Journal of Convex Analysis
Volume 30 (2023), No. 3, 937–950

By Bid-Ask Spreads towards Competitive Equilibrium

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Dedicated to Roger J-B Wets on the occasion of his 85th birthday.

Received: December 18, 2021
Accepted: January 6, 2023

Consider agents who just use markets and money to mediate trade and express economic interests. Suppose *bid-ask spreads* – derived from generalized differentials – drive and incite transactions. On that basis, this paper indicates good prospects for convergence towards *competitive equilibrium.* Presuming private property, main arguments hinge on three hooks: First, *Fenchel conjugation* to record profits or values added; second, *aggregate convexity* to support Pareto efficiency by prices; and third, *absence of externalities* to make total additions of value dwindle step by step.

Keywords: Conjugation, convexity, competitive equilibrium.

2010 Mathematics Subject Classification: 90C25, 91B24, 91B50.

1. Introduction

Two metaphors have long guided reasoning about markets. First, Adam Smith (1776) alluded to an "invisible hand". Second, Leon Walras (1874) suggested "tâtonnement" in prices. Both metaphors remain fictional, but both still hold some sway. They point respectively, to hidden versus central guidance. And broadly, either pits prices versus quantities.

Their lasting prominence may, however, stem from a more simple fact, namely: Trade generates processes – and their steady states, if any, might be exceptional. Yet, most microeconomic textbooks mainly study such states – and especially price-taking instances, called *competitive equilibria.*[1] These come idealized, institution-free and timeless. They mention no *money,* dispense with *bid-ask spreads,* and obviate *market mechanisms.* In fact, competitive equilibrium needs none of these agencies.

This notwithstanding, while the economy is out of equilibrium, the said items play chief roles. Intrigued by their absence, and by the reversal of natural order, this paper indicates how agents may use money to mediate trades and express economic

Research not funded.

[1] Using any equilibrium concept, "..*one has to keep it in its place, and its place is strictly in the preliminary stages of an analytic argument, not in the framing of hypotheses..*" [15] p. 78. Yet, precisely at this point, much economic theory got off on the wrong foot.

ISSN 0944-6532 / $ 2.50

interests. Coming naturally up then are bid-ask spreads. These drive trade, and they constitute the rationale for manifold market mechanisms.

Presuming that agents trade iteratively – each to improve own welfare - this paper indicates good prospects for convergence to competitive equilibrium. In doing so, it offers several novelties:

* a non-standard competitive equilibrium of *valuation sort* [4][2] – alongside a simple characterization (Definition 4.1 and Proposition 4.3),
* a formalized and informative version of bid-ask spread (Theorem 4.4),
* weakened assumptions on convexity and differentiability (Propositions 4.2&3), and
* a view on market-like sessions, leading agents towards competitive equilibrium (Theorem 6.3).

The paper addresses mathematically inclined optimers, system analysts and economists. It takes up topics at intersection of variational analysis, dynamical systems, and market theory.

Notations. Given a real vector space $\mathcal{X}$, let $\chi^* \in \mathcal{X}^*$ be shorthand for a *linear functional* $\chi \in \mathcal{X} \mapsto \chi^*\chi := \chi^*(\chi) \in R$, seen here as scheme for *pricing* or valuation. A *cost criterion* $c : \mathcal{X} \mapsto R \cup \{\pm\infty\}$ has *Fenchel conjugate*

$$\chi^* \in \mathcal{X}^* \mapsto c^*(\chi^*) := \sup\{\chi^*\chi - c(\chi) \mid \chi \in \mathcal{X}\}. \tag{1}$$

Its *subdifferential* $\partial c(\chi) \subseteq \mathcal{X}^*$ *at* χ is defined by

$$\chi^* \in \partial c(\chi) =: \check{\partial} c(\chi) \Longleftrightarrow \chi \in \arg\max(\chi^* - c) \text{ with finite value } c(\chi) \in R. \tag{2}$$

Clearly, $\partial c(\chi) \neq \varnothing \implies c$ must be *proper*, meaning *dom*$c := c^{-1}(R) \neq \varnothing$, and $c > -\infty$.

Similarly, turning things around, *benefit* $b : \mathcal{X} \mapsto R \cup \{\pm\infty\}$ has a corresponding *conjugate*

$$\chi^* \in \mathcal{X}^* \mapsto \hat{b}^*(\chi^*) := \sup\{b(\chi) - \chi^*\chi \mid \chi \in \mathcal{X}\} = (-b)^*(-\chi^*)$$

and a *supdifferential* $\hat{\partial} b(\chi) = -\partial(-b)(\chi) \subseteq \mathcal{X}^*$ at χ given by

$$\chi^* \in \hat{\partial} b(\chi) \Longleftrightarrow \chi \in \arg\max(b - \chi^*) \text{ with finite value } b(\chi) \in R. \tag{3}$$

2. Goods, money, pricing, preferences and payments

Goods, of various sorts, are presumed perfectly divisible, marketable, and transferable – with no fees or frictions. Bundles of such goods are construed and recorded as vectors in some real linear space $\mathcal{X}$, endowed with a locally convex, separating topology.[3] Any component of a bundle $\chi \in \mathcal{X}$ records the amount of some specific asset, commodity or contingent claim.

[2] It differs, except by chance, from its *Walrasian* counterpart in that each agent's wealth equals the value of his final holding – not that of his initial one.
[3] For finance or insurance, $\mathcal{X}$ can have infinite dimension, but typically, $\mathcal{X}$ is Euclidean.

Money maintains all customary properties here; it provides store of value, means of trade, and unit of account. But besides those features, it's also a special *commodity* [11]. Unlike ordinary goods – each more valuable in *use* than in *exchange* – money promotes either end.[4] Accordingly, by an *endowment* is meant an enlarged vector $x := (r, \chi)$ which comprises some *money reserve* (or liquid *bank roll*) $r \in R$ alongside a bundle $\chi \in \mathcal{X}$ of "*real goods*". Let $X := R \times \mathcal{X}$ denote the linear space of *endowments*.

Pricing or valuation – be it idiosyncratic, private, public or temporary – is always linear here.[5] Expression $\chi^* \in \mathcal{X}^*$ is shorthand for a continuous linear *price regime* $\chi^* : \mathcal{X} \to \mathbb{R}$. Quite similarly, valuation $x^* = (r^*, \chi^*) \in R^* \times \mathcal{X}^* =: X^*$ of endowments takes the form

$$x = (r, \chi) \in X \mapsto x^* x = (r^*, \chi^*)(r, \chi) = r^* r + \chi^* \chi \in R. \tag{4}$$

From first principles or elementary economics – and to be brought out later by Proposition 3.1 – actual money invariably commands unit price 1. That is, money trades for goods, never for itself, hence $r^* = 1$ in (4).

The preferences of a generic economic agent are represented by a *binary order* $\succsim$ over X. Write $x \in dom \succsim$ iff the upper level, preferred set

$$\{\tilde{x} \in X \mid \tilde{x} \succsim x\} \tag{5}$$

contains x. Thus, $\succsim$ is *reflexive* on *dom* $\succsim$. Further, it is presumed *transitive* there. Some subsequent arguments would simplify if each preferred set (5) comes closed convex. I shall seek though, to emphasize exactly where convexity imports.[6]

Payments, always in money, relate to indifference levels as follows. Let the generic agent have preference $\succsim$ and hold endowment $x \in dom \succsim$. If he contemplates to *supply* $\chi \in \mathcal{X}$, he *asks* no less money for it than

$$a(\chi \,|x) := \inf \{r \mid (r, -\chi) + x \succsim x\}. \tag{6}$$

[The convention $\inf \varnothing = +\infty$ applies naturally here.] Similarly, if *demanding* χ, he *bids* no more money for that bundle than

$$b(\chi \,|x) := \sup \{r \mid (-r, \chi) + x \succsim x\}. \tag{7}$$

To synthesize, *construe supply as negative demand* – and *regard revenue or benefit as negative expense.* Thereby, since

$$a(\chi \,|x) = -b(-\chi \,|x) =: c(\chi \,|x), \tag{8}$$

the *contingent* (*cost*) *criterion*

$$\chi \in \mathcal{X} \mapsto c(\chi \,|x) \tag{9}$$

[4] For perspectives on the nature and ascent of money, see [3], [7].
[5] Immediate bundling, unbundling or reversal of trades shouldn't generate *arbitrage* [13], [14].
[6] It merits mention that, by the Shapley-Folkman lemma, convolution (21) *contributes* towards convexity. Furher, the more agents, the merrier they are; that is, *in extremis*, non-atomic convolution *ensures* convexity [1].

becomes a chief and unifying object – one which represents preferences by indifference or threshold payments for trade. As defined, c facilitates monetary calculation and expression of economic interest. Acceptable payment substitutes for non-observable utility. And rather important: a bid or ask depends on the agent's endowment. When clear from the context, the endowment isn't mentioned.

Occasionally – for computational, practical or strategic reasons – instead of c, the agent may rather communicate a one-sided, shaded and preferable version

$$\check{c}(\cdot\,|x) \geq c(\cdot\,|x). \tag{10}$$

By hypothesis $c(\cdot\,|x)$ *never takes the value* $-\infty$ when $x \in dom \succsim$. Some properties obtain straightforwardly:

Proposition 2.1 (On the cost criterion (9)). *For each $x \in dom \succsim$ it holds that $c(0\,|x) \leq 0$ – to the effect that $c(\cdot\,|x)$ is proper, meaning finite somewhere.*
Provided the preferred set (5) be closed (convex), the criterion $\chi \in \mathcal{X} \mapsto c(\chi\,|x)$ also becomes closed[7] *(resp. convex).*
As function, $c(\cdot\,|x)$ extends additively in the money variable from $\mathcal{X}$ to X by

$$\tilde{x} = (\tilde{r}, \tilde{\chi}) \Longrightarrow c(\tilde{x}\,|x) := \inf\{r \mid (r, 0) - \tilde{x} + x \succsim x\} = \tilde{r} + c(\tilde{\chi}\,|x). \quad \square \tag{11}$$

A *utility function $U : X \to R \cup \{-\infty\}$ represents $\succsim$* in case

$$U(\tilde{x}) \geq U(x) \in R \Longleftrightarrow \tilde{x} \succsim x \;\; \& \;\; x \in dom \succsim.$$

Preferred sets (5) are then closed (convex) iff U is *upper semicontinuous* (resp. *quasi-concave*). *Quasi-linear* instances [16], $U(r, \chi) = r + u(\chi)$ are rather special, but convenient – and since long used for analysis of (public) costs versus benefits [12].

3. Profit and expense

Associate to a proper *cost* criterion $\chi \in \mathcal{X} \mapsto c(\chi\,|x) \in R \cup \{+\infty\}$ (9) its Fenchel *conjugate* (1):

$$\chi^* \in \mathcal{X}^* \mapsto c^*(\chi^*\,|x) := \sup\{\chi^*\chi - c(\chi\,|x) \mid \chi \in \mathcal{X}\} \in R \cup \{+\infty\}. \tag{12}$$

According to (3), provided $c(\chi\,|x)$ is finite, we define the (partial) *subdifferential* $\partial c(\chi\,|x) \subseteq \mathcal{X}^*$ by

$$\chi^* \in \partial c(\chi\,|x) =: \check{\partial} c(\chi\,|x) \Longleftrightarrow \chi^*\chi = c^*(\chi^*\,|x) + c(\chi\,|x). \tag{13}$$

For any firm or supplier, when $\chi^* \in \partial c(\chi\,|x)$, the last equality tells that *revenue* $\chi^*\chi$ comprises competitive *profit* $c^*(\chi^*\,|x)$ atop full cover of *cost* $c(\chi\,|x)$. Equivalently, what comes up here is a *generalized derivative* and corresponding *optimality condition*:

$$\chi^* \in \partial c(\chi\,|x) \Longleftrightarrow \chi \in \arg\max\{\chi^* - c(\cdot\,|x)\}. \tag{14}$$

[7] That is, *lower semicontinuous.*

As mirror image, *benefit* $b(\cdot\,|x)$, being negative cost, has a corresponding conjugate

$$\hat{b}^*(\chi^*\,|x) := \sup\{b(\chi\,|x) - \chi^*\chi \mid \chi \in \mathcal{X}\} = (-b)^*(-\chi^*\,|x)$$

which "maximizes" $b(\chi\,|x)$ less input cost $\chi^*\chi$. Following (3), the *supdifferential* $\hat{\partial} b := -\check{\partial}(-b)$ satisfies

$$\chi^* \in \hat{\partial} b(\chi\,|x) \Longleftrightarrow \chi \in \arg\max\{b(\cdot\,|x) - \chi^*\} \in R. \tag{15}$$

Proposition 3.1 (On price-taking profit and minimal expense).
Let $x^* = (r^*, \chi^*) \in X^*$ *denote any continuous linear "price", operating on* $X = R\times\mathcal{X}$ *by* (4). *Then,* competitive profit

$$c^*(x^*\,|x) := \sup\{x^*\tilde{x} - c(\tilde{x}\,|x) \mid \tilde{x} \in X\}$$

cannot be finite unless present money commands constant unit price $r^* = 1$. *On that premise,*

$$c^*(x^*\,|x) = \sup\{x^*(x - \tilde{x}) \mid \tilde{x} \succsim x\}, \tag{16}$$

and $\tilde{x}$ *is a best choice in* (16) *iff it solves the problem of* minimal expenditure [12]:

$$\mathcal{E}(x^*\,|x) := \inf\{x^*\bar{x} \mid \bar{x} \succsim x\} = x^*x - c^*(x^*\,|x). \tag{17}$$

Further, for any $x^* = (1, \chi^*) \in X^*$,

$$c^*(x^*\,|x) = c^*(\chi^*\,|x) \geq 0. \tag{18}$$

Proof from [9] is included for completeness. By (11) observe that

$$\begin{aligned}
c^*(x^*\,|x) &= \sup\{x^*\tilde{x} - r \mid \bar{x} := (r, 0) - \tilde{x} + x \succsim x,\ r \in R,\ \tilde{x} \in X\} \\
&= \sup\{x^*(x - \bar{x}) + (r^* - 1)r \mid r \in R,\ \bar{x} \succsim x\} \\
&= \begin{cases} \sup\{x^*(x - \bar{x}) \mid \bar{x} \succsim x\}, & \text{hence (16) holds, iff } r^* = 1, \\ +\infty & \text{otherwise (ignoring bounds on available money)} \end{cases} \\
&= \begin{cases} x^*x - \mathcal{E}(x^*\,|x) & \text{if } r^* = 1, \\ +\infty & \text{otherwise.} \end{cases}
\end{aligned}$$

Finally, (18) follows from (11), (12) and $c(0\,|x) \leq 0$. □

4. Bid-ask spread, efficiency and equilibrium

Accommodated henceforth is a fixed finite ensemble I of economic agents; $\#I \geq 2$. The members, not necessarily many, meet on various market-like platforms. This section mainly inquires: *When and why will agents trade*?

Member $i \in I$ has a reflexive and transitive order $\succsim_i$ on X, representing his preferences. As before, write $x_i \in dom \succsim_i$ iff $x_i \succsim_i x_i$.

While holding bundle $x_i \in dom \succsim_i$, member i has the *no-trade option* $\chi_i = 0$. Instead, he might contemplate a minor trade $\chi_i \approx 0$. Spurred then by intuition or by analytical inclination – or more likely, by own practical reasoning as to margins –

he could valuate a bundle with some "personal price" $\chi_i^* \in \partial c_i(0\,|x_i\,)$. In this optic, the event

$$\cap_{i\in I}\partial c_i(0\,|x_i\,) = \varnothing \tag{19}$$

stands out. It captures that agents differ in their valuations; some *spread* is seen among some parties. Being skilled, these would, at the spur of the moment, detect and execute attractive deals. Otherwise, in case all members face a *common price*

$$\chi^* \in \cap_{i\in I}\partial c_i(0\,|x_i\,), \tag{20}$$

nobody trades; each agent i rather contends with his actual endowment x_i. These loose ideas motivate the following:

Definition 4.1 (Competitive equilibrium means no spread). If $x_i \in$ *dom* $\succsim_i$ for each i, then any common price χ^* (20) and associated endowment profile $\mathbf{x} = (x_i)$ is declared a *competitive equilibrium* $(\chi^*, \mathbf{x})$. By contrast and logical negation, disequilibrium and a *bid-ask spread* are said to prevail iff (19) holds. □

As is well known, competitive equilibrium relates to economic welfare [12]. Brought out next are some aspects of such sort – organized here around Pareto efficiency. Write $\mathbf{x} := (x_i)$ and let the (conditional) *inf-convolution*

$$c_I(\chi\,|\mathbf{x}\,) := \inf\left\{\sum_{i\in I} c_i(\chi_i\,|x_i\,)\;\middle|\;\sum_{i\in I}\chi_i = \chi =: \chi_I\right\}, \tag{21}$$

model "cost-minimal" allocation of some specified aggregate $\chi \in \mathcal{X}$.

Proposition 4.2 (efficiency via equal margins [9]). *If allocation* (χ_i) *solves* (21), *then*

$$\partial c_I(\chi\,|\mathbf{x}\,) \subseteq \cap_{i\in I}\partial c_i(\chi_i\,|x_i\,).$$

Conversely, if $\sum_{i\in I}\chi_i = \chi$, *then* $\partial c_I(\chi\,|\mathbf{x}\,) \supseteq \cap_{i\in I}\partial c_i(\chi_i\,|x_i\,)$.
When moreover, $\cap_{i\in I}\partial c_i(\chi_i\,|x_i\,)$ *is non-empty,* (χ_i) *solves* (21), *and*

$$\partial c_I(\chi\,|\mathbf{x}\,) = \cap_{i\in I}\partial c_i(\chi_i\,|x_i\,).$$

Proof from [9] is included for completeness. Here, to simplify notations, omit mention of endowments. If $\chi^* \in \partial c_I(\chi)$ and (χ_i) solves (21), then, taken together, $\sum_{i\in I}\hat{\chi}_i =: \hat{\chi} \in \mathcal{X}$ and (13) imply

$$\sum_{i\in I} c_i(\hat{\chi}_i) \geq c_I(\hat{\chi}) \geq c_I(\chi) + \chi^*(\hat{\chi} - \chi) = \sum_{i\in I}\left[c_i(\chi_i) + \chi^*(\hat{\chi}_i - \chi_i)\right]. \tag{22}$$

In this string, posit $\hat{\chi}_j = \chi_j$ for each $j \in I \setminus i$ to get

$$c_i(\hat{\chi}_i) \geq c_i(\chi_i) + \chi^*(\hat{\chi}_i - \chi_i). \tag{23}$$

Since $i \in I$ and $\hat{\chi}_i \in \mathcal{X}$ were arbitrary, it follows from (12) and (13) that $\chi^* \in \partial c_i(\chi_i)$ for all $i \in I$, hence $\partial c_I(\chi) \subseteq \cap_{i\in I}\partial c_i(\chi_i)$.

For the turned-around inclusion, given any $\chi^* \in \cap_{i\in I}\partial c_i(\chi_i)$ with $\sum_{i\in I}\chi_i = \chi$, summation of (23) across I, subject to $\sum_{i\in I}\hat{\chi}_i = \chi$, proves the optimality of allocation (χ_i). Further, the same summation of (23), but now with $\sum_{i\in I}\hat{\chi}_i = \hat{\chi}$, gives the two inequalities in (22) – and thereby $\chi^* \in \partial c_I(\chi)$. This shows that $\cap_{i\in I}\partial c_i(\chi_i) \subseteq \partial c_I(\chi)$. □

The same result applies verbatim for any subset $\mathcal{I} \subseteq I$, $\#\mathcal{I} \geq 2$, with shaded versions $\check{c}_i(\cdot\,|x_i)$ in place of true cost $c_i(\cdot\,|x_i)$ (10) – and with corresponding convolution $\check{c}_{\mathcal{I}}(\cdot\,|\mathbf{x})$.

It's never presumed though, that the agents themselves expressly state or solve problem (21). Rather, for realism, they act individually; their coordination being accidental or eventual. So, statement of problem (21) helps here mainly to facilitate analysis and focus theory.

Note that Proposition 4.2 presumed no convexity or differentiability. It contended with weakened versions, just for the aggregate, convoluted "cost" $c_I(\cdot\,|\mathbf{x})$ – and just at χ. Admittedly, the result becomes vacuous if $\partial c_I(\chi\,|\mathbf{x})$ is empty. It deserves emphasis though, that convolution (21) has regularizing effects. For a most simple illustration, omitting mention of endowments, let $\mathcal{X} = R$, $I = \{0, 1\}$, and choose

$$c_0(\chi_0) = \chi_0 \text{ on } [0,1), \quad c_1(\chi_1) = \chi_1 \text{ on } \{0\} \cup [1,+\infty), \text{ both being } +\infty \text{ elsewhere.}$$

Then, c_0 is convex but not closed. By contrast, c_1 isn't convex but closed. Yet, $c_I(\chi) = \chi$ is fully regularized on its effective domain $[0, +\infty)$.

Invoking the convention (8) in (21), the instance $\chi = 0$ corresponds to material balance – whence to pure redistribution of goods. Then, supply equals demand, and markets clear under any price $\chi^* \in \partial c_I(0\,|\mathbf{x})$. Thus, competitive equilibrium, as defined above, becomes easy to characterize and recognize:

Proposition 4.3 (On characterizing and recognizing equilibrium) *Considering* (21), *naturally, suppose* $c_I(0\,|\mathbf{x}) = 0$. *Further, suppose that* $c_I(\cdot\,|\mathbf{x})$ *coincides with its closed convex envelope* $clconv c_I(\cdot\,|\mathbf{x})$ *at* 0.
Then, $(\chi^*, \mathbf{x})$ *is a competitive equilibrium iff all profits – or added values – are nil*:

$$c_i^*(\chi^*\,|x_i) = 0 \;\textit{ for each } i \in I, \;\textit{ and equivalently, }\; c_I^*(\chi^*\,|\mathbf{x}) = 0. \tag{24}$$

Proof. Consider a competitive equilibrium $(\chi^*, \mathbf{x})$ and fix any $i \in I$. Inclusion $\chi^* \in \partial c_i(0\,|x_i)$ implies that $c_i(\cdot\,|x_i)$ coincides with its closed convex envelope at $\chi_i = 0$. For that reason, by conjugation, $0 \in \partial c_i^*(\chi^*\,|x_i)$. Now, because $c_I(0\,|\mathbf{x}) = 0$, and $c_i(0\,|x_i) \leq 0$, it obtains that $c_i(0\,|x_i) = 0$. Since i was arbitrary, (24) follows from (13) and

$$0 = \chi^* 0 = c_i^*(\chi^*\,|x_i) + c_i(0\,|x_i) = c_i^*(\chi^*\,|x_i).$$

For the converse, recall that $c_i(0\,|x_i) \leq 0$ implies $c_i^*(\cdot\,|x_i) \geq 0$, hence $c_I^*(\cdot\,|\mathbf{x}_i) = \sum_{i\in I} c_i^*(\cdot\,|x_i) \geq 0$. Now invoke (24) to see that $\min c_I^*(\cdot\,|\mathbf{x}) = c_I^*(\chi^*\,|\mathbf{x})$ whence $0 \in \partial c_I^*(\chi^*\,|\mathbf{x})$ and thereby $\chi^* \in \partial c_I(0\,|\mathbf{x})$. Finally, from Proposition 4.2, and $c_I(0\,|\mathbf{x}) = \sum c_i(0\,|\mathbf{x})$, it follows that $\chi^* \in \partial c_i(0\,|x_i)$ for each i. □

As indicated ahead of Definition 4.1, agents may sometimes prefer to trade minor amounts. Then, *local convexity* might justify arguments which rest on that property – say, at closure of a market session or near equilibrium. This observation may lend some plausibility to the hypothesis that (eventually) $clconv c_I(\cdot\,|\mathbf{x}) = c_I(\cdot\,|\mathbf{x})$ at $\chi = 0$.

Along the same line, to emphasize *marginal* values – and to have a tractable representation of the bid-ask spread – let $B = -B$ be a weakly bounded, closed convex, symmetric and *small* neighborhood of 0 in $\mathcal{X}$ (say, a "closed unit ball").

Theorem 4.4 (The bid-ask spread) *Suppose each $\partial c_i(0\,|x_i)$ is non-empty. Then, with $\mathbf{x} := (x_i)$, the* bid-ask spread

$$\begin{aligned} S(\mathbf{x}) &:= -\sup_{(\chi_i^*)}\inf_{(\chi_i)}\left\{ \sum_{i\in I}\chi_i^*\chi_i \;\middle|\; \chi_i^* \in \partial c_i(0\,|x_i),\ \chi_i \in B\ \&\ \sum_{i\in I}\chi_i = 0 \right\} \\ &= \inf_{(\chi_i^*)}\sup_{(\chi_i)}\left\{ \sum_{i\in I}\chi_i^*\chi_i \;\middle|\; \chi_i^* \in \partial c_i(0\,|x_i),\ \chi_i \in B\ \&\ \sum_{i\in I}\chi_i = 0 \right\} \end{aligned} \tag{25}$$

is always non-negative, but nil at each equilibrium price $\chi^* \in \cap_{i\in I}\partial c_i(0\,|x_i)$.

For a converse, let at least one set $\partial c_i(0\,|x_i)$ be compact, and suppose the space $\mathcal{X}$ is reflexive. Then, $\cap_{i\in I}\partial c_i(0\,|x_i) = \varnothing \implies S(\mathbf{x}) > 0$. Thus, granted at least one compact subdifferential, and a reflexive commodity space $\mathcal{X}$,

$$\cap_{i\in I}\partial c_i(0\,|x_i) = \varnothing \iff S(\mathbf{x}) > 0.$$

Proof. To prove that $S(\mathbf{x}) \geq 0$, we just take each $\chi_i = 0$. Further, given any $\chi^* \in \cap_{i\in I}\partial c_i(0\,|x_i)$, simply note that $\sum_{i\in I}\chi_i = 0$ implies $\sum_{i\in I}\chi^*\chi_i = 0$.

For the converse, let $C \subset \mathcal{X}^{*I}$ equal the product set $\Pi_{i\in I}\partial c_i(0\,|x_i)$. So defined, C is closed convex and non-empty. If needed for compactness, intersect each $\partial c_i(0\,|x_i)$ with one among these which is compact. This done, C becomes compact. Now, $\cap_{i\in I}\partial c_i(0\,|x_i) = \varnothing$ iff C doesn't intersect the *diagonal* $D := \{(\chi_i^*) \mid \text{all } \chi_i^* \text{ are equal}\}$. Then, by reflexivity, C and D are strictly separated by some non-zero $(\chi_i) \in \mathcal{X}^I$, meaning

$$\sup\left\{\sum_{i\in I}\chi_i^*\chi_i \mid \chi_i^* \in \partial c_i(0\,|x_i)\right\} < \inf\left\{\sum_{i\in I}\chi^*\chi_i \mid \chi^* \in \mathcal{X}^*\right\}.$$

This inequality can't hold unless $\sum_{i\in I}\chi_i = 0$ – whence the right hand side equals 0. By suitable scaling (if necessary), it entails no loss to presume that each $\chi_i \in B$. Now conclude because by (25):

$$\begin{aligned} -S(\mathbf{x}) &= \sup_{(\chi_i^*)}\inf_{(\hat\chi_i)}\left\{\sum_{i\in I}\chi_i^*\hat\chi_i \;\middle|\; \chi_i^* \in \partial c_i(0\,|x_i),\ \hat\chi_i \in B\ \&\ \sum_{i\in I}\hat\chi_i = 0\right\} \\ &\leq \sup_{(\chi_i^*)}\left\{\sum_{i\in I}\chi_i^*\chi_i \mid \chi_i^* \in \partial c_i(0\,|x_i)\right\} < 0. \end{aligned}$$

□

Example 4.5 (On spread with smooth criteria) Let the space $\mathcal{X}$ be Hilbert here with norm $\|\cdot\|$ and unit ball $B := \{\chi \in \mathcal{X} \mid \|\chi\| \leq 1\}$. Suppose each i has an indifference criterion $c_i(\cdot\,|x_i)$ which is (classically) differentiable at his "no-trade option" $\chi_i = 0$. Accordingly, let the singleton $\{\chi_i^*\} = \partial c_i(0\,|x_i)$ comprise his unique "personal price". Denote by $\bar\chi^* := \sum_{i\in I}\chi_i^*/\#I$ the prevailing "average price", taken across I.

For the sake of improved efficiency, or for lower joint cost, if $\chi_i^* \neq \bar{\chi}^*$, agent i ought accept some commodity transfer $\chi_i \neq 0$ – subject, of course, to $\sum_{i \in I} \chi_i = 0$. Naturally, direct χ_i opposite to the price deviation $\chi_i^* - \bar{\chi}^*$. That is, posit $\chi_i = -\tau(\chi_i^* - \bar{\chi}^*)$ with $\tau > 0$ and $\|\chi_i\| \in (0, 1]$. Then, $\sum_{i \in I} \chi_i = 0$, and the *directional derivatives*

$$c_i'(0; \chi_i \,|x_i\,) := \lim_{\rho \to 0^+} [c_i(\rho\chi_i \,|x_i\,) - c_i(0 \,|x_i\,)]/\rho, \; i \in I,$$

satisfy

$$\begin{aligned} \sum_{i \in I} c_i'(0; \chi_i \,|x_i\,) &= \sum_{i \in I} \chi_i^* \chi_i = -\tau \sum_{i \in I} \chi_i^* (\chi_i^* - \bar{\chi}^*) \\ &= -\tau \sum_{i \in I} (\chi_i^* - \bar{\chi}^*)(\chi_i^* - \bar{\chi}^*) = -\tau \sum_{i \in I} \|\chi_i^* - \bar{\chi}^*\|^2 < 0. \end{aligned}$$

Thus overall cost is reduced. Otherwise, iff idiosyncratic valuations coincide – equivalently, iff all $\chi_i^* = \bar{\chi}^*$, and equilibrium reigns with common price $\bar{\chi}^* \in \cap_{i \in I} \partial c_i(0 \,|x_i\,)$ – then no agents see good reasons to undertake minor deals. $\Diamond$

As upshot, trade is driven by differences in agents' valuations of margins - manifested by some strictly positive *bid-ask spread* (25). Which economic institutions or mechanisms might eventually make all spreads disappear? That question is taken up next.

5. Market mechanisms

Exchange and trade happen at special arenas. There, to increasing degree, encounters and transactions are anonymous, computerized, impersonal or internet-based. Further, since platform competition tends to decrease transaction costs, many modern instances approach theoretical models of efficient markets.

So, this section briefly outlines *three* market mechanisms, all important, namely: *direct deals*,[8] *order markets,* and *double auctions.* For each, suppose participants exchange assets or goods for monetary side-payments. Further, every transaction should be voluntary and comply with incentives. That is, no party should ever see any set-back compared to his pre-trade position. The concerned parties vary – maybe randomly – in identities, numbers, proximity and in their focus on selected goods.

Direct deals are classical. For simplicity, suppose each agent trades just *one* real good at a time – and always for money. So here, to unburden notations, let $\mathcal{X} = R$.

It's convenient then to coach arguments in terms of various *orders.* A *limit order* $(\chi^*, \chi) \in R^2_{++}$ stands for some (maybe anonymous) submitter's *commitment* to trade (the single good in question) at positive unit price χ^* (or better) for whatever nonnegative quantity $\leq \chi$. That order is a *bid* if the submitter *pays* for receiving the quantity; it's an *ask* if he *is paid* for supplying the same. Limit orders are publicly *posted* and easily executed, modified, placed or withdrawn.

[8] For a remarkable paper on bilateral trading, see [6].

By contrast, a *market order* (χ^*, χ), be it a bid or ask, is *hidden* and held by some opportunistic (anonymous) fence-sitter, waiting for a worthwhile limit order to come up on the other side of the market.

Proposition 5.1 (On foresighted orders) *Suppose a supplier, having convex cost criterion* $a(\cdot) = a(\cdot\,|x)$ (6), *with* $a(0) \leq 0$, asks *unit price* $\chi^* \in \partial a(\chi)$ (13), *or more, up to some limit quantity* $\chi > 0$. *Thereby, he submits a one-sided, shaded cost* (10):

$$\check{a} := \chi^* \geq a \quad on\ [0, \chi], \quad +\infty \quad otherwise. \tag{26}$$

Thus, any $\check{\chi} \in [0, \chi]$ *gives him* producer's surplus

$$\chi^* \check{\chi} - a(\check{\chi}) \geq 0 \quad which\ is\ maximal\ at\ \check{\chi} = \chi.$$

Similarly, a consumer, having concave benefit criterion $b(\cdot) = b(\cdot\,|x)$ (7), *satisfying* $b(0) \geq 0$, bids *unit price* $\chi^* \in \hat{\partial} b(\chi)$ (15), *or less, up to some limit quantity* $\chi > 0$. *Thereby, he submits a one-sided, shaded benefit* (10)

$$\hat{b} := \chi^* \leq b \quad on\ [0, \chi], \quad -\infty \quad otherwise. \tag{27}$$

Thus, any $\hat{\chi} \in [0, \chi]$ *gives him* consumer's surplus

$$b(\hat{\chi}) - \chi^* \hat{\chi} \geq 0 \quad which\ is\ maximal\ at\ \hat{\chi} = \chi.$$

Proof. Recall convention (8). By convexity, on $[0, \chi]$ it holds $\partial c \leq \partial c(\chi)$. So, $\check{\chi} \in [0, \chi]$ implies

$$\chi^* \check{\chi} - c(\check{\chi}) \geq \chi^* \check{\chi} - [c(\check{\chi}) - c(0)] = \int_0^{\check{\chi}} [\chi^* - \partial c] \geq 0. \qquad \square$$

Proposition 5.2 (On surplus sharing and price by direct deals). *Suppose an ask order* (χ_a^*, χ_a), *be matched with a bid order* (χ_b^*, χ_b), *and* $\chi_a^* < \chi_b^*$. *Then, the quantity* $\chi := \min\{\chi_a, \chi_b\}$ *is efficiently traded at a unit price*

$$\chi^* \in \begin{cases} \{\chi_b^*\} & \text{when } \chi_a < \chi_b, \\ [\chi_a^*, \chi_b^*] & \text{when } \chi_a = \chi_b, \\ \{\chi_a^*\} & \text{when } \chi_a > \chi_b. \end{cases} \tag{28}$$

Proof. Define the shaded ask order $\check{a}$ (26) by $(\chi^*, \chi) = (\chi_a^*, \chi_a)$. Similarly, define the shaded bid order $\hat{b}$ (27) by $(\chi^*, \chi) = (\chi_b^*, \chi_b)$.

Now, χ maximizes the surplus $\hat{b} - \check{a}$ with a price χ^* belonging to $\hat{\partial}\hat{b}(\chi)$ (15) *and* to $\check{\partial}\check{a}(\chi)$ (13). Hence (28) obtains. Indeed, use (8), alongside Proposition 4.2, to see that $max\{\hat{b} - \check{a}\}$ corresponds to inf-convolution (21) with a two-agent ensemble I and $\chi_I = 0$. Finally, by Proposition 5.1, upon comparing with own pre-trade position, no party sees any set-back. $\square$

So, either party gets some surplus. But beyond that guaranteed amount, when $\chi_a^* < \chi_b^*$, any price $\chi^* \in [\chi_a^*, \chi_b^*]$ generates additional surplus $(\chi_b^* - \chi_a^*)\chi > 0$ – to be shared between the two interlocutors. The instance $\chi_a < \chi_b$ gives $\chi^* = \chi_b^*$, and the supplier takes all. By contrast, $\chi_a > \chi_b$ gives $\chi^* = \chi_a^*$, and the customer takes all. The "exceptional" instance $\chi_a = \chi_b$ opens for bilateral bargaining. Also exceptional is the instance $\chi_a^* = \chi_b^*$ which entails immediate clearing.

Proposition 5.2 brings out that direct deals invite *strategic behavior*: Indeed, *each party prefers to talk last*, using then a market order. Further, *each wants to commit smallest quantity.* These features, inspiring either rapid deals or some hesitation, also import when more than two parties meet:

Order markets accommodate many "players". Still suppose each among these trades at most one good at a time. So, in focus here is the order market for a single good.

A finite, possibly empty, occasionally varying list A (B) comprises the actually posted limit *ask* (resp. *bid*) orders. Each $\check{a} \in A$ is a shaded version (26) of some underlying cost criterion a (6). The members of A generate a convex, piecewise linear "supply curve"

$$\chi \in \mathcal{X} \mapsto \check{a}_A(\chi) := \inf \left\{ \sum_{\check{a} \in A} \check{a}(\check{\chi}_a) \;\middle|\; \sum_{\check{a} \in A} \check{\chi}_a = \chi \right\} \in R \cup \{+\infty\},$$

Its subdifferential $\check{\partial}\check{a}_A$ (13) places different ask prices on a staircase, leading upwards, step by step.[9]

Similarly, each $\hat{b} \in B$ is a shaded version (27) of some underlying benefit criterion b (7). These generate a concave, piece-wise linear "*demand curve*"

$$\chi \in \mathcal{X} \mapsto \hat{b}_B(\chi) := \sup \left\{ \sum_{\hat{b} \in B} \hat{b}(\hat{\chi}_b) \;\middle|\; \sum_{\hat{b} \in B} \hat{\chi}_b = \chi \right\} \in R \cup \{-\infty\}$$

Its supdifferential $\hat{\partial}\hat{b}_B$ (15) places different bid prices on a staircase, leading downwards, step by step.[10]

Often just *one* staircase $\check{\partial}\check{a}_A$ or $\hat{\partial}\hat{b}_B$ is easily visible. Otherwise, we notice that the *spread* $\max_{b \in B} \chi_b^* - \min_{a \in A} \chi_a^* \geq 0$ tends to disappear quickly by iterated, direct deals and matching.

Anyway, the problem $\max\{\hat{b}_B - \check{a}_A\}$ corresponds, via (8), to (21) with $I = A \cup B$, $c_{\check{a}} = \check{a}$, $c_{\hat{b}} = -\hat{b}(-)$ and $\chi_I = 0$.[11]

It falls outside the scope of this paper to discuss matching of orders and sharing of surplus.[12]

Double auctions [9] mobilize, at once, the entire agent ensemble I, to trade simultaneously – in all goods. Such auctions also mitigate strategic behavior. Moreover, they efficiently and immediately clear the market – in one shot, as follows:

[9] If perfectly discriminating suppliers, a buyer could pay $\check{a}_A(\chi)$ for quantity χ.

[10] If perfectly discriminating customers, a supplier could charge $\hat{b}_B(\chi)$ for quantity χ.

[11] An agent who posts no limit order, operates as if his limit quantity is nil.

[12] I might comprise market-makers, midlemen or specialists, each seeking some arbitrage.

Up front, each agent $i \in I$ submits his entire cost function $c_i(\cdot\,|x_i)$ – anonymously, silently and simultaneously – to an algorithm, central operator or auctioneer. Thereafter the latter solves problem (21), for a shadow price

$$\chi^* \in \partial c_I(0\,|\mathbf{x}) \subseteq \cap_{i\in I}\partial c_i(\chi_i^+\,|x_i),$$

alongside a best reallocation (χ_i^+), $\sum \chi_i^+ = 0$. Finally, he "charges" agent i the money amount $\chi^*(\chi_i^+ - \chi_i)$. From Proposition 3.2 follows straightforwardly:

Proposition 5.3. *A double auction immediately clears the market and Pareto improves the pre-auction endowment profile.* □

For each of the market-mechanisms, considered above, if agent $i \in I$ be part to a deal, and enters with endowment $x_i \in X$, he exits with an improved updated version

$$x_i^+ = (r_i, -\chi_i) + x_i \succsim_i x_i. \tag{29}$$

6. Convergence towards competitive equilibrium

For argument, suppose the market features non-overlapping, sequential *sessions*, each closing by clearance or clock. Also for argument, suppose that when a session closes, the very last transactions (29) are rationalized *ex post* – at closure time – by the parties themselves, and by Proposition 4.2, as follows:

Assumption 6.1 (On session closure). *Each market session closes by some last reallocation* (χ_i), $\sum_{i\in I}\chi_i = 0$, *supported by a clearing price* $\chi^* \in \cap_{i\in I}\partial c_i(\chi_i\,|x_i)$ *and revenues* $r_i = \chi^*\chi_i$ (29). *Immediately thereafter, agent i updates his holding to* x_i^+ (29). *With that update he enters the subsequent session. If worthwhile, the latter begins with* $\cap_{i\in I}\partial c_i(0\,\big|x_i^+) = \varnothing$.

What stands sharply out is the special case where $x_i^+ = x_i$ and $\cap_{i\in I}\partial c_i(0\,|x_i) \neq \varnothing$. Then, another session has no effect: the best option for every agent, given his endowment, is to stay put. *Can iterated sessions bring the agents towards such a state* - that is, towards some competitive equilibrium? For analysis, suppose agent i originally held endowment $x_i^0 \in dom \succsim_i$. Denote by

$$\mathbf{X} := \left\{ \mathbf{x} = (x_i) \;\middle|\; \sum_{i\in I} x_i = \sum_{i\in I} x_i^0 \right\}$$

the set of feasible allocations. Recall that, given any $\chi^* \in \mathcal{X}^*$ and $\mathbf{x} = (x_i) \in \mathbf{X}$, total profit equals

$$c_I^*(\chi^*\,|\mathbf{x}) = \sum_{i\in I} c_i^*(\chi^*\,|x_i) \;\text{ with each }\; c_i^*(\chi^*\,|x_i) \geq 0.$$

Specialize here to session closure with $\chi^* \in \cap_{i\in I}\partial c_i(\chi_i\,|x_i)$ and $\sum_{i\in I}\chi_i = 0$. So, to focus the above question, I first rather ask: *will total profit* $c_I^*(\chi^*\,|\mathbf{x})$ *decrease from one session to the subsequent*? Indeed, it does, as is confirmed next:

Proposition 6.2 (monotone decreasing profit). *Passing from the penultimate price-cum-endowment profile* $(\chi^*, \mathbf{x})$, *at the closure of one session, to its version* $(\chi^{*+1}, \mathbf{x}^{+1})$ *when the subsequent session closes, it holds* $c_I^*(\chi^{*+1}\,|\mathbf{x}^{+1}) \leq c_I^*(\chi^*\,|\mathbf{x})$.

Proof from [9]. By (29) $x_i^+ \succsim_i x_i$ for each $i \in I$. So, granted transitive preference orders, expenditures "increase": $\mathcal{E}_i(\cdot\,|x_i^+) \geq \mathcal{E}_i(\cdot\,|x_i)$ for all $i \in I$; see (17). The implication $\chi^* \in \partial c_I(0\,|\mathbf{x}) \implies 0 \in \partial c_I^*(\chi^*\,|\mathbf{x})$ tells us that $c^*(\cdot\,|\mathbf{x})$ is minimal at χ^*. Collecting these facts, and letting $\chi^0 := \sum_{i\in I}\chi_i^0$ with initial endowments $x_i^0 = (r_i^0, \chi_i^0)$, it follows from (17):

$$c_I^*(\chi^{*+}\,|\mathbf{x}^+) = \inf_{\hat{\chi}^*}\left\{\hat{\chi}^*\chi^0 - \sum_{i\in I}\mathcal{E}_i(\hat{\chi}^*\,|x_i^+)\right\}$$
$$\leq \left\{\chi^*\chi^0 - \sum_{i\in I}\mathcal{E}_i(\chi^*\,|x_i)\right\} = c_I^*(\chi^*\,|\mathbf{x}). \qquad \square$$

Theorem 6.3 (Convergence to competitive equilibrium). *For any* $\mathbf{x} \in \mathbf{X}$ *and* $\chi^* \in \partial c_I(0\,|\mathbf{x})$ *with* $c_I^*(\chi^*\,|\mathbf{x}) > 0$, *suppose that* $\mathcal{E}_i(\cdot\,|x_i^+) \geq \mathcal{E}_i(\cdot\,|x_i)$ $\forall i \in I$ *implies*

$$\inf_{\hat{\chi}^*} c_I^*(\hat{\chi}^*\,|\mathbf{x}^+) < c_I^*(\chi^*\,|\mathbf{x}). \tag{30}$$

Also suppose $c_I^*(\chi^*\,|\mathbf{x})$ *is jointly closed, meaning lower semicontinuous in* $(\chi^*, \mathbf{x})$. *Then, by iterated sessions, total profit* $c_I^*(\chi^*\,|\mathbf{x})$ *converges to* 0. *That is, each cluster point* $\mathbf{x}$ *of the generated sequence* $(\mathbf{x}^k)$ *qualifies as competitive equilibrium* $(\chi^*, \mathbf{x})$ *for any* $\chi^* \in \partial c_I(0\,|\mathbf{x})$.

Proof. Consider the sequence $(\mathbf{x}^k)$ emanating from $\mathbf{x}^0 = (x_i^0) \in \mathbf{X}$, where $\mathbf{x}^k = (x_i^k)$ is the penultimate endowment profile just prior to closure of session k. During that session each agent i may have secured finitely many "improving" updates (29). These can be seen as interim *spacer steps*; see [2] Theorem 7.3.4. By Proposition 5.1, the sequence $c_I^*(\chi^{k*}\,|\mathbf{x}^k)$ decreases monotonically. Being bounded below by 0, the values added converge to some limit $L \geq 0$.

Consider any cluster point $\mathbf{x}$ of $(\mathbf{x}^k)$. It suffices by Proposition 4.3 to show that $c_I^*(\chi^*\,|\mathbf{x}) = 0$ for each $\chi^* \in \partial c_I(0\,|\mathbf{x})$. But otherwise, (30) would yield the contradiction $\lim c_I^*(\chi^{*k+1}\,|\mathbf{x}^{k+1}) < L = \lim c_I^*(\chi^{k*}\,|\mathbf{x}^k)$. $\square$

7. Concluding remarks

Brought out above was the efficacy of many monetized markets. As commodity, money made preferences show up as willingness to bid or ask. Thereby many distinctions between customers and suppliers disappeared; modulo suitable choice of signs, they played symmetric roles. And convex analysis came more to the fore.

Admittedly, several hurdles were overlooked – say, on fees, frictions, money illusions, transaction costs or discrepancies between social versus private costs. The chosen optic comprises precisely those idealized settings where markets, oiled by money, operate well – whence deserve some praise.

Much economic theory – notably on decentralization and market equilibrium – presumes that each upper level, preferred set (5) be *convex* [5].[13] *Convolution* (21) of resulting criteria (Prop. 2.1) preserves, of course, convexity. But quite important:

[13] For extensions, see [17].

convolution also facilitates emergence of that property. Therefore, instead of lifting convexity – all the way, from the bottom up to the aggregate level – I rather choose to assume it, more weakly, at the top level. Then, if not vacuous, subject to local closure and upper boundedness, subdifferential calculus still applies.

Thanks are due a referee for most helpful remarks.

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