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Lebesgue Infinite Sums of Convex Functions: Subdifferential Calculus

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Dedicated to Roger J-B Wets on the occasion of his 85th birthday.

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We present a subdifferential analysis for a general concept of infinite sum $f := \sum_{i\in I} f_i$ of arbitrary collections of convex functions f_i, called Lebesgue infinite sum. Since this problem cannot be addressed, at least directly, through classical arguments from the theory of normal convex integrands, we perform a reduction analysis showing that the ε-subdifferential of f reduces to that of countable/finite subsums via appropriate lower limit and closure processes. Then, the usual calculus rules of (countable) integral functions give rise to characterizations of the ε-subdifferential of f, which are written exclusively by means of ε-subdifferentials of the data f_i. The resulting characterizations do not assume any qualification or boundedness condition.

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1. Introduction

Given a non-empty set I and a collection of convex functions $f_i : X \to \mathbb{R}_\infty$, $i \in I$, defined on a locally convex space X, we associate an optimization problem given in the form

$$\inf_{x\in X} f(x) := \sum_{i\in I} f_i(x),$$

which consists of minimizing a kind of an infinite sum that will be specified later. In particular, the objective function f reduces to the ordinary sums when I is finite or,

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more generally, countable. There have been many ways to give adequate meaning to this infinite sum. A first possible definition of f, for general sets I, is related to the so-called robust sum,

$$f(x) = \sup_{J \in \mathcal{F}(I)} \sum_{i \in J} f_i(x),$$

where $\mathcal{F}(I)$ represents the family of all finite subsets of I. In this case, f is well-defined and is known as the robust sum of the f_i's, therefore the associated optimization problem is called a robust problem and its solutions are referred to as robust optimal solutions. Generally, the robustness is a good device to approach the uncertainty in the choice of the finite sets J (see, e.g., [1, 6, 5] and references therein). However, the original optimization problem and its robust counterpart may have different interpretations. In fact, this robust approach responds to a very pessimistic attitude, where the process of finding optimal (robust) solutions might be a difficult task. Another difficulty comes here due to the frequent non-differentiability of the supremum function, and also because there may not exist solutions that are acceptable for all subfamilies assigned to finite subsets of I.

Another possibility is to endow $\mathcal{F}(I)$ with the partial order of upward inclusions, and then define f as the pointwise limit of the net $(\sum_{i \in J} f_i)_{J \in \mathcal{F}(I)}$. In other words, f now depends to a large extent on the existence of such a limit, which may not always be defined. Therefore, the associated optimal (limit) solutions may have a good robust-like interpretation, but cannot be properly analyzed because the given limit infinite sum function may not be defined everywhere.

In this paper, we introduce an alternative definition for the infinite sum that we call Lebesgue sum, in the spirit of the classical Lebesgue integral,

$$f := \sum_{i \in I} f_i^+ - \sum_{i \in I} f_i^-,$$

where f_i^+ and f_i^- are the positive and negative parts of f_i, respectively. Of course, since f_i^+ and f_i^- are obviously nonnegative, the associated sums $\sum_{i \in I} f_i^+$ and $\sum_{i \in I} f_i^-$ have a clear meaning and can be defined by each of the robust and limit sums. This concept has been also investigated in [15] by means of extensions of classical techniques of integration to infinite functions (see, also, [13, 17, 19, 20] for the case of countable sums (i.e., series of convex functions)). Our approach in the present work is different since we first perform a reduction step that allows us to go from infinite to countable sums, and then use the classical theory of normal convex integrals ([3, 4, 7, 8, 11, 12, 14] and others) adapted to the countable framework. By doing it this way, we could avoid a restrictive condition that is usually assumed when dealing with infinite sums and integral functions, asking that the functions involved be bounded below by a nice function (in L^1). This, in turn, is equivalent to the requirement that the functions involved be non-negative, but in such a case, the above three sums (robust, limit and Lebesgue) agree and their ε-subdifferentials can be directly addressed by applying the sum and the supremum subdifferential calculus rules from [9] and [10].

The main result of this paper is established in Theorem 5.1, which gives the ε-subdifferential $\partial_\varepsilon f(x)$ of the infinite sum $f := \sum_{i \in I} f_i$ at any point $x \in X$ and $\varepsilon > 0$. We assume no other assumption of continuity or boundedness, except that the data functions f_i's are proper, lsc, and convex. More precisely, we show that the set $\partial_\varepsilon f(x)$ is built on the following sums, given for sufficiently large finite subsets K of I (i.e., $K \in \mathcal{F}(I)$),

$$\sum_{i\in K}\partial_{\varepsilon_i} f_i(x) + \sum_{i\in J}\partial_{\varepsilon_i} f_i^+(x), \tag{1}$$

where J varies over countable subsets of $I\setminus K$ such that $\sum_{i\in J} f_i^+(x) = \sum_{i\in I\setminus K} f_i^+(x)$, for sufficiently small $\varepsilon_i \in [0,\varepsilon]$ (even with $\sum_{i\in K\cup J}\varepsilon_i \leq \varepsilon$). To make sure that K within the expression above is large enough, we take the lower limit over $K \in \mathcal{F}(I)$, provided this latter family is endowed with the partial order of upward inclusions. The above formula is made more precise because each of the sets $\partial_{\varepsilon_i} f_i^+(x)$ can be expressed in terms of the function f_i (and its effective domain), thanks to the relationship

$$\partial_{\varepsilon_i} f_i^+(x) = \bigcup_{0\leq\lambda\leq 1} \partial_{\varepsilon_i+\lambda f_i(x)-f_i^+(x)}(\lambda f_i)(x).$$

Of course, the above analysis also leads to the characterization of $\partial f(x)$ by making $\varepsilon \downarrow 0$. In the particular case when I is finite, the previous characterization only involves finite sets K, so we recover the well-known Hiriart-Urruty-Phelps formula ([10]).

It is also possible to replace, as we do in the same theorem giving the main result of the paper, the countable sums above with finite ones. Thus, instead of (1), $\partial_\varepsilon f(x)$ is also recovered from the following sets when K is large enough,

$$\sum_{i\in K}\partial_{\varepsilon_i} f_i(x) + \sum_{i\in J}\partial_{\varepsilon_i+\lambda_i f_i(x)-f_i^+(x)}(\lambda_i f_i)(x),$$

where now J is a finite subset of $I\setminus K$ such that $\sum_{i\in J} f_i^+(x) \geq \sum_{i\in I\setminus K} f_i^+(x) - \delta$ with $0 \leq \delta \leq \varepsilon$. These characterizations can be simplified under additional qualification conditions, as those used in [19, 20] and [17] (this last task will be the subject of a future work). In this way, the present characterizations provide natural extensions of the sum calculus rules known for finite ([9, 10, 16]) and countable ([4, 3, 12]) sums of convex functions.

This article is organized as follows: Section 2 collects the necessary notations and preliminary results that are used throughout the article. Section 3 presents the definition of the Lebesgue sum and establishes many of its properties, including its comparison with the robust and limit sums. Section 4 analyzes the reduction of the Lebesgue sum to finite/countable sums. This step provides the main ingredient that is used in the final section 5 where the characterizations of the ε-subdifferential of the Lebesgue sum are given in complete generality.

2. Notation and preliminary results

Let X be a real locally convex space (lcs, for short), and let X^* be its topological dual space with respect to a given dual pairing $\langle\cdot,\cdot\rangle$, defined on $X^*\times X$ by $\langle x^*, x\rangle := x^*(x)$, and endowed with a compatible topology for the pairing $\langle\cdot,\cdot\rangle$. Examples of such a topology on X^* are the weak* topology, the Mackey topology, and the topology of the norm when X is reflexive. By $\mathcal{N}_X$ we refer to the family of convex closed and balanced neighborhoods of the origin in X. We use the notation $\mathbb{R}_+ := [0, +\infty[$, $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, $\mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$, with the convention

$$0(+\infty) = +\infty, \text{ and } 0(-\infty) = 0, \ (+\infty) - (+\infty) = (+\infty). \tag{2}$$

The positive and negative parts of $\alpha \in \overline{\mathbb{R}}$ are $\alpha^+ := \max\{0, \alpha\}$ and $\alpha^- := \max\{0, -\alpha\}$, respectively, so that α^+ and α^- are the unique values that satisfy the system of equations $\alpha = \alpha^+ - \alpha^-$ and $|\alpha| = \alpha^+ + \alpha^-$.

Given a set $A \subset X$, we denote by $\mathrm{cl}(A)$ (or $\overline{A}$, indistinctly), $\mathrm{co}\, A$ and $\overline{\mathrm{co}}A$, the *closure*, the *convex hull* and the *closed convex hull* of A, respectively. The *indicator function* $\mathrm{I}_A : X \to \mathbb{R}_\infty$ of A is defined by $\mathrm{I}_A(x) := 0$ for $x \in A$, and $\mathrm{I}_A(x) := +\infty$ otherwise. The *support function* of A is the function $\sigma_A : X^* \to \overline{\mathbb{R}}$ defined by $\sigma_A(x) := \sup_{x^* \in A} \langle x^*, x\rangle$, with the convention $\sigma_\emptyset \equiv -\infty$. The *lower limit* of a net $(A_\alpha)_{\alpha\in\Lambda}$ of sets in X (or X^*), where $(\Lambda, \succcurlyeq)$ is a directed set, is defined as

$$\liminf_{\alpha\in\Lambda} A_\alpha = \bigcup_{\alpha_0\in\Lambda} \ \bigcap_{\alpha\in\Lambda,\ \alpha\succcurlyeq\alpha_0} A_\alpha.$$

The set $\liminf_{\alpha\in\Lambda} A_\alpha$ is convex whenever the sets A_α are. The (effective) *domain* and *epigraph* of a function $\varphi : X \to \overline{\mathbb{R}}$ are the sets $\mathrm{dom}\,\varphi := \{x \in X : \varphi(x) < +\infty\}$ and $\mathrm{epi}\,\varphi := \{(x, \lambda) \in X \times \mathbb{R} : \varphi(x) \leq \lambda\}$, respectively. The function φ is said to be *proper*, *convex*, and *lower semi-continuous* (lsc, for short), if $\varphi > -\infty$ and $\mathrm{dom}\,\varphi \neq \emptyset$, $\mathrm{epi}\,\varphi$ is convex, and $\mathrm{epi}\,\varphi$ is closed, respectively. The function φ is *sequentially lsc* at $x \in X$ if $\liminf_k \varphi(x_k) \geq \varphi(x)$ for every sequence $(x_k)_k$ that converges to x. As usual, $\Gamma_0(X)$ represents the family of proper convex and lsc functions defined on X. The *conjugate* of φ is the function $\varphi^* : X^* \to \overline{\mathbb{R}}$ defined as

$$\varphi^*(x^*) := \sup_{x\in X}\{\langle x^*, x\rangle - \varphi(x)\}.$$

The *ε-subdifferential* of φ at $x \in \varphi^{-1}(\mathbb{R})$ for $\varepsilon \geq 0$ is the set

$$\partial_\varepsilon\varphi(x) := \{x^* \in X^* : \langle x^*, y - x\rangle \leq \varphi(y) - \varphi(x) + \varepsilon \ \ \forall y \in X\};$$

when $x \notin \mathrm{dom}\,\varphi$, we set $\partial_\varepsilon\varphi(x) := \emptyset$. The ε-subdifferential is a good tool for studying the properties of convex functions (see, e.g., [2, 18]).

We need the following lemma giving an inner approximation of the ε-subdifferential ([2, Proposition 4.1.9]).

Lemma 2.1. *Given a convex function $\varphi : X \to \mathbb{R}_\infty$, $x \in \mathrm{dom}\,\varphi$ and $\varepsilon > 0$, we have*

$$\partial_\varepsilon\varphi(x) = \mathrm{cl}\Big(\bigcup_{0<\delta<\varepsilon} \partial_\delta\varphi(x)\Big), \tag{3}$$

whenever the last set is non-empty. Moreover, if X is a reflexive Banach space, then (3) also holds when the closure is taken with respect to the dual norm topology.

The following can be found in [2, Proposition 3.2.8].

Lemma 2.2. *Let $(A_i)_i \subset X$ be a non-increasing net of non-empty closed convex sets; that is, $i_1 \preccurlyeq i_2 \implies A_{i_2} \subset A_{i_1}$. Then $\cap_{i\in I} A_i \neq \emptyset$ if and only if the function $\mathrm{cl}\,(\inf_{i\in I} \sigma_{A_i})$ is proper and, under this assumption, we have*

$$\sigma_{\cap_{i\in I} A_i} = \mathrm{cl}\left(\inf_{i\in I} \sigma_{A_i}\right).$$

3. Infinite sums of functions

In this section we present and compare different concepts of infinite sums including the object of this article, the Lebesgue infinite sum. We give a non-empty set I and set

$$\mathcal{F}(I) := \{J \subset I : |J| < +\infty\} \text{ and } \mathfrak{C}(I) := \{J \subset I : J \text{ countable}\}, \tag{4}$$

where $|J|$ denotes the cardinal of J; we shall frequently consider $\mathcal{F}(I)$ as a directed set, endowed with the partial order of ascending inclusions. In particular, the lower limit of a net $(A_J)_{J\in\mathcal{F}(I)}$ of sets in X^* is written as

$$\liminf_{J\in\mathcal{F}(I)} A_J = \bigcup_{K\in\mathcal{F}(I)} \bigcap_{J\in\mathcal{F}(I),\, K\subset J} A_J.$$

Also, given elements $\alpha_i \in \mathbb{R}_\infty$, $i \in I$, and a set $J \in \mathcal{F}(I)$, for the sake of notational simplicity we denote

$$\alpha_J^- := \textstyle\sum_{i\in J}\alpha_i^-,\ \alpha_J^+ := \sum_{i\in J}\alpha_i^+, \text{ and } \alpha_J := \alpha_J^+ - \alpha_J^-. \tag{5}$$

We recall the concepts of limit and robust sums of given α_i, $i \in I$ (see [1, 6] and the references therein). The limit sum of the α_i's is the element $\sum_{i\in I}\alpha_i \in \overline{\mathbb{R}}$, defined as the limit of the net $(\alpha_J)_{J\in\mathcal{F}(I)}$ (when it exists in $\overline{\mathbb{R}}$; that is, when we have $\liminf_{J\in\mathcal{F}(I)}\alpha_J = \limsup_{J\in\mathcal{F}(I)}\alpha_J \in \overline{\mathbb{R}}$)

$$\sum_{i\in I}\alpha_i := \lim_{J\in\mathcal{F}(I)}\alpha_J = \liminf_{J\in\mathcal{F}(I)}\alpha_J = \limsup_{J\in\mathcal{F}(I)}\alpha_J. \tag{6}$$

The robust sum of the α_i's is the element $\sum_{i\in I}^{\mathcal{R}}\alpha_i \in \mathbb{R}_\infty$ defined as the pointwise supremum of the α_J's; that is,

$$\textstyle\sum_{i\in I}^{\mathcal{R}}\alpha_i := \sup\limits_{J\in\mathcal{F}(I)}\alpha_J.$$

Note that the robust sum is always defined as an element of $\mathbb{R}_\infty$, while the limit sum may not always exist as in the case $\alpha_n := (-1)^n$, $n \in I \equiv \mathbb{N}$, where $\limsup_{J\in\mathcal{F}(I)}\alpha_J = +\infty$ and $\liminf_{J\in\mathcal{F}(I)}\alpha_J = -\infty$. To overcome the difficulty encountered with the previous limit, we use here another sum that we call Lebesgue infinite sum. It is so named because of its similarity to the definition of the usual Lebesgue integral, so it will always make sense thanks to the convention $+\infty - (+\infty) = +\infty$ that we adopt throughout the text.

Definition 3.1. Given $\alpha_i \in \mathbb{R}_\infty$, $i \in I$, the *Lebesgue infinite sum* (or, simply, *Lebesgue sum*) of the α_i's is the element $\sum_{i\in I}^{\mathcal{L}}\alpha_i \in \overline{\mathbb{R}}$ defined by

$$\textstyle\sum_{i\in I}^{\mathcal{L}}\alpha_i := \sum_{i\in I}\alpha_i^+ - \sum_{i\in I}\alpha_i^-.$$

It is clear that the Lebesgue sum coincides with the ordinary sum when I is finite. More generally, when I is countable, it turns to be the Lebesgue integral of the function $i \in I \to \alpha(i) := \alpha_i$ with respect to the (σ-finite) counting measure on I. In our case, since we are requiring no measure or special structure on the set I, an integral representation of the Lebesgue sum with a σ-finite measure might not be possible. Nevertheless, it is possible to represent the Lebesgue sum over particular countable subsets of I, but these may depend on the α_i's.

Lemma 3.2. *Given $\alpha_i \in \mathbb{R}_\infty$, $i \in I$, there exists a countable set $J \subset I$ such that*

$$\textstyle\sum_{i\in I}^{\mathcal{L}}\alpha_i = \sum_{i\in J}^{\mathcal{L}}\alpha_i.$$

If, in addition, $\sum_{i\in I}^{\mathcal{L}}\alpha_i < +\infty$, then $\alpha_i = 0$ for all $i \in I \setminus J$.

Proof. Assume first that $\sum_{i\in I}^{\mathcal{L}}\alpha_i^+ < +\infty$, and denote

$$I_n := \left\{i \in I : \alpha_i^+ > 1/n\right\},\ n = 1, 2, \ldots.$$

Then $\alpha_i^+ = 0$ for all $i \notin J_0 := \cup_{n\geq 1} I_n$, so that $\sum_{i\in I}^{\mathcal{L}}\alpha_i^+ = \sum_{i\in J_0}^{\mathcal{L}}\alpha_i^+$. Moreover, for each $n \geq 1$, we have

$$\sum_{i\in I_n} 1/n \leq \sum_{i\in I_n}\alpha_i^+ \leq \textstyle\sum_{i\in I}^{\mathcal{L}}\alpha_i^+ < +\infty, \tag{7}$$

implying that $|I_n| < +\infty$, and the set J_0 is at most countable. In the other case, when $\sum_{i\in I}^{\mathcal{L}}\alpha_i^+ = +\infty$, we find sets $K_n \in \mathcal{F}(I)$, $n \geq 1$, such that $\sum_{i\in K_n}\alpha_i^+ \geq n$ and the countable set $J_0 := \cup_{n\geq 1} K_n$ satisfies $\sum_{i\in J_0}^{\mathcal{L}}\alpha_i^+ = +\infty = \sum_{i\in I}^{\mathcal{L}}\alpha_i^+$.

Similarly, we find an at most countable set $J_1 \subset I$ such that $\sum_{i\in I}^{\mathcal{L}}\alpha_i^- = \sum_{i\in I}^{\mathcal{L}}\alpha_i^-$, with $\alpha_i^- = 0$ for all $i \in I \setminus J_1$ in the case when $\sum_{i\in I}^{\mathcal{L}}\alpha_i^- < +\infty$. Consequently, taking $J := J_0 \cup J_1 \in \mathfrak{C}(I)$, we obtain

$$\textstyle\sum_{i\in I}^{\mathcal{L}}\alpha_i = \displaystyle\sum_{i\in I}\alpha_i^+ - \sum_{i\in I}\alpha_i^- = \textstyle\sum_{i\in J_0}\alpha_i^+ - \sum_{i\in J_1}\alpha_i^- = \sum_{i\in J}\alpha_i^+ - \sum_{i\in J}\alpha_i^- = \sum_{i\in J}^{\mathcal{L}}\alpha_i,$$

with the additional property that $\alpha_i = \alpha_i^+ - \alpha_i^- = 0$, for all $i \in I \setminus J$, provided that $\sum_{i\in I}^{\mathcal{L}}\alpha_i < +\infty$. □

The following proposition gives the linear property of the Lebesgue sum. Its proof is an easy consequence of Lemma 3.2.

Proposition 3.3. *For any α_i, $\beta_i \in \mathbb{R}_\infty$, $i \in I$, such that $\sum_{i\in I}^{\mathcal{L}}\alpha_i$, $\sum_{i\in I}^{\mathcal{L}}\beta_i \in \mathbb{R}$, and every $\gamma \in \mathbb{R}$ we have*

$$\textstyle\sum_{i\in I}^{\mathcal{L}}(\alpha_i + \gamma\beta_i) = \sum_{i\in I}^{\mathcal{L}}\alpha_i + \gamma\sum_{i\in I}^{\mathcal{L}}\beta_i. \tag{8}$$

Consequently, if $\alpha_i \geq \beta_i$ for all $i \in I$, then $\sum_{i\in I}^{\mathcal{L}}\alpha_i \geq \sum_{i\in I}^{\mathcal{L}}\beta_i$.

Lemma 3.4. *Given $\alpha_i \in \mathbb{R}_\infty$, $i \in I$, for all $K_0 \in \mathcal{F}(I)$ we have that*

$$\begin{aligned}\textstyle\sum_{i\in I}^{\mathcal{L}}\alpha_i &= \inf_{K\in\mathcal{F}(I)}\left(\alpha_K + \textstyle\sum_{i\in I\setminus K}\alpha_i^+\right)\\ &= \inf_{K\in\mathcal{F}(I)}\ \sup_{J\in\mathcal{F}(I\setminus K)}\left(\alpha_K + \alpha_J^+\right) = \inf_{K\in\mathcal{F}(I),\ K_0\subset K}\ \sup_{J\in\mathcal{F}(I\setminus K)}\left(\alpha_K + \alpha_J^+\right).\end{aligned} \tag{9}$$

Proof. We write

$$\begin{aligned}\textstyle\sum_{i\in I}^{\mathcal{L}}\alpha_i &= \textstyle\sum_{i\in I}\alpha_i^+ - \sum_{i\in I}\alpha_i^- = \sum_{i\in I}\alpha_i^+ - \displaystyle\sup_{K\in\mathcal{F}(I)}\alpha_K^-\\ &= \inf_{K\in\mathcal{F}(I)}\left(\textstyle\sum_{i\in I}\alpha_i^+ - \alpha_K^-\right) = \inf_{K\in\mathcal{F}(I)}\left(\alpha_K^+ + \textstyle\sum_{i\in I\setminus K}\alpha_i^+ - \alpha_K^-\right).\end{aligned}$$

Thus, since $\alpha_K = \sum_{i\in K}(\alpha_i^+ - \alpha_i^-) = \alpha_K^+ - \alpha_K^-$, for all $K \in \mathcal{F}(I)$, we obtain

$$\sum\nolimits_{i\in I}^{\mathcal{L}}\alpha_i = \inf_{K\in\mathcal{F}(I)} \left(\alpha_K + \sum\nolimits_{i\in I\setminus K}\alpha_i^+\right) = \inf_{K\in\mathcal{F}(I)} \sup_{J\in\mathcal{F}(I\setminus K)} \left(\alpha_K + \alpha_J^+\right).$$

Furthermore, since $\sup_{K\in\mathcal{F}(I)} \alpha_K^- = \sup_{K\in\mathcal{F}(I),\, K_0\subset K} \alpha_K^-$ for any $K_0 \in \mathcal{F}(I)$, the same reasoning as above entails

$$\sum\nolimits_{i\in I}^{\mathcal{L}}\alpha_i = \sum\nolimits_{i\in I}\alpha_i^+ - \sup_{K\in\mathcal{F}(I),\, K_0\subset K} \alpha_K^- = \inf_{K\in\mathcal{F}(I),\, K_0\subset K} \sup_{J\in\mathcal{F}(I\setminus K)} \left(\alpha_K + \alpha_J^+\right),$$

and we are done. □

To compare the previous three concepts of sums, we first note that they all agree when the α_i's are nonnegative. The following proposition gives another situation where the Lebesgue and the limit sums coincide, allowing an equivalent representation of the Lebesgue sum.

Proposition 3.5. *Let $\alpha_i \in \mathbb{R}_\infty$, $i \in I$, be given. If $\sum_{i\in I} \alpha_i^+$ or $\sum_{i\in I} \alpha_i^- \in \mathbb{R}_+$, then the Lebesgue and the limit sums coincide. In other words, we have*

$$\sum\nolimits_{i\in I}^{\mathcal{L}}\alpha_i = \begin{cases} \sum_{i\in I}\alpha_i, & \textit{if } \sum_{i\in I}\alpha_i^+ < +\infty, \\ +\infty, & \textit{if } \sum_{i\in I}\alpha_i^+ = +\infty. \end{cases}$$

Proof. The current assumption entails

$$\begin{aligned} \liminf_{J\in\mathcal{F}(I)} \alpha_J &= \liminf_{J\in\mathcal{F}(I)} \left(\alpha_J^+ - \alpha_J^-\right) \geq \liminf_{J\in\mathcal{F}(I)} \alpha_J^+ - \limsup_{J\in\mathcal{F}(I)} \alpha_J^- \\ &= \lim_{J\in\mathcal{F}(I)} \alpha_J^+ - \lim_{J\in\mathcal{F}(I)} \alpha_J^- = \sum\nolimits_{i\in I}\alpha_i^+ - \sum\nolimits_{i\in I}\alpha_i^- = \sum\nolimits_{i\in I}^{\mathcal{L}}\alpha_i, \end{aligned} \tag{10}$$

and

$$\limsup_{J\in\mathcal{F}(I)} \alpha_J \leq \limsup_{J\in\mathcal{F}(I)} \alpha_J^+ - \liminf_{J\in\mathcal{F}(I)} \alpha_J^- = \sum\nolimits_{i\in I}\alpha_i^+ - \sum\nolimits_{i\in I}\alpha_i^- = \sum\nolimits_{i\in I}^{\mathcal{L}}\alpha_i, \tag{11}$$

yielding $\sum_{i\in I}^{\mathcal{L}}\alpha_i = \lim_{J\in\mathcal{F}(I)} \alpha_J$. □

The condition in Proposition 3.5 is necessary: For example, considering the sequence $\alpha_n := (-1)^n$, $n \in I \equiv \mathbb{N}$, we have seen above that $\liminf_{J\in\mathcal{F}(I)} \alpha_J = -\infty$ whereas $\limsup_{J\in\mathcal{F}(I)} \alpha_J = +\infty$ and the limit sum $\sum_{n\in\mathbb{N}} \alpha_n = \lim_{J\in\mathcal{F}(I)} \alpha_J$ does not exist. At the same time, we have that $\sum_{n\in\mathbb{N}}^{\mathcal{L}}\alpha_n = +\infty$ as $\sum_{n\in\mathbb{N}} \alpha_n^+ = +\infty$. More generally, provided that $\sum_{i\in I} \alpha_i$ exists in $\overline{\mathbb{R}}$, Proposition 3.5 also shows that

$$\sum\nolimits_{i\in I}\alpha_i \leq \sum\nolimits_{i\in I}{}^{\mathcal{L}}\alpha_i. \tag{12}$$

This inequality may be strict even when $\sum_{i\in I}^{\mathcal{L}}\alpha_i$ is finite. For instance, we have

$$\sum\nolimits_{n\in\mathbb{N}}^{\mathcal{L}}(-1)^n/n^2 = -\pi^2/12$$

(which is the value of the series $\sum_{n\geq 1}(-1)^n/n^2$), while the limit sum

$$\textstyle\sum_{n\in\mathbb{N}}(-1)^n/n^2 := \lim_{J\in\mathcal{F}(\mathbb{N})} \sum_{n\in J}(-1)^n/n^2$$

does not exist as $\liminf_{J\in\mathcal{F}(I)} \alpha_J = -\pi^2/8$ and $\limsup_{J\in\mathcal{F}(I)} \alpha_J = \pi^2/24$.

Further properties of the Lebesgue and robust sums come next. In fact, assertion (v) below is an easy fact that follows from the definitions, while assertion (i) is given in [6, Lemma 2.2]. As for relation (14), it comes from the observation that (see, e.g., [6, Lemma 2.5])

$$\textstyle\sum_{i\in I}^{\mathcal{R}}\alpha_i = \begin{cases} \sum\limits_{i\in I}\alpha_i^+, & \text{if } \sup_{i\in I}\alpha_i \geq 0,\\ \sup_{i\in I}\alpha_i, & \text{if } \sup_{i\in I}\alpha_i \leq 0.\end{cases} \tag{13}$$

Proposition 3.6. *Let $\alpha_i \in \mathbb{R}_\infty$, $i \in I$, be given. Then we have*

$$\textstyle\sum_{i\in I}{}^{\mathcal{R}}\alpha_i = \sum_{i\in I}\alpha_i^+ - (\sup\limits_{i\in I}\alpha_i)^-, \tag{14}$$

together with the following properties:

(i) $(\sum_{i\in I}^{\mathcal{R}}\alpha_i)^+ = \sum_{i\in I}\alpha_i^+$.

(ii) $(\sum_{i\in I}^{\mathcal{R}}\alpha_i)^- = (\sup_{i\in I}\alpha_i)^-$.

(iii) $\sum_{i\in I}^{\mathcal{R}}\alpha_i = \left(\sum_{i\in I}\alpha_i^+ + \sup_{i\in I}\alpha_i\right) - \sup_{i\in I}\alpha_i^+$.

(iv) $\sum_{i\in I}^{\mathcal{R}}\alpha_i - \sum_{i\in I}\alpha_i^- = \sum_{i\in I}^{\mathcal{L}}\alpha_i - (\sup_{i\in I}\alpha_i)^-$.

(v) $\sum_{i\in I}^{\mathcal{L}}\alpha_i \leq \sum_{i\in I}^{\mathcal{R}}\alpha_i$.

Proof. Thanks to the comment before the proposition, only assertions (ii)–(iv) need to be checked. Indeed, (14) and assertion (i) imply that

$$\textstyle\sum_{i\in I}^{\mathcal{R}}\alpha_i = \sum_{i\in I}\alpha_i^+ - \left(\sup\limits_{i\in I}\alpha_i\right)^- = \left(\sum_{i\in I}^{\mathcal{R}}\alpha_i\right)^+ - \left(\sup\limits_{i\in I}\alpha_i\right)^-.$$

Thus, if $\sum_{i\in I}^{\mathcal{R}}\alpha_i < +\infty$, then $(\sup_{i\in I}\alpha_i)^- \in \mathbb{R}$ and we get

$$\textstyle\left(\sup\limits_{i\in I}\alpha_i\right)^- = \left(\sum_{i\in I}^{\mathcal{R}}\alpha_i\right)^+ - \sum_{i\in I}^{\mathcal{R}}\alpha_i = \left(\sum_{i\in I}^{\mathcal{R}}\alpha_i\right)^-.$$

Otherwise, if $\sum_{i\in I}^{\mathcal{R}}\alpha_i = +\infty$, then $(\sum_{i\in I}^{\mathcal{R}}\alpha_i)^- = (\sup_{i\in I}\alpha_i)^- = 0$ and (ii) holds true.

To show (iii) we first suppose that $(\sup_{i\in I}\alpha_i)^+ < +\infty$. This entails

$$\begin{aligned}\textstyle\sum_{i\in I}^{\mathcal{R}}\alpha_i &= \textstyle\sum_{i\in I}\alpha_i^+ - \left(\sup\limits_{i\in I}\alpha_i\right)^- + \left(\sup\limits_{i\in I}\alpha_i\right)^+ - \left(\sup\limits_{i\in I}\alpha_i\right)^+\\ &= \textstyle\left(\sum_{i\in I}\alpha_i^+ + \sup\limits_{i\in I}\alpha_i\right) - \left(\sup\limits_{i\in I}\alpha_i\right)^+,\end{aligned}$$

and (iii) follows because $(\sup_{i\in I}\alpha_i)^+ = \sup_{i\in I}\alpha_i^+$. Next, when $(\sup_{i\in I}\alpha_i)^+ = +\infty$, we have $\sum_{i\in I}^{\mathcal{R}}\alpha_i = \sum_{i\in I}\alpha_i^+ = +\infty$, and (iii) also holds because of the convention $+\infty - (+\infty) = +\infty$. $\square$

Observe that the inequality in Proposition 3.6(v) may be strict. For instance, if $I = \{1, 2\}$ and α_1, $\alpha_2 \in \mathbb{R}$ are such that $\alpha_1\alpha_2 < 0$, then we obtain

$$\textstyle\sum_{i\in I}^{\mathcal{R}}\alpha_i = \max\{\alpha_1, \alpha_2, \alpha_2 + \alpha_1\} = \max\{\alpha_1, \alpha_2\} > \alpha_1 + \alpha_2 = \sum_{i\in I}^{\mathcal{L}}\alpha_i.$$

We now extend the concept of Lebesgue sums to functions. Given a family of functions $f_i : X \to \mathbb{R}_\infty$, $i \in I$, the associated Lebesgue sum is the function

$$\textstyle\sum_{i\in I}^{\mathcal{L}} f_i : X \to \overline{\mathbb{R}} \;\text{ defined by }\; \left(\sum_{i\in I}^{\mathcal{L}} f_i\right)(x) := \sum_{i\in I}^{\mathcal{L}} f_i(x),\; x \in X;$$

that is, $\sum_{i\in I}^{\mathcal{L}} f_i = \sum_{i\in I} f_i^+ - \sum_{i\in I} f_i^-$, where f_i^+ and f_i^- are the positive and negative parts of f_i, respectively. Additionally, for the sake of simplicity, for each $J \in \mathcal{F}(I)$ we denote

$$\textstyle f_J := \sum_{i\in J} f_i,\; f_J^+ := \sum_{i\in J} f_i^+ \text{ and } f_J^- := \sum_{i\in J} f_i^-,$$

so that, by (9), for each $K_0 \in \mathcal{F}(I)$ the function $\sum_{i\in I}^{\mathcal{L}} f_i$ is equivalently written as

$$\begin{aligned}\textstyle\sum_{i\in I}^{\mathcal{L}} f_i &= \inf_{K\in\mathcal{F}(I)} \left(f_K + \textstyle\sum_{i\in I\setminus K} f_i^+\right) = \inf_{K\in\mathcal{F}(I)} \sup_{J\in\mathcal{F}(I\setminus K)} \left(f_K + \textstyle\sum_{i\in J} f_i^+\right) \\ &= \inf_{K\in\mathcal{F}(I),\, K_0\subset K} \sup_{J\in\mathcal{F}(I\setminus K)} \left(f_K + \textstyle\sum_{i\in J} f_i^+\right). \qquad (15)\end{aligned}$$

If we furthermore assume that the convex functions $\sup_{J\in\mathcal{F}(I\setminus K)}\left(f_K + \sum_{i\in J} f_i^+\right)$, $K \in \mathcal{F}(I)$, are proper, we have that (see, e.g., [2, Proposition 3.2.6])

$$\left(\sup_{J\in\mathcal{F}(I\setminus K)} \left(f_K + f_J^+\right)\right)^* = \overline{\mathrm{co}}\left(\inf_{J\in\mathcal{F}(I\setminus K)} \left(f_K + f_J^+\right)^*\right).$$

Thus, since each of the functions $\inf_{J\in\mathcal{F}(I\setminus K)}\left(f_K + f_J^+\right)^*$, $K \in \mathcal{F}(I)$, is convex (as the net of convex functions $(\left(f_K + f_J^+\right)^*)_{J\in\mathcal{F}(I\setminus K)}$ is non-increasing), by taking the conjugate of both sides of (15) and by using the second equality in Lemma 3.4, we obtain

$$\left(\textstyle\sum_{i\in I}^{\mathcal{L}} f_i\right)^* = \sup_{K\in\mathcal{F}(I)} \overline{\mathrm{co}}\left(\inf_{J\in\mathcal{F}(I\setminus K)} \left(f_K + f_J^+\right)^*\right) = \sup_{K\in\mathcal{F}(I)} \mathrm{cl}\left(\inf_{J\in\mathcal{F}(I\setminus K)} \left(f_K + f_J^+\right)^*\right). \qquad (16)$$

In addition, according to Proposition 3.5, we have that

$$\textstyle\sum_{i\in I}^{\mathcal{L}} f_i(x) = \begin{cases} \sum_{i\in I} f_i(x), & \text{if } \sum_{i\in I} f_i^+(x) < +\infty, \\ +\infty, & \text{otherwise.} \end{cases}$$

Let us also observe that, for each $x \in X$, Lemma 3.2 gives rise to a countable set $J := J(x) \subset I$ such that

$$\textstyle\sum_{i\in I}^{\mathcal{L}} f_i(x) = \sum_{i\in J} f_i^+(x) - \sum_{i\in J} f_i^-(x).$$

We emphasize here the fact that the countable set J in this last relation varies with x, and the function f does not reduce, in general, to a series. For comparative purposes, we also define the robust sum function of the f_i's, $\sum_{i\in I}^{\mathcal{R}} f_i : X \to \mathbb{R}_\infty$, as (see, i.e., [6] and references therein)

$$\textstyle\sum_{i\in I}^{\mathcal{R}} f_i = \sup_{J\in\mathcal{F}(I)} f_J.$$

Then, by Proposition 3.6, we have

$$\textstyle\sum_{i\in I}^{\mathcal{R}} f_i = \sum_{i\in I} f_i^+ - \left(\sup_{i\in I} f_i\right)^-, \tag{17}$$

and the functions $\sum_{i\in I}^{\mathcal{L}} f_i$ and $\sum_{i\in I}^{\mathcal{R}} f_i$ are related via the equality

$$\textstyle\sum_{i\in I}^{\mathcal{R}} f_i - \sum_{i\in I} f_i^- = \sum_{i\in I}^{\mathcal{L}} f_i - \left(\sup_{i\in I} f_i\right)^-.$$

Moreover, by Proposition 3.6(*iii*) we have that

$$\textstyle\sum_{i\in I}^{\mathcal{R}} f_i = \left(\sum_{i\in I} f_i^+ + \sup_{i\in I} f_i\right) - \sup_{i\in I} f_i^+,$$

implying that the robust sum is the difference of the two convex functions $\sum_{i\in I} f_i^+ + \sup_{i\in I} f_i$ and $\sup_{i\in I} f_i^+$. At the same time, since the functions f_K and $\sum_{i\in J} f_i^+$ are convex, the supremum function $\sup_{J\in\mathcal{F}(I\setminus K)} \left(f_K + \sum_{i\in J} f_i^+\right)$ is convex, and so is the Lebesgue sum function $\sum_{i\in I}^{\mathcal{L}} f_i$ (for being the infimum of a non-increasing net of convex functions). The following result gives a situation where $\sum_{i\in I}^{\mathcal{L}} f_i$ also inherits the lower semi-continuity from the f_i's, if any.

Proposition 3.7. *The Lebesgue sum function $f := \sum_{i\in I}^{\mathcal{L}} f_i$ is sequentially lsc provided that, for each countable set $J \subset I$, the function $\sum_{i\in J}^{\mathcal{L}} f_i$ is sequentially lsc. This happens, for instance, when all the f_i's are lsc and, for each countable set $J \subset I$, there are continuous functions $g_i : X \to \mathbb{R}$, $i \in J$, such that $f_i \geq g_i$, for all $i \in J$, and the function $x \mapsto \sum_{i\in J}^{\mathcal{L}} g_i(x)$ is continuous.*

Proof. Let $(x_n)_n \subset \operatorname{dom} f$ be a sequence with $x_n \to x \in X$. Then, by Lemma 3.2, we find countable sets $J_n \subset I$, $n \geq 0$, such that $f(x) = \sum_{i\in J_0} f_i^+(x) - \sum_{i\in J_0} f_i^-(x)$ and

$$\textstyle f(x_n) = \sum_{i\in J_n} f_i^+(x_n) - \sum_{i\in J_n} f_i^-(x_n), \text{ for all } n \geq 1.$$

Moreover, we have $f_i^+(x_n) = f_i^-(x_n) = 0$ for all $i \in I \setminus J$, where $J := \cup_{n\geq 0} J_n \subset I$, and the lower semi-continuity of the f_i's implies that $f_i^+(x) = 0$ for all $i \in I \setminus J$. Observe that J is countable and we have, for each $n \geq 1$,

$$\textstyle f(x_n) = \sum_{i\in J} f_i^+(x_n) - \sum_{i\in J} f_i^-(x_n) = \sum_{i\in J}^{\mathcal{L}} f_i(x_n).$$

So, Fatou's lemma gives rise to

$$\textstyle f(x) \leq \sum_{i\in J}^{\mathcal{L}} f_i(x) + \sum_{i\in I\setminus J}^{\mathcal{L}} f_i^+(x) = \sum_{i\in J}^{\mathcal{L}} f_i(x) \leq \liminf_{n\to+\infty} \sum_{i\in J}^{\mathcal{L}} f_i(x_n) = \liminf_{n\to+\infty} f(x_n);$$

that is, f is sequentially lsc at x. $\square$

4. Reduction to finite/countable sums

In this section, X is a lcs whose topological dual X^* is endowed with a compatible topology. We give a family of proper lsc convex functions $f_i : X \to \mathbb{R}_\infty$, $i \in I$, and denote the associated Lebesgue sum function by f,

$$f(x) := \textstyle\sum_{i\in I}^{\mathcal{L}} f_i(x) = \sum_{i\in I} f_i^+(x) - \sum_{i\in I} f_i^-(x),\ x \in X. \tag{18}$$

Before a full characterization of the subdifferential of f to be investigated in the next section, we show here that the ε-subdifferential of f, $\varepsilon \geq 0$, can be reduced to that of countable and finite sums. To this aim, we introduce the following families, for $x \in f^{-1}(\mathbb{R})$, $\varepsilon \geq 0$ and $J \subset I$,

$$\mathcal{F}_{x,\varepsilon}(J) = \left\{ K \in \mathcal{F}(J) : \sum_{i\in K} f_i^+(x) \geq \sum_{i\in J} f_i^+(x) - \varepsilon \right\} \tag{19}$$

and

$$\mathfrak{C}_x(J) = \left\{ K \in \mathfrak{C}(J) : \sum_{i\in K} f_i^+(x) = \sum_{i\in J} f_i^+(x) \right\}, \tag{20}$$

where $\mathcal{F}(J)$ and $\mathfrak{C}(J)$ were defined in (4). The first result gives a finite-dimensional representation of $\partial_\varepsilon f$ by means of the ε-subdifferential of the functions $f_K + \sum_{i\in J} f$, $K \in \mathcal{F}(I)$, $J \in \mathfrak{C}_x(I \setminus K)$.

Proposition 4.1. *Given $f_i \in \Gamma_0(\mathbb{R}^n)$, $i \in I$, for every $x \in \operatorname{dom} f$ and $\varepsilon > 0$ we have*

$$\partial_\varepsilon f(x) = \liminf_{K\in\mathcal{F}(I)} \Big(\bigcup_{J\in\mathfrak{C}_x(I\setminus K)} \partial_\varepsilon \Big(f_K + \sum_{i\in J} f_i^+ \Big)(x) \Big). \tag{21}$$

Proof. Given $x^* \in \partial_\varepsilon f(x)$, for every integer $m \geq 1$ we have that

$$f^*(x^*) + f(x) \leq \langle x, x^* \rangle + \varepsilon < \langle x^*, x \rangle + \varepsilon + 1/m, \tag{22}$$

and the function f is proper. Due to (15) and (16), f and f^* can also be written as

$$f = \inf_{K\in\mathcal{F}(I)} \sup_{J\in\mathcal{F}(I\setminus K)} \big(f_K + f_J^+ \big) = \inf_{K\in\mathcal{F}(I),\, K_0\subset K} \sup_{J\in\mathcal{F}(I\setminus K)} \big(f_K + f_J^+ \big) \tag{23}$$

for all $K_0 \in \mathcal{F}(I)$, and

$$f^* = \sup_{K\in\mathcal{F}(I)} \operatorname{cl} \left(\inf_{J\in\mathcal{F}(I\setminus K)} \big(f_K + f_J^+ \big)^* \right) \tag{24}$$

(here, since f is proper, we apply (15) only to functions $\sup_{J\in\mathcal{F}(I\setminus K)} \big(f_K + \sum_{i\in J} f_i^+ \big)$, $K \in \mathcal{F}(I)$, that are proper). So, (22) yields some $K_0 \in \mathcal{F}(I)$ (possibly depending on x and x^*) such that

$$\begin{aligned} f^*(x^*) + f_{K_0}(x) + \sum_{i\in I\setminus K_0} f_i^+(x) &= f^*(x^*) + \sup_{J\in\mathcal{F}(I\setminus K_0)} \big(f_{K_0}(x) + f_J^+(x) \big) \\ &< \langle x^*, x \rangle + \varepsilon + 1/m, \end{aligned}$$

and we get, for all $K \in \mathcal{F}(I)$,

$$\begin{aligned} f^*(x^*) + f_{K_0\cup K}(x) + \sum_{i\in I\setminus(K_0\cup K)} f_i^+(x) &\leq f^*(x^*) + f_{K_0}(x) + \sum_{i\in I\setminus K_0} f_i^+(x) \\ &< \langle x^*, x \rangle + \varepsilon + 1/m; \end{aligned}$$

that is, by (24), for all $K \in \mathcal{F}(I)$ such that $K_0 \subset K$ we have (taking K instead of $K_0 \cup K$ in the last inequality)

$$\mathrm{cl}\left(\inf_{J \in \mathcal{F}(I \setminus K)} \left(f_K + f_J^+\right)^*\right)(x^*) + f_K(x) + \sum_{i \in I \setminus K} f_i^+(x) < \langle x^*, x\rangle + \varepsilon + 1/m.$$

Therefore, for each $K \in \mathcal{F}(I)$ such that $K_0 \subset K$, we find sequences $(x^*_{k,m})_k \subset \mathbb{R}^n$ and associated sets $(J_{k,m})_k \subset \mathcal{F}(I \setminus K)$ such that $x^*_{k,m} \to_k x^*$ and

$$\left(f_K + f^+_{J_{k,m}}\right)^*(x^*_{k,m}) + f_K(x) + \sum_{i \in I \setminus K} f_i^+(x) < \left\langle x^*_{k,m}, x\right\rangle + \varepsilon + 1/m, \text{ for all } k \geq 1. \quad (25)$$

Moreover, by enlarging the sets $J_{k,m}$ (when necessary), as

$$\sum_{i \in I \setminus K} f_i^+(x) \leq \sum_{i \in I} f_i^+(x) < +\infty$$

we may suppose that each $J_{k,m}$ satisfies

$$\sum_{i \in I \setminus K} f_i^+(x) \leq f^+_{J_{k,m}}(x) + 1/k, \quad (26)$$

so that $J_0 := \cup_{m,k \geq 1} J_{k,m} \in \mathfrak{C}_x(I \setminus K)$. Therefore (25) reads, for all $k \geq 1$,

$$\left(f_K + f^+_{J_0}\right)^*(x^*_{k,m}) + f_K(x) + \sum_{i \in J_0} f_i^+(x) < \left\langle x^*_{k,m}, x\right\rangle + \varepsilon + 1/m, \quad (27)$$

and we obtain, making $k \to +\infty$ and after $m \to +\infty$,

$$\left(f_K + \sum_{i \in J_0} f_i^+\right)^*(x^*) + f_K(x) + \sum_{i \in J_0} f_i^+(x) \leq \langle x^*, x\rangle + \varepsilon. \quad (28)$$

Consequently, $x^* \in \partial_\varepsilon\left(f_K + \sum_{i \in J_0} f_i^+\right)(x)$ and the inclusion "$\subset$" in (21) follows.

Conversely, take any x^* in the right hand side of (21) and fix $y \in \mathrm{dom}\, f$. Then there is $K_0 \in \mathcal{F}(I)$ such that, for all $K \in \mathcal{F}(I)$ containing K_0, we find some $J_0 \in \mathfrak{C}_x(I \setminus K)$ such that $x^* \in \partial_\varepsilon(f_K + \sum_{i \in J_0} f_i^+)(x)$; that is,

$$\begin{aligned}\langle x^*, y - x\rangle &\leq f_K(y) + \sum_{i \in J_0} f_i^+(y) - f_K(x) - \sum_{i \in J_0} f_i^+(x) + \varepsilon \\ &= f_K(y) + \sum_{i \in J_0} f_i^+(y) - f_K(x) - \sum_{i \in I \setminus K} f_i^+(x) + \varepsilon \\ &\leq \sup_{J \in \mathcal{F}(I \setminus K)} \left(f_K(y) + f_J^+(y)\right) - f_K(x) + \varepsilon.\end{aligned}$$

Furthermore, since $\lim_{K \in \mathcal{F}(I)} f_K(x) = f(x)$ due to Proposition 3.5, for any $\delta > 0$ we may assume that $f_K(x) \geq f(x) - \delta$ for all $K \in \mathcal{F}(I)$ such that $K_0 \subset K$. Thus, the last inequality entails

$$\langle x^*, y - x\rangle \leq \sup_{J \in \mathcal{F}(I \setminus K)} \left(f_K(y) + f_J^+(y)\right) - f(x) + \varepsilon + \delta,$$

and we deduce, taking the infimum over all $K \in \mathcal{F}(I)$ satisfying $K_0 \subset K$, that

$$\begin{aligned}\langle x^*, y-x\rangle &\leq \inf_{K\in\mathcal{F}(I),\ K_0\subset K}\ \sup_{J\in\mathcal{F}(I\setminus K)} \left(f_K(y)+f_J^+(y)\right)-f(x)+\varepsilon+\delta\\ &= f(y)-f(x)+\varepsilon+\delta,\end{aligned}$$

where the last equality comes from (15). Finally, the desired conclusion follows as $\delta\downarrow 0$. □

The second result replaces the countable sets used in (21) with finite ones.

Proposition 4.2. *Given $f_i\in\Gamma_0(\mathbb{R}^n)$, $i\in I$, for every $x\in\operatorname{dom} f$ and $\varepsilon>0$ we have*

$$\partial_\varepsilon f(x)=\operatorname{cl}\Big(\liminf_{K\in\mathcal{F}(I)} \textstyle\bigcup_{\substack{J\in\mathcal{F}_{x,\delta}(I\setminus K)\\ 0<\delta\leq\varepsilon}} \partial_{\varepsilon-\delta}\left(f_K+f_J^+\right)(x)\Big). \tag{29}$$

Proof. Fix $x^*\in\partial_\varepsilon f(x)$ and, taking into account Lemma 2.1, choose $\rho>0$ with $0<\alpha<\varepsilon$ and $x^*_\alpha\in\partial_\alpha f(x)$ such that $x^*\in x^*_\alpha+\rho B_{\mathbb{R}^n}$ (the unit ball in $\mathbb{R}^n$). Then, reasoning as in the beginning of the proof of Proposition 4.1, we find some $K_\alpha\in\mathcal{F}(I)$ such that, for all $K\supset K_\alpha$ and all $0<\delta<(\varepsilon-\alpha)/3$,

$$\inf_{J\in\mathcal{F}(I\setminus K)}\left(f_K+f_J^+\right)^*(x^*_\alpha)+f_K(x)+\sum_{i\in I\setminus K} f_i^+(x)<\langle x^*,x\rangle+\alpha+\delta.$$

Observe that $\sum_{i\in I\setminus K} f_i^+(x)<+\infty$ and $\inf_{J\in\mathcal{F}(I\setminus K)}\left(f_K+f_J^+\right)^*(x^*_\alpha)<+\infty$ (as $x\in\operatorname{dom} f$ and f^* is proper). Next, for each $0<\delta<(\varepsilon-\alpha)/3$ such that $\delta\leq\varepsilon$, we take $J_{\alpha,K}\in\mathcal{F}(I\setminus K)$ such that

$$\left(f_K+f^+_{J_{\alpha,K}}\right)^*(x^*_\alpha)<\inf\nolimits_{J\in\mathcal{F}(I\setminus K)}\left(f_K+f_J^+\right)^*(x^*_\alpha)+\delta$$

and $J_{\alpha,K}\in\mathcal{F}_{x,\delta}(I\setminus K)$, which is possible because the net $(\left(f_K+f_J^+\right)^*)_{J\in\mathcal{F}(I\setminus K)}$ is non-increasing. Then we get, for all $K\supset K_\alpha$,

$$\left(f_K+f^+_{J_{\alpha,K}}\right)^*(x^*_\alpha)+f_K(x)+f^+_{J_{\alpha,K}}(x)<\langle x^*,x\rangle+\alpha+2\delta<\langle x^*,x\rangle+\varepsilon-\delta,$$

so that $x^*\in x^*_\alpha+\rho B_{\mathbb{R}^n}\subset\partial_{\varepsilon-\delta}\big(f_K+f^+_{J_{\alpha,K}}\big)(x)+\rho B_{\mathbb{R}^n}\subset D_K(x)+\rho B_{\mathbb{R}^n}$, where

$$D_K(x):=\bigcup_{\substack{J\in\mathcal{F}_{x,\delta}(I\setminus K)\\ 0<\delta\leq\varepsilon}}\partial_{\varepsilon-\delta}\left(f_K+f_J^+\right)(x).$$

But this last inclusion holds for all $K\supset K_\alpha$, so since the net of sets $(D_{x,K})_{K\in\mathcal{F}(I)}$ is non-increasing we get

$$x^*\in\textstyle\bigcup_{K_0\in\mathcal{F}(I)}\Big(\operatorname{cl}\Big(\displaystyle\bigcap_{K\in\mathcal{F}(I),\ K_0\subset K} D_K(x)\Big)+\rho B_{\mathbb{R}^n}\Big)\subset\operatorname{cl}\Big(\liminf_{K\in\mathcal{F}(I)} D_K(x)\Big)+\rho B_{\mathbb{R}^n},$$

which entails the inclusion "$\subset$" in (29) after taking the intersection over $\rho>0$.

Conversely, assume that $x^*\in\cap_{K\in\mathcal{F}(I),\ K_0\subset K} D_K(x)$ for some $K_0\in\mathcal{F}(I)$, and fix $y\in\operatorname{dom} f$. Then, for each $K\in\mathcal{F}(I)$ such that $K_0\subset K$, there exist some $0<\delta\leq\varepsilon$ and $J\in\mathcal{F}_{x,\delta}(I\setminus K)$ such that $x^*\in\partial_{\varepsilon-\delta}\left(f_K+f_J^+\right)(x)$; that is,

$$\begin{aligned}\langle x^*,y-x\rangle &\leq f_K(y)+f_J^+(y)-f_K(x)-f_J^+(x)+\varepsilon-\delta\\ &\leq f_K(y)+\sum_{i\in I\setminus K} f_i^+(y)-f_K(x)-\sum_{i\in I\setminus K} f_i^+(x)+\varepsilon\end{aligned}$$

$$= f_K(y) + \sum_{i\in I\setminus K} f_i^+(y) - f(x) - \sum_{i\in I\setminus K} f_i^-(x) + \varepsilon$$

$$\leq f_K(y) + \sum_{i\in I\setminus K} f_i^+(y) - f(x) + \varepsilon,$$

and (15) gives rise to $\langle x^*, y-x\rangle \leq f(y) - f(x) + \varepsilon$. In other words, $x^* \in \partial_\varepsilon f(x)$ and the inclusion "$\supset$" in (29) is obtained as $\partial_\varepsilon f(x)$ is closed. $\square$

We give now the representation of $\partial_\varepsilon f(x)$ in general infinite-dimensional spaces. More simpler characterizations are provided in Corollary 4.5 when the space X is reflexive.

Theorem 4.3. *Given $f_i \in \Gamma_0(X)$, $i \in I$, for every $x \in \operatorname{dom} f$ and $\varepsilon > 0$ we have*

$$\partial_\varepsilon f(x) = \operatorname{cl}\Big(\liminf_{K\in\mathcal{F}(I)}\ \operatorname{cl}\Big(\bigcup_{J\in\mathfrak{C}_x(I\setminus K)} \partial_\varepsilon\big(f_K + \sum_{i\in J} f_i^+\big)(x)\Big)\Big) \tag{30}$$

$$= \operatorname{cl}\Big(\liminf_{K\in\mathcal{F}(I)}\ \operatorname{cl}\Big(\bigcup_{\substack{0<\delta\leq\varepsilon\\ J\in\mathcal{F}_{x,\delta}(I\setminus K)}} \partial_{\varepsilon-\delta}\big(f_K + f_J^+\big)(x)\Big)\Big). \tag{31}$$

Proof. Fix $x^* \in \partial_\varepsilon f(x)$, and let $\alpha > 0$, $u \in X$ and $U \in \mathcal{N}_{X^*}$ such that $\sigma_U(u) \leq \alpha$. We also choose a finite-dimensional subspace $L \subset X$ containing x and such that $L^\perp \subset \frac{1}{2}U$; hence, $\partial_\varepsilon f(x) \subset \partial_\varepsilon(f + \mathrm{I}_L)(x)$. Next, we consider the restriction $\tilde{f}_i$, $i \in I$, of the function f_i to L, together with the associated Lebesgue sum $\tilde{f} := \sum_{i\in I}^{\mathcal{L}} \tilde{f}_i$, so that for each $J \subset I$

$$\partial_\varepsilon\left(\sum\nolimits_{i\in J}\tilde{f}_i\right)(x) = \left\{z^*_{|L} : z^* \in \partial_\varepsilon\left(\sum\nolimits_{i\in J} f_i + \mathrm{I}_L\right)(x)\right\}, \tag{32}$$

for all $J \in \mathcal{F}(I) \cup \mathfrak{C}_x(I\setminus K)$, and $\partial_\varepsilon \tilde{f}(x) = \{z^*_{|L} : z^* \in \partial_\varepsilon(f + \mathrm{I}_L)(x)\}$, where $z^*_{|L}$ denotes the restriction of $z^* \in X^*$ to L $(\subset X)$. Then, applying Proposition 4.1, we obtain that

$$\partial_\varepsilon \tilde{f}(x) = \liminf_{K\in\mathcal{F}(I)} \ \bigcup\nolimits_{J\in\mathfrak{C}_x(I\setminus K)} \partial_\varepsilon\left(\tilde{f}_K + \sum_{i\in J}\tilde{f}_i^+\right)(x), \tag{33}$$

where $\tilde{f}_K := \sum_{i\in K}\tilde{f}_i$, and $\mathfrak{C}_x(I\setminus K)$ is the same as in (20). Therefore, using (32), there exists some $K_0 \in \mathcal{F}(I)$ such that, for each $K \in \mathcal{F}(I)$ containing K_0, we can find $J \in \mathfrak{C}_x(I\setminus K)$ and $y^* \in \partial_\varepsilon\left(f_K + \sum_{i\in J} f_i^+ + \mathrm{I}_L\right)(x)$ such that (using the ε-subdifferential sum rule of [10])

$$x^* \in y^* + L^\perp \subset \partial_\varepsilon\Big(f_K + \sum_{i\in J} f_i^+\Big)(x) + L^\perp + \frac{1}{2}U \subset A_K + U,$$

where
$$A_K := \bigcup_{J\in\mathfrak{C}_x(I\setminus K)} \partial_\varepsilon\Big(f_K + \sum_{i\in J} f_i^+\Big)(x)$$

(this set is easily shown to be convex). Then the inequality above yields

$$\langle x^*, u\rangle \leq \inf_{K\in\mathcal{F}(I),\ K_0\subset K} \sigma_{A_K+U}(u) \leq \inf_{K\in\mathcal{F}(I),\ K_0\subset K} \sigma_{A_K}(u) + \alpha,$$

and the arbitrariness of $\alpha > 0$ entails

$$\langle x^*, u\rangle \leq \operatorname{cl}\Big(\inf_{K\in\mathcal{F}(I),\ K_0\subset K} \sigma_{A_K}\Big)(u) \ \text{ for all } u \in X;$$

hence, in particular, the function $\mathrm{cl}\left(\inf_{K\in\mathcal{F}(I),\, K_0\subset K}\sigma_{A_K}\right)$ is proper. Thus, since the net $(A_K)_{K\in\mathcal{F}(I)}$ is non-increasing and each set A_K is convex, Lemma 2.2 entails

$$\langle x^*, u\rangle \leq \mathrm{cl}\left(\inf_{K\in\mathcal{F}(I),\, K_0\subset K}\sigma_{\mathrm{cl}(A_K)}\right)(u) = \sigma_{\liminf\limits_{K\in\mathcal{F}(I)}\mathrm{cl}(A_K)}(u) \text{ for all } u\in X.$$

Consequently, since the lim inf preserves the convexity, we conclude that

$$x^* \in \mathrm{cl}\left(\liminf_{K\in\mathcal{F}(I)}\mathrm{cl}\,(A_K)\right),$$

and the inclusion "$\subset$" in (30) follows. The converse inclusion is checked with the same reasoning as in the proof of Propositions 4.1 and 4.2.

To prove (31) we proceed as above but, instead of (33) we use Proposition 4.2, which leads us to

$$\partial_\varepsilon \tilde{f}(x) = \mathrm{cl}\left(\liminf_{K\in\mathcal{F}(I)} \bigcup\nolimits_{J\in\mathcal{F}_{x,\delta}(I\setminus K),\, 0<\delta\leq\varepsilon} \partial_{\varepsilon-\delta}\left(\tilde{f}_K + \tilde{f}_J^+\right)(x)\right),$$

where $\tilde{f}_K$ is defined as above and $\tilde{f}_J^+ := \sum_{i\in J}\tilde{f}_i^+$. We also find some $K_0\in\mathcal{F}(I)$ such that, for each $K\in\mathcal{F}(I)$ containing K_0, there exists $0<\delta\leq\varepsilon$, $J\in\mathcal{F}_{x,\delta}(I\setminus K)$, and $y^*\in\partial_{\varepsilon-\delta}\left(f_K+\tilde{f}_J^+ + \mathrm{I}_L\right)(x)$ such that

$$x^* \in y^* + L^\perp + \frac{1}{3}U \subset \partial_\varepsilon\left(f_K + \tilde{f}_J^+\right)(x) + L^\perp + \frac{2}{3}U \subset B_K + U,$$

where
$$B_K := \bigcup_{\substack{J\in\mathcal{F}_{x,\delta}(I\setminus K)\\ 0<\delta\leq\varepsilon}} \partial_{\varepsilon-\delta}\left(\tilde{f}_K + \tilde{f}_J^+\right)(x)$$

(observe that B_K is convex and the net $(B_K)_{K\in\mathcal{F}(I)}$ is non-increasing). Then, proceeding as in the previous paragraph, we prove that

$$\langle x^*, u\rangle \leq \sigma_{\liminf\limits_{K\in\mathcal{F}(I)}\mathrm{cl}(B_K)} \text{ for all } u\in X,$$

which leads us to (31). □

Theorem 4.3 takes a simple form for nonnegative functions.

Corollary 4.4. *Given nonnegative functions $f_i\in\Gamma_0(X)$, $i\in I$, for every $x\in\mathrm{dom}\, f$ and $\varepsilon>0$ we have*

$$\partial_\varepsilon f(x) = \mathrm{cl}\left(\bigcup_{J\in\mathfrak{C}_x(I)}\partial_\varepsilon\left(\sum_{i\in J}f_i\right)(x)\right) = \mathrm{cl}\left(\bigcup_{\substack{0<\delta\leq\varepsilon\\ K\in\mathcal{F}_{x,\delta}(I)}}\partial_{\varepsilon-\delta}f_K(x)\right). \tag{34}$$

Proof. For every $K\in\mathcal{F}(I)$ and $J_0\in\mathfrak{C}_x(I\setminus K)$ we have $J := K\cup J_0\in\mathfrak{C}_x(I)$,

$$f_K + \sum_{i\in J_0} f_i^+ = \sum_{i\in J} f_i \text{ and } \sum_{i\in K} f_i^+(x) = f(x).$$

So, (30) entails

$$\partial_\varepsilon f(x) \subset \mathrm{cl}\left(\liminf_{K\in\mathcal{F}(I)}\ \mathrm{cl}\left(\bigcup\nolimits_{J\in\mathfrak{C}_x(I)}\partial_\varepsilon f_J(x)\right)\right) = \mathrm{cl}\left(\bigcup\nolimits_{J\in\mathfrak{C}_x(I)}\partial_\varepsilon f_J(x)\right),$$

and we get the first equality in (34) as the opposite inclusion is straightforward.

The second equality in (34) follows similarly by using (31) instead of (30). □

The following corollary presents a situation where the inner closure in (30) can be removed.

Corollary 4.5. *If X is a reflexive Banach space, then for every $x\in\operatorname{dom} f$ and $\varepsilon>0$*

$$\partial_\varepsilon f(x) = \operatorname{cl}\Big(\liminf_{K\in\mathcal{F}(I)} \bigcup_{J\in\mathfrak{C}_x(I\setminus K)} \partial_\varepsilon\big(f_K+\sum_{i\in J} f_i^+\big)(x)\Big). \tag{35}$$

Proof. Suppose $x\in f^{-1}(\mathbb{R})$ and $\varepsilon>\delta>0$. According to Theorem 4.3, and taking into account Mazur's theorem, the convexity of the sets

$$A_{K,\delta} := \bigcup\nolimits_{J\in\mathfrak{C}_x(I\setminus K)}\partial_\delta\left(f_K+\sum\nolimits_{i\in J} f_i^+\right)(x)$$

and $\liminf_{K\in\mathcal{F}(I)} \operatorname{cl}^{\|\cdot\|}(A_{K,\delta})$ implies that

$$\partial_\delta f(x)\subset \operatorname{cl}^{\|\cdot\|}\big(\liminf_{K\in\mathcal{F}(I)} \operatorname{cl}^{\|\cdot\|}(A_{K,\delta})\big).$$

Choose $K_{-1}\in\mathcal{F}(I)$ such that $\sum_{i\in I\setminus K_{-1}} f_i^-(x)\le\varepsilon-\delta$ (as $\sum_{i\in I} f_i^-(x)<+\infty$; see Lemma 3.2). Let $K_0\in\mathcal{F}(I)$ such that $K_0\supset K_{-1}$ and consider a sequence $(x_n^*)_n\subset \liminf_{K\in\mathcal{F}(I)}\operatorname{cl}^{\|\cdot\|}(A_{K,\delta})$ which norm converges to $x^*\in X^*$. Then, for each $n\ge 1$, there exists $K_n\in\mathcal{F}(I)$ such that, for all $K\in\mathcal{F}(I)$ satisfying $K\supset K_n$, we have

$$x_n^*\in\bigcup\nolimits_{J\in\mathfrak{C}_x(I\setminus K)}\partial_\delta\Big(f_K+\sum_{i\in J} f_i^+\Big)(x). \tag{36}$$

We may assume without loss of generality that the sequence $(K_n)_n$ is non-decreasing and that $K_n\supset K_0$ ($\supset K_{-1}$). Consequently, for all $K\in\mathcal{F}(I)$ with $K\supset K_n$ and $J\in\mathfrak{C}_x(I\setminus K)$, we have $(K\setminus K_0)\cup J\in\mathfrak{C}_x(I\setminus K_0)$,

$$f^-_{K\setminus K_0}(x)\le f^-_{K\setminus K_{-1}}(x)\le\sum\nolimits_{i\in I\setminus K_{-1}} f_i^-(x)\le\varepsilon-\delta,$$

$$\begin{aligned} f_K+\sum_{i\in J} f_i^+ &= f_{K_0}+f_{K\setminus K_0}+\sum_{i\in J} f_i^+ = f_{K_0}+f^+_{K\setminus K_0}-f^-_{K\setminus K_0}+\sum_{i\in J} f_i^+\\ &= f_{K_0}-f^-_{K\setminus K_0}+\sum_{i\in(K\setminus K_0)\cup J} f_i^+\le f_{K_0}+\sum_{i\in(K\setminus K_0)\cup J} f_i^+, \end{aligned}$$

and

$$\Big(f_{K_0}+\sum_{i\in(K\setminus K_0)\cup J} f_i^+\Big)(x)=\Big(f_K+\sum_{i\in J} f_i^+\Big)(x)+f^-_{K\setminus K_0}(x)\le\Big(f_K+\sum_{i\in J} f_i^+\Big)(x)+\varepsilon-\delta.$$

Thus, (36) gives rise, for all $K\in\mathcal{F}(I)$ satisfying $K\supset K_n$, to

$$\begin{aligned} x_n^* &\in\bigcup\nolimits_{J\in\mathfrak{C}_x(I\setminus K)}\partial_\delta\Big(f_K+\sum_{i\in J} f_i^+\Big)(x)\subset\bigcup\nolimits_{J\in\mathfrak{C}_x(I\setminus K)}\partial_\varepsilon\Big(f_{K_0}+\sum_{i\in(K\setminus K_0)\cup J} f_i^+\Big)(x)\\ &\subset\bigcup\nolimits_{J\in\mathfrak{C}_x(I\setminus K_0)}\partial_\varepsilon\Big(f_{K_0}+\sum_{i\in J} f_i^+\Big)(x). \end{aligned}$$

In other words, for each $n\ge 1$, there exists some $J_n\in\mathfrak{C}_x(I\setminus K_0)$ such that

$$x_n^*\in\partial_\varepsilon\big(f_{K_0}+\sum_{i\in J_n} f_i^+\big)(x).$$

Thus, the set $J_0 := \cup_{n\geq 1} J_n$ belongs to $\mathfrak{C}_x(I \setminus K_0)$ and we have that

$$(x_n^*)_n \subset \partial_\varepsilon \left(f_{K_0} + \sum\nolimits_{i\in J_0} f_i^+ \right)(x). \tag{37}$$

Therefore, $x^* \in \partial_\varepsilon \left(f_{K_0} + \sum_{i\in J_0} f_i^+ \right)(x)$ for all $K_0 \in \mathcal{F}(I)$ with $K_0 \supset K_{-1}$, and Lemma 2.1 yields

$$\partial_\varepsilon f(x) = \mathrm{cl}^{\|\cdot\|} \left(\bigcup\nolimits_{0<\delta<\varepsilon} \partial_\delta f(x) \right) \subset \mathrm{cl}^{\|\cdot\|} \left(\liminf_{K\in\mathcal{F}(I)} \bigcup\nolimits_{J\in\mathfrak{C}_x(I\setminus K)} \partial_\varepsilon \left(f_K + \sum\nolimits_{i\in J} f_i^+ \right)(x) \right),$$

which is the inclusion "$\subset$" in (35). This finishes the proof of (35) because the opposite inclusion is an immediate consequence of Theorem 4.3. □

The following remark is also useful.

Remark 4.6. As the proof above shows (see (37)), in the setting of reflexive Banach spaces, given any $0 < \delta < \varepsilon$ and $x^* \in \partial_\delta f(x)$, for each $K_{-1} \in \mathcal{F}(I)$ satisfying $\sum_{i\in I\setminus K_{-1}} f_i^-(x) \leq \varepsilon - \delta$ and each $K_0 \in \mathcal{F}(I)$ with $K_{-1} \subset K_0$, there exists $J_0 \in \mathfrak{C}_x(I \setminus K_0)$ such that $x^* \in \partial_\varepsilon \left(f_{K_0} + \sum_{i\in J_0} f_i^+ \right)(x)$. In other words, if $x^* \in \partial_\varepsilon f(x)$, then for each $n \geq 1$, $K_n \in \mathcal{F}(I)$ satisfying $\sum_{i\in I\setminus K_n} f_i^-(x) \leq 1/n$, and each $K \in \mathcal{F}(I)$ containing K_n, there exists $J \in \mathfrak{C}_x(I \setminus K)$ such that

$$x^* \in \partial_{\varepsilon+1/n} \left(f_K + \sum\nolimits_{i\in J} f_i^+ \right)(x).$$

Indeed, this relation also characterizes the fact that $x^* \in \partial_\varepsilon f(x)$: To see that let the net of sets $(K_n)_{n\geq 1}$ be as above and fix $y \in \operatorname{dom} f$. Then, for each n and $K \in \mathcal{F}(I)$ containing K_n, there exists $J \in \mathfrak{C}_x(I \setminus K)$ such that $x^* \in \partial_{\varepsilon+1/n} \left(f_K + \sum_{i\in J} f_i^+ \right)(x)$; i.e.,

$$\begin{aligned}
\langle x^*, y - x\rangle &\leq f_K(y) + \sum\nolimits_{i\in J} f_i^+(y) - f_K(x) - \sum\nolimits_{i\in J} f_i^+(x) + \varepsilon + 1/n \\
&\leq f_K(y) + \sum\nolimits_{i\in J} f_i^+(y) - \sum\nolimits_{i\in J\cup K} f_i^+(x) + \varepsilon + 1/n \\
&\leq \sup_{J\in\mathfrak{C}_x(I\setminus K)} \left(f_K(y) + \sum\nolimits_{i\in J} f_i^+(y) \right) - \sum\nolimits_{i\in I} f_i^+(x) + \varepsilon + 1/n.
\end{aligned}$$

Thus, taking the infimum over $K \in \mathcal{F}(I)$ containing K_n, we get

$$\langle x^*, y - x\rangle \leq f(y) - f(x) + \varepsilon + 1/n,$$

and we get $x^* \in \partial_\varepsilon f(x)$ as $n \uparrow +\infty$.

5. Subdifferential of the Lebesgue sum

We establish in this section general subdifferential formulas for the Lebesgue sum of convex functions defined on an lcs X. As in the previous section, the topological dual X^* of X is endowed with a locally convex topology compatible for the duality pairing $\langle x^*, x\rangle := x^*(x)$ defined on $X^* \times X$. We have already reduced the problem of the ε-subdifferential of Lebesgue sums in the previous section to at most countable sums. The aim now is to go further and produce complete characterizations of this Lebesgue sum function, given exclusively by means of the data. To do this, we specify to our setting the results of [8], which provide subdifferential formulas for integral functions on locally convex Suslin spaces. In fact, since we only need to apply these results to countable sums at most, it is not necessary to assume the Suslin structure (see [3]).

Consider a countable set J and a function $x^*(\cdot) : J \to X^*$, that we also represent through its components by denoting $x_i^* := x^*(i)$, $i \in J$. The function $x^*(\cdot)$ is said to be weakly summable if $\sum_{j\in J}^{\mathcal{L}} \left\langle x_j^*, x\right\rangle \in \mathbb{R}$ for all $x \in X$, and for every $J_0 \subset J$ there exists $x_{J_0}^* \in X^*$ such that

$$\left\langle x_{J_0}^*, x\right\rangle = \sum_{j\in J_0}^{\mathcal{L}} \left\langle x_j^*, x\right\rangle := \sum_{j\in J_0} \left\langle x_j^*, x\right\rangle^+ - \sum_{j\in J_0} \left\langle x_j^*, x\right\rangle^-, \quad \text{for all } x \in X.$$

The function $x^*(\cdot)$ is said to be (strongly) summable, written $x^*(\cdot) \in L^1(J, X)$, if there exists some $V \in \mathcal{N}_X$ such that $\sum_{j\in J}\sigma_V(x_j^*) < +\infty$.

Since every summable function is weakly summable, we call strong or, simply, sum of $x^*(\cdot)$ the element of X^* given by $\sum_{j\in J} x_j^* := x_J^*$.

If X is finite dimensional, say $X = \mathbb{R}^n$, then (strong) summability and weak summability coincide. Given a set-valued map $F : J \rightrightarrows X^*$, we call sum of F the set

$$\sum_{j\in J} F(j) := \left\{ \sum_{j\in J} x_j^* : x^*(j) \in F(j), \text{ for all } j \in J \right\}.$$

We give the main result of this section in which we denote

$$\mathcal{A}_{J,\varepsilon} := \left\{ \varepsilon(\cdot) := (\varepsilon_j)_{j\in J} : \varepsilon_j \geq 0,\ \sum_{j\in J}\varepsilon \leq \varepsilon \right\},\ J \subset I.$$

Remember that $\mathcal{F}(J)$, $\mathfrak{C}_x(J \setminus K)$ and $\mathcal{F}_{x,\delta}(J \setminus K)$ have been defined in (4), (20) and (19), respectively.

Theorem 5.1. *Given a family $f_i \in \Gamma_0(X)$, $i \in I$, we denote $f := \sum_{i\in I}^{\mathcal{L}} f_i$. Then, for every $x \in X$ and $\varepsilon > 0$, we have*

$$\partial_\varepsilon f(x) = \operatorname{cl}\left(\liminf_{K\in\mathcal{F}(I)} \operatorname{cl}\left(\bigcup_{\substack{J\in\mathfrak{C}_x(I\setminus K)\\ (\varepsilon_i)\in\mathcal{A}_{K\cup J,\varepsilon}\\ 0\leq\lambda_i\leq 1,\ i\in J}} \sum_{i\in K} \partial_{\varepsilon_i} f_i(x) + \sum_{i\in J} \partial_{\varepsilon_i+\lambda_i f_i(x) - f_i^+(x)}(\lambda_i f_i)(x) \right)\right)$$

$$= \operatorname{cl}\left(\liminf_{K\in\mathcal{F}(I)} \operatorname{cl}\left(\bigcup_{\substack{0<\delta\leq\varepsilon\\ J\in\mathcal{F}_{x,\delta}(I\setminus K)\\ (\varepsilon_i)\in\mathcal{A}_{K\cup J,\varepsilon-\delta}\\ 0\leq\lambda_i\leq 1,\ i\in J}} \sum_{i\in K} \partial_{\varepsilon_i} f_i(x) + \sum_{i\in J} \partial_{\varepsilon_i+\lambda_i f_i(x) - f_i^+(x)}(\lambda_i f_i)(x) \right)\right). \tag{38}$$

Proof. Fix $x \in \operatorname{dom} f$ and $\varepsilon > 0$. Then, by Theorem 4.3, we have

$$\begin{aligned}\partial_\varepsilon f(x) &= \operatorname{cl}\left(\liminf_{K\in\mathcal{F}(I)} \ \operatorname{cl}\left(\bigcup\nolimits_{J\in\mathfrak{C}_x(I\setminus K)} \partial_\varepsilon \left(f_K + \sum_{i\in J} f_i^+\right)(x)\right)\right) \\ &= \operatorname{cl}\left(\liminf_{K\in\mathcal{F}(I)} \ \operatorname{cl}\left(\bigcup\nolimits_{J\in\mathcal{F}_{x,\delta}(I\setminus K),\ 0<\delta\leq\varepsilon} \partial_\varepsilon \left(f_K + \sum_{i\in J} f_i^+\right)(x)\right)\right).\end{aligned}$$

Thus, for each $K \in \mathcal{F}(I)$ and $J \in \mathfrak{C}_x(I \setminus K)$, by applying [8, Theorem 6] (taking into account that J is countable and $f_i^+ \geq 0$ for all $i \in J$) and [10], for all $\varepsilon_1, \varepsilon_2 \geq 0$ such that $\varepsilon_1 + \varepsilon_2 \leq \varepsilon$ we obtain that

$$\partial_{\varepsilon_1}\Big(\sum_{i\in J} f_i^+\Big)(x) = \mathrm{cl}\Big(\bigcup\nolimits_{\varepsilon(\cdot)\in\mathcal{A}_{J,\varepsilon_1}} \sum_{i\in J}\partial_{\varepsilon_i} f_i^+(x)\Big),$$

where (see, e.g., [2, Example 5.1.9]),

$$\partial_{\varepsilon_i} f_i^+(x) = \bigcup\nolimits_{0\leq\lambda\leq 1}\partial_{\varepsilon_i+\lambda f_i(x)-f_i^+(x)}(\lambda f_i)(x) \text{ for all } i \in J,$$

and
$$\partial_{\varepsilon_2}\Big(\sum_{i\in K} f_i\Big)(x) = \mathrm{cl}\Big(\bigcup\nolimits_{\varepsilon(\cdot)\in\mathcal{A}_{K,\varepsilon_2}} \sum_{i\in K}\partial_{\varepsilon_i} f_i(x)\Big).$$

Formula (38) follows then by combining all these expressions, and using [10] which ensures that

$$\partial_\varepsilon\left(f_K + \sum_{i\in J} f_i^+\right)(x) = \mathrm{cl}\left(\bigcup\nolimits_{\substack{\varepsilon_1,\varepsilon_2\geq 0\\ \varepsilon_1+\varepsilon_2\leq\varepsilon}} \partial_{\varepsilon_1}\left(\sum_{i\in J} f_i^+\right)(x) + \partial_{\varepsilon_2}\left(\sum_{i\in K} f_i\right)(x)\right). \qquad \square$$

If I is finite in Theorem 5.1, then formula (38) reduces to

$$\partial_\varepsilon f(x) = \mathrm{cl}\left(\bigcup_{(\varepsilon_i)_{i\in I}\in\mathcal{A}_{I,\varepsilon}} \sum_{i\in I}\partial_{\varepsilon_i} f_i(x)\right), \text{ for all } x \in X \text{ and } \varepsilon > 0,$$

giving rise to the well-known formula in [10]. In the case of nonnegative functions, we can argue as in the proof above, but with the use of Corollary 4.4 instead of Theorem 4.3 to show that

$$\partial_\varepsilon f(x) = \mathrm{cl}\left(\bigcup_{\substack{J\in\mathfrak{C}_x(I)\\ (\varepsilon_i)_{i\in J}\in\mathcal{A}_{J,\varepsilon}}} \sum_{i\in J}\partial_{\varepsilon_i} f_i(x)\right) = \mathrm{cl}\left(\bigcup_{\substack{K\in\mathcal{F}_{x,\delta}(I),\ 0<\delta\leq\varepsilon\\ (\varepsilon_i)_{i\in K}\in\mathcal{A}_{K,\varepsilon-\delta}}} \sum_{i\in K}\partial_{\varepsilon_i} f_i(x)\right),$$

for all $x \in X$ and $\varepsilon > 0$. The following results gives another representation of $\partial_\varepsilon f(x)$ in reflexive Banach spaces. Its proof is done in a similar way to that of Theorem 5.1, based on Remark 4.6.

Corollary 5.2. *Given a family $f_i \in \Gamma_0(X)$, $i \in I$, we denote $f := \sum_{i\in I}^{\mathcal{L}} f_i$. Assume that X is a reflexive Banach space and take $x \in X$ and $\varepsilon > 0$. Then $x^* \in \partial_\varepsilon f(x)$ if and only if, for each $n \geq 1$, $K_n \in \mathcal{F}(I)$ satisfying $\sum_{i\in I\setminus K_n} f_i^-(x) \leq 1/n$, and each $K \in \mathcal{F}(I)$ containing K_n, there exists $J \in \mathfrak{C}_x(I \setminus K)$ such that*

$$x^* \in \mathrm{cl}^{\|\cdot\|}\left(\bigcup_{\substack{J\in\mathfrak{C}_x(I\setminus K)\\ (\varepsilon_i)_{i\in I}\in\mathcal{A}_{K\cup J,\varepsilon+1/n},\ 0\leq\lambda_i\leq 1}} \sum_{i\in K}\partial_{\varepsilon_i} f_i(x) + \sum_{i\in J}\partial_{\varepsilon_i+\lambda_i f_i(x)-f_i^+(x)}(\lambda_i f_i)(x)\right).$$

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