coupled with the sweeping process with Lipschitz perturbation.

Theorem 10.5. *Let* E *be a separable Hilbert space,* $I := [0,1]$*, and let* $r \in]0,+\infty]$*. Let* $C\colon I \rightrightarrows E$ *be a multimapping which is absolutely continuous with* $v(\cdot)$ *as a control function of absolute continuity and such that* $C(t)$ *is ball-compact and* r*-prox-regular for all* $t \in I$*. Let* $f\colon I \times E \times E \to E$ *be a Carathéodory mapping for which there is* $M \in L^1_{\mathbb{R}_+}(I)$ *such that*

(i) $\|f(t,x,y)\| \le M(t)$ *for all* $t \in I$*,* $x, y \in E$*;*

(ii) $\|f(t,x,y) - f(t,x,z)\| \le M(t)\|y - z\|$ *for all* $t \in I$*,* $x, y, z \in E$*.*

Then, for each $(x_0, u_0) \in E \times C(0)$ *there is a* $W_E^{2,1}(I)$ *mapping* $u\colon I \to E$ *and an absolutely continuous mapping* $\zeta\colon I \to E$ *satisfying*

$$\begin{cases} \ddot u(t) + \gamma \dot u(t) = \zeta(t),\ t \in I \\ u(0) = x_0,\ u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \\ -\frac{d\zeta}{dt}(t) \in N_{C(t)} \zeta(t) + f(t, u(t), \zeta(t)) \quad \text{a.e. } t \in I \\ \zeta(0) = u_0. \end{cases}$$

Proof. Let $\beta, \rho \in L^1_{\mathbb{R}_+}(I)$ be defined by

$$\beta(t) = M(t) + |\dot v(t)| \ \text{ and } \ \rho(t) = \max\left\{2\beta(t), \|u_0\| + 2\int_0^t \beta(s)\, ds\right\} \text{ for all } t \in I.$$

Let also for each $t \in I$: $C_0(t) := C(t) \cap \rho(t)\mathbb{B}_E$ and $X(t) := \overline{\mathrm{co}}\, C_0(t)$.

Then (as seen in the proof of Theorem 10.1) the multimapping X is compact convex valued, $\mathcal{L}(I)$-measurable and integrably bounded.

Now we proceed as in Theorem 10.1 with the adapted Green function considered in Lemma 10.2. Let $\mathcal{X}$ be the set (depending on $\rho(\cdot)$) defined by

$$\mathcal{X} := \{y_\ell \colon I \to E,\ y_\ell(t) = e_{x_0}(t) + \int_0^1 G(t,s)\ell(s)\,ds,\ \ell \in S^1_X\},$$

so $\mathcal{X}$ is the solution set to the second-order differential inclusion with m-points boundary conditions

$$\begin{cases} \ddot{u}(t) + \gamma\dot{u}(t) \in X(t),\ t \in I \\ u(0) = x_0,\ u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i). \end{cases}$$

By Proposition 10.4 the set $\mathcal{X}$ is convex, equicontinuous and compact in $\mathcal{C}_E(I)$.

For every $h \in \mathcal{X}$, the mapping f_h defined by $f_h(t,x) = f(t,h(t),x)$ is Carathéodory, Lipschitz with respect to $x \in E$ and satisfies $\|f_h(t,x)\| \le M(t)$. Consequently, by Theorem 3.3, for every $h \in \mathcal{X}$, there is a unique absolutely continuous solution u_h to the evolution inclusion

$$\begin{cases} -\dot{u}_h(t) \in N_{C(t)}u_h(t) + f(t,h(t),u_h(t)),\ \text{a.e. } t \in I \\ u_h(0) = u_0, \end{cases}$$

and one has the velocity $\|\dot{u}_h(t)\| \le \rho(t)$ for a.e. $t \in I$.

We define now a mapping Φ from $\mathcal{X}$ into $\mathcal{C}_E(I)$ by setting for each $h \in \mathcal{X}$

$$\Phi(h)(t) = e_{x_0}(t) + \int_0^1 G(t,s)u_h(s)\,ds, \text{ for all } t \in I.$$

Since $u_h(s) \in C_0(s) \subset X(s) \subset \rho(s)\mathbb{B}_E$ for all $s \in I$, it is clear that $\Phi(h) \in \mathcal{X}$. We claim that $\Phi\colon \mathcal{X} \to \mathcal{X}$ is continuous. First, by Lemma 3.4 we can see that if $h_n \to h$ in $\mathcal{X}$ uniformly on I, the absolutely continuous solution u_{h_n} of the differential inclusion

$$\begin{cases} -\dot{u}_{h_n}(t) \in N_{C(t)}u_{h_n}(t) + f(t,h_n(t),u_{h_n}(t)),\ \text{a.e. } t \in I \\ u_{h_n}(0) = u_0, \end{cases}$$

converges uniformly to the absolutely continuous solution u_h of the differential inclusion

$$\begin{cases} -\dot{u}_h(t) \in N_{C(t)}u_h(t) + f(t,h(t),u_h(t)),\ \text{a.e. } t \in I \\ u_h(0) = u_0. \end{cases}$$

Then using the uniform convergence of $(u_{h_n})_n$ to u_h we obtain that

$$\sup_{t\in I} \|\Phi(h_n)(t) - \Phi(h)(t)\| \le \int_0^1 M_G \|u_{h_n}(s) - u_h(s)\|\,ds \to 0,$$

so $\Phi(h_n) \to \Phi(h)$ uniformly on I, confirming the continuity of $\Phi\colon \mathcal{X} \to \mathcal{X}$ with respect to the topology of uniform convergence. Since the convex set $\mathcal{X}$ is compact with respect to the topology of uniform convergence, then Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}$, which means

$$h(t) = \Phi(h)(t) = e_{x_0}(t) + \int_0^1 G(t,s)u_h(s)\,ds,\ t \in I,$$

$$\begin{cases} u_h(0) = u_0,\ u_h(t) \in C(t),\ \forall t \in I \\ -\dfrac{du_h}{dt}(t) \in N_{C(t)}u_h(t) + f(t, h(t), u_h(t)) \text{ a.e. } t \in I \end{cases}$$

By Lemma 10.2 and Proposition 10.3, this gives

$$\begin{cases} \ddot{h}(t) + \gamma\dot{h}(t) = u_h(t),\ t \in I, \\ h(0) = x_0,\ h(1) = \displaystyle\sum_{i=1}^{m-2} \alpha_i h(\eta_i) \\ u_h(0) = u_0,\ u_h(t) \in C(t),\ \forall t \in I \\ -\dfrac{du_h}{dt}(t) \in N_{C(t)}u_h(t) + f(t, h(t), u_h(t)) \text{ a.e. } t \in I. \end{cases}$$

Taking $u(\cdot) = h(\cdot)$ and $\zeta(\cdot) = u_h(\cdot)$, the proof is complete. □

11. Periodic solution for the sweeping process

We provide in this section various existence and uniqueness results of periodic solution for perturbed sweeping processes. We will begin with the setting of closed convex sets $C(t)$ in finite dimensions.

Theorem 11.1. *Let $E = \mathbb{R}^e$ and I be either $[0,T]$ or $\mathbb{R}$. Let $C\colon I \rightrightarrows E$ be a multimapping with nonempty closed convex values which is absolutely continuous on $[0,T]$ with $v(\cdot)$ as a control function of absolute continuity on $[0,T]$. Let $\Gamma\colon I \times E \rightrightarrows E$ be a multimapping with nonempty closed convex values which is $\mathcal{L}(I) \otimes \mathcal{B}(E)$ measurable with respect to (t,x) and upper semicontinuous with respect to x such that:*

(i) *there exists a Lebesgue integrable function $m\colon [0,T] \to \mathbb{R}_+$ such that for all $t \in [0,T]$ and all $x \in E$ we have $\max\{\|y\| : y \in \Gamma(t,x)\} \le m(t)$;*

(ii) *there exists a function $\gamma\colon [0,T] \to]0,+\infty[$ Lebesgue integrable on $[0,T]$ such that*

$$\langle y_1 - y_2, x_1 - x_2\rangle \ge \gamma(t)\|x_1 - x_2\|^2$$

for all $y_1 \in \Gamma(t,x_1)$, $y_2 \in \Gamma(t,x_2)$ and $t \in [0,T]$.

(I) *Assume that $I = [0,T]$ and that $C(T) \subset C(0)$. Then the differential inclusion*

$$\begin{cases} u(t) \in C(t),\ \forall t \in I \\ -\dot{u}(t) \in N_{C(t)}u(t) + \Gamma(t, u(t)) \quad \text{a.e. } t \in I \end{cases}$$

has a unique absolutely continuous solution $u(\cdot)$ on I satisfying $u(0) = u(T)$.

(II) *Assume that $I = \mathbb{R}$ and that the multimappings $C(\cdot)$ and $\Gamma(\cdot,x)$ (for all $x \in E$) are T-periodic. Then the differential inclusion*

$$\begin{cases} u(t) \in C(t),\ \forall t \in \mathbb{R} \\ -\dot{u}(t) \in N_{C(t)}u(t) + \Gamma(t, u(t)) \quad \text{a.e. } t \in \mathbb{R} \end{cases}$$

has a unique locally absolutely continuous solution on $\mathbb{R}$ which is T-periodic.

Proof. (I) We start with the case $I = [0, T]$.

(A) *A uniqueness property.*

First we show under condition (ii) that for any $a \in C(0)$ the differential inclusion

$$\begin{cases} -\dot{u}(t) \in N_{C(t)}u(t) + \Gamma(t, u(t)) \quad \text{a.e. } t \in I \\ u(0) = a, \ u(t) \in C(t) \ \forall t \in I \end{cases} \tag{11.1}$$

admits a unique solution. Let u_1 and u_2 be two solutions of the latter differential inclusion whose existence is guaranteed by Theorem 3.2. Then, there exist two $L^1_E(I)$-mappings $h_1(\cdot)$ and $h_2(\cdot)$ such that for all $t \in I$

$$h_1(t) \in \Gamma(t, u_1(t)) \text{ and } h_2(t) \in \Gamma(t, u_2(t))$$

and for almost all $t \in I$

$$-\dot{u}_1(t) - h_1(t) \in N_{C(t)}u_1(t) \ \text{ and } \ -\dot{u}_2(t) - h_2(t) \in N_{C(t)}u_2(t).$$

The *monotone* condition (ii) on Γ yields

$$\langle h_1(t) - h_2(t), u_1(t) - u_2(t)\rangle \geq \gamma(t)\|u_1(t) - u_2(t)\|^2.$$

Moreover, the monotone property of $N_{C(t)}$ ensures that

$$\langle -\dot{u}_1(t) - h_1(t) - (-\dot{u}_2(t) - h_2(t)), u_1(t) - u_2(t)\rangle \geq 0,$$

hence $$\langle \dot{u}_1(t) - \dot{u}_2(t), u_1(t) - u_2(t)\rangle \leq -\gamma(t)\|u_1(t) - u_2(t)\|^2,$$

that is, $$\frac{1}{2}\frac{d}{dt}\|u_1(t) - u_2(t)\|^2 \leq -\gamma(t)\|u_1(t) - u_2(t)\|^2 \ \text{ for a.e. } t \in I. \tag{11.2}$$

Remark that $\frac{d}{dt}\|u_1(t) - u_2(t)\|^2 \leq 0$, since by assumption γ is a non-negative function. This along with the fact that $\|u_1(0) - u_2(0)\| = 0$, furnish by integrating that $u_1 = u_2$. Thus, the uniqueness of the solution for the problem (11.1) follows.

(B) *The equality for the trajectory at times* 0 *and* T.

Let $a, b \in C(0)$ and denote according to (A) by u_a (resp. u_b) the unique solution of the problem (11.1) with the initial condition a (resp. b) in $C(0)$. Then, apply inequality (11.2) by taking $u_1 = u_a$ and $u_2 = u_b$. For each fixed $t \in I$ the Gronwall lemma (see Lemma 2.2) entails that

$$\|u_a(t) - u_b(t)\|^2 \leq \|a - b\|^2 \exp\big(-2\int_0^t \gamma(s)\, ds\big),$$

hence in particular for $\ell := \exp\big(-\int_0^T \gamma(s)\, ds\big) < 1$

$$\|u_a(T) - u_b(T)\| \leq \ell\|a - b\| \quad \text{for all } a, b \text{ in } C(0).$$

This and the inclusion $C(T) \subset C(0)$ tell us that the mapping $a \mapsto u_a(T)$ on $C(0)$ is an ℓ-contraction from $C(0)$ into itself with $0 \leq \ell < 1$. Then it follows that this mapping has a unique fixed point $a_0 \in C(0)$, and the mapping u_{a_0} is an AC solution of the differential inclusion in (I) with $u_{a_0}(0) = u_{a_0}(T)$.

(C) *Uniqueness of solution of the differential inclusion in (I).*

Regarding the uniqueness of the solution, let $U\colon I \to E$ be any AC solution of the differential inclusion in (I) with $U(0) = U(T)$. Put $b_0 := U(0)$. Then $b_0 \in C(0)$ and U is an AC solution of the differential inclusion (11.1) for $a = b_0$. The uniqueness property in (A) assures us that $U = u_{b_0}$, which entails $b_0 = u_{b_0}(T)$. Keeping in mind that a_0 is the unique fixed point of the mapping $a \mapsto u_a(T)$ from $C(0)$ into $C(0)$ (as seen above), it ensues that $b_0 = a_0$, thus $U = u_{a_0}$. This justifies the uniqueness of AC solution of the differential inclusion in (I) with equality of the trajectory at times 0 and T.

(II) Assume now the hypotheses in (II) hold. By (I) there exists an absolutely continuous mapping $u_0\colon [0,T] \to E$ with $u_0(0) = u_0(T)$ and $u_0(t) \in C(t)$ for every $t \in [0,T]$ and such that there is a Lebesgue negligible set $N_0 \subset [0,T]$ for which

$$-\dot{u}_0(t) \in N_{C(t)}u_0(t) + \Gamma(t, u_0(t)) \text{ for all } t \in [0,T] \setminus N_0.$$

For each integer $k \in \mathbb{Z} \setminus \{0\}$ put $N_k := (kT + N_0) \cup \{kT, (k+1)T\}$ and define in the classical way $u_k : [kT, (k+1)T]$ by

$$u_k(t) = u_0(t - kT) \quad \text{for all } t \in [kT, (k+1)T].$$

For every $t \in [kT, (k+1)T]$ we have by T-periodicity of $C(\cdot)$

$$u_k(t) = u_0(t - kT) \in C(t - kT) = C(t),$$

and for every $t \in [kT, (k+1)T] \setminus N_k$ the derivative $\dot{u}_0(t - kT)$ exists, hence $\dot{u}_k(t)$ exists and

$$\begin{aligned} -\dot{u}_k(t) = -\dot{u}_0(t-kT) &\in N_{C(t-kT)}u_0(t-kT) + \Gamma(t-kT, u_0(t-kT)) \\ &= N_{C(t)}u_k(t) + \Gamma(t, u_k(t)), \end{aligned}$$

since $\Gamma(t-kT, u_0(t-kT)) = \Gamma(t, u_0(t-kT)) = \Gamma(t, u_k(t))$. We can define $u\colon \mathbb{R} \to E$ by setting

$$u(t) = u_k(t) \quad \text{if } t \in [kT, (k+1)T],\ k \in \mathbb{Z}.$$

The mapping u is well-defined on $\mathbb{R}$, T-periodic, locally absolutely continuous and for every $t \in \mathbb{R} \setminus \bigcup_{k\in\mathbb{Z}} N_k$

$$-\dot{u}(t) \in N_{C(t)}u(t) + \Gamma(t, u(t)),$$

so u is a desired T-periodic solution.

Finally, the uniqueness of the periodic solution follows from (A) in (I) above. □

Remark 11.2. The proof of (I.A-B) is inspired by arguments in the beginning of Section 4 of Moreau [46] and mainly by the arguments in Castaing et al [14] dealing with the differential inclusion

$$\begin{cases} -\dot{u}(t) \in A(t)u(t) + \Gamma(t, u(t)),\ \text{a.e. } t \in I \\ u(0) = u_0 \in \mathrm{Dom}\,(A(0)), \end{cases}$$

where $A(t)\colon \mathrm{Dom}\,(A(t)) \subset E \rightrightarrows E$ is a time dependent maximal monotone operator "*absolutely continuous in variation*" (see, e.g, [14] for definition) in a separable Hilbert space E with $\mathrm{Dom}\,(A(t))$ ball-compact for every $t \in I$.

The second theorem of the above type, say Theorem 11.4, will consider the situation of convex sets $C(t)$ in infinite dimensions. Before proceeding to that theorem let us state the following theorem whose existence result follows from Edmond and Thibault [27, Theorem 5.1] whereas the equivalence with the differential inclusion involving the distance function is a consequence of Thibault [55, Corollary 3.2]. Recall that a multimapping F between two normed spaces X, Y is *scalarly upper semicontinuous* provided that for each $y^* \in Y^*$ the support function $x \mapsto \delta^*(y^*, F(x))$ is upper semicontinuous.

Theorem 11.3. *Let E be a separable Hilbert space, $r \in]0, +\infty]$ and $I := [0, T]$. Let $C\colon I \rightrightarrows E$ be a multimapping which is absolutely continuous on $[0, T]$ with $v(\cdot)$ as a control function of absolute continuity on $[0, T]$ and such that $C(t)$ is r-prox-regular for all $t \in [0, T]$. Let $\Gamma\colon I \times E \rightrightarrows E$ be a multimapping with nonempty compact convex values which is $\mathcal{L}(I) \otimes \mathcal{B}(E)$ measurable with respect to (t, x) and scalarly upper semicontinuous with respect to x and for which there is a compact set $K \subset \mathbb{B}_E$ and a Lebesgue integrable function $m\colon I \to \mathbb{R}_+$ such that for all $t \in I$ and all $x \in C(t)$ we have $\Gamma(t, x) \subset m(t)K$.*

Then, given $a \in C(0)$, the differential inclusion

$$\begin{cases} u(0) = a, \ u(t) \in C(t) \ \forall t \in I \\ -\frac{du}{dt}(t) \in N_{C(t)}u(t) + \Gamma(t, u(t)) \ a.e. \ t \in I \end{cases}$$

admits at least an AC solution.

Further, an AC mapping $u\colon I \to E$ is a solution of the latter differential inclusion if and only if it is a solution of the unconstrained differential inclusion

$$\begin{cases} u(0) = a \\ -\frac{du}{dt}(t) \in \big(|\dot{v}(t)| + m(t)\big)\partial d_{C(t)}(u(t)) + \Gamma(t, u(t)) \ a.e. \ t \in I. \end{cases}$$

Using Theorem 11.3 instead of Theorem 3.2, the proof of the next Theorem 11.4 is mutatis mutandis the same as that of Theorem 11.1 above.

Theorem 11.4. *Let E be a separable Hilbert space and let I be either $[0, T]$ or the real line $\mathbb{R}$. Let $C\colon I \rightrightarrows E$ be a multimapping with nonempty closed convex values which is absolutely continuous on $[0, T]$ with $v(\cdot)$ as a control function of absolute continuity on $[0, T]$. Suppose also that C satisfies the condition $C(T) \subset C(0)$. Let $\Gamma\colon I \times E \rightrightarrows E$ be a multimapping with nonempty compact convex values which is $\mathcal{L}(I) \otimes \mathcal{B}(E)$ measurable with respect to (t, x) and scalarly upper semicontinuous with respect to x and for which there is a compact set $K \subset \mathbb{B}_E$ such that*

(i) *there exists a Lebesgue integrable function $m\colon [0, T] \to \mathbb{R}_+$ such that for all $t \in [0, T]$ and all $x \in C(t)$ we have $\Gamma(t, x) \subset m(t)K$;*

(ii) *there exists for $I = [0, T]$ (resp. $I = \mathbb{R}$) a function $\gamma\colon I \to \mathbb{R}_+$ Lebesgue integrable on $I = [0, T]$ (resp. locally Lebesgue integrable on $I = \mathbb{R}$) such that*

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq \gamma(t) \|x_1 - x_2\|^2$$

for all $y_1 \in \Gamma(t, x_1)$, $y_2 \in \Gamma(t, x_2)$ and $t \in I$.

(I) *Assume that $I = [0, T]$ and that $C(T) \subset C(0)$. Then the differential inclusion*

$$\begin{cases} u(t) \in C(t), \ \forall t \in I \\ -\dot{u}(t) \in N_{C(t)}u(t) + \Gamma(t, u(t)) \quad \text{a.e. } t \in I \end{cases}$$

has a unique absolutely continuous solution $u(\cdot)$ on I satisfying $u(0) = u(T)$.

(II) *Assume that $I = \mathbb{R}$ and that the multimappings $C(\cdot)$ and $\Gamma(\cdot, x)$ (for all $x \in E$) are T-periodic. Then the differential inclusion*

$$\begin{cases} u(t) \in C(t), \ \forall t \in \mathbb{R} \\ -\dot{u}(t) \in N_{C(t)}u(t) + \Gamma(t, u(t)) \quad \text{a.e. } t \in \mathbb{R} \end{cases}$$

has a unique locally absolutely continuous solution on $\mathbb{R}$ which is T-periodic.

Concerning the case of prox-regular sets we have the following theorem.

Theorem 11.5. *Let E be a separable Hilbert space, let $r \in]0, +\infty]$ and let I be either $[0, T]$ or $\mathbb{R}$. Let $C\colon I \rightrightarrows E$ be a multimapping with r-prox-regular values which is κ-Lipschitz on $[0, T]$ for some real $\kappa > 0$. Suppose also that C satisfies the condition $C(T) \subset C(0)$. Let Γ be a multimapping from $I \times E$ into E with nonempty closed convex values which is $\mathcal{L}(I) \otimes \mathcal{B}(E)$ measurable with respect to (t, x) and upper semicontinuous with respect to x (resp. scalarly upper semicontinuous with respect to x). Assume*

(i) *there exists a real $\mu > 0$ and a compact set $K \subset \mathbb{B}_E$ such that for all $t \in [0, T]$ and all $x \in C(t)$ we have $\Gamma(t, x) \subset \mu K$;*

(ii) *for $I = [0, T]$ along with $I = \mathbb{R}$ there is a real $\gamma > (\mu + \kappa)/r$ (γ depending on I) such that*
$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq \gamma \|x_1 - x_2\|^2$$
for all $y_1 \in \Gamma(t, x_1)$, $y_2 \in \Gamma(t, x_2)$ and $t \in I$.

(I) *Assume that $I = [0, T]$ and that $C(T) \subset C(0)$. Then the differential inclusion*

$$\begin{cases} u(t) \in C(t), \ \forall t \in I \\ -\dot{u}(t) \in N_{C(t)}u(t) + \Gamma(t, u(t)) \quad \text{a.e. } t \in I \end{cases}$$

has a unique absolutely continuous solution $u(\cdot)$ on $[0, T]$ satisfying $u(0) = u(T)$.

(II) *Assume that $I = \mathbb{R}$ and that the multimappings $C(\cdot)$ and $\Gamma(\cdot, x)$ (for all $x \in E$) are T-periodic. Then the differential inclusion*

$$\begin{cases} u(t) \in C(t), \ \forall t \in \mathbb{R} \\ -\dot{u}(t) \in N_{C(t)}u(t) + \Gamma(t, u(t)) \quad \text{a.e. } t \in \mathbb{R} \end{cases}$$

has a unique locally absolutely continuous solution on $\mathbb{R}$ which is T-periodic.

Proof. We follow the main lines of the arguments for the proof of Theorem 11.1.
(I) Consider $I = [0, T]$.

(A) *A uniqueness property*

First we show under condition (ii) that for any $a \in C(0)$ the differential inclusion

$$\begin{cases} -\dot{u}(t) \in N_{C(t)}u(t) + \Gamma(t, u(t)) & \text{a.e. } t \in I \\ u(0) = a \in C(0), \ u(t) \in C(t) \ \forall t \in I \end{cases} \tag{11.3}$$

admits a unique solution. Let u_1 and u_2 be two solutions of that differential inclusion whose existence is justified by Theorem 11.3 recalled above. Then, there exist two $L^1_E(I)$-mappings $h_1(\cdot)$ and $h_2(\cdot)$ such that for all $t \in I$

$$h_1(t) \in \Gamma(t, u_1(t)) \text{ and } h_2(t) \in \Gamma(t, u_2(t))$$

and for almost all $t \in I$

$$-\dot{u}_1(t) - h_1(t) \in (\kappa + \mu)\partial d_{C(t)}(u_1(t)) \subset N_{C(t)}u_1(t)$$

as well as
$$-\dot{u}_2(t) - h_2(t) \in (\kappa + \mu)\partial d_{C(t)}(u_2(t)) \subset N_{C(t)}u_2(t).$$

The *monotone* condition (ii) on Γ yields

$$\langle h_1(t) - h_2(t), u_1(t) - u_2(t) \rangle \geq \gamma \|u_1(t) - u_2(t)\|^2.$$

Moreover, the *hypomonotone* property of $N_{C(t)}$ ensures that for $\beta = (\mu + \kappa)/r$

$$\langle -\dot{u}_1(t) - h_1(t) - (-\dot{u}_2(t) - h_2(t)), u_1(t) - u_2(t) \rangle \geq -\beta \|u_1(t) - u_2(t)\|^2,$$

hence for $\alpha := \gamma - \beta > 0$

$$\langle \dot{u}_1(t) - \dot{u}_2(t), u_1(t) - u_2(t) \rangle \leq -\alpha \|u_1(t) - u_2(t)\|^2,$$

that is,
$$\frac{1}{2}\frac{d}{dt}\|u_1(t) - u_2(t)\|^2 \leq -\alpha \|u_1(t) - u_2(t)\|^2 \text{ for a.e. } t \in I. \tag{11.4}$$

This says in particular that $\frac{d}{dt}\|u_1(t) - u_2(t)\|^2 \leq 0$, which implies that we have $\|u_1(t) - u_2(t)\|^2 \leq 0$ according to the equality $\|u_1(0) - u_2(0)\| = 0$, hence $u_1 = u_2$. Thus, the uniqueness of the solution for the differential inclusion (11.3) follows.

(B) *The equality for the trajectory at times* 0 *and* T.

Let $a, b \in C(0)$ and denote according to (A) by u_a (resp. u_b) the unique solution of (11.3) with the initial condition a (resp. b) in $C(0)$. Then, applying inequality (11.4) by taking $u_1 = u_a$ and $u_2 = u_b$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_a(t) - u_b(t)\|^2 \leq -\alpha \|u_a(t) - u_b(t)\|^2 \text{ for a.e. } t \in I. \tag{11.5}$$

The Gronwall lemma (see Lemma 2.2) yields

$$\|u_a(T) - u_b(T)\|^2 \leq \|a - b\|^2 \exp\big(-2\alpha T\big),$$

hence $\|u_a(T) - u_b(T)\| \leq \big(\exp(-\alpha T)\big)\|a - b\|$. This and the inclusion $C(T) \subset C(0)$ tell us that the mapping $a \mapsto u_a(T)$ on $C(0)$ is from $C(0)$ into itself an ℓ-contraction with $\ell < 1$. Then it follows that this mapping has a unique fixed point $a_0 \in C(0)$, and the mapping u_{a_0} is an AC solution of the differential inclusion in (I) with $u_{a_0}(0) = u_{a_0}(T)$.

(C) *Uniqueness of solution of the differential inclusion in (I).*

Regarding the uniqueness of the solution, as in the proof of Theorem 11.1, we assume $U\colon I \to E$ to be any AC solution of the differential inclusion in (I) with $U(0) = U(T)$. Put $b_0 := U(0)$. Then $b_0 \in C(0)$ and U is an AC solution of the differential inclusion (11.1) for $a = b_0$. The uniqueness property in (A) assures us that $U = u_{b_0}$, which entails $b_0 = u_{b_0}(T)$. Keeping in mind that a_0 is the unique fixed point of the mapping $a \mapsto u_a(T)$ from $C(0)$ into $C(0)$ (as seen above), it ensues $b_0 = a_0$, thus $U = u_{a_0}$. This justifies the uniqueness of AC solution of the differential inclusion in (I) with equality of the trajectory at times 0 and T.

(II) The arguments for (II) of the theorem under consideration are the same ones for (II) of Theorem 11.1. □

Combining the above arguments with Theorem 9.2 also yields to the following theorem with mix perturbation.

Theorem 11.6. *Consider $I = [0, T]$ and the standard finite-dimensional Euclidean space $E = \mathbb{R}^e$. Let $C\colon I \rightrightarrows E$ be a multimapping with nonempty closed bounded convex values which is absolutely continuous with $v(\cdot)$ as a control function of absolute continuity. Let $\Gamma\colon I \times E \rightrightarrows E$ be a compact valued multimapping satisfying the properties* (i), (ii) *and* (iii) *in Theorem* 9.2. *Assume that there exists a real $\gamma > 0$ such that*

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq \gamma \|x_1 - x_2\|^2$$

for all $t \in I$ and all x_i, y_i with $y_i \in \Gamma(t, x_i)$, $i = 1, 2$. Assume also that $C(T) \subset C(0)$. Then, the differential inclusion

$$\begin{cases} u(t) \in C(t), \ \forall t \in I \\ -\dot{u}(t) \in N_{C(t)}u(t) + \Gamma(t, u(t)) \quad \text{a.e. } t \in I \end{cases}$$

has a unique absolutely continuous solution $u(\cdot)$ on $[0, T]$ satisfying $u(0) = u(T)$.

12. Towards asymptotic properties of the sweeping process

In the periodicity theorems in Section 11 the analysis is for a great part based on the assumption $C(T) \subset C(0)$. Since the multimapping $C(\cdot)$ therein is in addition Hausdorff-Pompeiu continuous (by absolute continuity), the inclusion $C(T) \subset C(0)$ can be reformulated as the condition

$$\operatorname*{Lim\,sup}_{t \uparrow T} C(t) \subset C(0).$$

In this section, via $\operatorname*{Lim\,sup}_{t\uparrow+\infty} C(t)$ and $\operatorname*{Lim\,inf}_{t\uparrow+\infty} C(t)$ we will study diverse asymptotic properties of solutions on $[T_0, +\infty[$. Given a topology τ on E, by ${}^\tau \operatorname*{Lim\,sup}_{t\uparrow+\infty} C(t)$ we mean the τ-sequential outer limit (or limit superior), that is, the set of τ-limits of sequential sequences $(x_n)_n$ with $x_n \in C(t_n)$ and $t_n \to +\infty$ as $n \to \infty$. When τ is the topology associated to the norm of E we will merely write $\operatorname*{Lim\,sup}_{t\uparrow+\infty} C(t)$. Similarly, ${}^\tau \operatorname*{Lim\,inf}_{t\uparrow+\infty} C(t)$ is defined as the set of $x \in E$ such that for every sequence $(t_n)_n$ in

$[0,+\infty[$ with $t_n \to +\infty$ there is a sequence $(x_n)_n$ in E which τ-converges to x with $x_n \in C(t_n)$ for n large enough. If τ is the topology associated to the norm of E we will merely write $\operatorname*{Lim\,inf}_{t\uparrow+\infty} C(t)$.

When $\operatorname*{Lim\,inf}_{t\uparrow+\infty} C(t) = \operatorname*{Lim\,sup}_{t\uparrow+\infty} C(t)$ (resp. $\operatorname*{Lim\,inf}_{t\to+\infty} C(t) = {}^w\operatorname*{Lim\,sup}_{t\uparrow+\infty} C(t)$) one says that $\big(C(t)\big)_t$ *Painlevé-Kuratowski* (resp. *Mosco*) converges to the common set given by the equality. Another useful convergence is the *Attouch-Wets* convergence saying that $\big(C(t)\big)_t$ Attouch-Wets converges to a set C^∞ in E if for each real $\rho > 0$

$$\operatorname{haus}_\rho\big(C(t), C^\infty\big) \to 0 \quad \text{as } t \to +\infty,$$

where $\operatorname{haus}_\rho(S_1,S_2) = \max\{\operatorname{exc}(S_1 \cap \rho S_2), \operatorname{exc}(S_2 \cap \rho S_1)\}$ for subsets S_1, S_2 of E (see Section 2 for the definition of the excess $\operatorname{exc}(\cdot,\cdot)$).

Given a multimapping $C\colon [T_0,+\infty[\rightrightarrows E$ defined on the unbounded interval $[T_0,+\infty[$ with nonempty values in the Hilbert space E and given a map $f\colon [T_0,+\infty[\times E \to E$, recall that by a global solution on $[T_0,+\infty[$ of the differential inclusion (resp. with initial condition $u(T_0) = a \in C(T_0)$)

$$-\dot u(t) \in N_{C(t)}u(t) + f(t,u(t)) \text{ a.e. } t \in [T_0,+\infty[,$$

one means any locally absolutely continuous mapping $u\colon [T_0,+\infty[\to E$ such that

$$-\dot u(t) \in N_{C(t)}u(t) + f(t,u(t)) \text{ a.e. } t \in [T_0,+\infty[$$

and $u(t) \in C(t)$ for all $t \in [T_0,+\infty[$ (resp. along with the equality $u(T_0) = a$).

Theorem 12.1. *Let E be a separable Hilbert space and let $r \in]0,+\infty]$.*
Let $C\colon [0,+\infty[\rightrightarrows E$ be a multimapping with nonempty r-prox-regular values which is absolutely continuous on $[0,+\infty[$ with $v(\cdot)$ as a control function of absolute continuity satisfying $\dot v \in L^1_{\mathbb{R}+}([0,+\infty[)$. Let $a \in C(0)$ and $f\colon [0,+\infty[\times E \to E$ be a mapping for which there are functions $\beta, \lambda\colon [0,+\infty[\to \mathbb{R}_+$ in $L^1_{\mathbb{R}_+}([0,+\infty[)$ such that

$$\|f(t,x) - f(t,y)\| \le \lambda(t)\|x-y\| \quad \textit{for all } t \in [0,+\infty[,\ x,y \in E$$

and $$\|f(t,x)\| \le \beta(t)(1+\|x\|) \quad \textit{for all } t \in [0,+\infty[,\ x \in \bigcup_{s\in[0,+\infty[} C(s).$$

Then the differential inclusion

$$\begin{cases} u(0) = a,\ u(t) \in C(t)\ \forall t \in [0,+\infty[\\ -\dot u(t) \in N_{C(t)}u(t) + f(t,u(t)) \ \textit{a.e. } t \in [0,+\infty[\end{cases} \tag{12.1}$$

admits a unique global solution $u\colon [0,+\infty[\to E$. Further, this solution is absolutely continuous on the whole interval $[0,+\infty[$ and for

$$\kappa := \|a\| + \left(\exp\left\{2\int_0^\infty \beta(s)\,ds\right\}\right)\int_0^\infty \big[2\beta(s)(1+\|a\|) + |\dot v(s)|\big]\,ds,$$

one has $\|u(\cdot)\|_\infty \le \kappa$ along with

$$\|\dot{u}(t)\| \le |\dot{v}(t)| + 2(1+\kappa)\beta(t) \quad a.e.\ t \in [0,+\infty[,$$

and one also has: $z := \lim\limits_{t\to+\infty} u(t)$ *exists in* E *and* $z \in \operatorname*{Lim\,inf}\limits_{t\uparrow+\infty} C(t)$.

Proof. The first assertion is a direct consequence of Theorem 3.3. Namely, there is a unique global solution $u\colon [0,\infty[\to E$ to the differential inclusion (12.1), and this solution is absolutely continuous on the whole interval $[0,+\infty[$ and satisfies the property $\|\dot{u}(t)\| \le \gamma(t)$ a.e. $t \in [0,+\infty[$ with $\gamma(\cdot) := (1+\ell)\beta(\cdot) + |\dot{v}(\cdot)|$ and with ℓ as in the statement of the theorem, so $\gamma \in L^1_{\mathbb{R}^+}([0,+\infty[)$. Therefore, writing

$$\|u(t) - u(s)\| = \left\| \int_s^t \dot{u}(\tau)\, d\tau \right\| \le \left| \int_s^t \gamma(\tau)\, d\tau \right|,$$

we see that $z := \lim_{t\to+\infty} u(t)$ exists in the complete space E by Cauchy property since $\gamma(\cdot)$ is Lebesgue integrable on $[0,+\infty[$. Finally, the inclusion $z \in \operatorname*{Lim\,inf}\limits_{t\uparrow+\infty} C(t)$ follows from the definition of the latter set since $u(t) \in C(t)$ for all $t \in [0,+\infty[$. $\square$

The next asymptotic result in Theorem 12.3 will utilize the following theorem which follows from Theorem 9.4 in Bernard-Thibault-Zlateva [7] (see also [57, Theorem 15.126]) and from Theorem 3.17 and Corollary 4.11 in Thibault-Zakaryan [58].

Theorem 12.2. *Let* $r \in]0,+\infty]$ *and* $(S_n)_n$ *be a sequence of* r*-prox-regular sets of a Hilbert space* E *Mosco converging (resp. Attouch-Wets converging) in* E *to a set* S *in* E. *Then the set* S *is* r*-prox-regular and the sequence of graphs* $(\operatorname{gph} N_{S_n})_n$ *Painlevé-Kuratowski converges in* $E \times E$ *to* $\operatorname{gph} N_S$.

Given $f\colon [0,+\infty[\times E \to E$ and denoting $f_t := f(t,\cdot)$, we say following Rockafellar-Wets [54, Definition 5.41] that $(f_t)_t$ *continuously converges* to a mapping $f_\infty\colon E \to E$ as $t \to +\infty$ provided that, for any $x \in E$ one has

$$f(t_n, x_n) \to f_\infty(x) \quad \text{as } n \to \infty,$$

for every sequence $(t_n)_n$ in $[0,+\infty[$ tending to $+\infty$ and for every sequence $(x_n)_n$ in E converging to x. This can be reformulated as $\lim\limits_{t\to\infty, y\to x} f_t(y) = f_\infty(x)$ for every element $x \in E$.

Theorem 12.3. *Let* E *be a separable Hilbert space and let* $r \in]0,+\infty]$.
Let $C\colon [0,+\infty[\rightrightarrows E$ *be a multimapping with nonempty* r*-prox-regular values which is locally absolutely continuous on* $[0,+\infty[$ *with* $v(\cdot)$ *as a control function of local absolute continuity satisfying* $\dot{v}$ *locally Lebesgue integrable on* $[0,+\infty[$.
Let $a \in C(0)$ *and* $f\colon [0,+\infty[\times E \to E$ *be a mapping for which there are functions* $\beta, \lambda\colon [0,+\infty[\to \mathbb{R}_+$ *locally Lebesgue integrable on* $[0,+\infty[$ *such that*

$$\|f(t,x) - f(t,y)\| \le \lambda(t)\|x-y\| \quad \textit{for all}\, t \in [0,+\infty[,\ x,y \in E$$

and $$\|f(t,x)\| \le \beta(t)(1+\|x\|) \quad \textit{for all}\, t \in [0,+\infty[,\ x \in \bigcup_{s\in[0,+\infty[} C(s).$$

(I) *Then the differential inclusion*

$$\begin{cases} u(0) = a,\ u(t) \in C(t)\ \forall t \in [0,+\infty[\\ -\dot{u}(t) \in N_{C(t)}u(t) + f(t,u(t))\ a.e.\ t \in [0,+\infty[\end{cases} \tag{12.2}$$

admits a unique locally absolutely continuous solution $u\colon [0,+\infty[\to E$ *on* $[0,+\infty[$.

(II) *Assume that* $(C(t))_t$ *Mosco converges (resp. Attouch-Wets converges) to a set* C^∞ *in* E, *that* $(f_t)_t$ *continuously converges to a mapping* $f_\infty\colon E \to E$ *and that the derivative* $\dot u$ *of the solution* u *of* (12.2) *is Lebesgue integrable on* $[0,+\infty[$.
Then $z := \lim\limits_{t\to+\infty} u(t)$ *exists in* E *and* $z \in C^\infty$ *along with* $0 \in N_{C^\infty} z + f_\infty(z)$.

Proof. The r-prox-regularity of C^∞ is due to Theorem 12.2.

(I) The existence and uniqueness of the locally absolutely continuous solution u on $[0,+\infty[$ of the differential inclusion (12.2) follows from Theorem 12.1.

(II) Now assume the assumptions in (II) are satisfied, so $\|\dot u(\cdot)\|$ is Lebesgue integrable on $[0,+\infty[$. Writing (as in the proof of Theorem 12.1)

$$\|u(t) - u(s)\| = \left\| \int_s^t \dot u(\tau)\, d\tau \right\| \le \left| \int_s^t \|\dot u(\tau)\|\, d\tau \right|,$$

we have that $z := \lim_{t\to+\infty} u(t)$ exists in the complete space E by Cauchy property according to the Lebesgue integrability of $\|\dot u(\cdot)\|$ on $[0,+\infty[$. Consider now the set $D := \{t \in [0,+\infty[\colon \frac{dv}{dt}(t) \text{ exists in } E\}$. The property $\int_0^\infty \|\dot u(t)\|\, dt < +\infty$ entails that there exists a sequence $(t_n)_n$ in D with $t_n \to +\infty$ as $n \to \infty$ such that $\dot u(t_n)$ exists,

$$\zeta_n := -\dot u(t_n) - f_{t_n}(u(t_n)) \in N_{C(t_n)} u(t_n) \quad \text{for all } n \in \mathbb{N},$$

and $\dot u(t_n) \to 0$ as $n \to +\infty$. Then we have $\zeta_n \to -f_\infty(z)$, so Theorem 12.2 yields

$$-f_\infty(z) \in N_{C^\infty} z, \ \text{ i.e., } 0 \in N_{C^\infty} z + f_\infty(z).$$

All the conclusions of the theorem are then justified. □

Corollary 12.4. *Let* E *be a separable Hilbert space and let* $r \in]0,+\infty]$.
Let $C\colon [0,+\infty[\rightrightarrows E$ *be a multimapping with nonempty* r*-prox-regular values which is absolutely continuous on* $[0,+\infty[$ *with* $v(\cdot)$ *as a control function of absolute continuity satisfying* $\dot v \in L^1([0,+\infty[)$, *and for which* $(C(t))_t$ *Mosco converges (resp. Attouch-Wets converges) to a set* C^∞ *in* E. *Let* $a \in C(0)$ *and let* $f\colon [0,+\infty[\times E \to E$ *be a mapping for which there are functions* $\beta, \lambda\colon [0,+\infty[\to \mathbb{R}_+$ *locally Lebesgue integrable on* $[0,+\infty[$ *such that*

$$\|f(t,x) - f(t,y)\| \le \lambda(t)\|x - y\| \quad \textit{for all } t \in [0,+\infty[,\ x, y \in E$$

and $$\|f(t,x)\| \le \beta(t)(1 + \|x\|) \quad \textit{for all } t \in [0,+\infty[,\ x \in \bigcup_{s\in[0,+\infty[} C(s).$$

Let $u\colon [0,+\infty[\to E$ *be, by Theorem* 3.3, *the locally absolutely continuous solution on* $[0,+\infty[$ *of the differential inclusion* (12.2). *Assume that* $t \mapsto f(t,u(t))$ *is Lebesgue integrable on* $[0,+\infty[$.
Then $z := \lim\limits_{t\to+\infty} u(t)$ *exists in* E *and* $z \in C^\infty$ *along with* $0 \in N_{C^\infty} z + f_\infty(z)$.

Proof. By Theorem 3.3 one has

$$\|\dot u(t)\| \le |\dot v(t)| + 2\|f(t,u(t))\| \ \text{ a.e. } t \in]0,+\infty[,$$

which ensures that $\dot{u}(\cdot)$ is Lebesgue integrable on $[0,+\infty[$ according to the assumptions of integrability on $[0,+\infty[$ of $\dot{v}(\cdot)$ and $f(\cdot,u(\cdot))$. The corollary then follows from Theorem 12.3. $\square$

Example 12.5. We illustrate Theorem 12.3 with the basic perturbed sweeping process

$$-\dot{u}(t) \in N_{C(t)}u(t) + \varepsilon(t)u(t), \text{ a.e. } t \in I, \tag{12.3}$$

where either $I = [0,T]$ or $I = [0,+\infty[$, the multimapping $C\colon I \rightrightarrows E$ from I into the separable Hilbert space E is locally absolutely continuous on I with weakly compact convex values, and $\varepsilon(\cdot)$ is either Lebesgue integrable on $[0,T]$ or locally Lebesgue integrable on $[0,+\infty[$. Assume that $C(\cdot)$ is bounded, that is, for some real $M > 0$

$$\sup\{\|x\| : x \in \bigcup_{t\in I} C(t)\} \le M.$$

First, regarding the existence/uniqueness of solution, instead of evoking the general result in Theorem 3.3, it is worth observing that (12.3) can be reduced to an unperturbed sweeping process. For that purpose, let us define, for $t \in I$

$$\varphi(t) := \int_0^t \varepsilon(s)\,ds \quad \text{and} \quad D(t) := e^{\varphi(t)}C(t).$$

Then φ is locally absolutely continuous from I into $\mathbb{R}$ and the multimapping D is locally absolutely continuous on I. Indeed, fixing any $\theta \in I$ and denoting $\beta(\theta) := \max_{\tau\in[0,\theta]} e^{\varphi(\tau)}$, we have for $0 \le s \le t \le \theta$

$$\begin{aligned}
\operatorname{haus}(D(t), D(s)) &= \operatorname{haus}(e^{\varphi(t)}C(t), e^{\varphi(s)}C(s)) \\
&\le \beta(\theta)\operatorname{haus}\big(C(t), C(s)\big) + M|e^{\varphi(t)} - e^{\varphi(s)}| \\
&\le \beta(\theta)\operatorname{haus}\big(C(t), C(s)\big) + M\int_s^t |\frac{d\varphi}{dt}(\tau)|e^{\varphi(\tau)}\,d\tau.
\end{aligned}$$

Thus, for $s \le t$ in $[0,\theta]$

$$\operatorname{haus}(D(t), D(s)) \le \beta(\theta)\operatorname{haus}\big(C(t), C(s)\big) + g(t) - g(s),$$

where
$$g(t) = M\int_0^t |\varepsilon(\tau)|e^{\varphi(\tau)}\,d\tau,$$

so g is a non decreasing locally absolutely continuous function from I into $[0,\infty[$ with $\dot{g}(t) = M\,|\varepsilon(t)|e^{\varphi(t)}$ for a.e. $t \in I$. Now, for each $x \in D(0) = C(0)$ let us consider the sweeping process associated with the locally absolutely continuous multimapping $D(\cdot)$

$$\begin{cases} -\dot{v}(t) \in N_{D(t)}v(t), \text{ a.e. } t \in I \\ v(0) = x \in D(0) = C(0). \end{cases}$$

Then, for every $x \in \ \in D(0) = C(0)$, let v_x be a locally absolutely continuous solution to the latter differential inclusion, otherwise stated

$$\begin{cases} -\dot{v}_x(t) \in N_{D(t)} v_x(t), \text{ a.e. } t \in I \\ v_x(0) = x \in D(0) = C(0). \end{cases}$$

Putting
$$u_x(t) = e^{-\varphi(t)} v_x(t) \tag{12.4}$$

we see that u_x is a locally absolutely continuous solution to

$$\begin{cases} -\dot{u}_x(t) \in N_{C(t)} u_x(t) + \varepsilon(t) u_x(t), \text{ a.e. } t \in I \\ u_x(0) = x \in C(0) \end{cases}$$

by noting that for a.e. $t \in I$

$$\frac{dv_x}{dt}(t) = e^{\varphi(t)}\left(\frac{du_x}{dt}(t) + \varepsilon(t) u_x(t)\right)$$

and
$$-\frac{dv_x}{dt}(t) \in N_{D(t)} v_x(t) = N_{C(t)} u_x(t)$$

because we are dealing with cones. Reversing the arguments, for u_x and v_x linked as in (12.4), we have shown that u_x solves the differential inclusion

$$\begin{cases} -\dot{u}(t) \in N_{C(t)} u(t) + \varepsilon(t) u(t), \text{ a.e. } t \in I \\ u(0) = x \in C(0) \end{cases}$$

if and only if v_x solves

$$\begin{cases} -\dot{v}(t) \in N_{D(t)} v(t), \text{ a.e. } t \in I \\ v(0) = x \in D(0) = C(0). \end{cases}$$

Consequences

• Equality at initial and final times

If $C(0) = C(T)$ and $\int_0^T \varepsilon(t)\, dt > 0$, then there is a unique absolutely continuous solution on $[0, T]$ to the differential inclusion

$$-\dot{u}(t) \in N_{C(t)} u(t) + \varepsilon(t) u(t), \text{ a.e. } t \in [0, T]$$

satisfying the equality $u(0) = u(T)$.

Proof. Indeed, using the monotonicity of the normal cone $N_{D(t)}$, with the above notation we have the estimate

$$\|v_x(t) - v_y(t)\| \leq \|x - y\| \quad \text{for all } t \in [0, T] \text{ and } x, y \in D(0) = C(0).$$

From this and the equality $u_x(t) = e^{-\varphi(t)} v_x(t)$ in (12.4) we deduce that

$$\|u_x(T) - u_y(T)\| \le e^{-\varphi(T)} \|x - y\|$$

for all $x, y \in C(0)$. Since $u_x(T) \in C(T) = C(0)$ and $\varphi(T) > 0$, the mapping $U: x \mapsto u_x(T)$ from $C(0)$ into itself is a strict contraction on $C(0)$, so U admits a fixed point $\overline{x}$, i.e., $U(\overline{x}) = \overline{x}$. Clearly, $u_{\overline{x}}(0) = \overline{x} = u_{\overline{x}}(T)$, so $u_{\overline{x}}$ is the desired solution. Finally, the uniqueness is easily verified. □

- **Periodicity property**

If $C(\cdot)$ and $\varepsilon(\cdot)$ are T-periodic with $\int_0^T \varepsilon(t)\,dt > 0$, then there is a unique locally absolutely continuous solution to the differential inclusion

$$-\dot{u}(t) \in N_{C(t)} u(t) + \varepsilon(t) u(t), \text{ a.e. } t \in [0, \infty[$$

which is T-periodic.

Proof. The proof lies on the main tool developed above along with the argument given in the proof of Theorem 11.1. □

- **Asymptotic property**

If $\big(C(t)\big)_t$ Mosco converges to C_∞ as $t \to +\infty$ and if $\lim_{t\to\infty} \varepsilon(t) = \varepsilon_\infty \in \mathbb{R}$ and the derivative $\dot{u}$ of the solution u of

$$\begin{cases} -\dot{u}(t) \in N_C u(t) + \varepsilon(t) u(t), \text{ a.e. } t \in [0, \infty[\\ u(0) = a \in C \end{cases}$$

is Lebesgue integrable on $[0, \infty[$, then $\lim_{t\to\infty} u(t) = z \in C$ with $0 \in N_{C_\infty} z + \varepsilon_\infty z$.

Proof. The property follows from Theorem 12.3. □

In (II) of Theorem 12.3 the curve $u(\cdot)$ is assumed to be of finite length, that is, $\int_0^\infty \|\dot{u}(t)\|\,dt < \infty$. We consider next the case of infinite length, or in other words, $\int_0^\infty \|\dot{u}(t)\|\,dt = +\infty$. A large part of the analysis will be based on stronly convex sets, a concept which was be probably firstly employed in variational analysis by Polyak [52] and by Levitin and Polyak [39]. Recall first that, given a real $r > 0$, a subset S of a Hilbert space E is *r-strongly convex* (see, e.g., [2, 6, 29, 32, 37, 39, 50, 51, 52, 65] and references therein) provided that it is the intersection of a collection of closed balls with the common radius r, i.e., $S = \bigcap_{j\in J} B[x_j, r]$ for some family $(x_i)_{j\in J}$ of points in E. Such a set is clearly convex and bounded; it was called a convex set with *curvature lower bounded by* $1/r$ in Moreau [46]. A remarkable characterization is the following Proposition 12.6 (see [2, 32, 51, 65]); we mention that Proposition 2 in Moreau [46] also contains the fact that the r-strong convexity implies the property in Proposition 12.6. The result in Proposition 12.6 can also be found in [57, Theorem 16.11].

Proposition 12.6. *Given a real $r > 0$, a nonempty closed set S of a Hilbert space E is r-strongly convex if and only if for any $x, y \in S$ and any $\zeta \in N_S x$ with $\|\zeta\| = 1$ one has*

$$\langle \zeta, y - x \rangle \le -\frac{1}{2r} \|x - y\|^2.$$

Using this proposition we have the following asymptotic result in the line of Moreau [46, Proposition 3].

Theorem 12.7. *Let E be a separable Hilbert space and let $r\in]0,+\infty[$.*
Let $C\colon[0,+\infty[\rightrightarrows E$ be a multimapping with nonempty r-strongly convex values in E.
Let $f\colon [0,+\infty[\times E\to E$ be a mapping for which there is a function $\gamma\colon [0,+\infty[\to\mathbb{R}_+$ such that for all $t\in[0,+\infty[$ and $x,y\in\bigcup_{s\in[0,+\infty[}C(s)$

$$\langle f(t,x)-f(t,y),x-y\rangle\geq\gamma(t)\|x-y\|^2.$$

Assume that the differential inclusion

$$\begin{cases} u(t)\in C(t)\ \forall t\in[0,+\infty[\\ -\dot u(t)\in N_{C(t)}u(t)+f(t,u(t))\ a.e.\ t\in[0,+\infty[\end{cases}\tag{12.5}$$

admits a locally absolutely continuous solution $u\colon[0,+\infty[\to E$ on $[0,+\infty[$ such that $\int_0^\infty\|\dot u(t)\|\,dt=+\infty$ along with

$$\|f(t,u(t))\|\leq r\gamma(t)\quad a.e.\ t\in[0,+\infty[.$$

Then, for any other locally absolutely continuous solution $U\colon[0,+\infty[\to E$ of (12.5) on $[0,+\infty[$ satisfying $\|f(t,U(t))\|\leq r\gamma(t)$ a.e. $t\in[0,+\infty[$, one has

$$\lim_{t\to+\infty}\|U(t)-u(t)\|=0.$$

Proof. Let $u(\cdot)$ and $U(\cdot)$ be as in the statement. For a.e. $t\in[0,+\infty[$ we have by Proposition 12.6 that

$$\langle-\dot u(t)-f(t,u(t)),U(t)-u(t)\rangle\leq-\frac{1}{2r}(\|\dot u(t)+f(t,u(t))\|)\|U(t)-u(t)\|^2,$$

which gives

$$\begin{aligned}&\langle-\dot u(t),U(t)-u(t)\rangle\\ &=\langle-\dot u(t)-f(t,u(t)),U(t)-u(t)\rangle+\langle f(t,u(t)),U(t)-u(t)\rangle\\ &\leq\frac{-1}{2r}(\|\dot u(t)+f(t,u(t))\|)\|U(t)-u(t)\|^2+\langle f(t,u(t)),U(t)-u(t)\rangle.\end{aligned}$$

From this inequality and the similar one for $\langle-\dot U(t),u(t)-U(t)\rangle$ it ensues that for a.e. $t\in[0,+\infty[$

$$\begin{aligned}&\langle\dot U(t)-\dot u(t),U(t)-u(t)\rangle\\ &\leq\tfrac{-1}{2r}(\|\dot u(t)\|+\|\dot U(t)\|)\|U(t)-u(t)\|^2+\tfrac{\|f(t,u(t))\|+\|f(t,U(t))\|}{2r}\|U(t)-u(t)\|^2\\ &\quad-\langle f(t,U(t))-f(t,u(t)),U(t)-u(t)\rangle,\end{aligned}$$

hence we have

$$\begin{aligned}\langle\dot U(t)-\dot u(t),U(t)-u(t)\rangle\leq{}&\tfrac{-1}{2r}\big(\|\dot u(t)\|+\|\dot U(t)\|\big)\|U(t)-u(t)\|^2\\ &-\left(\gamma(t)-\frac{\|f(t,u(t))\|+\|f(t,U(t))\|}{2r}\right)\|U(t)-u(t)\|^2.\end{aligned}$$

Then according to the inequalities $\|f(t,u(t))\| \le r\gamma(t)$ and $\|f(t,U(t))\| \le r\gamma(t)$, we obtain

$$\langle \dot U(t) - \dot u(t), U(t) - u(t)\rangle \le \frac{-1}{2r}(\|\dot u(t)\| + \|\dot U(t)\|)\|U(t) - u(t)\|^2,$$

which is equivalent to

$$\frac{d}{dt}\|U(t) - u(t)\|^2 \le -\frac{1}{r}(\|\dot u(t)\| + \|\dot U(t)\|)\|U(t) - u(t)\|^2.$$

By Gronwall lemma we obtain for all $s \le t$ in $[0, +\infty[$

$$\begin{aligned}\|U(t) - u(t)\|^2 &\le \|U(s) - u(s)\|^2 \exp\left(-\frac{1}{2r}\left[\int_s^t \|\dot u(\tau)\|\, d\tau + \int_s^t \|\dot U(\tau)\|\, d\tau\right]\right)\\ &\le \|U(s) - u(s)\|^2 \exp\left(-\frac{1}{2r}\int_s^t \|\dot u(\tau)\|\, d\tau\right).\end{aligned}$$

Since $\int_s^t \|\dot u(\tau)\|\, d\tau \to +\infty$ as $t \to +\infty$ (by assumption), it results that

$$\|U(t) - u(t)\|^2 \to 0 \quad \text{as } t \to +\infty,$$

which finishes the proof of the theorem. □

Corollary 12.8. *Let E be a separable Hilbert space and let $r \in]0, +\infty[$. Let $C\colon [0, +\infty[\rightrightarrows E$ be a multimapping with nonempty r-strongly convex values in E. Let $f\colon [0, +\infty[\times E \to E$ be a map for which there are functions $\beta, \gamma\colon [0, +\infty[\to \mathbb{R}_+$ with $\beta(t) \le r\gamma(t)$ for all $t \in [0, +\infty[$ such that we have for all $t \in [0, +\infty[$ and $x, y \in \bigcup_{s\in[0,+\infty[} C(s)$*

$$\|f(t,x)\| \le \beta(t) \quad \textit{and} \quad \langle f(t,x) - f(t,y), x - y\rangle \ge \gamma(t)\|x - y\|^2.$$

If the differential inclusion (12.5) *above admits a locally absolutely continuous solution $u\colon [0, +\infty[\to E$ whose limit exists in E as $t \to +\infty$, then any other locally absolutely continuous solution $U\colon [0, +\infty[\to E$ also admits a limit as $t \to +\infty$.*

Proof. Let U be any other locally absolutely continuous solution.

If $\int_0^\infty \|\dot U(t)\|\, dt < +\infty$, then the inequality

$$\|U(t) - U(s)\| \le \left|\int_s^t \|\dot U(\tau)\|\, d\tau\right|$$

ensures that the limit $\lim_{t\to+\infty} U(t)$ exists in E by Cauchy property. So suppose that $\int_0^\infty \|\dot U(t)\|\, dt = +\infty$. In this case, it is easily seen that the assumptions of the corollary entail those in Theorem 12.7 for $U(\cdot)$. Since $u(\cdot)$ is a solution by assumption, Theorem 12.7 says that $\|u(t) - U(t)\| \to 0$ as $t \to +\infty$. This and the assumption of the existence of $\lim\limits_{t\to+\infty} u(t)$ imply that $\lim\limits_{t\to+\infty} U(t)$ exists in E. □

Taking $f(\cdot,\cdot) \equiv 0$ and $\gamma(\cdot) \equiv 0$ in Theorem 12.7 we obtain the following second corollary.

Corollary 12.9. (Moreau [46]) *Let E be a separable Hilbert space. Let $r\in]0,+\infty[$ and $C\colon [0,+\infty[\rightrightarrows E$ be a multimapping with nonempty r-strongly convex values in E. Assume that the differential inclusion*

$$\begin{cases} u(t) \in C(t) \ \forall t \in [0,+\infty[\\ -\dot{u}(t) \in N_{C(t)}u(t) \ a.e.\ t \in [0,+\infty[\end{cases} \tag{12.6}$$

admits a locally absolutely continuous solution $u\colon [0,+\infty[\to E$ on $[0,+\infty[$ with infinite length, that is, $\int_0^\infty \|\dot{u}(t)\|\,dt = +\infty$ (which holds in particular if $\lim_{t\to+\infty} u(t)$ does not exist in E). Then, for any other locally absolutely continuous solution $U\colon [0,+\infty[\to E$ of (12.6) on $[0,+\infty[$ one has

$$\lim_{t\to+\infty} \|U(t) - u(t)\| = 0.$$

Taking again $f(\cdot,\cdot) \equiv 0$ and $\gamma(\cdot) \equiv 0$, Corollary 12.8 directly yields the following.

Corollary 12.10. (Moreau [46]) *Let E be a separable Hilbert space. Let $r\in]0,+\infty[$ and $C\colon [0,+\infty[\rightrightarrows E$ be a multimapping with nonempty r-strongly convex values in E. If the differential inclusion (12.6) above admits a locally absolutely continuous solution $u\colon [0,+\infty[\to E$ whose limit exists in E as $t\to+\infty$, then any other locally absolutely continuous solution $U\colon [0,+\infty[\to E$ also admits a limit as $t\to+\infty$.*

For other related results on the asymptotic behavior of evolution problems, we refer to Attouch, Cabot and Czarnecki [3] and to Furuya, Miashiba and Kenmochi [31].

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