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Duality Between Lagrangians and Rockafellians

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Dedicated to Roger J-B Wets on the occasion of his 85th birthday.

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In his monograph *Conjugate Duality and Optimization* [CBMS-NSF Regional Conference Series in Applied Mathematics Vol. 16, Society for Industrial and Applied Mathematics, Philadelphia (1974)], R. T. Rockafellar puts forward a "perturbation + duality" method to obtain a dual problem for an original minimization problem. First, one embeds the minimization problem into a family of perturbed problems (thus giving a so-called perturbation function); the perturbation of the original function to be minimized has recently been called a Rockafellian. Second, when the perturbation variable belongs to a primal vector space paired, by a bilinear form, with a dual vector space, one builds a Lagrangian from a Rockafellian; one also obtains a so-called dual function (and a dual problem). The method has been extended from Fenchel duality to generalized convexity: when the perturbation belongs to a primal set paired, by a coupling function, with a dual set, one also builds a Rockafellian from a Lagrangian. Following these paths, we highlight a duality between Lagrangians and Rockafellians. Where the material mentioned above mostly focuses on moving from Rockafellian to Lagrangian, we treat them equally and display formulas that go both ways. We propose a definition of Lagrangian-Rockafellian couples. We characterize these latter as dual functions, with respect to a coupling, and also in terms of generalized convex functions. The duality between perturbation and dual functions is not as clear cut.

Keywords: Lagrangian, Rockafellian, duality, generalized convexity.

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1. Introduction

In [11], Rockafellar developed in a systematic way the "perturbation + duality" method (see also [3, 4, 5]). First, one embeds a minimization problem into a family of perturbed problems, thus giving a so-called perturbation function; the bivariate perturbation of the original function to be minimized has recently been called a *Rockafellian* [13, Chapter 5.A]. Second, when the perturbation variable belongs to a primal vector space paired, by a bilinear form, with a dual vector space, Rockafellar showed how one can build a *Lagrangian* from a Rockafellian; one also obtains a so-called dual function (and a dual problem).

A large part of the theory has been developed in the convex bifunction case [11, Section 29], that is, when the Rockafellian is jointly convex in both variables, namely decision and perturbation, or at least when the decision set is a linear space (see [11, Sect. 30], [12, Chap. 11.H-I], [13, Chapter 5]). This has much to do with obtaining strong duality results, for which convex analysis offers powerful tools. In the nonconvex case, it seems that the extension to generalized convexity has begun

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with [1] that makes use of couplings and conjugacies (see also [16], [6, Sect. 3] and references therein).

In this paper, we simply want to stress a duality between Lagrangians and Rockafellians, in the classic Fenchel bilinear pairing case and also in the more general coupling case. Our contribution is modest and formal. It is modest in that most of the material can be traced back to [11] and then [1] – and that we follow and slightly extend their paths – and also as we do not focus on strong duality. It is formal in that we treat Lagrangians and Rockafellians in a symmetric fashion, and highlight a duality between them (Theorem 3.4). So, there is no real novelty in the paper, but for a symmetric examination of two-way relationships between Lagrangians and Rockafellians when convexity is not assumed.

The paper is organized as follows. In Section 2, we consider the classic Fenchel bilinear pairing case. We revisit some of the results in [11] and try to reformulate the "perturbation + duality" method with as little convexity as possible in the assumptions. We recall and sketch how one can build a Lagrangian from a Rockafellian, and we highlight a converse construction. In Sect. 3, we recall how one can build a Lagrangian from a Rockafellian, in the case where the perturbation belongs to a primal set paired, by a coupling function, with a dual set. We also propose a converse construction, from Lagrangian to Rockafellian. Finally, we propose a notion of Lagrangian-Rockafellian couple, and we formally express duality between Lagrangians and Rockafellians.

2. Lagrangians and Rockafellians: bilinear pairing case

In § 2.1, we provide background on the classic Fenchel conjugacy. In § 2.2, we sketch, in one table, how one can build a Lagrangian from a Rockafellian as developed in [11]. In § 2.3, we sketch, in one table, how one can build a Rockafellian from a Lagrangian.

2.1. The bilinear pairing case

We consider two real vector spaces $\mathcal{U}$ and $\mathcal{V}$ paired, in the sense of convex analysis, by a bilinear form $\langle \cdot , \cdot \rangle : \mathcal{U} \times \mathcal{V} \to \mathbb{R}$. The classic Fenchel conjugacy $\star$ is defined, for any functions $f : \mathcal{U} \to \overline{\mathbb{R}}$ and $g : \mathcal{V} \to \overline{\mathbb{R}}$, by [1]

$$f^{\star}(v) = \sup_{u \in \mathcal{U}} \big(\langle u , v \rangle + (-f(u)) \big) \; , \;\; \forall v \in \mathcal{V} \; , \tag{1a}$$

$$g^{\star'}(u) = \sup_{v \in \mathcal{V}} \big(\langle u , v \rangle + (-g(v)) \big) \; , \;\; \forall u \in \mathcal{U} \; , \tag{1b}$$

$$f^{\star\star'}(u) = \sup_{v \in \mathcal{V}} \big(\langle u , v \rangle + (-f^{\star}(v)) \big) \; , \;\; \forall u \in \mathcal{U} \; , \tag{1c}$$

$$g^{\star'\star}(v) = \sup_{u \in \mathcal{U}} \big(\langle u , v \rangle + (-g^{\star'}(u)) \big) \; , \;\; \forall v \in \mathcal{V} \; . \tag{1d}$$

A function $f : \mathcal{U} \to \overline{\mathbb{R}}$ is said to be $\star$-convex if $f^{\star\star'} = f$ and a function $g : \mathcal{V} \to \overline{\mathbb{R}}$ is said to be $\star'$-convex if $g^{\star'\star} = g$. In both cases, this is equivalent to the function

[1] In convex analysis, one does not use the notation $\star'$, but simply $\star$. We use $\star'$ to be consistent with the notation (2c) for general conjugacies. Also the $+(-\cdots)$ expression is here to stress the proximity with the $\dot{+}\,(-\cdots)$ expression in Equations (2).

being either one of the two constant functions $-\infty$ and $+\infty$ or proper convex lsc (lower semi continuous) [10, Corollary 12.2.1].

2.2. From Rockafellians to Lagrangians: bilinear pairing

We sketch below the "perturbation + duality" method developed in [11]. More precisely, we follow [11, Sect. 4], but with less structure. Indeed, in [11, Sect. 4] it is supposed that the decision set $\mathcal{X}$ below is a linear space, in duality with another linear space. However, for this § 2.2, and for the whole paper, we do not require any assumption on the decision set $\mathcal{X}$.

Original minimization problem. Suppose given a nonempty set $\mathcal{X}$, a function $h : \mathcal{X} \to \overline{\mathbb{R}}$ and consider the *original minimization problem* $\inf_{x\in\mathcal{X}} h(x)$.

Perturbation scheme, Rockafellian. Let be given a vector space $\mathcal{U}$ and a function $R : \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$ such that $h(x) = R(x, 0)$, $\forall x \in \mathcal{X}$. The variables in $\mathcal{U}$ are called perturbations, and the function R is called a *Rockafellian* [13, Chapter 5] (a *dualizing parameterization* in [12, Definition 11.45], but with the requirement that $R(x, u)$ be convex lsc in the perturbation variable u).

Perturbation function. The original minimization problem is now embedded in a family given by the *perturbation function* [10, p. 295] (called *min-value function* in [13, p. 264]) $\varphi : \mathcal{U} \to \overline{\mathbb{R}}$ defined by $\varphi(u) = \inf_{x\in\mathcal{X}} R(x, u)$, $\forall u \in \mathcal{U}$. The original minimization problem corresponds to $\varphi(0) = \inf_{x\in\mathcal{X}} R(x, 0) = \inf_{x\in\mathcal{X}} h(x)$.

Dual problem, dual function, Lagrangian. Suppose that there exists a vector space $\mathcal{V}$ such that $\mathcal{U}$ and $\mathcal{V}$ are paired by a bilinear form $\langle \cdot, \cdot \rangle : \mathcal{U} \times \mathcal{V} \to \mathbb{R}$. Then, we obtain a *dual problem* by $\sup_{v\in\mathcal{V}}(-\varphi^\star(v)) = \varphi^{\star\star'}(0) \leq \varphi(0) = \inf_{x\in\mathcal{X}} h(x)$. In the dual problem appears the so-called *dual function* $\psi = -\varphi^\star : \mathcal{V} \to \overline{\mathbb{R}}$, also given by $\psi(v) = \inf_{x\in\mathcal{X}} L(x, v)$, $\forall v \in \mathcal{V}$, where the *Lagrangian* $L : \mathcal{X} \times \mathcal{V} \to \overline{\mathbb{R}}$ is defined by [2] $L(x, v) = \inf_{u\in\mathcal{U}}\{R(x, u) - \langle u, v\rangle\}$, $\forall (x, v) \in \mathcal{X} \times \mathcal{V}$.

Thus, in the classical setting of [11], one can build a Lagrangian from a Rockafellian as summarized and sketched in Table 2.1 (where usc stands for upper semi continuous). We do not provide a proof of the properties stated in Table 2.1, as these properties are well-known [11] (and as their proof will be a specialization of the proof of Proposition 3.1 in § 3.2 in the special case of the Fenchel bilinear pairing).

For any bivariate functions $L : \mathcal{X} \times \mathcal{V} \to \overline{\mathbb{R}}$ and $R : \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$, we denote, for all $x \in \mathcal{X}$, by $L(x, \cdot) : \mathcal{V} \to \overline{\mathbb{R}}$ and $R(x, \cdot) : \mathcal{U} \to \overline{\mathbb{R}}$ the partial functions obtained by "freezing" x. In [10, Sect. 29], Rockafellar uses the vocable of *bifunction* and the notation $L(x, \cdot) = (Lx)$, $R(x, \cdot) = (Rx)$.

2.3. From Lagrangians to Rockafellians: bilinear pairing

Conversely, one can build a Rockafellian from a Lagrangian as sketched in Table 2.2. This seems to be less classical than the opposite construction recalled in § 2.2, although the formula $R(x, u) = \sup_{v\in\mathcal{V}}\{L(x, v) + \langle u, v\rangle\}$ appears in [12, Equa-

[2] The expression of the Lagrangian is given in [11, Equation (4.2)] but with $+ \langle u, v\rangle$ instead of $- \langle u, v\rangle$. The expression we adopt is the one in [12, Equation 11(16)], but without requiring that $R(x, u)$ be convex lsc in the perturbation variable u.

sets	optimization set $\mathcal{X}$	primal space $\mathcal{U}$	pairing $\mathcal{U} \overset{\langle\cdot,\,\cdot\rangle}{\leftrightarrow} \mathcal{V}$	dual space $\mathcal{V}$
variables	decision $x \in \mathcal{X}$	perturbation $u \in \mathcal{U}$	$\langle u,\ v\rangle \in \mathbb{R}$	sensitivity $v \in \mathcal{V}$
bivariate functions		Rockafellian $R : \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$		Lagrangian $L : \mathcal{X} \times \mathcal{V} \to \overline{\mathbb{R}}$
definition				$L(x,v) = \inf_{u\in\mathcal{U}} \{R(x,u) - \langle u,\ v\rangle\}$
property				$-L(x,\cdot) = (R(x,\cdot))^{\star}$
property				$-L(x,\cdot)$ is $\star'$-convex (hence $L(x,\cdot)$ is concave usc)
univariate functions		perturbation function $\varphi : \mathcal{U} \to \overline{\mathbb{R}}$		dual function $\psi : \mathcal{V} \to \overline{\mathbb{R}}$
definition		$\varphi(u) = \inf_{x\in\mathcal{X}} R(x,u)$		$\psi(v) = \inf_{x\in\mathcal{X}} L(x,v)$
property				$-\psi = \varphi^{\star}$

Table 2.1: From Rockafellians to Lagrangians: bilinear pairing

tion 11(17)], as a consequence of [12, Equation 11(16)] and of the requirement that $R(x,u)$ be convex lsc in the perturbation variable u. Note also that [10, Chapter 33] establishes a relationship between the class of concave-convex saddle functions (Lagrangians of convex programs being a subclass) and the class of convex Rockafellians via such formula (see especially [10, Corollary 33.12]).

The properties stated in Table 2.2 will be a consequence of the results of Proposition 3.2 in § 3.3, in the special case of the Fenchel bilinear pairing.

3. Lagrangians and Rockafellians: general coupling case

In § 3.1, we provide background on couplings and Fenchel-Moreau conjugacies. In § 3.2, we recall and sketch, in one table, how one can build a Lagrangian from a Rockafellian [1]. In § 3.3, we sketch, in one table, how one can build a Rockafellian from a Lagrangian. Finally, in § 3.4, we propose a notion of Lagrangian-Rockafellian couple, and we formally express duality between Lagrangians and Rockafellians.

3.1. The general coupling case

When we manipulate functions with values in $\overline{\mathbb{R}} = [-\infty, +\infty]$, we adopt the Moreau *lower addition* [7, 8] that extends the usual addition with $(+\infty) \underset{\cdot}{+} (-\infty) = (-\infty) \underset{\cdot}{+} (+\infty) = -\infty$, and the Moreau *upper addition* that extends the usual addition with $(+\infty) \dot{+} (-\infty) = (-\infty) \dot{+} (+\infty) = +\infty$.

For background on coupling and conjugacies, we refer the reader to [15, 14, 6]. We consider two sets $\mathcal{U}$ and $\mathcal{V}$ paired by a *coupling* $c : \mathcal{U} \times \mathcal{V} \to \overline{\mathbb{R}}$. Then, one associates *conjugacies* from the set $\overline{\mathbb{R}}^{\mathcal{U}}$ of functions $\mathcal{U} \to \overline{\mathbb{R}}$ to the set $\overline{\mathbb{R}}^{\mathcal{V}}$ of functions $\mathcal{V} \to \overline{\mathbb{R}}$, and from $\overline{\mathbb{R}}^{\mathcal{V}}$ to $\overline{\mathbb{R}}^{\mathcal{U}}$ as follows.

sets	optimization set $\mathcal{X}$	primal space $\mathcal{U}$	pairing $\mathcal{U} \overset{\langle\cdot,\cdot\rangle}{\leftrightarrow} \mathcal{V}$	dual space $\mathcal{V}$
variables	decision $x \in \mathcal{X}$	perturbation $u \in \mathcal{U}$	$\langle u,\ v\rangle \in \mathbb{R}$	sensitivity $v \in \mathcal{V}$
bivariate functions		Rockafellian $R : \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$		Lagrangian $L : \mathcal{X} \times \mathcal{V} \to \overline{\mathbb{R}}$
definition		$R(x,u) = \sup_{v\in\mathcal{V}}\{L(x,v)+\langle u,\ v\rangle\}$		
property		$R(x,\cdot) = (-L(x,\cdot))^{\star'}$		
property		$R(x,\cdot)$ is $\star$-convex (hence $R(x,\cdot)$ is convex lsc)		
univariate functions		perturbation function $\varphi : \mathcal{U} \to \overline{\mathbb{R}}$		dual function $\psi : \mathcal{V} \to \overline{\mathbb{R}}$
definition		$\varphi(u) = \inf_{x\in\mathcal{X}} R(x,u)$		$\psi(v) = \inf_{x\in\mathcal{X}} L(x,v)$
property		$\varphi \geq (-\psi)^{\star'}$		

Table 2.2: From Lagrangians to Rockafellians: bilinear pairing

The *c-Fenchel-Moreau conjugate* of a function $f : \mathcal{U} \to \overline{\mathbb{R}}$ is the function $f^c : \mathcal{V} \to \overline{\mathbb{R}}$ defined by

$$f^c(v) = \sup_{u\in\mathcal{U}} \Big(c(u,v) \dotplus \big(-f(u)\big) \Big) \ , \ \ \forall v \in \mathcal{V} \ . \tag{2a}$$

With the coupling c, we associate the *reverse coupling* c' defined by

$$c' : \mathcal{V} \times \mathcal{U} \to \overline{\mathbb{R}} \ , \ \ c'(v,u) = c(u,v) \ , \ \ \forall (v,u) \in \mathcal{V} \times \mathcal{U} \ . \tag{2b}$$

The *c'-Fenchel-Moreau conjugate* of a function $g : \mathcal{V} \to \overline{\mathbb{R}}$ is the function $g^{c'} : \mathcal{U} \to \overline{\mathbb{R}}$ defined by

$$g^{c'}(u) = \sup_{v\in\mathcal{V}} \Big(c'(v,u) \dotplus \big(-g(v)\big) \Big) = \sup_{v\in\mathcal{V}} \Big(c(u,v) \dotplus \big(-g(v)\big) \Big) \ , \ \ \forall u \in \mathcal{U} \ . \tag{2c}$$

Then, the *c-Fenchel-Moreau biconjugate* of a function $f : \mathcal{U} \to \overline{\mathbb{R}}$ is the function $f^{cc'} : \mathcal{U} \to \overline{\mathbb{R}}$ defined by

$$f^{cc'}(u) = \big(f^c\big)^{c'}(u) = \sup_{v\in\mathcal{V}} \Big(c(u,v) \dotplus \big(-f^c(v)\big) \Big) \ , \ \ \forall u \in \mathcal{U} \ . \tag{2d}$$

Finally, the *c'-Fenchel-Moreau biconjugate* of a function $g : \mathcal{V} \to \overline{\mathbb{R}}$ is the function $g^{c'c} : \mathcal{V} \to \overline{\mathbb{R}}$ defined by

$$g^{c'c}(v) = \big(g^{c'}\big)^{c}(v) = \sup_{u\in\mathcal{U}} \Big(c(u,v) \dotplus \big(-g^{c'}(u)\big) \Big) \ , \ \ \forall v \in \mathcal{V} \ . \tag{2e}$$

A function $f : \mathcal{U} \to \overline{\mathbb{R}}$ is said to be *c-convex* if $f^{cc'} = f$ (which is equivalent to $f = g^{c'}$ for some $g : \mathcal{V} \to \overline{\mathbb{R}}$). A function $g : \mathcal{V} \to \overline{\mathbb{R}}$ is said to be *c'-convex* if $g^{c'c} = g$ (which is equivalent to $g = f^c$ for some $f : \mathcal{U} \to \overline{\mathbb{R}}$).

3.2. From Rockafellians to Lagrangians: general coupling

Following [1] – and taking inspiration from § 2.2 (see also [6, Sect. 3]) – one can build a Lagrangian from a Rockafellian as sketched in Table 3.1. The formal statement is given in Proposition 3.1. The result is not new [6, Sect. 3], but we give a proof for the sake of completeness and symmetry in the exposition.

sets	optimization set $\mathcal{X}$	primal set $\mathcal{U}$	coupling $\mathcal{U} \overset{c}{\leftrightarrow} \mathcal{V}$	dual set $\mathcal{V}$
variables	decision $x \in \mathcal{X}$	perturbation $u \in \mathcal{U}$	$c(u,v)$ $\in \overline{\mathbb{R}}$	sensitivity $v \in \mathcal{V}$
bivariate functions		Rockafellian $R : \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$		Lagrangian $L : \mathcal{X} \times \mathcal{V} \to \overline{\mathbb{R}}$
definition				$L(x,v) = \inf_{u \in \mathcal{U}} \big\{ R(x,u) \dot{+} (-c(u,v)) \big\}$
property				$-L(x,\cdot) = (R(x,\cdot))^c$
property				$-L(x,\cdot)$ is c'-convex
univariate functions		perturbation function $\varphi : \mathcal{U} \to \overline{\mathbb{R}}$		dual function $\psi : \mathcal{V} \to \overline{\mathbb{R}}$
definition		$\varphi(u) = \inf_{x \in \mathcal{X}} R(x,u)$		$\psi(v) = \inf_{x \in \mathcal{X}} L(x,v)$
property				$-\psi = \varphi^c$

Table 3.1: From Rockafellians to Lagrangians: general coupling

Proposition 3.1. (from [1, 6]) *We consider a set $\mathcal{X}$, and two sets $\mathcal{U}$ and $\mathcal{V}$ paired by a coupling $c : \mathcal{U} \times \mathcal{V} \to \overline{\mathbb{R}}$. We consider a Rockafellian $R : \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$ and we define the* Lagrangian $L : \mathcal{X} \times \mathcal{V} \to \overline{\mathbb{R}}$ *by*[3]

$$L(x,v) = \inf_{u \in \mathcal{U}} \big\{ R(x,u) \dot{+} (-c(u,v)) \big\} \ , \ \ \forall (x,v) \in \mathcal{X} \times \mathcal{V} \ , \tag{3}$$

the perturbation function $\varphi : \mathcal{U} \to \overline{\mathbb{R}}$ *and the* dual function $\psi : \mathcal{V} \to \overline{\mathbb{R}}$ *by*

$$\varphi(u) = \inf_{x \in \mathcal{X}} R(x,u) \ , \ \ \forall u \in \mathcal{U} \quad \textit{and} \quad \psi(v) = \inf_{x \in \mathcal{X}} L(x,v) \ , \ \ \forall v \in \mathcal{V} \ . \tag{4}$$

Then, we have that $-L(x,\cdot) = (R(x,\cdot))^c$ and $-L(x,\cdot)$ is c'-convex, for all $x \in \mathcal{X}$, and that $-\psi = \varphi^c$.

Proof. The equality $-L(x,\cdot) = (R(x,\cdot))^c$, for all $x \in \mathcal{X}$, is a straightforward application of definition (2a) with $f = R(x,\cdot)$ and of

$$-L(x,v) = \sup_{u \in \mathcal{U}} \big\{ c(u,v) \underset{\cdot}{+} (-R(x,u)) \big\}, \ \text{ for all } (x,v) \in \mathcal{X} \times \mathcal{V}.$$

[3] Our definition is close to [6, p. 252].

By property of conjugacies [6, Proposition 6.1 (ii)], we have that

$$(-L(x,\cdot))^{c'c} = (R(x,\cdot))^{cc'c} = (R(x,\cdot))^{c} = -L(x,\cdot),$$

for all $x \in \mathcal{X}$. Hence, $-L(x,\cdot)$ is c'-convex, for all $x \in \mathcal{X}$.

Finally, as conjugacies turn an infimum into a supremum, we have, by (4),

$$\varphi^{c} = (\inf_{x\in\mathcal{X}} R(x,\cdot))^{c} = \sup_{x\in\mathcal{X}}(R(x,\cdot))^{c} = \sup_{x\in\mathcal{X}}(-L(x,\cdot)) = -\inf_{x\in\mathcal{X}} L(x,\cdot) = -\psi. \qquad \square$$

The dual problem is

$$\varphi^{cc'}(u) = (-\psi)^{c'}(u) = \sup_{v\in\mathcal{V}}(c(u,v) \dotplus \psi(v)).$$

In the Fenchel bilinear pairing case, and in [11], we have that $u = 0$ and $c(0,v) = \langle 0,\ v\rangle = 0$, $\forall v \in \mathcal{V}$. This is generally no longer the case in the general coupling case.

3.3. From Lagrangians to Rockafellians: general coupling

Taking inspiration from § 2.3, we show one can build a Rockafellian from a Lagrangian as sketched in Table 3.2. The formal statement is given in Proposition 3.2, which seems to be new.

sets	optimization set $\mathcal{X}$	primal set $\mathcal{U}$	coupling $\mathcal{U} \overset{c}{\leftrightarrow} \mathcal{V}$	dual set $\mathcal{V}$
variables	decision $x \in \mathcal{X}$	perturbation $u \in \mathcal{U}$	$c(u,v) \in \overline{\mathbb{R}}$	sensitivity $v \in \mathcal{V}$
bivariate functions		Rockafellian $R : \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$		Lagrangian $L : \mathcal{X} \times \mathcal{V} \to \overline{\mathbb{R}}$
definition		$R(x,u) = \sup_{v\in\mathcal{V}}\{L(x,v) \dotplus c(u,v)\}$		
property		$R(x,\cdot) = (-L(x,\cdot))^{c'}$		
property		$R(x,\cdot)$ is c-convex		
univariate functions		perturbation function $\varphi : \mathcal{U} \to \overline{\mathbb{R}}$		dual function $\psi : \mathcal{V} \to \overline{\mathbb{R}}$
definition		$\varphi(u) = \inf_{x\in\mathcal{X}} R(x,u)$		$\psi(v) = \inf_{x\in\mathcal{X}} L(x,v)$
property		$\varphi \geq (-\psi)^{c'}$		

Table 3.2: From Lagrangians to Rockafellians: general coupling

Contrarily to Proposition 3.1 – where we obtained the equality $-\psi = \varphi^{c}$ between dual and perturbation functions – we now only obtain the inequality $\varphi \geq -\psi^{c'}$. This is because the definition (5) of a Rockafellian built from a Lagrangian involves a supremum operation, which does not behave, with a conjugacy, as nicely as an infimum operation does.

Proposition 3.2. *We consider a set $\mathcal{X}$, and two sets $\mathcal{U}$ and $\mathcal{V}$ paired by a coupling $c : \mathcal{U} \times \mathcal{V} \to \overline{\mathbb{R}}$. We consider a Lagrangian $L : \mathcal{X} \times \mathcal{V} \to \overline{\mathbb{R}}$ and we define the* Rockafellian $R : \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$ *by*

$$R(x,u) = \sup_{v\in\mathcal{V}}\{L(x,v) \dotplus c(u,v)\} \ , \ \ \forall (x,u) \in \mathcal{X} \times \mathcal{U} \ , \tag{5}$$

and the perturbation function $\varphi : \mathcal{U} \to \overline{\mathbb{R}}$ *and the* dual function $\psi : \mathcal{V} \to \overline{\mathbb{R}}$ *as in* (4)*. Then, we have that $R(x,\cdot) = (-L(x,\cdot))^{c'}$ and $R(x,\cdot)$ is c-convex, for all $x \in \mathcal{X}$, and that $\varphi \geq -\psi^{c'}$.*

Proof. The equality $R(x,\cdot) = (-L(x,\cdot))^{c'}$ for all $x \in \mathcal{X}$, is a straightforward application of definition (2c) with $g = -L(x,\cdot)$ and $R(x,\cdot)$ given by (5). By property of conjugacies [6, Proposition 6.1 (ii)], we have that $(R(x,\cdot))^{cc'} = (-L(x,\cdot))^{c'cc'} = (-L(x,\cdot))^{c'} = R(x,\cdot)$, for all $x \in \mathcal{X}$. Hence, $R(x,\cdot)$ is c-convex, for all $x \in \mathcal{X}$.

Finally, as conjugacies turn an infimum into a supremum, for all $x \in \mathcal{X}$ we have that $\varphi^c = (\inf_{x\in\mathcal{X}} R(x,\cdot))^c = \sup_{x\in\mathcal{X}}(R(x,\cdot))^c = \sup_{x\in\mathcal{X}}(-L(x,\cdot))^{cc'} \leq \sup_{x\in\mathcal{X}}(-L(x,\cdot)) = -\inf_{x\in\mathcal{X}} L(x,\cdot) = -\psi$ by (4), and where we have used that $(-L(x,\cdot))^{cc'} \leq -L(x,\cdot)$. Then, we get that $\varphi^{cc'} \geq (-\psi)^{c'}$, hence that $\varphi \geq \varphi^{cc'} \geq (-\psi)^{c'}$. □

3.4. Duality between Lagrangians and Rockafellians

To formally express duality between Lagrangians and Rockafellians, we take inspiration from [9, Sect. 8], especially the notions of *fonctions sur-duales* and of *fonctions duales* (dual functions), that is, minimal elements in a generalized Fenchel-Young type inequality (see also [10, p. 104–105]). The following definition and theorem are new.

Definition 3.3. We consider a set $\mathcal{X}$, and two sets $\mathcal{U}$ and $\mathcal{V}$ paired by a coupling $c : \mathcal{U} \times \mathcal{V} \to \overline{\mathbb{R}}$. We equip $\overline{\mathbb{R}}^{\mathcal{X}\times\mathcal{U}} \times \overline{\mathbb{R}}^{\mathcal{X}\times\mathcal{V}}$ with the natural ordering. We say that the functions $L : \mathcal{X} \times \mathcal{V} \to \overline{\mathbb{R}}$ and $R : \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$ form a *Lagrangian-Rockafellian* couple (L,R) if $(-L,R)$ is minimal in the inequality

$$(-L(x,v)) \dotplus R(x,u) \geq c(u,v) \ , \ \ \forall (x,u,v) \in \mathcal{X} \times \mathcal{U} \times \mathcal{V} \ . \tag{6}$$

We can now state our main result, which establishes a duality between Lagrangians and Rockafellians, expressed in different equivalent forms: as minimal elements in a generalized Fenchel-Young type inequality; by means of Fenchel-Moreau conjugates dual pairs; and also by means of generalized convex functions (equal to their Fenchel-Moreau biconjugate).

Theorem 3.4. *We consider a set $\mathcal{X}$, two sets $\mathcal{U}$ and $\mathcal{V}$ paired by a coupling function $c : \mathcal{U} \times \mathcal{V} \to \overline{\mathbb{R}}$, and two functions $L : \mathcal{X} \times \mathcal{V} \to \overline{\mathbb{R}}$ and $R : \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$. Then, the following are equivalent.*

(a) *The functions L and R form a Lagrangian-Rockafellian couple (L,R).*

(b) *Equality* (3) *and Equality* (5) *hold true, that is,*

$$L(x,v) = \inf_{u\in\mathcal{U}} \big\{R(x,u) \dotplus (-c(u,v))\big\} \ , \ \ \forall (x,v) \in \mathcal{X} \times \mathcal{V} \ , \tag{7a}$$

$$R(x,u) = \sup_{v\in\mathcal{V}}\{L(x,v) \dotplus c(u,v)\} \ , \ \ \forall (x,u) \in \mathcal{X} \times \mathcal{U} \ . \tag{7b}$$

(c) *The functions $-L$ and R are dual functions, with respect to the coupling c, in the sense that*

$$-L(x,\cdot) = \big(R(x,\cdot)\big)^{c} \quad \textit{and} \quad R(x,\cdot) = \big(-L(x,\cdot)\big)^{c'} \ , \ \ \forall x \in \mathcal{X} \ . \tag{8}$$

(d) *We have that*

$$-L(x,\cdot) = \big(R(x,\cdot)\big)^{c} \quad \textit{and} \quad \big(R(x,\cdot)\big)^{cc'} = R(x,\cdot) \ , \ \ \forall x \in \mathcal{X} \ . \tag{9}$$

(e) *We have that*

$$R(x,\cdot) = \big(-L(x,\cdot)\big)^{c'} \quad \textit{and} \quad \big(-L(x,\cdot)\big)^{c'c} = -L(x,\cdot) \ , \ \ \forall x \in \mathcal{X} \ . \tag{10}$$

The above equivalences are formal, in that they rather easily follow from definitions (Definition 3.3, Equation (2) in § 3.1) and from basic properties of conjugacies. They simply establish, in different equivalent ways, the duality between Lagrangians and Rockafellians. In particular, in (a), (b) and (c), Lagrangians and Rockafellians are characterized in a balanced fashion, by contrast with (d) and (e).

Proof. (a) $\iff$ (b): The equivalence follows from Definition 3.3 and from the following equivalences [7, 8] (see also [2, Appendix A.1]): for any $(x,u,v) \in \mathcal{X} \times \mathcal{U} \times \mathcal{V}$,

$$\big(-L(x,v)\big) \dotplus R(x,u) \geq c(u,v) \ , \tag{11a}$$

$$\iff L(x,v) \leq R(x,u) \dotplus \big(-c(u,v)\big) \ , \tag{11b}$$

$$\iff R(x,u) \geq L(x,v) \underset{\cdot}{+} c(u,v) \ . \tag{11c}$$

(b) $\iff$ (c): Equations (7) are equivalent to Equations (8) by definitions (2a) and (2c).

(c) $\implies$ (d): Equations (8) imply that $-L(x,\cdot) = \big(R(x,\cdot)\big)^{c}$ and $\big(R(x,\cdot)\big)^{cc'} = \Big(\big(R(x,\cdot)\big)^{c}\Big)^{c'} = \big(-L(x,\cdot)\big)^{c'} = R(x,\cdot)$, for all $x \in \mathcal{X}$. Hence, we obtain (9).

(d) $\implies$ (c): Equations (9) imply that $-L(x,\cdot) = \big(R(x,\cdot)\big)^{c}$ and $R(x,\cdot) = \big(R(x,\cdot)\big)^{cc'} = \Big(\big(R(x,\cdot)\big)^{c}\Big)^{c'} = \big(-L(x,\cdot)\big)^{c'}$, for all $x \in \mathcal{X}$. Hence, we obtain Equations (8).

(c) $\iff$ (e): This is shown by two implications: (8) $\implies$ (10) and (10) $\implies$ (8) as above. □

As a consequence, for a Lagrangian-Rockafellian couple (L,R), one necessarily has that $-L(x,\cdot)$ is c'-convex and $R(x,\cdot)$ is c-convex, for all $x \in \mathcal{X}$.

4. Conclusion

In this paper, we have highlighted a duality between Lagrangians and Rockafellians, as these two functions appear in the "perturbation + duality" method of [11]. We have treated both functions equally, and have provided formulas that go both way: from Rockafellian to Lagrangian (classical); from Lagrangian to Rockafellian (less classical). The setting is the one of so-called abstract or generalized convexity, where the perturbation belongs to a primal set paired, by a coupling function, with a dual set. It encompasses the classical Fenchel duality setting.

We have proposed a notion of Lagrangian-Rockafellian couple – minimal elements in a generalized Fenchel-Young type inequality – and we have formally expressed duality between Lagrangians and Rockafellians by means of Fenchel-Moreau conjugates dual pairs, and also by means of generalized convex functions (equal to their Fenchel-Moreau biconjugate).

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