Journal of Convex Analysis
Volume 30 (2023), No. 3, 851–885

Stochastic Elliptic Inverse Problems. Solvability, Convergence Rates, Discretization, and Applications

Marc Dambrine
Université de Pau et des Pays de l'Adour,
Lab. de Mathématiques et de leurs Applications, Pau, France
marc.dambrine@univ-pau.fr

Akhtar A. Khan
School of Math. Sciences, Rochester Institute of Technology, Rochester, New York, U.S.A.
aaksma@rit.edu

Miguel Sama
Dep. de Matemática Aplicada, Universidad Nacional de Educación a Distancia, Madrid, Spain
msama@ind.uned.es

Hans-Jörg Starkloff
Institute of Stochastics, Technische Universität Bergakademie Freiberg, Germany
hjstark@math.tu-freiberg.de

Dedicated to Roger J-B Wets on the occasion of his 85th birthday.

Received: July 19, 2022
Accepted: April 26, 2023

Motivated by the necessity to identify stochastic parameters in a wide range of stochastic partial differential equations, an abstract inversion framework is designed. The stochastic inverse problem is studied in a stochastic optimization framework. The essential properties of the solution map are derived and used to prove the solvability of the stochastic optimization problems. Novel convergence rates for the stochastic inverse problem are presented in the abstract formulation without requiring the so-called smallness condition. Under the assumption of finite-dimensional noise, the stochastic inverse problem is parametrized and solved by using the Stochastic Galerkin discretization scheme. The developed framework is applied to estimate stochastic Lamé parameters in the system of linear elasticity. We present numerical results that are quite encouraging and show the feasibility and efficacy of the developed framework.

Keywords: Stochastic inverse problems, partial differential equations with random data, stochastic Galerkin method, regularization, finite-dimensional noise, convergence rates.

2000 Mathematics Subject Classification: 35R30, 49N45, 65J20, 65J22, 65M30.

1. Introduction

The overarching objective of this work is to develop an abstract framework for stochastic inverse problems of identifying random parameters in stochastic elliptic variational problems. One of the primary motivations for devising the abstract inversion approach is to study the stochastic inverse problem of identifying random Lamé parameters describing the elastic properties of an isotropic heterogeneous elastic body or a membrane under a stochastic load.

ISSN 0944-6532 / $ 2.50

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a complete probability space, where Ω is a nonempty set of elementary events, $\mathbb{F}$ is a σ-algebra of subsets of Ω, and $\mathbb{P} : \mathbb{F} \to [0,1]$ is a probability measure. Let $D \subset \mathbb{R}^2$ be a bounded domain and let ∂D be its sufficiently smooth boundary $\partial D = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$. The following stochastic linear elasticity system models the displacement in the elastic body induced by the stochastic load f and the stochastic boundary traction h:

$$-\nabla \cdot [2\mu(\omega,x)\epsilon_u + \lambda(\omega,x)\mathrm{tr}\,(\epsilon_u)\, I] = f(\omega,x), \text{ in } D, \tag{1a}$$

$$u(\omega,x) = 0, \text{ on } \Gamma_1, \tag{1b}$$

$$[2\mu(\omega,x)\epsilon_u + \lambda(\omega,x)\mathrm{tr}\,(\epsilon_u)\, I]\, n = h(\omega,x), \text{ on } \Gamma_2. \tag{1c}$$

Here the random field $u(\omega,x)$ is the displacement vector, I is the identity map, n is the outward-pointing unit normal to ∂D, and the random fields $\mu : \Omega \times D \to \mathbb{R}$ and $\lambda : \Omega \times D \to \mathbb{R}$ are the Lamé parameters. Denoting by ∇u, the gradient of the vector-valued random field u, the linearized strain tensor

$$\epsilon_u := \frac{1}{2}\left(\nabla u(\omega,x) + \nabla u(\omega,x)^\top\right),$$

is the local deformation of the elastic body, whereas $\mathrm{tr}(\epsilon_u)$ is the trace of ϵ_u. We note that the derivatives in the elasticity system are with respect to x. The stress tensor σ measures the response of the elastic object described by the strain. In the context of (1), our focus will be on the stochastic inverse problem of estimating the stochastic Lamé parameters $\mu(\omega,x)$ and $\lambda(\omega,x)$ from some statistical information concerning the corresponding solution u.

Given the probability space $(\Omega, \mathbb{F}, \mathbb{P})$, a bounded domain $D \subset \mathbb{R}^n$ with smooth boundary ∂D, and two random fields $a : \Omega \times D \to \mathbb{R}$ and $f : \Omega \times D \to \mathbb{R}$, the prototypical stochastic partial differential equation (PDE) seeks a random field $u : \Omega \times D \to \mathbb{R}$ such that, almost surely,

$$-\nabla \cdot (a(\omega,x)\nabla u(\omega,x)) = f(\omega,x), \text{ in } D, \quad u(\omega,x) = 0, \text{ on } \partial D. \tag{2}$$

Over the past two decades, the above stochastic PDE has been widely studied. An extensively used approach for (2) is the sampling-based Monte Carlo Finite Element Method (MC-FE), where a finite element discretization scheme is used for the deterministic part. The key idea of the MC-FE method is to generate s realizations of the random variable, and for each realization, ω_j, compute a solution $u_j = u(\omega_j, x)$ of (2). From the solution samples, the mean and the variance of the solution $u(\omega,x)$ can be computed. The MC-FE approach is stable, readily implementable, and does not impose regularity constraints on the data. However, a glaring pitfall is slow convergence, with asymptotic order $\frac{1}{\sqrt{s}}$, requiring a large sample size for an acceptable approximation. On the other hand, the stochastic Galerkin scheme discretizes the deterministic and stochastic components simultaneously and is more involved from a computational standpoint but results in a stable and fast numerical procedure, see, e.g., [39].

Different viewpoints can be used to interpret the identification of stochastic parameters in PDEs. Since a usual way of defining a stochastic model consists in defining the distributions of the random variables, the direct problem consists of computing the distribution of the state given the joint distribution of the parameters and source term. For the inverse problem, one possibility is to estimate the realization of the coefficient based on the knowledge of the realization of the source term and the data. Another possibility for the inverse problem consists of computing the distribution of the coefficient given the joint distribution of the source term and the data. The nonlinear stochastic inverse problem associated with (2) seeks an approximation of the random parameter $a(\omega, x)$ from some statistical information of the solution $u(\omega, x)$. We remark that besides the inverse problem of identifying $a(\omega, x)$, another inverse problem related to (2) is the linear stochastic inverse problem of estimating the right-hand side term $f(\omega, x)$ from a measurement of the solution $u(\omega, x)$, see, e.g. [34]. If f is the control variable, this linear inverse problem becomes the widely studied stochastic optimal control problem, see, e.g., [10]. There have also been studies on the stochastic inverse problem of estimating boundary data in PDEs similar to (2) from some probabilistic knowledge of the solution $u(\omega, x)$, see, e.g., [18].

Much research has recently been conducted on the vibrant and expanding domains of stochastic PDEs and associated inverse and control problems. It is partly driven by the fact that in recent years an abundance of applied models has been studied as PDEs with stochastic parameters, where these parameters characterize some physical features of the model. The typical examples include the stochastic diffusion coefficient in the second-order PDEs, the stochastic rigidity coefficient in certain fourth-order PDEs, and stochastic Lamé parameters in linear elasticity. Numerous causes have led to incorporating stochasticity in these parameters. For instance, these parameters are estimated based on experiments that involve noise, and the natural uncertainty can be compensated by treating them as random variables. In many engineering applications, random parameters are also used to describe the influence of fluctuations in system parameters on overall performance, see, e.g., [4]. It is also worth noting that the usual deterministic approaches, such as the Tikhonov regularization, can be derived from stochastic models; see the celebrated work by Franklin [20] on this viewpoint. Another relevant setting often described in recent works is identifying the optimal cooling of a super-computing facility where the heat source from the supercomputers may change over time, and the heat conduction properties of the machines may not be entirely known. Since the material properties or boundary conditions involve uncertainty, it is reasonable to consider identification problems where the material parameters or the boundary conditions in the PDEs are random fields, see, e.g., [42].

For inverse problems involving uncertainty, Bayesian approaches have been dominant. A Bayesian approach, however, is impractical if the underlying direct problem is computationally expensive, as is often the case with identification in PDEs. It is because sampling through Markov Chain Monte Carlo requires solving the underlying PDE iteratively, and the discrete parameter space can result in a high-dimensional sampling density function. There has been extensive exploration of an alternative methodology called the variational approach to circumvent the deficiencies of the Bayesian approach. Variational approaches solve the stochastic inverse problem by posing a stochastic optimization problem whose solutions provide statistical information about the unknown parameter. In a variational approach, the stochastic inverse problem can be parameterized to yield a high-dimensional deterministic problem that can be solved with traditional methods readily available for deterministic inverse problems.

Before describing the main contributions of this work, we briefly discuss some of the available related results. In one of the earlier contributions on the subject, in [3], the authors studied the inverse heat conduction problems in the presence of uncertainty in input data. They extended the iterative regularization technique from the deterministic framework to the stochastic setting. They developed an optimization-based approach for the inverse problem and computed the gradient of the output least-squares (OLS)-type objective functional by the adjoint approach. They used the spectral stochastic finite element method to solve the parametric stochastic direct, sensitivity, and adjoint subproblems. In a related paper, [61], the focus is on a semi-linear stochastic transport equation with three unknown parameters. The authors showed that the initial displacement, the terminal state, and the random term in diffusion are uniquely determined by the state on the partial boundary. They also prove the Lipschitz stability of the inverse problem. In [19], the authors proposed a stochastic optimization formulation to understand the impact of the input parameter uncertainties on the electrocardiography imaging problem. Assuming that the material parameters are uncertain, the focus of [18] is on the inverse problem of identifying boundary conditions on an inaccessible part of a solid body boundary from the knowledge of over-specified data given on an accessible part of this boundary. The source of the stochasticity is the material properties.

In [40], the authors developed a novel approach for inverse problems using a Bayesian inference method. In [41], the authors introduced the Stochastic Newton method in which Markov Chain Monte Carlo is accelerated by sampling from a density built upon a local Gaussian approximation based on the local gradient and the Hessian of the log posterior density. The Bayesian solution of an inverse problem illustrates the stochastic Newton method. In [47], the authors studied the inverse problem to identify a non-Gaussian positive matrix-valued random field in a stochastic elliptic boundary value problem. Soize [52] studied the identification of Bayesian posteriors for the random coefficients of the high-dimension polynomial chaos expansions of non-Gaussian tensor-valued random fields using partial and limited experimental data. Sternfels and Earls [53] developed a probabilistic framework that identifies model parameters in PDE-based models from noisy response measurements. The key ingredients are a Bayesian approach, an efficient sequential Monte Carlo sampling scheme, and adaptive reduced-order models. Wang and Matthies [59] studied the parameter identification with epistemic uncertainty and developed a novel interval theory-based inverse analysis method. Some of the related results are available in [1, 5–7, 9, 10, 14–17, 19, 28, 35, 36, 40, 41, 43–46, 50, 54–56, 62] and the references therein. Another fruitful research direction for stochastic inverse problems that rendered encouraging results in recent years is the stochastic approximation, see, e.g., [24,27,31]. For applications of stochastic approximation, see, e.g., [12,51,63,64].

The main contributions of this work are as follows:

(i) Motivated by the fact that most of the research on stochastic inverse problems so far has concentrated around the diffusion equation (cf. (2)), we develop a general inversion approach for an abstract stochastic variational problem in this work.

It is shown that the abstract framework applies to many important stochastic PDEs. The developed approach is inspired by an analogous framework that has been quite fruitful for deterministic inverse problems, see [21]. Although the critical ideas of the approach are akin to the overall strategy of [21], a transition from deterministic to stochastic inverse problems demands careful analysis and significantly more complicated computational methods. The proposed energy least-squares (ELS) functional extends a convex functional studied in [29] for the stochastic diffusion equation. It also generalizes a convex functional studied in [22] for an inverse problem associated with certain fourth-order stochastic PDEs.

(ii) We derive a second-order derivative characterization, which is crucial in developing Hessian-based numerical schemes for inverse problems. The second-order characterization is new, even for the inverse problems of estimating the random diffusivity coefficient in scalar PDEs. We also develop a pathwise approach for solving stochastic inverse problems.

(iii) For the first time, in the context of stochastic inverse problems, we derive the convergence rates for the ELS objective. We emphasize that even for the deterministic inverse problems, the convergence rates for the ELS are known only for the scalar PDE, and since the given results are in an abstract framework, they apply to a broad class of PDEs.

1.1. Notation

We conclude this section by recalling some function spaces. Given the domain D, for $1 \leq p < \infty$, by $L^p(D)$, we denote the space of pth Lebesgue integrable (equivalence classes of) functions. The space $L^\infty(D)$ consists of (equivalence classes of) measurable functions that are bounded almost everywhere (a.e.) on D. We also recall that the Sobolev spaces are given by

$$H^1(D) = \left\{y \in L^2(D), \;\; \partial_{x_i} y \in L^2(D), \; i = 1, \dots, n\right\},$$
$$H_0^1(D) = \left\{y \in H^1(D), \;\; y|_{\partial D} = 0\right\},$$

and $H^{-1}(D) = (H_0^1(D))^*$ is the topological dual of $H_0^1(D)$. For $m \in \mathbb{N}$, higher-order Sobolev spaces $H^m(D)$ consist of $L^2(D)$ functions with all partial derivatives up to order m reside in $L^2(D)$. In accordance, by $H_0^m(D)$, we represent the functions y in $H^m(D)$ whose boundary traces of the derivatives of up to order less than m are zero.

Bochner spaces of random variables render a useful framework for variational problems arising from stochastic partial differential equations, see, e.g., [39]. Given a real Banach space X, a probability space $(\Omega, \mathbb{F}, \mathbb{P})$, and $p \in [1, \infty)$, the Bochner space $L^p(\Omega; X)$ contains (equivalence classes of) Bochner integrable functions $u : \Omega \to X$ with finite p-th moment, that is,

$$\|u\|_{L^p(\Omega;X)} := \left(\int_\Omega \|u(\omega)\|_X^p d\mathbb{P}(\omega)\right)^{1/p} =: \mathbb{E}\left[\|u\|_X^p\right]^{1/p} < \infty,$$

where $\mathbb{E}$ is the expectation operator. If $p = \infty$, then $L^\infty(\Omega; X)$ is the space of (equivalence classes of) Bochner measurable functions $u : \Omega \to X$ satisfying the condition $\operatorname{ess\,sup}_{\omega\in\Omega} \|u(\omega)\|_X < \infty$, see, [11].

The representation of the random fields by a finite number of mutually independent random variables is a vital aspect of the study of stochastic inverse problems. Random fields that can be expressed as a finite number of random variables are known as finite-dimensional noise. To be specific, a function $v \in L^2(\Omega; L^2(D))$ is a finite-dimensional noise, if it has the form $v(x, \xi(\omega))$ for $x \in D$ and $\omega \in \Omega$, where $\xi_k : \Omega \mapsto \Gamma_k$, for $k = 1, \dots, M$, with $M < \infty$, are real-valued random variables, $\xi = (\xi_1, \xi_2, \dots, \xi_M) : \Omega \mapsto \Gamma \subset \mathbb{R}^M$ and $\Gamma := \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_M$. For details, see [39].

The main advantage of dealing with finite-dimensional noise random fields is that a change of variables can be performed for evaluating expectations. To elucidate, let $v(x, \xi)$ be a finite-dimensional noise random field, and let ρ be the joint density of ξ. Then,

$$\|v\|^2_{L^2(\Omega;L^2(D))} = \mathbb{E}\left[\|v\|^2_{L^2(D)}\right] = \int_\Gamma \rho(y)\|v(y,\cdot)\|^2_{L^2(D)} dy.$$

By defining $y_k := \xi_k(\omega)$ and taking $y = (y_1, y_2, \dots, y_M)$, we relate the finite-dimensional noise random field $v(x, \xi)$ with a function $v(x, y)$ in the weighted L^2 space

$$L^2_\rho(\Gamma; L^2(D)) := \left\{ v : D \times \Gamma \to \mathbb{R} : \int_\Gamma \rho(y)\|v(\cdot, y)\|^2_{L^2(D)} dy < \infty \right\}.$$

2. Stochastic elliptic variational problems

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a complete probability space, let B be a Banach space, let V be a separable Hilbert space with V^* as its topological dual, and let $K \subset B$ be nonempty. Given the parameter space B, the admissible parameters $\ell : \Omega \times B \to \mathbb{R}$ are such that for each $\omega \in \Omega$, we have $\ell(\omega, \cdot) \in K$. This means that for each $\omega \in \Omega$, $\ell(\omega, \cdot)$ belongs to a function space satisfying certain feasibility criteria. Let $m : \Omega \times V \to \mathbb{R}$ be such that for every $\omega \in \Omega$, we have $m(\omega, \cdot) \in V^*$. This means that for each $\omega \in \Omega$, $m(\omega, \cdot)$ is a linear and continuous map. For notational simplicity, we set $m_\omega(\cdot) \equiv m(\omega, \cdot)$, and $\ell_\omega(\cdot) \equiv \ell(\omega, \cdot)$. Let $T : B \times V \times V \to \mathbb{R}$ be a trilinear map, that is $T(\cdot, \cdot, \cdot)$ is linear in each of the three arguments with the other two being fixed. We assume that for each $\omega \in \Omega$, there are constants $\alpha(\omega) > 0$ and $\beta(\omega) > 0$ such that the following symmetry, continuity, and ellipticity conditions hold:

$$T(\ell_\omega, u, v) = T(\ell_\omega, v, u), \quad \text{for all } u, v \in V, \ \ell_\omega \in B, \tag{3}$$

$$|T(\ell_\omega, u, v)| \le \beta(\omega)\|\ell_\omega\|_B \|u\|_V \|v\|_V, \quad \text{for all } u, v \in V, \ \ell_\omega \in B, \tag{4}$$

$$T(\ell_\omega, u, u) \ge \alpha(\omega)\|u\|^2_V, \ \text{for all } u \in V, \ \ell_\omega \in K. \tag{5}$$

Moreover, there are constants $\alpha > 0$ and $\beta > 0$ such that for each $\omega \in \Omega$, we have

$$\alpha \le \alpha(\omega) \le \beta(\omega) \le \beta < \infty. \tag{6}$$

Given the above data, for any $\omega \in \Omega$, and for a fixed parameter $\ell_\omega \in K$, we consider the pathwise stochastic elliptic variational problem of finding $u_\omega(\ell_\omega) \in V$ such that

$$T(\ell_\omega, u_\omega, v) = \langle m_\omega, v \rangle, \quad \text{for all } v \in V. \tag{7}$$

We have the following result concerning the solvability of the above variational problem.

Proposition 2.1. *For any fixed $\omega \in \Omega$, for any $\ell_\omega \in K$, the pathwise stochastic elliptic variational problem* (7) *has a unique solution $u_\omega(\ell_\omega)$ that satisfies*

$$\|u_\omega(\ell_\omega)\|_V \leq \alpha(\omega)^{-1}\|m_\omega\|_{V^*}. \tag{8}$$

Moreover, if $m \in L^2(\Omega; V^*)$, then $u \in L^2(\Omega; V)$, and the following inequality holds:

$$\|u(\ell)\|_{L^2(\Omega;V)} \leq \alpha^{-1}\|m\|_{L^2(\Omega;V^*)}. \tag{9}$$

Proof. For any fixed $\omega \in \Omega$, and for any $\ell_\omega \in K$, the existence of the unique solution $u_\omega(\ell_\omega)$ follows directly from using the Lax-Milgram Lemma. Moreover, by taking $v = u_\omega(\ell_\omega)$ in (7), we get $T(\ell_\omega, u_\omega(\ell_\omega), u_\omega(\ell_\omega)) = \langle m_\omega, u_\omega(\ell_\omega)\rangle$, and (5) gives $\alpha(\omega)\|u_\omega(\ell_\omega)\|_V^2 \leq \|m_\omega\|_{V^*}\|u_\omega(\ell_\omega)\|_V$, proving (8). Since $\mathbb{E}\left[\|m_\omega\|_{V^*}^2\right] < \infty$ by assumption, we get (9) by using (6) and (8). □

Remark 2.2. In Proposition 2.1, for every $\omega \in \Omega$, we proved the existence of a deterministic solution $u_\omega(\ell_\omega)$ satisfying (8). We now discuss the implication of this result in establishing that the map $\omega \to u_\omega(\ell_\omega)$ is measurable (with respect to the $\sigma-$algebras $\mathbb{F}$ and the Borel $\sigma-$algebra $\mathcal{B}(V)$). This follows from the continuity of the deterministic map $B \ni \ell \mapsto u(\ell) \in V$ if B is a separable Banach space (for example a closed separable subspace of a nonseparable Banach space). For a general non-separable Banach space one can argue in the case of a deterministic $m \in V^*$ as follows: Assuming that the random variable ℓ is strongly measurable, also called Bochner measurable, ℓ is the almost sure limit of a sequence of simple random variables with values in B (i.e. random variables of the form $\Omega \ni \omega \mapsto \sum_{i=1}^n \ell_i \mathbb{1}_{\Omega_i}(\omega)$ with sets $\Omega_i \in \mathbb{F}$, $\ell_i \in B$ and $\mathbb{1}_{\Omega_i}(\omega) := 1$ if $\omega \in \Omega_i$ and $= 0$ otherwise).
For such a simple random coefficient the solution is $\omega \mapsto \sum_{i=1}^n u(\ell_i)\mathbb{1}_{\Omega_i}(\omega)$, hence it is measurable. In the case of a continuous solution map the solutions to the almost surely converging simple random variables as coefficients converge almost surely to the solution of the given random coefficient. As the probability space is assumed to be complete, this limit map is also measurable. On the other hand, if also m is random, one can for example approximate pathwise (i.e. pointwise) m by a sequence of simple random variables (now in a separable space, which is always possible) and take the limit. The measurability of the map $\omega \to u_\omega(\ell_\omega)$ can also be deduced directly from [23, Proposition 1.1], where the focus is on random variational inequalities. □

In proving (9), the uniform boundedness assumption (6) played a central role. In the following, we give a variant of the above result without assuming (6).

Proposition 2.3. *Assume that $m \in V^*$ in* (7) *is deterministic. Then, for $\omega \in \Omega$, for any $\ell_\omega \in K$,* (7) *has a unique solution that satisfies* (8). *Moreover, if we have $\alpha(\omega)^{-1} \in L^2(\Omega)$, then $u \in L^2(\Omega; V)$, and the following inequality holds:*

$$\|u(\ell)\|_{L^2(\Omega;V)} \leq \sqrt{\mathbb{E}\left[\alpha(\omega)^{-2}\right]}\|m\|_{V^*}. \tag{10}$$

Proof. The proof follows from the proof of Proposition 2.1 by making evident changes. □

As a crucial consequence of the measurability of the unique solution $u_\omega(\ell_\omega)$ of (7), which follows from Proposition 2.1, we consider the integral stochastic elliptic variational problem:

Given $\ell \in L^\infty(\Omega; B)$, find $u \in L^2(\Omega; V)$ with

$$\mathbb{E}\left[T(\ell, u(\ell), v)\right] = \mathbb{E}\left[\langle m, v\rangle\right], \quad \text{for all } v \in L^2(\Omega; V). \tag{11}$$

For each admissible parameter $\ell \in L^\infty(\Omega; B)$, stochastic variational problem (11) is uniquely solvable by the aid of Lax-Milgram Lemma in the Hilbert space $L^2(\Omega; V)$. Note that the uniform bounds on the admissible parameters are imposed through (6).

Remark 2.4. In this work, the primary source of uncertainty in the trilinear form T is through the random fields ℓ_ω that we aim to identify. The sources of uncertainty in m_ω are due to the random fields appearing on the right-hand sides of stochastic boundary value problems and the stochastic boundary conditions. In the following, because we will work with the integral formulation for the stochastic variational problems, we assume that (6) holds. In other words, we assume that (4) and (5) hold for fixed positive constants α and β. □

Remark 2.5. Although it will be clear from the context, we note that we will often drop the subscript ω. This is commonly done when we specify an element in a Bochner space. □

3. Lipschitz continuity and differentiability of the solution map

We begin by proving the Lipschitz continuity of the parameter-to-solution map $\ell \mapsto u(\ell)$ assigning to ℓ, the unique solution $u(\ell)$ of the stochastic variational problem (11). In the following, $\mathbb{K} \subset L^\infty(\Omega; B)$ is the feasible set such that for $\ell \in \mathbb{K}$, the realization ℓ_ω are in $K \subset B$.

Theorem 3.1. *Let $m \in L^2(\Omega; V^*)$.*
For any $\ell_1, \ell_2 \in \mathbb{K} \subset L^\infty(\Omega; B)$, the following estimates hold:

$$\|u(\ell_1) - u(\ell_2)\|_{L^2(\Omega;V)} \leq \frac{\beta}{\alpha}\|u(\ell_1)\|_{L^2(\Omega;V)}\|\ell_1 - \ell_2\|_{L^\infty(\Omega;B)}. \tag{12}$$

$$\|u(\ell_1) - u(\ell_2)\|_{L^2(\Omega;V)} \leq \frac{\beta}{\alpha}\|u(\ell_2)\|_{L^2(\Omega;V)}\|\ell_1 - \ell_2\|_{L^\infty(\Omega;B)}. \tag{13}$$

$$\|u(\ell_1) - u(\ell_2)\|_{L^2(\Omega;V)} \leq \frac{\beta}{\alpha^2}\|m\|_{L^2(\Omega;V^*)}\|\ell_1 - \ell_2\|_{L^\infty(\Omega;B)}. \tag{14}$$

Consequently, for any $\ell \in \mathbb{K}$, the map $\ell \mapsto u(\ell)$ is Lipschitz continuous.

Proof. Assume that $u(\ell_1) \in L^2(\Omega; V)$ is the solution of (11) for $\ell_1 \in \mathbb{K}$ and let $u(\ell_2) \in L^2(\Omega; V)$ be the solution of (11) for $\ell_2 \in \mathbb{K}$. Then, by the definitions of $u(\ell_1)$ and $u(\ell_2)$, we have

$$\mathbb{E}\left[T(\ell_1, u(\ell_1), v)\right] = \mathbb{E}\left[\langle m, v\rangle\right], \quad \text{for every } v \in L^2(\Omega; V), \tag{15}$$

$$\mathbb{E}\left[T(\ell_2, u(\ell_2), v)\right] = \mathbb{E}\left[\langle m, v\rangle\right], \quad \text{for every } v \in L^2(\Omega; V). \tag{16}$$

We combine (15) and (16), and rearrange the resulting equation in the following form:

$$\mathbb{E}\left[T(\ell_1 - \ell_2, u(\ell_1), v)\right] = -\mathbb{E}\left[T(\ell_2, u(\ell_1) - u(\ell_2), v)\right], \quad \text{for every } v \in L^2(\Omega; V).$$

By setting $v = u(\ell_1) - u(\ell_2)$ in the above equation, we obtain

$$\begin{aligned}
\alpha\|u(\ell_1) - u(\ell_2)\|^2_{L^2(\Omega;V)} &= \alpha\mathbb{E}\left[\|u(\ell_1) - u(\ell_2)\|^2_V\right] \\
&\leq \mathbb{E}\left[T(\ell_2, u(\ell_1) - u(\ell_2), u(\ell_1) - u(\ell_2))\right] \\
&= \mathbb{E}\left[T(\ell_1 - \ell_2, u(\ell_1), u(\ell_2) - u(\ell_1))\right] \\
&\leq \mathbb{E}\left[|\, T(\ell_1 - \ell_2, u(\ell_1), u(\ell_2) - u(\ell_1))\,|\right] \\
&\leq \beta\mathbb{E}\left[\|\ell_1 - \ell_2\|_B\|u(\ell_1)\|_V \; \|u(\ell_1) - u(\ell_2)\|_V\right] \\
&\leq \beta\|\ell_1 - \ell_2\|_{L^\infty(\Omega;B)}\|u(\ell_1)\|_{L^2(\Omega;V)} \; \|u(\ell_1) - u(\ell_2)\|_{L^2(\Omega;V)},
\end{aligned}$$

which proves (12). The bound (13) can be established in an entirely analogous fashion. Finally, by using (9), we get (14), which also proves the Lipschitz continuity of the solution map. □

We will now prove the first- and second-order derivative characterizations of the solution map. For the following result, we assume that $\mathbb{K}$ has a nonempty interior.

Theorem 3.2. *Let the interior of the set $\mathbb{K} \subset L^\infty(\Omega; B)$ be nonempty. For the given ℓ, let $\delta\ell$ be such that $\ell + \delta\ell \in \mathbb{K}$. The first-order derivative of u at ℓ in a direction $\delta\ell$, denoted by $\delta u = Du(\ell)\delta\ell \in L^2(\Omega; V)$ is the unique solution of the stochastic variational problem:*

$$\mathbb{E}\left[T(\ell, \delta u, v)\right] = -\mathbb{E}\left[T(\delta\ell, u(\ell), v)\right], \quad \textit{for all } v \in L^2(\Omega; V). \tag{17}$$

The second-order derivative of u at ℓ in a direction $(\delta\ell_1, \delta\ell_2)$, where $\ell + \delta\ell_1$ and $\ell+\delta\ell_2 \in \mathbb{K}$, denoted by $\delta^2 u = D^2u(\ell)(\delta\ell_1, \delta\ell_2)$, is the unique solution of the stochastic variational problem:

$$\begin{aligned}
\mathbb{E}\left[T(\ell, \delta^2 u, v)\right] = &-\mathbb{E}\left[T(\delta\ell_2, Du(\ell)\delta\ell_1, v)\right] \\
&- \mathbb{E}\left[T(\delta\ell_1, Du(\ell)\delta\ell_2, v)\right], \; \textit{for all } v \in L^2(\Omega, V).
\end{aligned} \tag{18}$$

Moreover, the following bounds hold:

$$\|Du(\ell)\| \leq \frac{\beta}{\alpha}\|u(\ell)\|_{L^2(\Omega;V)} \leq \frac{\beta}{\alpha^2}\|m\|_{L^2(\Omega,V^*)}. \tag{19}$$

$$\left\|D^2u(\ell)\right\| \leq \frac{2\beta^2}{\alpha^2}\|u(\ell)\|_{L^2(\Omega;V)} \leq \frac{2\beta^2}{\alpha^3}\|m\|_{L^2(\Omega,V^*)}. \tag{20}$$

Proof. Since $-\mathbb{E}\left[T(\delta\ell, u(\ell), \cdot)\right]$ defines a bounded linear functional on $L^2(\Omega; V)$, the stochastic variational problem (17) has a unique solution $\delta u \in L^2(\Omega; V)$. Since $\ell+\delta\ell \in \mathbb{K}$, we have the following two uniquely solvable integral stochastic variational problems:

$$\mathbb{E}\left[T(\ell, u(\ell), v)\right] = \mathbb{E}\left[\langle m, v\rangle\right], \quad \text{for all } v \in L^2(\Omega; V), \tag{21}$$

$$\mathbb{E}\left[T(\ell + \delta\ell, u(\ell + \delta\ell), v)\right] = \mathbb{E}\left[\langle m, v\rangle\right], \quad \text{for all } v \in L^2(\Omega; V). \tag{22}$$

We subtract (21) from (22) and rearrange the resulting equation as follows

$$\mathbb{E}\left[T(\ell + \delta\ell, u(\ell + \delta\ell) - u(\ell), v)\right] = -\mathbb{E}\left[T(\delta\ell, u(\ell), v)\right], \quad \text{for all } v \in L^2(\Omega; V).$$

Then, by subtracting (17) from the above equation, we obtain

$$\mathbb{E}\left[T(\ell+\delta\ell, u(\ell+\delta\ell)-u(\ell), v)\right] - \mathbb{E}\left[T(\ell,\delta u, v)\right] = 0, \quad \text{for all } v \in L^2(\Omega;V),$$

or equivalently, for all $v \in L^2(\Omega;V)$, we have

$$\mathbb{E}\left[T(\ell, u(\ell+\delta\ell)-u(\ell)-\delta u, v)\right] = -\mathbb{E}\left[T(\delta\ell, u(\ell+\delta\ell)-u(\ell), v)\right].$$

We now set $v = u(\ell+\delta\ell)-u(\ell)-\delta u$ in the above equation to get

$$\begin{aligned}&\mathbb{E}\left[T(\ell, u(\ell+\delta\ell)-u(\ell)-\delta u, u(\ell+\delta\ell)-u(\ell)-\delta u)\right]\\ &= -\mathbb{E}\left[T(\delta\ell, u(\ell+\delta\ell)-u(\ell), u(\ell+\delta\ell)-u(\ell)-\delta u)\right],\end{aligned}$$

which, after a simple calculation as before, yields

$$\begin{aligned}\|u(\ell+\delta\ell)-u(\ell)-\delta u\|_{L^2(\Omega;V)} &\le \frac{\beta}{\alpha}\|\delta\ell\|_{L^\infty(\Omega;B)}\|u(\ell+\delta\ell)-u(\ell)\|_{L^2(\Omega;V)}\\ &\le \frac{\beta^2}{\alpha^2}\|u(\ell)\|_{L^2(\Omega;V)}\|\delta\ell\|^2_{L^\infty(\Omega;B)},\end{aligned} \tag{23}$$

where, in the last inequality, we used the Lipschitz continuity of the solution map. Therefore,

$$\frac{\|u(\ell+\delta\ell)-u(\ell)-\delta u\|_{L^2(\Omega;V)}}{\|\delta\ell\|_{L^\infty(\Omega;B)}} \le \frac{\beta^2}{\alpha^2}\|u(\ell)\|_{L^2(\Omega;V)}\|\delta\ell\|_{L^\infty(\Omega;B)} = O\left(\|\delta\ell\|_{L^\infty(\Omega;B)}\right).$$

Therefore, u is differentiable at ℓ and $Du(\ell)\delta\ell = \delta u$, as it was claimed.

We now proceed to derive the second-order derivative characterization (18) for the parameter-to-solution map. Since $\ell+\delta\ell_1 \in \mathbb{K}$ and $\ell+\delta\ell_2 \in \mathbb{K}$, by (17), we have

$$\mathbb{E}\left[T(\ell+\delta\ell_1, Du(\ell+\delta\ell_1)\delta\ell_2, v)\right] = -\mathbb{E}\left[T(\delta\ell_2, u(\ell+\delta\ell_1), v)\right], \text{ for all } v \in L^2(\Omega;V),$$

which after some manipulation of terms implies that for every $v\in L^2(\Omega;V)$, we have

$$\begin{aligned}&\mathbb{E}\left[T(\ell+\delta\ell_1, Du(\ell+\delta\ell_1)\delta\ell_2 - Du(\ell)\delta\ell_2, v)\right] = -\mathbb{E}\left[T(\delta\ell_2, Du(\ell)\delta\ell_1, v)\right]\\ &\quad - \mathbb{E}\left[T(\delta\ell_1, Du(\ell)\delta\ell_2, v)\right] - \mathbb{E}\left[T(\delta\ell_2, u(\ell+\delta\ell_1)-u(\ell)-Du(\ell)\delta\ell_1, v)\right].\end{aligned} \tag{24}$$

By subtracting (18) from (24) and rearranging the resulting equation, we obtain

$$\begin{aligned}&\mathbb{E}\left[T(\ell, Du(\ell+\delta\ell_1)\delta\ell_2 - Du(\ell)\delta\ell_2 - \delta^2 u, v)\right]\\ &= -\mathbb{E}\left[T(\delta\ell_2, u(\ell+\delta\ell_1)-u(\ell)-Du(\ell)\delta\ell_1, v)\right]\\ &\quad - \mathbb{E}\left[T(\delta\ell_1, Du(\ell+\delta\ell_1)\delta\ell_2 - Du(\ell)\delta\ell_2, v)\right], \text{ for all } v \in L^2(\Omega;V).\end{aligned}$$

We now set $v = Du(\ell+\delta\ell_1)\delta\ell_2 - Du(\ell)\delta\ell_2 - \delta^2 u$ in the above equation and obtain

$$\begin{aligned}&\left\|Du(\ell+\delta\ell_1)\delta\ell_2 - Du(\ell)\delta\ell_2 - \delta^2 u\right\|_{L^2(\Omega,V)}\\ &\le \frac{\beta}{\alpha}\Big(\|\delta\ell_2\|_{L^\infty(\Omega;B)}\|u(\ell+\delta\ell_1)-u(\ell)-Du(\ell)\delta\ell_1\|_{L^2(\Omega,V)}\\ &\quad + \|\delta\ell_1\|_{L^\infty(\Omega;B)}\,\|Du(\ell+\delta\ell_1)\delta\ell_2 - Du(\ell)\delta\ell_2\|_{L^2(\Omega;V)}\Big).\end{aligned} \tag{25}$$

To find a bound on the last term in (25), we take $v = Du(\ell+\delta\ell_1)\delta\ell_2 - Du(\ell)\delta\ell_2$ in (24) and by performing a simple calculation obtain

$$\begin{aligned}
&\alpha\left\|Du(\ell+\delta\ell_1)\delta\ell_2 - Du(\ell)\delta\ell_2\right\|_{L^2(\Omega;V)}\\
&\le \beta\Big(\|\delta\ell_2\|_{L^\infty(\Omega;B)}\|Du(\ell)\delta\ell_1\|_{L^2(\Omega;B)} + \|\delta\ell_1\|_{L^\infty(\Omega;B)}\|Du(\ell)\delta\ell_2\|_{L^2(\Omega;V)}\\
&\qquad + \|\delta\ell_2\|_{L^\infty(\Omega;B)}\|u(\ell+\delta\ell_1) - u(\ell) - Du(\ell)\delta\ell_1\|_{L^2(\Omega;V)}\Big)\\
&\le \beta\Big(\frac{2\beta}{\alpha}\|u(\ell)\|_{L^2(\Omega;V)}\|\delta\ell_1\|_{L^\infty(\Omega;B)}\|\delta\ell_2\|_{L^\infty(\Omega;B)}\\
&\qquad + \frac{\beta^2}{\alpha^2}\|u(\ell)\|_{L^2(\Omega;V)}\|\delta\ell_1\|^2_{L^\infty(\Omega;B)}\|\delta\ell_2\|_{L^\infty(\Omega;B)}\Big),
\end{aligned}$$

where we used the bound

$$\|u(\ell+\delta\ell_1) - u(\ell) - Du(\ell)\delta\ell_1\|_{L^2(\Omega;V)} \le \frac{\beta^2}{\alpha^2}\|u(\ell)\|_{L^2(\Omega;V)}\|\delta\ell_1\|^2_{L^\infty(\Omega;B)}. \tag{26}$$

Therefore,

$$\begin{aligned}
&\left\|Du(\ell+\delta\ell_1)\delta\ell_2 - Du(\ell)\delta\ell_2\right\|_{L^2(\Omega;V)}\\
&\le \frac{\beta^2}{\alpha^2}\|u(\ell)\|_{L^2(\Omega;V)}\left(2\|\delta\ell_1\|_{L^\infty(\Omega;B)}\|\delta\ell_2\|_{L^\infty(\Omega;B)} + \frac{\beta}{\alpha}\|\delta\ell_1\|^2_{L^\infty(\Omega;B)}\|\delta\ell_2\|_{L^\infty(\Omega;B)}\right).
\end{aligned}$$

Now, with the aid of the above bound and (26), we return to (25):

$$\begin{aligned}
&\left\|Du(\ell+\delta\ell_1)\delta\ell_2 - Du(\ell)\delta\ell_2 - \delta^2 u\right\|_{L^2(\Omega;V)}\\
&\le \frac{\beta}{\alpha}\Big(\frac{\beta^2}{\alpha^2}\|u(\ell)\|_{L^2(\Omega;V)}\|\delta\ell_1\|^2_{L^\infty(\Omega;B)}\|\delta\ell_2\|_{L^\infty(\Omega;B)}\\
&\quad + \|\delta\ell_1\|_{L^\infty(\Omega;B)}\frac{\beta^2}{\alpha^2}\|u(\ell)\|_{L^2(\Omega;V)}\Big(2\|\delta\ell_1\|_{L^\infty(\Omega;B)}\|\delta\ell_2\|_{L^\infty(\Omega;B)}\\
&\quad + \frac{\beta}{\alpha}\|\delta\ell_1\|^2_{L^\infty(\Omega;B)}\|\delta\ell_2\|_{L^\infty(\Omega;B)}\Big)\Big)\\
&= \frac{\beta^3}{\alpha^3}\|u(\ell)\|_{L^2(\Omega;V)}\Big(3\|\delta\ell_1\|^2_{L^\infty(\Omega;B)}\|\delta\ell_2\|_{L^\infty(\Omega;B)} + \frac{\beta}{\alpha}\|\delta\ell_1\|^3_{L^\infty(\Omega;B)}\|\delta\ell_2\|_{L^\infty(\Omega;B)}\Big)\\
&= o\left(\|\delta\ell_1\|_{L^\infty(\Omega;B)}\|\delta\ell_2\|_{L^\infty(\Omega;B)}\right),
\end{aligned}$$

which shows that u is twice-differentiable at ℓ, and $\delta^2 u = Du(\ell)(\delta\ell_1, \delta\ell_2)$.

The first inequality in (19) follows from an elementary calculation after substituting $v = \delta u$ in (17). The second inequality in (19) then follows from using (9).

By the usual reasoning (18) implies that

$$\left\|\delta^2 u\right\|_{L^2(\Omega;V)} \le \frac{2\beta}{\alpha}\|Du(\ell)\|_{L^2(\Omega;V)}\|\delta\ell_1\|_{L^\infty(\Omega;B)}\|\delta\ell_2\|_{L^\infty(\Omega;B)},$$

and consequently

$$\left\|\delta^2 u\right\|_{L^2(\Omega;V)} \le \frac{2\beta^2}{\alpha^2}\|u(\ell)\|_{L^2(\Omega;V)}\|\delta\ell_1\|_{L^\infty(\Omega;B)}\|\delta\ell_2\|_{L^\infty(\Omega;B)},$$

proving the first inequality of (20). The second inequality of (20) now follows from (9). □

4. Optimization formulations for the stochastic inverse problem

In this work, we will study two stochastic optimization formulations for the stochastic elliptic inverse problem. The first optimization formulation is the classical output least-squares (OLS)-based optimization problem that seeks $\ell \in \mathbb{K}$ by solving the stochastic optimization problem:

$$\min_{\ell \in \mathbb{K}} J(\ell) := \frac{1}{2}\mathbb{E}\left[\|u(\ell) - z\|_Z^2\right], \tag{27}$$

where $u(\ell)$ is the solution of stochastic variational problem (7), $z \in L^2(\Omega; Z)$ is the data with Z a suitable Hilbert space, $\mathbb{E}[\cdot]$ is the expectation, and $\mathbb{K} \subset L^\infty(\Omega, B)$.

The second optimization formulation is a new energy least-squares (ELS)-based optimization problem that seeks $\ell \in \mathbb{K}$ by solving the following stochastic optimization problem:

$$\min_{\ell \in \mathbb{K}} \mathbb{J}(\ell) := \frac{1}{2}\mathbb{E}\left[T(\ell, u(\ell) - z, u(\ell) - z)\right], \tag{28}$$

where $T(\cdot,\cdot,\cdot)$ is the trilinear form, $u(\ell)$ is the solution of stochastic variational problem (7), $z \in L^2(\Omega; V)$ is the data, $\mathbb{E}[\cdot]$ is the expectation, and $\mathbb{K} \subset L^\infty(\Omega, B)$.

The inherited non-convexity of the OLS functional is a significant obstacle to both theoretical and computational aspects of the study of inverse problems, and can result in only finding local solutions to the OLS-based stochastic optimization problem. By contrast, the ELS -function is convex, as shown in the following result:

Theorem 4.1. *The ELS functional* $\mathbb{J}$*, defined in* (28) *is convex.*

Proof. We will compute the first derivative of the ELS functional $\mathbb{J}$ at point ℓ in any direction $\delta\ell$ by resorting to the chain rule as follows

$$D\mathbb{J}(\ell)(\delta\ell) = \frac{1}{2}\mathbb{E}\left[T(\delta\ell, u(\ell) - z, u(\ell) - z)\right] + \mathbb{E}\left[T(\ell, Du(\ell)(\delta\ell), u(\ell) - z)\right], \tag{29}$$

where $Du(\ell)(\delta\ell)$ is the derivative of $u(\ell)$ in the direction $\delta\ell$.

By Theorem 3.2, we have the following identity

$$\mathbb{E}\left[T(\ell, Du(\ell)(\delta\ell), u(\ell) - z)\right] = -\mathbb{E}\left[T(\delta\ell, u(\ell), u(\ell) - z)\right],$$

which when substituted in (29) yields

$$\begin{aligned} D\mathbb{J}(\ell)(\delta\ell) &= \frac{1}{2}\mathbb{E}\left[T(\delta\ell, u(\ell) - z, u(\ell) - z)\right] - \mathbb{E}\left[T(\delta\ell, u(\ell), u(\ell) - z)\right] \\ &= -\frac{1}{2}\mathbb{E}\left[T(\delta\ell, u(\ell) + z, u(\ell) - z)\right]. \end{aligned}$$

By the above derivative formula, we can readily compute the second-order derivative:

$$\begin{aligned} D^2\mathbb{J}(\ell)(\delta\ell, \delta\ell) &= -\frac{1}{2}\mathbb{E}\left[T(\delta\ell, Du(\ell)(\delta\ell), u(\ell) - z)\right] - \mathbb{E}\left[T(\delta\ell, u(\ell) + z, Du(\ell)(\delta\ell))\right] \\ &= -\mathbb{E}\left[T(\delta\ell, u(\ell), Du(\ell)(\delta\ell))\right] = \mathbb{E}\left[T(\ell, Du(\ell)(\delta\ell), Du(\ell)(\delta\ell))\right], \end{aligned}$$

where we again used Theorem 3.2. Consequently,

$$D^2\mathbb{J}(\ell)(\delta\ell, \delta\ell) \geq \alpha \|Du(\ell)(\delta\ell)\|^2_{L^2(\Omega; V)}, \tag{30}$$

proving that the ELS functional $\mathbb{J}$ is convex on $\mathbb{K}$. □

The inverse problems are known to be ill-posed, so any stable approximation ought to incorporate some regularization. For this we assume that $\mathbb{H}$ is a Hilbert space that is compactly embedded in $\mathbb{B} := L^\infty(\Omega; B)$. We denote the set of feasible parameters by $\widetilde{\mathbb{K}} \subset \mathbb{H} \cap \mathbb{K}$ which we assume to be a nonempty, closed, and convex set.

We shall now introduce minimization problems with the regularized objective functions. The following stochastic optimization problem uses the regularized ELS functional:

$$\min_{\ell \in \widetilde{\mathbb{K}}} \mathbb{J}_\kappa(\ell) := \frac{1}{2}\mathbb{E}\left[T(\ell, u(\ell) - z, u(\ell) - z)\right] + \frac{\kappa}{2}\|\ell\|_{\mathbb{H}}^2, \tag{31}$$

where $u(\ell)$ is the solution of (7) for ℓ, $z \in L^2(\Omega; V)$ is the data, $\kappa > 0$ is a fixed regularization parameter, and $\|\cdot\|_{\mathbb{H}}^2$ is the quadratic regularizer.

The following stochastic optimization problem uses the regularized OLS functional:

$$\min_{\ell \in \widetilde{\mathbb{K}}} J_\kappa(\ell) := \frac{1}{2}\mathbb{E}\left[\|u(\ell) - z\|_Z^2\right] + \frac{\kappa}{2}\|\ell\|_{\mathbb{H}}^2, \tag{32}$$

where $u(\ell)$ is the solution of (7) for ℓ, $z \in L^2(\Omega; Z)$ is the data with Z a Hilbert space, $\kappa > 0$ is a fixed regularization parameter, and $\|\cdot\|_{\mathbb{H}}^2$ is the quadratic regularizer.

The following result shows that stochastic optimization problem (31) is uniquely solvable. Similar arguments prove the existence of a (nonunique) solution of (32).

Theorem 4.2. *For each $\kappa > 0$, the ELS based optimization problem* (31) *has a unique solution.*

Proof. The proof is based on exploiting the properties of the trilinear form and the solution map. It follows by the definition of the regularized ELS functional that for every feasible $\ell \in \widetilde{\mathbb{K}}$, we have $\mathbb{J}_\kappa(\ell) \geq 0$. Therefore, there exists a minimizing sequence $\{\ell_n\}$ in $\widetilde{\mathbb{K}}$ satisfying

$$\lim_{n\to\infty} \mathbb{J}_\kappa(\ell_n) = \inf\{\mathbb{J}_\kappa(\ell) |\, \ell \in \widetilde{\mathbb{K}}\}.$$

This confirms that the sequence $\{\mathbb{J}_\kappa(\ell_n)\}$ is bounded, and since $\frac{\kappa}{2}\|\ell_n\|_{\mathbb{H}}^2 \leq \mathbb{J}_\kappa(\ell_n)$ for every $n \in \mathbb{N}$, we deduce that the sequence $\{\ell_n\}$ is bounded in $\|\cdot\|_{\mathbb{H}}$. However, by the assumption, the space $\mathbb{H}$ is compactly embedded into $\mathbb{B} = L^\infty(\Omega, B)$ and hence the sequence $\{\ell_n\}$ has a subsequence, which converges to some $\bar{\ell} \in \widetilde{\mathbb{K}}$. In the following discussion, we shall keep the same notation for the subsequences. For $n \in \mathbb{N}$, let $u_n = u(\ell_n)$ be the solution of the stochastic variational problem that corresponds to the parameter ℓ_n. Then, we have

$$\mathbb{E}\left[T(\ell_n, u_n, v)\right] = \mathbb{E}\left[\langle m, v\rangle\right], \quad \text{for every } v \in L^2(\Omega; V), \tag{33}$$

and taking $v = u_n$ in (33), we get $\mathbb{E}\left[T(\ell_n, u_n, u_n)\right] = \mathbb{E}\left[\langle m, u_n\rangle\right]$. As in the proof of Proposition 2.3, we can show that the sequence $\{u_n\}$ is bounded in $L^2(\Omega; V)$. Let $\{u_n\}$ be a subsequence that converges weakly to some $\bar{u} \in L^2(\Omega; V)$. We will prove that $\bar{u} = u(\bar{\ell})$. Next, we rearrange (33) so that for every $v \in L^2(\Omega, V)$, we have

$$\mathbb{E}\left[T(\bar{\ell}, \bar{u}, v)\right] - \mathbb{E}\left[\langle m, v\rangle\right] = -\mathbb{E}\left[T(\ell_n - \bar{\ell}, u_n, v)\right] - \mathbb{E}\left[T(\bar{\ell}, u_n - \bar{u}, v)\right]. \tag{34}$$

By passing the above equation to limit $n \to \infty$, and using the facts that $\{\ell_n\}$ converges to $\bar{\ell}$ in $\mathbb{B}$ and $\{u_n\}$ converges weakly to $\bar{u} \in L^2(\Omega, V)$, we obtain

$$\mathbb{E}\left[T(\bar{\ell}, \bar{u}, v)\right] = \mathbb{E}\left[\langle m, v\rangle\right], \quad \text{for every } v \in L^2(\Omega; V), \tag{35}$$

which, due to the fact that (35) is uniquely solvable, confirms that $\bar{u} = u(\bar{\ell})$.

We claim that for the sequence $\{\ell_n\}$ converging to $\bar{\ell}$ in $\mathbb{B}$, we have $\mathbb{J}(\ell_n) \to \mathbb{J}(\bar{\ell})$. For this, we set $v = u_n - z$ in (33) and $v = \bar{u} - z$ in (35), and rearrange the resulting equations to get

$$\begin{aligned} \mathbb{E}\left[T(\ell_n, u_n - z, u_n - z)\right] &= \mathbb{E}\left[\langle m, u_n - z\rangle\right] - \mathbb{E}\left[T(\ell_n, z, u_n - z)\right] \\ \mathbb{E}\left[T(\bar{\ell}, \bar{u} - z, \bar{u} - z)\right] &= \mathbb{E}\left[\langle m, \bar{u} - z\rangle\right] - \mathbb{E}\left[T(\bar{\ell}, z, \bar{u} - z)\right], \end{aligned}$$

which, due to

$$\mathbb{E}\left[T(\ell_n, z, u_n - z)\right] - \mathbb{E}\left[T(\bar{\ell}, z, \bar{u} - z)\right] = \mathbb{E}\left[T(\ell_n - \bar{\ell}, z, u_n - z)\right] + \mathbb{E}\left[T(\bar{\ell}, z, u_n - \bar{u})\right],$$

implies that $\mathbb{E}\left[T(\ell_n, u_n - z, u_n - z)\right] \to \mathbb{E}\left[T(\bar{\ell}, \bar{u} - z, \bar{u} - z)\right]$, as $n \to \infty$.

Equipped with the above preparation, we now return to the regularized ELS. We have

$$\begin{aligned} \mathbb{J}_\kappa(\bar{\ell}) &= \frac{1}{2}\mathbb{E}\left[T(\bar{\ell}, \bar{u} - z, \bar{u} - z)\right] + \frac{\kappa}{2}\|\bar{\ell}\|^2_{\mathbb{H}} \\ &\leq \lim_{n\to\infty} \frac{1}{2}\mathbb{E}\left[T(\ell_n, u(\ell_n) - z, u(\ell_n) - z)\right] + \liminf_{n\to\infty} \frac{\kappa}{2}\|\ell_n\|^2_{\mathbb{H}} \\ &\leq \liminf_{n\to\infty} \left\{\frac{1}{2}\mathbb{E}\left[T(\ell_n, u(\ell_n) - z, u(\ell_n) - z)\right] + \frac{\kappa}{2}\|\ell_n\|^2_{\mathbb{H}}\right\} \\ &= \inf\left\{\mathbb{J}_\kappa(\ell) \mid \ell \in \widetilde{\mathbb{K}}\right\}. \end{aligned}$$

Therefore, $\bar{\ell}$ is a solution of (28). The uniqueness follows from the strong convexity of the regularizer. The proof is complete. □

4.1. A pathwise approach for inverse problems

The compact embedding required on the Bochner spaces in the proof of Theorem 4.1 is quite stringent, and simple counterexamples can show that it does not always hold. Nonetheless, recent developments in this delicate area of research reveal that such embeddings require conditions on the random component, which are not desirable in the present context of inverse problems, see, e.g., [2, 57, 58, 60]. Therefore, we develop a path-wise approach for stochastic inverse problems in this subsection, which allows us to bypass the compact embedding of the Bochner spaces. The idea is to solve the inverse problem to get a unique solution ℓ_ω for an arbitrary but fixed $\omega \in \Omega$, defining the map $\omega \to \ell_\omega$ and then proving its measurability for obtaining the integral formulation.

We recall that B is a Banach space and $\emptyset \neq K \subset B$ is such that for any $\ell \in \mathbb{K} \subset L^\infty(\Omega; B)$, the corresponding realizations $\ell_\omega \in K$. For measurability, we will additionally assume that K is bounded. Now, let H be a separable Hilbert space compactly embedded into B. Let for any $\ell \in \mathbb{K}$, the realization ℓ_ω be in an open subset $\mathcal{O} \subset H \cap K$. Let $\emptyset \neq \mathcal{K} \subset \mathcal{O}$ be closed, convex, and bounded. For a fixed $\omega \in \Omega$, we introduce the pathwise ELS functional

$$\mathcal{J}(\ell_\omega) := \frac{1}{2} T(\ell_\omega, u_\omega(\ell_\omega) - z_\omega, u_\omega(\ell_\omega) - z_\omega), \tag{36}$$

and formulate the pathwise regularized ELS-based stochastic optimization problem:

$$\min_{\ell \in \mathcal{K}} \mathcal{J}^\kappa(\ell_\omega) := \frac{1}{2} T(\ell_\omega, u_\omega(\ell_\omega) - z_\omega, u_\omega(\ell_\omega) - z_\omega) + \frac{\kappa}{2} \|\ell_\omega\|_H^2, \tag{37}$$

where $z_\omega \in V$ is the data realization, $\kappa > 0$ is a regularization parameter, $\|\cdot\|_H^2$ is the regularizer, and $u_\omega(\ell_\omega) \in V$ is the unique solution of the pathwise variational problem

$$T(\ell_\omega, u_\omega, v) = \langle m_\omega, v \rangle, \quad \text{for all } v \in V. \tag{38}$$

We have the following result concerning the above optimization problem.

Theorem 4.3. *For any fixed $\omega \in \Omega$, (37) has a unique minimizer satisfying the following variational inequality as a necessary and sufficient optimality condition:*

$$\langle \nabla \mathcal{J}(\bar{\ell}_\omega), \ell - \bar{\ell}_\omega \rangle_H + \kappa \langle \bar{\ell}_\omega, \ell - \bar{\ell}_\omega \rangle_H \geq 0, \quad \textit{for all } \ell \in \mathcal{K}, \tag{39}$$

where the derivative $\nabla \mathcal{J}$ of $\mathcal{J}$ at $\bar{\ell}_\omega$ in a direction ℓ_ω is given by

$$\langle \nabla \mathcal{J}(\bar{\ell}_\omega), \ell_\omega \rangle_H = -\frac{1}{2} T(\ell_\omega, u_\omega(\ell_\omega) + z_\omega, u_\omega(\ell_\omega) - z_\omega). \tag{40}$$

Assume that $m \in L^2(\Omega; V^*)$, $z \in L^2(\Omega; V)$, and there exists $\ell^0 \in L^2(\Omega; H)$ such that any realization $\ell_\omega^0 \in \mathcal{K}$. Then $\bar{\ell} \in L^2(\Omega; H)$.

Proof. The unique solvability of (37) can be proved by the arguments employed in the proof of Theorem 4.2. Since the regularized ELS functional is differentiable and strongly convex, the variational inequality (39) is a necessary and sufficient optimality condition for (37).

For the given $\ell_\omega^0 \in \mathcal{K}$, by the strong monotonicity of $\nabla \mathcal{J} + \kappa I$, where I is the identity, we get

$$\langle (\nabla \mathcal{J} + \kappa I)(\ell_\omega^0) - (\nabla \mathcal{J} + \kappa I)(\bar{\ell}_\omega), \ell_\omega^0 - \bar{\ell}_\omega \rangle_H \geq \kappa \|\bar{\ell}_\omega - \ell_\omega^0\|_H^2,$$

which implies that

$$\langle (\nabla \mathcal{J} + \kappa I)(\ell_\omega^0), \ell_\omega^0 - \bar{\ell}_\omega \rangle_H \geq \kappa \|\bar{\ell}_\omega - \ell_\omega^0\|_H^2 + \langle (\nabla \mathcal{J} + \kappa I)(\bar{\ell}_\omega), \ell_\omega^0 - \bar{\ell}_\omega \rangle_H \geq \kappa \|\bar{\ell}_\omega - \ell_\omega^0\|_H^2,$$

where we used (39). We simplify the above inequality to write it as

$$\langle \nabla \mathcal{J}(\ell_\omega^0), \ell_\omega^0 - \bar{\ell}_\omega \rangle_H + \kappa \langle \ell_\omega^0, \ell_\omega^0 \rangle_H - \kappa \langle \ell_\omega^0, \bar{\ell}_\omega \rangle_H \geq \kappa \Big(\|\bar{\ell}_\omega\|_H^2 + \|\ell_\omega^0\|_H^2 - 2 \langle \bar{\ell}_\omega, \ell_\omega^0 \rangle_H \Big),$$

or equivalently

$$\kappa \|\bar{\ell}_\omega\|_H^2 \leq \langle \nabla \mathcal{J} \ell_\omega^0, \ell_\omega^0 - \bar{\ell}_\omega \rangle_H + \kappa \langle \ell_\omega^0, \bar{\ell}_\omega \rangle_H \leq \langle \nabla \mathcal{J}(\ell_\omega^0), \ell_\omega^0 - \bar{\ell}_\omega \rangle_H + \frac{\kappa}{2} \|\ell_\omega^0\|_H^2 + \frac{\kappa}{2} \|\bar{\ell}_\omega\|_H^2,$$

and by using (40), we get

$$\begin{aligned}\frac{\kappa}{2}\|\bar{\ell}_\omega\|_H^2 &\leq \frac{\kappa}{2}\|\ell_\omega^0\|_H^2 - \frac{1}{2}T(\ell_\omega^0 - \bar{\ell}_\omega, u_\omega(\ell_\omega^0) + z_\omega, u_\omega(\ell_\omega^0) - z_\omega)\\ &\leq \frac{\kappa}{2}\|\ell_\omega^0\|_H^2 + \frac{1}{2}\|\ell_\omega^0 - \bar{\ell}_\omega\|_B\|u_\omega(\ell_\omega^0) + z_\omega\|_V\|u_\omega(\ell_\omega^0) - z_\omega\|_V\\ &\leq \frac{\kappa}{2}\|\ell_\omega^0\|_H^2 + (\|\ell_\omega^0\|_B + \|\bar{\ell}_\omega\|_B)(\|u_\omega(\ell_\omega^0)\|_V^2 + \|z_\omega\|_V^2). \end{aligned} \tag{41}$$

By taking expectation, and using $\|u(\ell^0)\|_{L^2(\Omega;V)} \leq \alpha^{-1}\|m\|_{L^2(\Omega;V^*)}$ as $m \in L^2(\Omega;V^*)$, we get

$$\begin{aligned}\frac{\kappa}{2}\mathbb{E}\left[\|\bar{\ell}\|_H^2\right] &\leq \frac{\kappa}{2}\|\ell^0\|_{L^2(\Omega;H)}^2 + (\|\ell^0\|_{L^\infty(\Omega;B)} + \|\bar{\ell}\|_{L^\infty(\Omega;B)})(\|u(\ell_0)\|_{L^2(\Omega;V)}^2 + \|z\|_{L^2(\Omega,V)}^2),\\ &\leq \frac{\kappa}{2}\|\ell^0\|_{L^2(\Omega;H)}^2 + (\|\ell^0\|_{L^\infty(\Omega;B)} + \|\bar{\ell}\|_{L^\infty(\Omega;B)})(\alpha^{-2}\|m\|_{L^2(\Omega;V^*)}^2 + \|z\|_{L^2(\Omega,V)}^2),\end{aligned}$$

which proves that $\bar{\ell} \in L^2(\Omega;H)$. □

Remark 4.4. For the measurability of the minimizer obtained in Theorem 4.3, we can argue analogously as in Remark 2.2. For the deterministic/pathwise case, utilizing the continuity of the map $z \to \bar{\ell}$, we can regard at first simple random variables $z = \sum_{i=1}^n z_i \mathbb{1}_{\Omega_i}$. Here the minimizer is $\bar{\ell} = \sum_{i=1}^n \bar{\ell}_i \mathbb{1}_{\Omega_i}$, hence measurable. As before in the general situation the measurability follows from the fact that pointwise limits of a sequence of measurable mappings are measurable (at least in separable Banach spaces). If corresponding moment conditions on the random variables are fulfilled, the pathwise minimizer defines also the minimizer for the regularized ELS functional (31). Since the set K is bounded, the second term on the right hand side of (37) is uniformly bounded by some constant (also the expectation of it) and also taking expectations of the first term there gives bounded values by continuity of T and assumed second order moments of z (and $u(\ell)$ which follows from the assumptions on m and (9)). The measurability of the minimizer can also be derived directly from [25]. □

As a consequence of the above result, we can consider the following integral-form variant of the optimality condition (39):

$$\mathbb{E}\left[-\frac{1}{2}T(\ell - \bar{\ell}_\omega, u_\omega(\bar{\ell}_\omega) + z_\omega, u_\omega(\bar{\ell}_\omega) - z_\omega)\right] + \kappa\mathbb{E}\left[\langle\bar{\ell}_\omega, \ell - \bar{\ell}_\omega\rangle_H\right] \geq 0, \tag{42}$$

for all $\ell \in \widetilde{\mathcal{K}}$, where $\widetilde{\mathcal{K}} \in L^2(\Omega;H)$ is such that for $\ell \in \widetilde{\mathcal{K}}$, the realizations belong to $\mathcal{K} \subset H$.

Remark 4.5. We note that a variant of (42) can also be obtained directly by using the Hadamard derivative, which would allow us to bypass the assumption that $\widetilde{\mathbb{K}}$ has a nonempty interior and also avoid the mechanism of using a path-wise formulation. In that case, the gradient can be computed from the discrete objective ELS/OLS functionals, following the philosophy of the discretize-then-optimize framework. □

5. Convergence rates

The study of convergence rates for inverse problems, relating the regularization parameter to the noise level in the data, is an important topic with numerous theoretical and practical implications. To signify the main contribution of this section, we briefly recall a celebrated result by Engl, Kunisch, and Neubauer [13]. Let X and Y be two Hilbert spaces and let $F : D(F) \subset X \to Y$ be a nonlinear map. Given $y_0 \in Y$, we consider the problem of finding $x \in D(F)$ such that

$$F(x) = y_0.$$

Let S be the set of solutions to this problem. The above operator equation is commonly studied as the following optimization problem:

$$\min \|F(x) - \bar{y}\|_Y^2 + \alpha\|x - \hat{x}\|_X^2,$$

where $\alpha > 0$ and $\hat{x}$ is a priori estimate of the unknown solution. We recall that $\bar{x} \in S$ is called a $\hat{x}$-minimal norm solution, if $\|\bar{x} - \hat{x}\| \leq \|x - \hat{x}\|$, for every $x \in S$.

We now recall the following result from [13]:

Theorem 5.1. *Let X and Y be Hilbert spaces, let $F : D(F) \subset X \to Y$ be a given map with the convex domain $D(F)$. Let $y_0 \in Y$ and for $\delta > 0$, let y_δ be such that $\|y_\delta - y_0\|_Y \leq \delta$. Let $\bar{x}$ be an $\hat{x}-$minimum norm solution. Assume that the following conditions hold:*

(i) *The map F is Fréchet differentiable with F' denoting the Fréchet derivative.*

(ii) *There exists $L > 0$ such that $\|F'(x_0) - F'(z)\| \leq L\|x_0 - z\|$, for every $z \in D(F)$.*

(iii) *There exists $w \in Y$ such that $\bar{x} - \hat{x} = F'(x_0)^* w$, where $F'(x_0)^*$ is the adjoint of $F'(x_0)$.*

(iv) $L\|w\|_Y < 1$.

Let $x_{\delta,\alpha}$ be a solution of the following optimization problem with noisy data:

$$\min \|F(x) - y_\delta\|_Y^2 + \alpha\|x - \hat{x}\|_X^2.$$

Then for the choices $\alpha \sim \delta$, we have $|x_{\delta,\alpha} - x_0\| = O(\sqrt{\delta})$.

In the above result, the most stringent and hard-to-verify assumption is $L\|w\|_Y < 1$, which is often referred to as the ”smallness condition”. Many research activities have followed the above result, and it has been extended in numerous directions. Notable extensions include convergence rates in Banach spaces involving nonsmooth regularizers. On the other hand, many attempts have been made to relax the smallness condition. See Burger and Osher [8], Kaltenbacher and Hofmann [32], Kügler and Sincich [37], Resmerita [48], and Resmerita and Scherzer [49], as a small sample of the available results. The recent paper by Kaltenbacher and Van [33] contains an updated bibliography.

We now discuss one of the extensions of Theorem 5.1. Hào and Quyen [26] considered the inverse problem of identifying the parameter q in the elliptic boundary value problem (BVP)

$$-\nabla \cdot (q\nabla u) = f \text{ in } \Omega, \; u = 0 \text{ on } \partial\Omega, \tag{43}$$

where Ω is a suitable domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ and $\partial\Omega$ is its smooth boundary.

They employed the following deterministic ELS functional

$$q \to \int q\nabla(u(q) - z) \cdot \nabla(u(q) - z), \tag{44}$$

where z is the data (the measurement of u) and $u(q)$ solves the variational formulation of (43).

For the above functional, Hào and Quyen [26] proved convergence rates completely analogous to those given in Theorem 5.1, but without requiring the smallness condition.

We note that the stochastic version of (44) was recently studied in [30]. In the following, in an abstract framework, we prove the same convergence rate as in Theorem 5.1 or as in [26], without requiring the smallness condition. Not only is this the first convergence rate result for stochastic inverse problems, but it is also a new result for a wide range of deterministic BVPs other than the diffusion equation considered in [26].

We now return to the stochastic inverse problems. In the following presentation we assume that the data $z \in L^2(\Omega, V)$ is attainable, that is, there exists a feasible parameter $\ell \in \mathbb{K} \subset L^\infty(\Omega; B)$ such that $z = u(\ell)$. We assume that instead of z, the contaminated data z^δ is available which satisfies

$$\|z - z^\delta\|_{L^2(\Omega;V)} \leq \delta. \tag{45}$$

We now introduce the regularized ELS-based optimization problem that involves z^δ:

$$\min_{\ell \in \widetilde{\mathbb{K}}} \mathbb{J}_\delta^\kappa(\ell) := \frac{1}{2}\mathbb{E}\left[T(\ell, u(\ell) - z^\delta, u(\ell) - z^\delta)\right] + \frac{\kappa}{2}\mathbb{E}\left[\|\ell - \hat{\ell}\|_H^2\right], \tag{46}$$

where $u(\ell) \in L^2(\Omega; V)$ solves variational problem (7) and $\hat{\ell} \in L^\infty(\Omega; B)$ is a priori estimate for the unknown solution.

We have the following result concerning the convergence rates.

Theorem 5.2. *Let $\ell_{\kappa,\delta}$ be the unique regularized solution of* (46) *and let ℓ^* be an $\hat{\ell}$-minimal norm solution that is attainable. Assume that there exists $w \in L^2(\Omega; V)$ such that*

$$\ell^* - \hat{\ell} = Du(\ell^*)^* w, \tag{47}$$

where $Du(\ell^*)^* : V \to H$ is the adjoint operator of $Du(\ell^*)$ such that

$$\langle Du(\ell^*)\ell, v\rangle_V = \langle \ell, Du(\ell^*)^*(v)\rangle_H, \quad \text{for every } \ell \in H,\ v \in V.$$

Then, for the choices $\kappa \sim \delta$, we have

$$\|\ell_{\kappa,\delta} - \ell^*\|_{L^2(\Omega;H)} = O(\sqrt{\delta}), \quad \text{and} \quad \|u(\ell_{\kappa,\delta}) - z^\delta\|_{L^2(\Omega;V)} = O(\delta).$$

Proof. Since $\ell_{\kappa,\delta} \in \mathbb{K}$ is the regularized solution, we have

$$\mathbb{J}_\delta(\ell_{\kappa,\delta}) + \frac{\kappa}{2}\mathbb{E}\left[\|\ell_{\kappa,\delta} - \hat{\ell}\|_H^2\right] \leq \mathbb{J}_\delta(\ell) + \frac{\kappa}{2}\mathbb{E}\left[\|\ell - \hat{\ell}\|_H^2\right], \quad \text{for every } \ell \in \mathbb{K}, \tag{48}$$

where
$$\mathbb{J}_\delta(\ell) = \frac{1}{2}\mathbb{E}\left[T(\ell, u(\ell) - z^\delta, u(\ell) - z^\delta)\right].$$

Note that it follows from (5) that

$$\begin{aligned}\mathbb{J}_\delta(\ell_{\kappa,\delta}) &= \frac{1}{2}\mathbb{E}\left[T(\ell_{\kappa,\delta}, u(\ell_{\kappa,\delta}) - z^\delta, u(\ell_{\kappa,\delta}) - z^\delta)\right] \\ &\geq \frac{\alpha}{2}\mathbb{E}\left[\|u(\ell_{\kappa,\delta}) - z^\delta\|_V^2\right] = \frac{\alpha}{2}\|u(\ell_{\kappa,\delta}) - z^\delta\|_{L^2(\Omega;V)}^2.\end{aligned} \tag{49}$$

By taking $\ell = \ell^*$ in (48), we obtain

$$\mathbb{J}_\delta(\ell_{\kappa,\delta}) + \frac{\kappa}{2}\mathbb{E}\left[\|\ell_{\kappa,\delta} - \hat{\ell}\|_H^2\right] \le \mathbb{J}_\delta(\ell^*) + \frac{\kappa}{2}\mathbb{E}\left[\|\ell^* - \hat{\ell}\|_H^2\right].$$

which can be rearranged as follows

$$\begin{aligned}
&\mathbb{J}_\delta(\ell_{\kappa,\delta}) + \frac{\kappa}{2}\mathbb{E}\left[\|\ell_{\kappa,\delta} - \ell^*\|_H^2\right] \\
&\le \mathbb{J}_\delta(\ell^*) + \frac{\kappa}{2}\left[\mathbb{E}\left[\|\ell^* - \hat{\ell}\|_H^2\right] + \mathbb{E}\left[\|\ell_{\kappa,\delta} - \ell^*\|_H^2\right] - \mathbb{E}\left[\|\ell_{\kappa,\delta} - \hat{\ell}\|_H^2\right]\right] \\
&= \mathbb{J}_\delta(\ell^*) + \kappa\mathbb{E}\left[\langle \ell^* - \hat{\ell}, \ell^* - \ell_{\kappa,\delta}\rangle_H\right] \le c_1\delta^2 + \kappa\mathbb{E}\left[\langle \ell^* - \hat{\ell}, \ell^* - \ell_{\kappa,\delta}\rangle_H\right] \\
&= c_1\delta^2 + \kappa\mathbb{E}\left[\langle w, Du(\ell^*)(\ell^* - \ell_{\kappa,\delta})\rangle_V\right],
\end{aligned} \tag{50}$$

where we used (47), the fact that due to (4) with $c_1 = \frac{2}{\beta}\|\ell^*\|_{L^\infty(\Omega;B)}$, we have

$$\mathbb{J}_\delta(\ell^*) = \frac{1}{2}\mathbb{E}\left[T(\ell^*, u(\ell^*) - z^\delta, u(\ell^*) - z^\delta)\right] \le \frac{\beta}{2}\|\ell^*\|_{L^\infty(\Omega;B)}\|u(\ell^*) - z^\delta\|_{L^2(\Omega;V)}^2 \le c_1\delta^2,$$

and the following identity that holds for all $a, b, c \in H$,

$$2\langle a - b, a - c\rangle_H = \|a - c\|_H^2 + \|a - b\|_H^2 - \|b - c\|_H^2.$$

Given $w \in L^2(\Omega, V)$, we now consider the stochastic variational problem of finding $\widetilde{w} \in L^2(\Omega; V)$ such that

$$\mathbb{E}\left[T(\ell^*, \widetilde{w}, v)\right] = \mathbb{E}\left[\langle w, v\rangle_V\right], \quad \text{for every } v \in L^2(\Omega; V). \tag{51}$$

By standard arguments, (51) is uniquely solvable and $\|\widetilde{w}\|_{L^2(\Omega;V)}$ is bounded by a constant involving a bound on $\|w\|_{L^2(\Omega;V)}$ We take $v = Du(\ell^*)(\ell^* - \ell_{\kappa,\delta})$ in (51) which leads to

$$\mathbb{E}\left[T(\ell^*, \widetilde{w}, Du(\ell^*)(\ell^* - \ell_{\kappa,\delta}))\right] = \mathbb{E}\left[\langle w, Du(\ell^*)(\ell^* - \ell_{\kappa,\delta})\rangle_V\right],$$

and by using the symmetry of the trilinear form (see (3)), obtain

$$\mathbb{E}\left[T(\ell^*, Du(\ell^*)(\ell^* - \ell_{\kappa,\delta}), \widetilde{w})\right] = \mathbb{E}\left[\langle w, Du(\ell^*)(\ell^* - \ell_{\kappa,\delta})\rangle_V\right].$$

Next, we apply derivative characterization (17) to the left-hand side of the above identity to get

$$\mathbb{E}\left[T(\ell_{\kappa,\delta} - \ell^*, u(\ell^*), \widetilde{w})\right] = \mathbb{E}\left[\langle w, Du(\ell^*)(\ell^* - \ell_{\kappa,\delta})\rangle_V\right]. \tag{52}$$

Since the following two stochastic variational problems are uniquely solvable

$$\begin{aligned}
\mathbb{E}\left[T(\ell^*, u(\ell^*), v)\right] &= \mathbb{E}\left[\langle m, v\rangle\right], \quad \text{for every } v \in L^2(\Omega; V), \\
\mathbb{E}\left[T(\ell_{\kappa,\delta}, u(\ell_{\kappa,\delta}), v)\right] &= \mathbb{E}\left[\langle m, v\rangle\right], \quad \text{for every } v \in L^2(\Omega; V),
\end{aligned}$$

we have the following valuable identity:

$$\mathbb{E}\left[T(\ell^*, u(\ell^*), \widetilde{w})\right] = \mathbb{E}\left[T(\ell_{\kappa,\delta}, u(\ell_{\kappa,\delta}), \widetilde{w})\right].$$

Now, with the help of the above identity, we now return to (52) to obtain

$$\begin{aligned}
\mathbb{E}\left[T(\ell_{\kappa,\delta}-\ell^*,u(\ell^*),\widetilde{w})\right] &= \mathbb{E}\left[T(\ell_{\kappa,\delta},u(\ell^*),\widetilde{w})\right]-\mathbb{E}\left[T(\ell^*,u(\ell^*),\widetilde{w})\right]\\
&= \mathbb{E}\left[T(\ell_{\kappa,\delta},u(\ell^*),\widetilde{w})\right]-\mathbb{E}\left[T(\ell_{\kappa,\delta},u(\ell_{\kappa,\delta}),\widetilde{w})\right]\\
&= \mathbb{E}\left[T(\ell_{\kappa,\delta},u(\ell^*)-z^\delta,\widetilde{w})\right]+\mathbb{E}\left[T(\ell_{\kappa,\delta},z^\delta-u(\ell_{\kappa,\delta}),\widetilde{w})\right]\\
&\leq \mathbb{E}\left[\beta\|\ell_{\kappa,\delta}\|_B\|u(\ell^*)-z^\delta\|_V\|\|\widetilde{w}\|_V\right]\\
&\quad+\mathbb{E}\left[\beta\|\ell_{\kappa,\delta}\|_B\|u(\ell_{\kappa,\delta})-z^\delta\|_V\|\|\widetilde{w}\|_V\right]\\
&\leq \beta\|\ell_{\kappa,\delta}\|_{L^\infty(\Omega;B)}\|u(\ell^*)-z^\delta\|_{L^2(\Omega;V)}\|\widetilde{w}\|_{L^2(\Omega;V)}\\
&\quad+\beta\|\ell_{\kappa,\delta}\|_{L^\infty(\Omega;B)}\|u(\ell_{\kappa,\delta})-z^\delta\|_{L^2(\Omega;V)}\|\widetilde{w}\|_{L^2(\Omega;V)}\\
&\leq \beta\delta\|\ell_{\kappa,\delta}\|_{L^\infty(\Omega;B)}\|\widetilde{w}\|_{L^2(\Omega;V)}+\frac{\kappa\beta}{\alpha}\|\ell_{\kappa,\delta}\|^2_{L^\infty(\Omega;B)}\|\widetilde{w}\|^2_{L^2(\Omega;V)}\\
&\quad+\frac{\alpha\beta}{4\kappa}\|u(\ell_{\kappa,\delta})-z^\delta\|^2_{L^2(\Omega;V)}\\
&\leq c\delta+c\kappa+\frac{\alpha}{4\kappa}\|u(\ell_{\kappa,\delta})-z^\delta\|^2_{L^2(\Omega;V)},
\end{aligned}\tag{53}$$

where c is a positive constant involving bounds on β, $\|\ell_{\kappa,\delta}\|_{L^\infty(\Omega;B)}$, and $\|\widetilde{w}\|_{L^2(\Omega;V)}$. We now combine (49), (50), (52), and (53) to obtain

$$\frac{\alpha}{2}\|u(\ell_{\kappa,\delta})-z^\delta\|^2_{L^2(\Omega;V)}+\frac{\kappa}{2}\mathbb{E}\left[\|\ell_{\kappa,\delta}-\ell^*\|^2_H\right]\leq c_1\delta^2+c\kappa\delta+c\kappa^2+\frac{\alpha}{4}\|u(\ell_{\kappa,\delta})-z^\delta\|^2_{L^2(\Omega;V)},$$

which yields
$$\|u(\ell_{\kappa,\delta})-z^\delta\|^2_{L^2(\Omega;V)}\leq C(\delta^2+\kappa\delta+\kappa^2),$$
and
$$\|\ell_{\kappa,\delta}-\ell^*\|^2_{L^2(\Omega;H)}\leq C\frac{(\delta^2+\kappa\delta+\kappa^2)}{\kappa},$$

where $C>0$ is a constant. Thus, as a consequence of the above inequalities, for $\kappa\sim\delta$, we have $\|\ell_{\kappa,\delta}-\ell^*\|_{L^2(\Omega;H)}=O(\sqrt{\delta})$ and $\|u(\ell_{\kappa,\delta})-z^\delta\|_{L^2(\Omega;V)}=O(\delta)$. The proof is thus complete. □

Remark 5.3. Since our focus is on the convergence rates, we assume the existence of a regularized solution with the perturbed data and the regularizer with a priori shift. However, the unique solvability of this regularized problem can be proved by the arguments used in Theorem 4.2. □

6. Discretization of finite-dimensional noise problems

In this work, we assume that the parameter $\ell(\omega,x)$ and the function $m(\omega,x)$ are finite-dimensional noises and are given by

$$\ell(\omega,x)=\ell_0(x)+\sum_{k=1}^{P}\ell_k(x)\xi_k(\omega)\quad\text{and}\quad m(\omega,x)=f_0(x)+\sum_{k=1}^{G}f_k(x)\eta_k(\omega),$$

where the real-valued functions ℓ_k and f_k are uniformly bounded. For notational simplicity, in the following, we assume that ξ_k and η_k have the same distribution and set $\xi_k=\eta_k$.

We note that for the well-posed of the direct problem and other results, we assume that ℓ_k $(k=0,\dots,P)$ satisfies the feasibility constraints imposed on $\ell(\omega,x)$.

Then, the variational problem (11) reduces to the following parametric deterministic variational problem: Find $u(y,x) \in V_\rho := L^2_\rho(\Gamma;V)$ such that for every $v(y,x) \in V_\rho$,

$$\int_\Gamma \rho(y) T(\ell(y,\cdot), u(y,\cdot), v(y,\cdot))\, dy = \int_\Gamma \rho(y) \langle m(y,\cdot), v(y,\cdot)\rangle\, dy, \text{ for all } v \in V_\rho.$$

It follows from the Doob-Dynkin lemma that a solution of the above variational problem is finite-dimensional noise and u is a function of ξ where $\xi = (\xi_1, \xi_2, \ldots, \xi_M) : \Omega \mapsto \Gamma$ and $M := \max\{P, G\}$, see [39].

Let V_{hk} be a finite-dimensional subspace of V_ρ. An element $u_{hk} \in V_{hk}$ is the stochastic Galerkin solution if

$$\int_\Gamma \rho(y) T(\ell(y,\cdot), u_{hk}(y,\cdot), v(y,\cdot)) dy = \int_\Gamma \rho(y) \langle m(y,\cdot), v(y,\cdot)\rangle\, dy, \tag{54}$$

for all $v \in V_{hk}$. We adapt the discretization scheme given in [30] to the considered abstract problem. Let V_h be an N-dimensional subspace of V and let S_k be a Q-dimensional subspace of $L^2_\rho(\Gamma)$ with

$$V_h = \operatorname{span}\{\phi_1, \phi_2, \ldots, \phi_N\} \quad \text{and} \quad S_k = \operatorname{span}\{\psi_1, \psi_2, \ldots, \psi_Q\}.$$

We assume that the basis $\{\psi_1, \psi_2, \ldots, \psi_Q\}$ is orthonormal with respect to the density ρ, that is,

$$\int_\Gamma \rho(y) \psi_n(y) \psi_m(y) dy = \delta_{nm},$$

where δ_{nm} is the Kronecker delta: $\delta_{nm} = 1$ for $n = m$, $\delta_{nm} = 0$ for $n \neq m$. We will follow the standard approach of constructing a finite-dimensional subspace of V_ρ by taking the tensor product of the basis functions ϕ_i and ψ_j. That is, we will use the following NQ-dimensional subspace for solving the discrete variational problem:

$$V_{hk} := V_h \otimes S_k := \operatorname{span}\{\phi_i \psi_j |\ i = 1, \ldots, N,\ j = 1, \ldots, Q\}.$$

Then, any $v \in V_h \otimes S_k$ has the representation

$$v(y,x) = \sum_{i=1}^N \sum_{j=1}^Q V_{ij} \phi_i(x) \psi_j(y) = \sum_{j=1}^Q \left[\sum_{i=1}^N V_{ij} \phi_i(x) \right] \psi_j(y) = \sum_{j=1}^Q V_j(x) \psi_j(y),$$

where
$$V_j(x) \equiv \sum_{i=1}^N V_{ij} \phi_i(x).$$

It is convenient to introduce the following vectorized notation

$$V = \operatorname{vec}(V_{ij}) = [V_1\ V_2 \cdots V_Q]^\top \in \mathbb{R}^{NQ}, \text{ with } V_j := [V_{1j}\ V_{2j} \cdots V_{Nj}]^\top \in \mathbb{R}^N. \tag{55}$$

Analogously, since the unknown random field has a finite linear expansion, we get

$$\ell(y,x) = \ell_0(x) + \sum_{s=1}^M y_s \ell_s(x) = \sum_{s=0}^M y_s \ell_s(x) = \sum_{s=0}^M y_s \sum_{i=1}^P L_{is} \varphi_i(x) = \sum_{s=0}^M L_s y_s, \tag{56}$$

where, by convention, we take $y_0 = 1$, and $L_s(x) = \sum_{i=1}^P L_{is} \varphi_i(x)$ is the discretization of $\ell_s(x)$ in a P-dimensional space $L_h = \operatorname{span}\{\varphi_1, ..., \varphi_P\}$, for $s = 0, \ldots, M$.

We will again use the vectorized notation:

$$L = [L_0 \; L_1 \; \cdots L_M]^\top \in \mathbb{R}^{P(M+1)}.$$

The above preparation leads to the following equivalent form of the discrete variational problem (54): Find $u_{hk}(y,x) \in V_h \otimes S_Q$ such that

$$\int_\Gamma \rho(y)\psi_n(y) T\left(\sum_{s=0}^{M} L_s y_s, u_{hk}(y,\cdot), \phi_i(\cdot)\right) dy = \int_\Gamma \rho(y)\psi_n(y) \langle m(y,\cdot), \phi_i(x)\rangle \, dy,$$

for every $i = 1, \dots, N, \; n = 1, \dots, Q$.

By standard reasoning (see [30]), the above discrete variational problem leads to a linear system which can be expressed in the following compact form by using Kronecker product $\otimes$: Find $U \in \mathbb{R}^{NQ}$ such that

$$\left[\sum_{s=0}^{M} G^s \otimes K(L_s)\right] U = F. \tag{57}$$

where $K(L_s) \in \mathbb{R}^{N\times N}$, $G^s = (g^s_{nm}) \in \mathbb{R}^{Q\times Q}$, $F = \mathrm{vec}(F_{in}) \in \mathbb{R}^{NQ}$ are defined by

$$K(L_s)_{i,k} = T(L_s(\cdot), \phi_k(\cdot), \phi_i(\cdot)), \quad g^s_{nm} = \int_\Gamma \rho(y)\psi_n(y)\psi_m(y) y_s dy,$$
$$F_{in} = \int_\Gamma \rho(y)\psi_n(y) \langle m(y,\cdot), \phi_i(\cdot)\rangle \, dy, \quad \text{for every } s = 0, \cdots, M.$$

6.1. Discrete ELS

We consider the following parameterized regularized ELS functional:

$$\begin{aligned}\min_{\ell\in\mathbb{K}} J_\kappa(\ell) = \frac{1}{2}\int_\Gamma \rho(y) T(\ell(y,\cdot), u(y,\cdot) - z(y,\cdot), u(y,\cdot) - z(y,\cdot))\, dy \\ + \frac{\kappa}{2}\int_\Gamma \rho(y)\, \|\ell(y,\cdot)\|^2_H \, dy,\end{aligned} \tag{58}$$

where $u \equiv u(y,x)$ is the solution of the finite-dimensional noise variational problem

$$\int_\Gamma \rho(y) T(\ell(y,\cdot), u(y,\cdot), v(y,\cdot)) dy = \int_\Gamma \rho(y) \langle m(y,\cdot), v(y,\cdot)\rangle \, dy, \text{ for every } v \in V_\rho.$$

For the inverse problem, we will assume that the data z depends, via ξ, on the finite-dimensional noise variables $\{\xi_i\}_{i=1}^M$. Assuming finite linear expansion (56) for the unknown coefficient and by following the discretization approach used before, we obtain

$$J_\kappa(L) = \frac{1}{2}(U - Z)^\top \left(\sum_{s=0}^{M} G^s \otimes K(L_s)\right)(U - Z) + \frac{\kappa}{2} L^\top (\Psi \otimes H_L) L$$

where $\Psi \in \mathbb{R}^{(M+1)\times(M+1)}$, $H_L \in \mathbb{R}^{P\times P}$ are given by

$$\Psi_{s,t} = \int_\Gamma \rho(y) y_s y_t dy, \text{ for every } s,t = 0,\dots,M,$$
$$(H_L)_{i,j} = \langle L_i(\cdot), L_j(\cdot)\rangle_H, \text{ for every } i,j = 1,\dots,P.$$

We note that the derivative formula (without the regularizer) in parametric form is given by

$$DJ_0(\ell)(b) = -\frac{1}{2}\int_\Gamma \rho(y) T(b, u(y,\cdot)+z(y,\cdot), u(y,\cdot)-z(y,\cdot))dy,$$

and consequently,

$$DJ_0(L)(B) = -\tfrac{1}{2}(U+Z)^\top \left[\sum_{s=0}^M G^s \otimes K(B_s)\right](U-Z).$$

Therefore, the gradient of the regularized ELS functional takes the following form:

$$\nabla J_\kappa(L) = -\frac{1}{2}\Big[(U+Z)^\top \left(G^0 \otimes I_N\right) S(U-Z)\dots(U+Z)^\top\left(G^M\otimes I_N\right) S\,(U-Z)\Big]$$
$$+\,\kappa L^\top (\Psi \otimes H_L),$$

where $S(U-Z) = [S(U_1-Z_1)\ S(U_2-Z_2)\ \cdots S(U_Q-Z_Q)]^\top$ and $S(\cdot)\in\mathbb{R}^{N\times P}$, is the adjoint stiffness matrix satisfying

$$S(V)B = K(B)V, \text{ for every } B\in\mathbb{R}^P,\ V\in\mathbb{R}^N.$$

6.2. Discrete OLS

Defining an inner product $\langle\cdot,\cdot\rangle_X$ on the physical space, we consider the following parametric OLS functional under the finite dimensional noise assumption:

$$\min_{a\in A}\widehat{J}_\kappa(\ell) =$$
$$= \frac{1}{2}\int_\Gamma \rho(y)\langle u(y,\cdot)-z(y,\cdot), u(y,\cdot)-z(y,\cdot)\rangle_X\, dy + \frac{\kappa}{2}\int_\Gamma \rho(y)\,\|\ell(y,x)\|_H^2\, dy. \quad (59)$$

In this case, denoting $(X_U)_{i,j} = \langle\phi_i(\cdot),\phi_j(\cdot)\rangle_X$, we have the corresponding discrete formula

$$\widehat{J}_k(A) = \tfrac{1}{2}(U-Z)^\top (I_Q\otimes X_U)(U-Z) + \tfrac{\kappa}{2}L^\top(\Psi\otimes H_L)L.$$

By the standard usage of the adjoint approach, the derivative of the above objective is given by

$$D\widehat{J}_0(\ell)(b) = \int_\Gamma \rho(y) T(b(y,\cdot), u(y,\cdot), w(y,\cdot))dy, \quad (60)$$

where $w\in V$ satisfies, for every $v\in V$, the adjoint equation:

$$\int_\Gamma \rho(y) T(\ell(y,\cdot), w(y,\cdot), v(y,\cdot))\,dy = \int_\Gamma \rho(y)\langle z(y,\cdot)-u(y,\cdot), v(y,\cdot)\rangle_X\, dy.$$

By an analogous discretization scheme that was used for the ELS functional, we obtain

$$\left[\sum_{s=0}^M G^s\otimes K(L_s)\right] W = (I_Q\otimes X_U)(Z-U).$$

The corresponding gradient is given by

$$\nabla \widehat{J}_0(A) = \left[U^\top \left(G^0 \otimes I_N\right) S(W) \dots U^\top \left(G^M \otimes I_N\right) S\left(W\right)\right].$$

Summarizing, we have the following discrete formula for the regularized OLS:

$$\nabla \widehat{J}_\kappa(A) = \left[U^\top \left(G^0 \otimes I_N\right) S(W) \ \dots \ U^\top \left(G^M \otimes I_N\right) S\left(W\right)\right] + \kappa L^\top \left(\Psi \otimes H_L\right)).$$

7. Applications and examples

This section illustrates the abstract framework by providing concrete examples of stochastic elliptic boundary value problems, the corresponding trilinear forms, and the associated energy least-squares functionals.

Example 7.1. Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space, and let $D \subset \mathbb{R}^n$ be a bounded domain with ∂D as the sufficiently smooth boundary of D. Given random fields $a : \Omega \times D \to \mathbb{R}$, $b : \Omega \times D \to \mathbb{R}$, and $f : \Omega \times D \to \mathbb{R}$, we consider the stochastic boundary value problem (BVP) of finding a random field $u : \Omega \times D \to \mathbb{R}$ that almost surely satisfies

$$-\nabla \cdot (a(\omega,x)\nabla u(\omega,x)) + b(\omega,x)u(\omega,x) = f(\omega,x), \quad \text{in} \quad D, \tag{61a}$$

$$u(\omega,x) = 0, \quad \text{on} \quad \partial D. \tag{61b}$$

In the context of (61), we are interested in the simultaneous identification of the random parameters $a(\omega,x) \in \mathbb{A}_1$ and $b(\omega,x) \in \mathbb{A}_2$, where, given constants a_1, a_2, b_1, b_2, we define

$$\mathbb{A}_1 := \{a(\omega,x) \in L^\infty(\Omega; L^\infty(D)) : \ 0 < a_1 \le a(\omega,x) \le a_2\},$$
$$\mathbb{A}_2 := \{b(\omega,x) \in L^\infty(\Omega; L^\infty(D)) : \ 0 < b_1 \le b(\omega,x) \le b_2\}.$$

The stochastic variational formulation of (61) (in the integral form) seeks some $u \in L^2(\Omega, H_0^1(D))$ such that for every $v \in L^2(\Omega, H_0^1(D))$, we have

$$\begin{aligned}\mathbb{E}\left[\int_D [a(\omega,x)\nabla u(\omega,x)\nabla v(\omega,x) + b(\omega,x)u(\omega,x)v(\omega,x)]\right]\\ = \mathbb{E}\left[\int_D f(\omega,x)v(\omega,x)\right].\end{aligned} \tag{62}$$

The trilinear form $T : L^\infty(D)^2 \times H_0^1(D) \times H_0^1(D) \to \mathbb{R}$ takes the following form

$$T(\ell,u,v) = \int_D \left[a\nabla u \nabla v + buv\right], \quad \ell = (a,b).$$

Then, the ELS functional for the inverse problem of identifying $\ell = (a,b)$ is given by

$$\begin{aligned}\mathbb{J}(\ell) = \frac{1}{2}\mathbb{E}\Big[\int_D \Big[&a(\omega,x)\nabla(u_\ell(\omega,x) - z(\omega,x)) \cdot \nabla(u_\ell(\omega,x) - z(\omega,x))\\ &+ b(\omega,x)(u_\ell(\omega,x) - z(\omega,x))(u_\ell(\omega,x) - z(\omega,x))\Big]\Big],\end{aligned}$$

where $z \in L^2(\Omega; H_0^1(D))$ is the data and $u_\ell \in L^2(\Omega, H_0^1(D))$ solves (7.1). $\square$

Example 7.2. Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space, and let $D \subset \mathbb{R}^n$ be a bounded domain with ∂D as the sufficiently smooth boundary of D. Given random fields $a : \Omega \times D \to \mathbb{R}$ and $f : \Omega \times D \to \mathbb{R}$, we consider the problem of finding a random field $u : \Omega \times D \to \mathbb{R}$ that almost surely satisfies the following BVP:

$$\Delta(a(\omega, x)\Delta u(\omega, x)) = f(\omega, x), \quad \text{in} \quad D, \tag{63a}$$

$$u(\omega, x) = 0, \quad \text{on} \quad \partial D, \tag{63b}$$

$$\partial_n u(\omega, x) = 0, \quad \text{on} \quad \partial D, \tag{63c}$$

where Δ is the Laplace operator and $\partial_n u$ is the normal derivative.

Fourth-order BVP (63) models the pure bending of a Kirchhoff plate that occupies the region D. The parameter a is the flexural rigidity, and the solution u is the lateral deflection under the force f per unit area. The parameter a is related to the thickness of the plate, Young's modulus of elasticity, and the Poisson ratio of the material.

The stochastic variational formulation of (63) (in the integral form) seeks some $u \in L^2(\Omega, H_0^2(D))$ such that for every $v \in L^2(\Omega, H_0^2(D))$, we have

$$\mathbb{E}\Big[\int_D a(\omega, x)\Delta u(\omega, x)\Delta v(\omega, x)\Big] = \mathbb{E}\Big[\int_D f(\omega, x)v(\omega, x)\Big]. \tag{64}$$

To estimate the coefficient a, given constants a_1, a_2, we define

$$\mathbb{A} := \{a \in L^\infty(\Omega, L^\infty(D))|\ 0 < a_1 \le a(\omega, x) \le a_2\}.$$

The trilinear form $T : L^\infty(D) \times H_0^2(D) \times H_0^2(D) \to \mathbb{R}$ is defined by

$$T(a, u, v) = \int_D a\Delta u\Delta v,$$

and consequently the ELS functional for the inverse problem of identifying $a \in \mathbb{A}$, with the data $z \in L^2(\Omega, H_0^2(\Omega))$, is given by:

$$\mathbb{J}(a) = \mathbb{E}\left[\int_\Omega a(\omega, x)\Delta(u_a(\omega, x) - z(\omega, x)\Delta(u_a(\omega, x) - z(\omega, x))\right],$$

where u_a solves (64).

Example 7.3. Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space, and let $D \subset \mathbb{R}^n$ be a bounded domain with ∂D as the sufficiently smooth boundary of D. Given random fields $a\colon \Omega\times D \to \mathbb{R}$ and $f\colon \Omega\times D \to \mathbb{R}$, we consider the problem of finding a random field $u\colon \Omega \times D \to \mathbb{R}$ that almost surely satisfies the following variant of the BVP (63):

$$\begin{aligned} \mu\Delta(a(\omega, x)\Delta u(\omega, x)) + (1-\mu)((a(\omega, x)u_{xx}(\omega, x))_{xx} \\ +2(a(\omega, x)u_{xy}(\omega, x))_{xy} + (au_{yy})_{yy}) = f(\omega, x), \quad \text{in} \quad D, \qquad (65) \\ u(\omega, x) = 0, \quad \text{on} \quad \partial D, \\ \partial_n u(\omega, x) = 0, \quad \text{on} \quad \partial D, \end{aligned}$$

where $\mu > 0$ is a known constant.

The stochastic variational formulation of (65) (in the integral form) seeks some $u \in L^2(\Omega, H_0^2(D))$ such that for every $v \in L^2(\Omega, H_0^2(D))$, we have

$$\begin{aligned}\mathbb{E}\Big[\int_D \Big[&\mu a(\omega,x)\Delta u(\omega,x)\Delta v(\omega,x) + (1-\mu)(a(\omega,x)u_{xx}(\omega,x)v_{xx}(\omega,x)\\ &+ 2a(\omega,x)u_{xy}(\omega,x)v_{xy}(\omega,x) + a(\omega,x)u_{yy}(\omega,x)v_{yy}(\omega,x))\Big]\Big]\\ = \mathbb{E}\Big[\int_\Omega & f(\omega,x)v(\omega,x)\Big]. \end{aligned} \tag{66}$$

For the inverse problem of identifying the variable coefficient a, we define

$$\mathbb{A} := \{a \in L^\infty(\Omega, L^\infty(D)) |\ 0 < a_1 \le a(\omega,x) \le a_2\}$$

where a_1, a_2 are constants. The trilinear form $T : L^\infty(D) \times H_0^2(D) \times H_0^2(D) \to \mathbb{R}$ then reads

$$T(a,u,v) = \int_D [\mu a \Delta u \Delta v + (1-\mu)(a u_{xx} v_{xx} + 2a u_{xy} v_{xy} + a u_{yy} v_{yy})]\,.$$

The ELS functional for identifying $a \in \mathbb{A}$, with $z \in L^2(\Omega, H_0^2(D))$, is given by:

$$\begin{aligned}\mathbb{J}(a) = \mathbb{E}\Big[\int_\Omega \Big[&\mu a(\omega,x)\Delta(u(\omega,x)-z(\omega,x))\Delta(u(\omega,x)-z(\omega,x))\\ &+ (1-\mu)(a(\omega,x)(u(\omega,x)-z(\omega,x))_{xx}(u(\omega,x)-z(\omega,x))_{xx}\\ &+ 2a(\omega,x)(u(\omega,x)-z(\omega,x))_{xy}(u(\omega,x)-z(\omega,x))_{xy}\\ &+ a(\omega,x)(u(\omega,x)-z(\omega,x))_{yy}(u(\omega,x)-z(\omega,x))_{yy})\Big]\Big].\end{aligned}$$

Example 7.4. We now return to stochastic linear elasticity system described in (1). Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space. Let $D \subset \mathbb{R}^2$ be a bounded domain and let ∂D be its sufficiently smooth boundary $\partial D = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$. We consider the stochastic linear elasticity system:

$$-\nabla \cdot [2\mu(\omega,x)\epsilon_u + \lambda(\omega,x)\mathrm{tr}\,(\epsilon_u)\, I] = f(\omega,x), \text{ in } D, \tag{67a}$$

$$u(\omega,x) = 0, \text{ on } \Gamma_1, \tag{67b}$$

$$[2\mu(\omega,x)\epsilon_u + \lambda(\omega,x)\mathrm{tr}\,(\epsilon_u)\, I]\, n = h(\omega,x), \text{ on } \Gamma_2. \tag{67c}$$

The stochastic variational formulation of (67) (in the integral form) seeks some $u \in L^2(\Omega, Y)$ such that for every $v \in L^2(\Omega, Y)$, we have

$$\int_\Omega 2\mu(\omega,x)\epsilon(u(\omega,x)) \cdot \epsilon(v(\omega,x)) + \lambda(\omega,x)\mathrm{div}u(\omega,x)\mathrm{div}v(\omega,x) = \int_\Omega f(\omega,x)v(\omega,x).$$

where Y is the test space defined by

$$Y = \left\{u : D \to \mathbb{R}^2 \,:\, u_1, u_2 \in H^1(D),\ u_1 = u_2 = 0 \text{ on } \Gamma_1\right\}.$$

To identify the variable coefficients μ and λ, we define

$$\mathbb{A}_1 := \{\mu \in L^\infty(\Omega, L^\infty(D)) :\ 0 < \mu_1 \le \mu(\omega,x) \le \mu_2\},$$
$$\mathbb{A}_2 := \{\lambda \in L^\infty(\Omega, L^\infty(D)) :\ 0 < \lambda_1 \le \lambda(\omega,x) \le \lambda_2\},$$

where $\mu_1, \mu_2, \lambda_1, \lambda_2$ are constants.

The trilinear form $T : L^\infty(\Omega)^2 \times Y \times Y \to \mathbb{R}$ takes the following form

$$T(\ell, u, v) = \int_\Omega \left[2\mu\epsilon(u) \cdot \epsilon(v) + \lambda\nabla \cdot u\nabla \cdot v\right], \quad \ell = (\mu, \lambda),$$

and the ELS objective for identifying the Lamé coefficients $\ell = (\mu, \lambda)$ reads

$$\mathbb{J}(\mu, \lambda) = \frac{1}{2}\mathbb{E}\Big[\int_D 2\mu(\omega, x)|\epsilon(u(\omega, x)-z(\omega, x))|^2+\lambda(\omega, x)|\nabla \cdot (u(\omega, x)-z(\omega, x))|^2\Big].$$

8. A computational example

In this section, we apply the developed framework to identify the stochastic Láme parameters in the system of linear elasticity.

For the numerical computations, we take $D = (0, 1) \times (0, 1)$ and for $Y_1(\omega) \sim U[0, 1]$ uniformly distributed over $[0, 1]$, we define the random parameter

$$\bar{\ell}(\omega, x) = (\bar{\mu}(\omega, x), \bar{\lambda}(\omega, x)) = (1 + x_1^2 + Y_1(\omega), 3 + Y_1(\omega)(x_1^2 + x_2^2)),$$

and the random displacement $\bar{u} = (\bar{u}_1(\omega, x), \bar{u}_2(\omega, x))$ given by

$$\begin{aligned}\bar{u}_1(\omega, x) &= (\cos(2\pi x_1) - 1)\sin(2\pi x_2) + Y_1(\omega)\sin(\pi x_1)\sin(\pi x_2)\\ \bar{u}_2(\omega, x) &= (1 - \cos(2\pi x_2))\sin(2\pi x_1) + \sin(\pi x_1)\sin(\pi x_2),\end{aligned}$$

while $f(\omega, x) = (f_1(\omega, x), f_2(\omega, x))$ is computed accordingly. This example corresponds to full Dirichlet conditions $\Gamma_1 = \partial D$. We consider $\mathbb{P}_2$ Lagrange finite element for the discrete displacement space V_h and $\mathbb{P}_1$ elements for the discrete parameter space L_h. In this case, $M = 1$, due to the uniform distribution $\sigma(y) = 1$ and we use Legendre polynomial defined on $[0, 1]$. All numerical experiments were done in Python by using finite element library FEniCS (see [38]).

$\dim V_h / \dim A_h$	$\frac{\mathbb{E}\left(\int_D (\bar{u}_h(\omega,x)-u_h(y,x))^2 dx\right)}{\mathbb{E}\left(\int_D (\bar{u}_h(\omega,x)^2 dx\right)}$	$\frac{\mathrm{Var}\left(\int_D (\bar{u}_h(\omega,x)-u_h(\omega,x)^2 dx)\right)}{\mathrm{Var}\left(\int_D (\bar{u}_h(\omega,x))^2 dx)\right)}$
882/242	2.9818e-03	3.2073e-03
1250/338	1.6428e-03	2.0422e-03
1682/450	1.0098e-03	1.4121e-03
2178/578	6.7381e-04	1.0353e-03
2738/772	4.7872e-04	7.9249e-04
3362/882	3.5701e-04	6.2689e-04

Table 1: Stochastic Galerkin discretization error for Example 7.1.

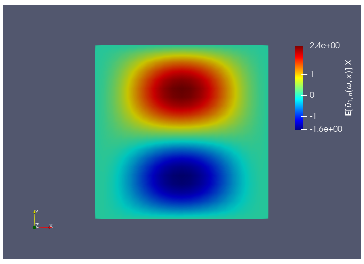
For the chosen discretization level, the Stochastic Galerkin gives good approximation for the direct problem as we can see in Table 1 and Figures 1 and 2. We measure the expectation and the variance of the identification error via the (relative) error functional. Denoting by $\ell_h^{ELS} = \left(\mu_h^{ELS}, \lambda_h^{ELS}\right)$, the numerical solution to the ELS problem (58), we measure the error separately. For example, for the ELS objective functional, we estimate the identification error by the quantities

$$\varepsilon_{\text{mean}}^{ELS}(\mu) = \frac{\sqrt{\int_D (\mathbb{E}[\bar{\mu}_h(\omega,x)] - \mathbb{E}[\mu_h^{ELS}(\omega,x)])^2 dx}}{\sqrt{\int_D \mathbb{E}\left[\bar{\mu}_h(\omega,x)\right]^2 dx}},$$

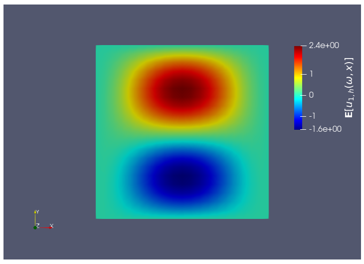
$$\varepsilon_{\text{var}}^{ELS}(\mu) = \frac{\sqrt{\int_D (\text{Var}\left[\bar{\mu}_h(\omega,x)\right] - \text{Var}[\mu_h^{ELS}(\omega,x)])^2 dx}}{\sqrt{\int_D \text{Var}\left[\bar{\mu}_h(\omega,x)\right]^2 dx}}.$$

$\frac{\dim V_h}{\dim(A_h)}$	$\varepsilon_{\text{mean}}^{ELS}(\mu)$	$\varepsilon_{\text{var}}^{ELS}(\mu)$	$\varepsilon_{\text{mean}}^{ELS}(\lambda)$	$\varepsilon_{\text{var}}^{ELS}(\lambda)$	$\varepsilon_{\text{mean}}^{ELS}(u)$	$\varepsilon_{\text{var}}^{ELS}(u)$
882/242	1.4570e-02	4.3279e-02	2.5738e-02	3.1505e-01	1.0625e-03	2.4366e-03
1250/338	8.1752e-03	2.3983e-02	1.3073e-02	1.7302e-01	5.2964e-04	1.2922e-03
1682/450	5.1086e-03	1.6655e-02	8.8290e-03	1.2810e-01	3.0260e-04	7.8048e-04
2178/578	3.6639e-03	1.3930e-02	6.9702e-03	1.0694e-01	1.9336e-04	5.9069e-04
2738/772	3.1454e-03	1.2892e-02	6.1073e-03	9.6859e-01	1.4345e-04	4.8495e-04
3362/882	2.9329e-03	1.2957e-02	6.0540e-03	9.6032e-01	1.1444e-04	4.2228e-04

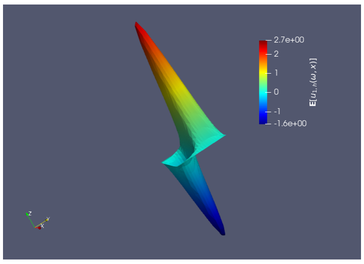
Table 2: Numerical errors for the ELS approach recorded using κ =1e-04 in Ex. 7.1.



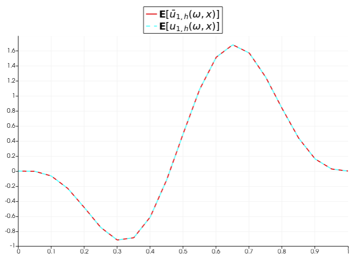
(a) SG approximation $\mathbb{E}[u_{1,h}(\omega,x)]$



(b) Exact $\mathbb{E}[\bar{u}_{1,h}(\omega,x)]$



(c) 3D reconstruction of $\mathbb{E}[u_{1,h}(\omega,x)]$



(d) Line reconstruction $\mathbb{E}[u_{1,h}(\omega,x,x)]$

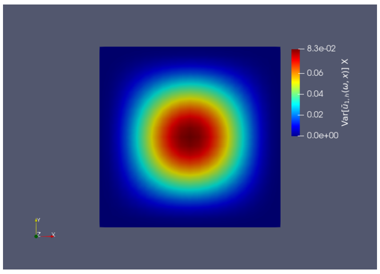
Figure 1: Direct problem. Stochastic Galerkin approximation of the mean of the displacement u_1 ($\dim V_h = 3362/\dim A_h = 882$).

Similarly, we measure the simulated data error by the quantities

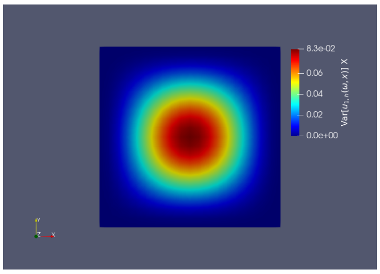
$$\varepsilon_{\text{mean}}^{ELS}(u) = \frac{\sqrt{\int_D (\mathbb{E}[\bar{u}_h(\omega,x)] - \mathbb{E}[u_h(\ell_h^{ELS})(\omega,x)])^2 dx}}{\sqrt{\int_D \mathbb{E}\left[\bar{u}_h(\omega,x)\right]^2 dx}},$$

$$\varepsilon_{\text{var}}^{ELS}(u) = \frac{\sqrt{\int_D (\text{Var}\left[\bar{u}_h(\omega,x)\right] - \text{Var}[u_h(\ell_h^{ELS})(\omega,x)])^2 dx}}{\sqrt{\int_D \text{Var}\left[\bar{u}_h(\omega,x)\right]^2 dx}},$$

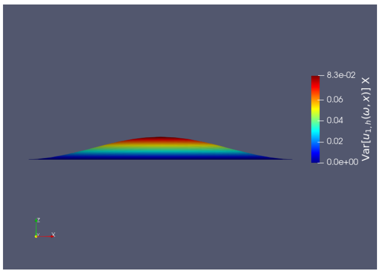
where $u_h(\ell_h^{ELS})(\omega, x)$ solves the Stochastic Galerkin system (57) for the estimated ℓ_h^{ELS}.



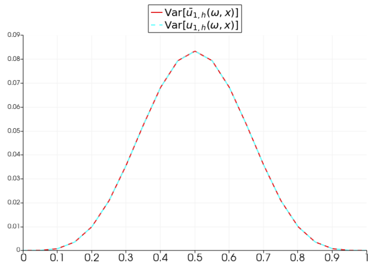
(a) SG approximation $\text{Var}[u_{1,h}(\omega, x)]$



(b) Exact $\text{Var}[\bar{u}_{1,h}(\omega, x)]$

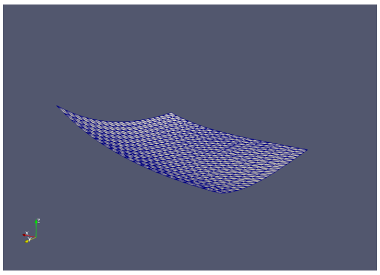


(c) 3D reconstruction of $\text{Var}[u_{1,h}(\omega, x)]$

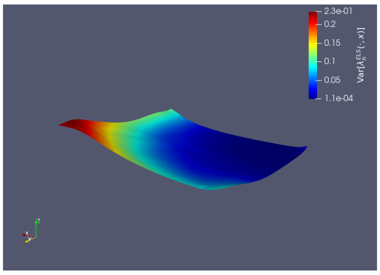


(d) Line reconstruction $\text{Var}[u_{1,h}(\omega, x, x)]$

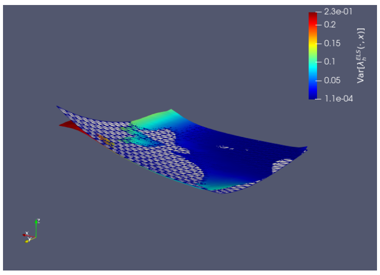
Figure 2: Direct problem. Stochastic Galerkin approximation of the variance of displacement u_1 ($\dim V_h = 3362/\dim A_h = 882$).



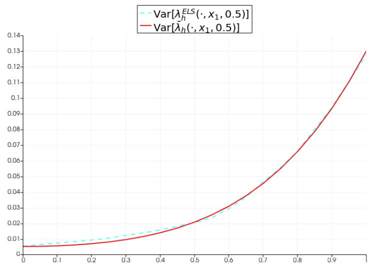
(a) 3D Exact $\text{Var}[\bar{\lambda}_h^{ELS}(\omega, x)]$



(b) 3D reconstruction of $\text{Var}[\lambda_h^{ELS}(\omega, x)]$



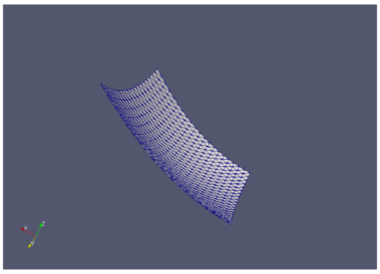
(c) 3D comparison ($\varepsilon_{\text{var}}^{ELS}(\lambda)$=9.6023e-01)



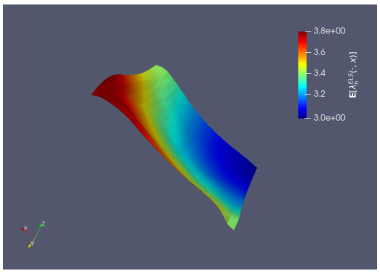
(d) Line reconstruction $\text{Var}[\lambda_h^{ELS}(\omega, x_1, 0.5)]$

Figure 3: Variance identification for parameter $\bar{\lambda}(\omega, x)$ by ELS ($\dim V_h = 3362$ / $\dim A_h = 882$).

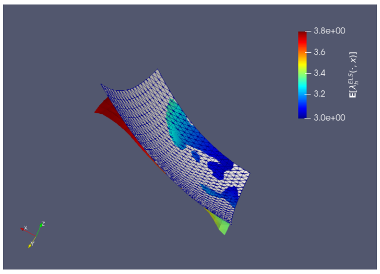
The numerical results for the identification of both parameters given in Table 2 for different discretization levels correspond to the regularization parameter κ =1e-04. The identification results are excellent for the mean recovery and good for the variance recovery of both the parameters, see Figures 3, 4, 5 and 6. Table 2 shows that the simulated data gives an excellent reconstruction.



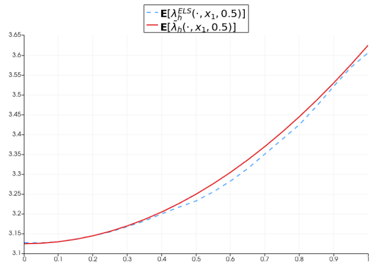
(a) 3D Exact $\mathbb{E}[\bar{\lambda}_h^{ELS}(\omega, x)]$



(b) 3D reconstruction of $\mathbb{E}[\lambda_h^{ELS}(\omega, x)]$

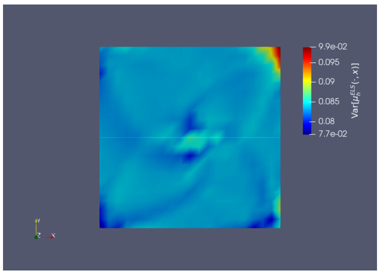


(c) 3D comparison ($\varepsilon_{\text{mean}}^{ELS}(\lambda)$=6.0540e-03)

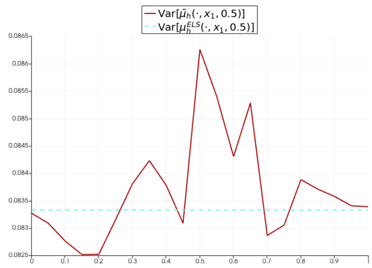


(d) Line reconstruction $\mathbb{E}[\lambda_h^{ELS}(\omega, x_1, 0.5)]$

Figure 4: Mean identification for parameter $\bar{\lambda}(\omega, x)$ by ELS ($\dim V_h = 3362$ / $\dim A_h = 882$).



(a) Reconstruction $\text{Var}[\mu_h^{ELS}(\omega, x)]$ ($\varepsilon_{\text{var}}^{ELS}(\mu)$ =1.2957e-02)



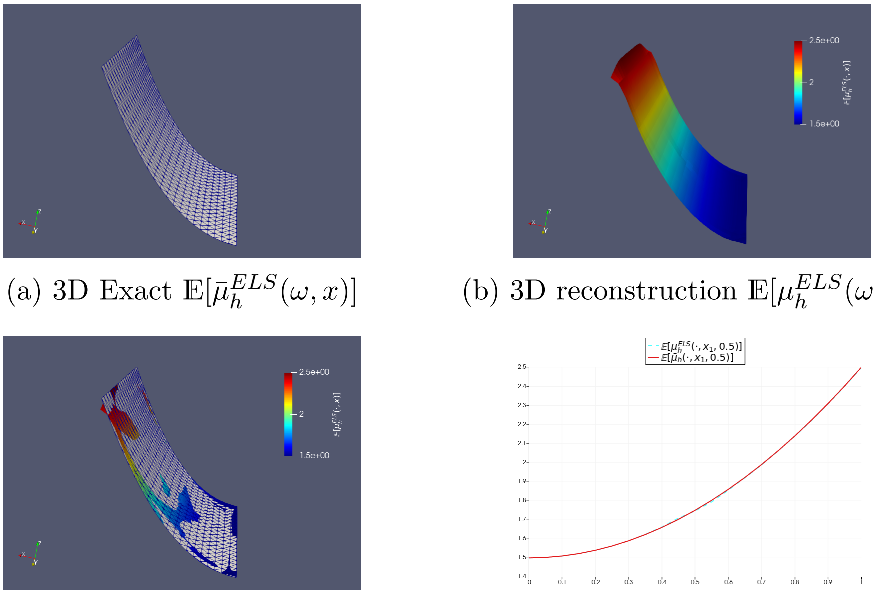
(b) Line reconstruction $\text{Var}[\mu_h^{ELS}(\omega, x_1, 0.5)]$

Figure 5: Variance identification for parameter $\bar{\lambda}(\omega, x)$ by ELS ($\dim V_h = 3362$ / $\dim A_h = 882$).

9. Concluding remarks

We developed an abstract framework for solving stochastic elliptic inverse problems. It would be of interest to derive error estimates for stochastic inverse problems. Another critical research direction we would like to pursue is the identification of rapidly varying or even discontinuous parameters. In recent years, second-order stochastic approximation approaches have been used extensively. It would be of interest to employ such methods for inverse problems.

Acknowledgements: Project PID2020-112491GB-I00/AEI/10.13039/501100011033 through the Ministerio de Ciencia e Innovacion, Agencia Estatal de Investigacion, Spain (Khan, Sama) partially supported this research. A. A. Khan is also supported



(a) 3D Exact $\mathbb{E}[\bar{\mu}_h^{ELS}(\omega, x)]$

(b) 3D reconstruction $\mathbb{E}[\mu_h^{ELS}(\omega, x)]$

(c) 3D comparison ($\varepsilon_{\text{mean}}^{ELS}(\mu)$=2.9329e-03)

(d) Line reconstruction $\mathbb{E}[\mu_h^{ELS}(\omega, x_1, 0.5)]$

Figure 6: Mean identification for parameter $\bar{\mu}(\omega, x)$ by ELS ($\dim V_h = 3362$ / $\dim A_h = 882$).

by FEAD AY-22-23 Grant. The authors are grateful to the referees for their carefulreading and corrections, which substantially improved the paper. Miguel Sama acknowledges the support of the Escuela Técnica Superior de Ingenieros Industriales (UNED) of Spain, project 2023-ETSII-UNED-01.

References

[1] A. Alexanderian, N. Petra, G. Stadler, O. Ghattas: *Mean-variance risk-averse optimal control of systems governed by PDEs with random parameter fields using quadratic approximations*, SIAM/ASA J. Uncert. Quant. 5/1 (2017) 1166–1192.

[2] W. Arendt, M. Kreuter: *Mapping theorems for Sobolev spaces of vector-valued functions*, Studia Math. 240/3 (2018) 275–299.

[3] V. A. Badri Narayanan, N. Zabaras: *Stochastic inverse heat conduction using a spectral approach*, Int. J. Num. Methods Eng. 60/9 (2004) 1569–1593.

[4] A. T. Bharucha-Reid: *Random Integral Equations*, Mathematics in Science and Engineering Vol. 96, Academic Press, New York (1972).

[5] J. Borggaard, H.-W. van Wyk: *Gradient-based estimation of uncertain parameters for elliptic partial differential equations*, Inverse Problems 31/6 (2015), art. no. 065008, 33 p.

[6] F. M. Borodich, B. A. Galanov: *Non-direct estimations of adhesive and elastic properties of materials by depth-sensing indentation*, Proc. Royal Soc. London Ser. A Math. Phys. Eng. Sci. 464 (2008) 2759–2776.

[7] J. Breidt, T. Butler, D. Estep: *A measure-theoretic computational method for inverse sensitivity problems. I: Method and analysis*, SIAM J. Num. Analysis 49/5 (2011) 1836–1859.

[8] M. Burger, S. Osher: *Convergence rates of convex variational regularization*, Inverse Problems 20/5 (2004) 1411–1421.

[9] T. Butler, D. Estep, S. Tavener, C. Dawson, J. J. Westerink: *A measure-theoretic computational method for inverse sensitivity problems. III: Multiple quantities of interest*, SIAM/ASA J. Uncert. Quant. 2/1 (2014) 174–202.

[10] P. Chen, A. Quarteroni, G. Rozza: *Multilevel and weighted reduced basis method for stochastic optimal control problems constrained by Stokes equations*, Num. Math. 133/1 (2016) 67–102.

[11] D. L. Cohn: *Measure Theory*, 2nd ed., Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser, New York (2013).

[12] A. Dieuleveut, A. Durmus, F. Bach: *Bridging the gap between constant step size stochastic gradient descent and Markov chains*, Ann. Statist. 48/3 (2020) 1348–1382.

[13] H. W. Engl, K. Kunisch, A. Neubauer: *Convergence rates for Tikhonov regularisation of nonlinear ill-posed problems*, Inverse Problems 5/4 (1989) 523–540.

[14] O. G. Ernst, A. Mugler, H.-J. Starkloff, E. Ullmann: *On the convergence of generalized polynomial chaos expansions*, ESAIM Math. Model. Numer. Analysis 46/2 (2012) 317–339.

[15] O. G. Ernst, B. Sprungk, H.-J. Starkloff: *Bayesian inverse problems and Kalman filters*, in: *Extraction of Quantifiable Information from Complex Systems*, Lect. Notes Comput. Sci. Eng. 102, Springer, Cham (2014) 133–159.

[16] O. G. Ernst, B. Sprungk, H.-J. Starkloff: *Analysis of the ensemble and polynomial chaos Kalman filters in Bayesian inverse problems*, SIAM/ASA J. Uncert. Quant. 3/1 (2015) 823–851.

[17] S. Esmaili, M. R. Eslahchi: *Application of fixed point-collocation method for solving an optimal control problem of a parabolic-hyperbolic free boundary problem modeling the growth of tumor with drug application*, Comput. Math. Appl. 75/7 (2018) 2193–2216.

[18] B. Faverjon, B. Puig, T. N. Baranger: *Identification of boundary conditions by solving Cauchy problem in linear elasticity with material uncertainties*, Comput. Math. Appl. 73/3 (2017) 494–504.

[19] N. Fikal, R. Aboulaich, E. M. E. Guarmah, N. Zemzemi: *Propagation of two independent sources of uncertainty in the electrocardiography imaging inverse solution*, Math. Model. Nat. Phenomena 14/2 (2019), art. no. 206, 16 p.

[20] J. N. Franklin: *Well-posed stochastic extensions of ill-posed linear problems*, J. Math. Analysis Appl. 31 (1970) 682–716.

[21] M. S. Gockenbach, A. A. Khan: *An abstract framework for elliptic inverse problems: Part 1. an output least-squares approach*, Math. Mech. Solids 12/3 (2007) 259–276.

[22] W. Grecksch, B. Jadamba, A. Khan, M. Sama, C. Tammer: *Inverse problem of estimating the stochastic flexural rigidity in fourth-order models*, Pure Appl. Funct. Analysis 6/6 (2021) 1273–1301.

[23] J. Gwinner: *A class of random variational inequalities and simple random unilateral boundary value problems – existence, discretization, finite element approximation*, Stochastic Analysis Appl. 18/6 (2000) 967–993.

[24] J. Gwinner, B. Jadamba, A. A. Khan, F. Raciti: *Uncertainty Quantification in Variational Inequalities*, CRC Press, Boca Raton (2022).

[25] J. Gwinner, F. Raciti: *On monotone variational inequalities with random data*, J. Math. Inequalities 3/3 (2009) 443–453.

[26] D. N. Hào, T. N. T. Quyen: *Convergence rates for Tikhonov regularization of coefficient identification problems in Laplace-type equations*, Inverse Problems 26/12 (2010), art. no. 125014, 23 p.

[27] R. Hawks, B. Jadamba, A. A. Khan, M. Sama, Y. Yang: *A variational inequality based stochastic approximation for inverse problems in stochastic partial differential equations*, in: *Nonlinear Analysis and Global Optimization*, T. Rassias et al. (eds.), Springer, Cham (2021) 207–226.

[28] M. Heinkenschloss, B. Kramer, T. Takhtaganov: *Adaptive reduced-order model construction for conditional value-at-risk estimation*, SIAM/ASA J. Uncert. Quant. 8/2 (2020) 668–692.

[29] B. Jadamba, A. A. Khan, G. Rus, M. Sama, B. Winkler: *A new convex inversion framework for parameter identification in saddle point problems with an application to the elasticity imaging inverse problem of predicting tumor location*, SIAM J. Appl. Math. 74/5 (2014) 1486–1510.

[30] B. Jadamba, A. A. Khan, M. Sama, H.-J. Starkloff, C. Tammer: *A convex optimization framework for the inverse problem of identifying a random parameter in a stochastic partial differential equation*, SIAM/ASA J. Uncert. Quant. 9/2 (2021) 922–952.

[31] B. Jadamba, A. A. Khan, M. Sama, Y. Yang: *An iteratively regularized stochastic gradient method for estimating a random parameter in a stochastic PDE. A variational inequality approach*, J. Nonlinear Var. Analysis 5 (2021) 865–880.

[32] B. Kaltenbacher, B. Hofmann: *Convergence rates for the iteratively regularized Gauss-Newton method in Banach spaces*, Inverse Problems 26/3 (2010), art. no. 035007, 21 p.

[33] B. Kaltenbacher, K. Van Huynh: *Iterative regularization for constrained minimization formulations of nonlinear inverse problems*, Comput. Optim. Appl. 81/2 (2022) 569–611.

[34] M. Keyanpour, A. M. Nehrani: *Optimal thickness of a cylindrical shell subject to stochastic forces*, J. Optim. Theory Appl. 167/3 (2015) 1032–1050.

[35] D. P. Kouri, M. Heinkenschloss, D. Ridzal, B. G. van Bloemen Waanders: *A trust-region algorithm with adaptive stochastic collocation for PDE optimization under uncertainty*, SIAM J. Sci. Comput. 35/4 (2013) A1847–A1879.

[36] P. S. Koutsourelakis: *Variational Bayesian strategies for high-dimensional, stochastic design problems*, J. Comput. Physics 308 (2016) 124–152.

[37] P. Kügler,E. Sincich: *Logarithmic convergence rates for the identification of a nonlinear Robin coefficient*, J. Math. Analysis Appl. 359/2 (2009) 451–463.

[38] H. P. Langtangen, A. Logg: *Solving PDEs in Python. The FEniCS Tutorial I*, Simula SpringerBriefs on Computing 3, Springer, Cham (2016.

[39] G. J. Lord, C. E. Powell, T. Shardlow: *An Introduction to Computational Stochastic PDEs*, Cambridge Texts in Applied Mathematics, Cambridge University Press, New York (2014).

[40] X. Ma, N. Zabaras: *An efficient Bayesian inference approach to inverse problems based on an adaptive sparse grid collocation method*, Inverse Problems 25/3 (2009), art. no. 035013, 27 p.

[41] J. Martin, L. C. Wilcox, C. Burstedde, O. Ghattas: *A stochastic Newton MCMC method for large-scale statistical inverse problems with application to seismic inversion*, SIAM J. Sci. Comput. 34/3 (2012) A1460–A1487.

[42] M. Martin, S. Krumschield, F. Nobile: *Analysis of stochastic gradient methods for PDE-constrained optimal control problems with uncertain parameters*, preprint (2018), 39 p.

[43] L. Mathelin, C. Desceliers, M. Y. Hussaini: *Stochastic data assimilation of the random shallow water model loads with uncertain experimental measurements*, Comput. Mech. 47/6 (2011) 603–616.

[44] A. Mugler, H.-J. Starkloff: *On elliptic partial differential equations with random coefficients*, Stud. Univ. Babeş-Bolyai Math. 56/2 (2011) 473–487.

[45] A. Mugler, H.-J. Starkloff: *On the convergence of the stochastic Galerkin method for random elliptic partial differential equations*, ESAIM Math. Model. Num. Analysis 47/5 (2013) 1237–1263.

[46] B. Notghi, M. Ahmadpoor, J. C. Brigham: *Adaptive reduced-basis generation for reduced-order modeling for the solution of stochastic nondestructive evaluation problems*, Comp. Methods Appl. Mech. Eng. 310 (2016) 172–188.

[47] A. Nouy, C. Soize: *Random field representations for stochastic elliptic boundary value problems and statistical inverse problems*, Eur. J. Appl. Math. 25/3 (2014) 339–373.

[48] E. Resmerita: *Regularization of ill-posed problems in Banach spaces: convergence rates*, Inverse Problems 21/4 (2005) 1303–1314.

[49] E. Resmerita, O. Scherzer: *Error estimates for non-quadratic regularization and the relation to enhancement*, Inverse Problems 22/3 (2006) 801–814.

[50] E. Rosseel, G. N. Wells: *Optimal control with stochastic PDE constraints and uncertain controls*, Comp. Methods Appl. Mech. Eng. 213/216 (2012) 152–167.

[51] K. Scaman, F. Bach, S. Bubeck, Y. T. Lee, L. Massoulié: *Optimal convergence rates for convex distributed optimization in networks*, J. Mach. Learning Res. 20 (2019), art. no. 159, 31p.

[52] C. Soize: *A computational inverse method for identification of non-Gaussian random fields using the Bayesian approach in very high dimension*, Comp. Methods Appl. Mech. Eng. 200/45-46 (2011) 3083–3099.

[53] R. Sternfels, C. J. Earls: *Reduced-order model tracking and interpolation to solve PDE-based Bayesian inverse problems*, Inverse Problems 29/7 (2013), art. no. 075014, 19p.

[54] R. E. Tanase: *Parameter estimation for partial differential equations using stochastic methods*, Ph. D. Thesis, University of Pittsburgh, ProQuest LLC, Ann Arbor (2016).

[55] H. Tiesler, R. M. Kirby, D. Xiu, T. Preusser: *Stochastic collocation for optimal control problems with stochastic PDE constraints*, SIAM J. Control Optimization 50/5 (2012) 2659–2682.

[56] W. I. T. Uy, M. Grigoriu: *Specification of additional information for solving stochastic inverse problems*, SIAM J. Sci. Comput. 41/5 (2019) A2880–A2910.

[57] J. van Neerven: *Compactness in vector-valued Banach function spaces*, Positivity 11/3 (2007) 461–467.

[58] J. van Neerven: *Compactness in the Lebesgue-Bochner spaces $L^p(\mu; X)$*, Indag. Math. (N.S.) 25/2 (2014) 389–394.

[59] C. Wang, H. G. Matthies: *Novel interval theory-based parameter identification method for engineering heat transfer systems with epistemic uncertainty*, Int. J. Num. Methods Eng. 115/6 (2018) 756–770.

[60] Y. Wang, X. Zhu, P. Kloeden: *Compactness in Lebesgue-Bochner spaces of random variables and the existence of mean-square random attractors*, Stoch. Dynamics 19/4 (2019), art. no. 1950032, 16p.

[61] Z. Yin: *Lipschitz stability for a semi-linear inverse stochastic transport problem*, J. Inverse Ill-Posed Problems 28/2 (2020) 185–193.

[62] N. Zabaras: *Solving stochastic inverse problems: a sparse grid collocation approach*, in: *Large-Scale Inverse Problems and Quantification of Uncertainty*, L. Biegler et al. (eds.), Wiley, Hoboken (2011) 291–319.

[63] X.-J. Zhang, X.-W. Du, Z,-P. Yang, G.-H. Lin: *An infeasible stochastic approximation and projection algorithm for stochastic variational inequalities*, J. Optim. Theory Appl. 183/3 (2019) 1053–1076.

[64] J. Zhu, J. C. James: *Stochastic approximation with nondecaying gain: error bound and data-driven gain-tuning*, Int. J. Robust Nonlinear Control 30/15 (2020) 5820–5870.