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Periodic and Attraction-Stifling Behavior in Numerical Minimization

Adam B. Levy
Dept. of Mathematics, Bowdoin College, 8600 College Station, Brunswick, U.S.A.
alevy@bowdoin.edu

Dedicated to Roger J-B Wets on the occasion of his 85th birthday.

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We introduce and study concepts of "periodic" and "nearly-periodic" to describe and understand specific iterative behaviors related to attraction in numerical minimization. We also introduce a notion of "attraction-stifling" which can signal slow progress toward an attractor. We explore connections between these concepts, as well as how attraction-stifling relates to the categorization presented in a recent paper of the author [*Categorizing with catastrophic radii in numerical minimization*, Set-Valued Var. Analysis 29 (2021) 1–28].

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1. Introduction

Numerical minimization methods may carry more than one point from iteration to iteration as they attempt to identify a minimizer. This situation is relatively common in direct methods, including simplex methods (like the Nelder-Mead method) and evolutionary methods (like the genetic algorithm). We call these collections of points "iterate-multisets" (after [8]), and the "multi" here signals that these may include repeated elements. The terminology of "attractors" was introduced in [8] to describe the points at which numerical minimization methods cluster, and conditions were developed there to ensure when attractors are minimizers. The "basin of attraction" was subsequently defined to be the collection of initializations from which the method clusters at a particular attractor.

Attractors and their basins of attraction have been studied widely in dynamical systems (e.g., [1], [2], [3], [4], [5], [6], [12], [13], [14]), and [8, Chapter 3] describes these and other notions from dynamical systems that are adapted in [8] to numerical minimization. In particular, these objects are used in [8] to assess the significance of several examples of badly performing method/objective combinations, by determining whether a non-minimizing attractor has a relatively small or large basin of attraction.

The results in the present paper complement the existing work on attraction in numerical minimization by identifying and analyzing specific iterative behaviors.

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In particular, we introduce a notion of "periodic" iterate-multisets to describe the situation when the same iterate-multiset is revisited after a finite number of iterations. These objects are extensions to iterate-multisets of the classical concept from discrete dynamical systems (e.g., [2]). We show that a periodic iterate-multiset is only a member of one (finite) basin of attraction, and we explore how notions of monotonicity can rule out periodicity.

We also introduce "nearly periodic" iterate-multisets which have a near-return to the original iterate-multiset after a finite number of iterations, where "nearness" is defined via a pseudo-distance function we develop here for iterate-multisets. The periodic and nearly periodic properties have no restrictions on the distance from the intervening iterate-multisets to the attractor, but we also introduce the concept of "attraction-stifling" iterate-multisets which instead requires arbitrarily small changes in the distance to an attractor for a finite number of iterate-multisets in a row. We explore how these notions are all connected and we also show that attraction-stifling iterate-multisets identify the "catastrophic radii" used in the categorization from [9].

2. Background

Multisets are collections X of elements x from a normed vector space $\mathcal{X}$ where the same elements may appear in multiple instances, and a *multiset mapping* $S : \mathcal{X} \rightrightarrows \mathcal{Y}$ between two normed vector spaces $\mathcal{X}$ and $\mathcal{Y}$ maps multisets X of elements from $\mathcal{X}$ to multisets $S(X)$ of elements from $\mathcal{Y}$ (see [8]). The *domain* of a multiset mapping S is the collection of multisets X for which $S(X)$ is non-empty:

$$\mathrm{dom}(S) := \{X : S(X) \neq \emptyset\},$$

and the *range* is the set of elements appearing in some image-multiset:

$$\mathrm{range}(S) := \{y : \exists\, X \text{ with } y \in S(X)\}.$$

A minimization method $\mathcal{M}$ applied to an objective function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ defines a special multiset mapping $I_{\mathcal{M},f} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ called an *iteration mapping.* Iteration mappings were introduced in [8] and operate on *iterate-multisets* $X \subseteq \mathbb{R}^n$ with the same fixed number of elements $m \geq 1$ (where m is determined by the particular minimization method $\mathcal{M}$). When $m = 1$, we simply refer to the iterate-multisets as *iterates* and denote them in lower-case x^k.

To illustrate some of our results, we will use the steepest-descent iteration mapping

$$I_{\mathrm{SD}_4,f}(x) = x - \alpha_4(x)\, \nabla f(x), \tag{1}$$

with backtracking line-search (strict descent): $\alpha_4(x) = 1$ when $\nabla f(x) = \vec{0}$, and otherwise

$$\alpha_4(x) = \max\Big(\underset{\alpha \in \{2^{-j} : j \in \mathbb{N}\}}{\operatorname{argmax}} \delta_{(0,\infty)} \left(f(x) - f\left(x - \alpha \nabla f(x)\right)\right) \Big),$$

where $\delta_X(x)$ is the *indicator function* returning 1 for elements $x \in X$, and 0 otherwise. We have used the subscript "4" to keep consistency with the list of possible step-size functions in [8]. We will also use the steepest-descent iteration mapping

$$I_{\mathrm{SD}_1,f}(x) = x - \nabla f(x), \tag{2}$$

without a line-search, and the steepest-descent iteration mapping

$$I_{\mathrm{SD}_5,f}(x) = x - \alpha_5(x)\,\nabla f(x), \tag{3}$$

with backtracking line-search (simple descent):

$$\alpha_5(x) = \max\Big(\operatorname*{argmax}_{\alpha\in\{2^{-j}:j\in\mathbb{N}\}} \delta_{[0,\infty)}\left(f(x) - f\left(x - \alpha\,\nabla f(x)\right)\right) \Big).$$

Note that the difference between $\alpha_5(x)$ and $\alpha_4(x)$ is the indicator of the non-negative numbers $[0,\infty)$ versus the indicator of the positive numbers $(0,\infty)$.

We use the notation $\mathcal{N}_\infty^\# := \{N \subseteq \mathbb{N} : N \text{ infinite}\}$ to denote the infinite subsets of $\mathbb{N}$ (e.g, see [11]), and we use $\mathbb{N}^+ := \{1, 2, 3, \ldots\}$ to exclude 0. For a sequence $\{X^k\}$ of multisets, the *outer limit* was defined in [8] to be the set

$$\limsup_{k\to\infty} X^k := \{x^\infty : \exists\, N \in \mathcal{N}_\infty^\#,\ \exists\, x^k \in X^k\ (k \in N) \text{ with } x^k \overset{N}{\to} x^\infty\},$$

which collects the cluster points of sequences of elements $x^k \in X^k$ when the multisets X^k are all non-empty. As in our previous paper [8], we define the *attractor mapping* $A_{I_{\mathcal{M},f}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ by

$$A_{I_{\mathcal{M},f}}\big(X\big) := \limsup_{k\to\infty} \operatorname{argmin}_f \left(I^{(k)}_{\mathcal{M},f}\big(X\big)\right),$$

using compositions of the iteration mapping

$$I^{(k)}_{\mathcal{M},f}\big(X\big) := \underbrace{I_{\mathcal{M},f}\circ I_{\mathcal{M},f}\circ \ldots \circ I_{\mathcal{M},f}}_{k \text{ times}}(X)$$

and the set of lowest f-valued elements

$$\operatorname{argmin}_f(X) := \operatorname*{argmin}_{x\in X} f(x) = \left\{ x' \in X : f(x') = \min_{x\in X} f(x) \right\}.$$

Any element $x^\infty \in \operatorname{range}\left(A_{I_{\mathcal{M},f}}\right)$ in the range of the attractor mapping $A_{I_{\mathcal{M},f}}$ is an *attractor* for $I_{\mathcal{M},f}$, and the *basin of attraction* $\mathrm{Basin}_{\mathcal{M},f}(x^\infty)$ associated with an attractor x^∞ for $I_{\mathcal{M},f}$ is the set of all iterate-multisets for which x^∞ is in the image of the attractor mapping:

$$\mathrm{Basin}_{\mathcal{M},f}(x^\infty) := \left\{X : x^\infty \in A_{I_{\mathcal{M},f}}\big(X\big)\right\}.$$

2.1. Periodicity in discrete dynamical systems

Periodicity has been studied for some time already in discrete dynamical systems (e.g., [2]). A discrete dynamical system is a map F from a *state space* S to itself, and the *orbit* of any element $x^0 \in S$ is the sequence of states x^0, x^1, x^2, ... obtained by repeated iteration of the map F:

$$x^1 = F(x^0), \quad x^2 = F(x^1) = F^{(2)}(x^0), \ldots, x^j = F^{(j)}\left(x^0\right), \ldots$$

A state x^0 is a *periodic point for F of period k* if k iterations of the map F return to x^0: $F^{(k)}(x^0) = x^0$. The smallest value of k for which this is true is called the *prime period* and the set of states

$$\{x^0, F(x^0),\ F^{(2)}(x^0), \ldots,\ F^{(k-1)}\left(x^0\right)\}$$

is called the *k-cycle of* x^0. If a state x^0 has prime period $k = 1$, x^0 is called a *fixed point* for the map F.

For instance, the map $F(x) = -x^3$ on the state space $S = \mathbb{R}$ has a periodic point of prime period 1 at $x^0 = 0$, as well two periodic points of prime period 2 at $x^0 = 1$ and $x^0 = -1$. Our first notion of periodic behavior for iterate-multisets is a direct extension of periodic points in dynamical systems.

3. Periodic iterate-multisets

Definition 3.1. A *periodic iterate-multiset for* $I_{\mathcal{M},f}$ is one which satisfies the condition $I^{(k)}_{\mathcal{M},f}(X) = X$ for some $k \in \mathbb{N}^+$. The smallest value of k for which this is true is called the *prime period*, and we call the collection $P^{(k)}_{\mathcal{M},f}$ of all such iterate-multisets the *k-periodic iterate-multisets for* $I_{\mathcal{M},f}$:

$$P^{(k)}_{\mathcal{M},f} := \left\{ X : I^{(k)}_{\mathcal{M},f}(X) = X,\ I^{(j)}_{\mathcal{M},f}(X) \neq X \text{ for } j \in (0,k) \right\}.$$

In the case of $k = 1$ we refer to these as *fixed iterate-multisets for* $I_{\mathcal{M},f}$. For every $X \in P^{(k)}_{\mathcal{M},f}$, we refer to the collection of iterate-multisets

$$C^{(k)}_{\mathcal{M},f}(X) := \{X, I_{\mathcal{M},f}(X), I^{(2)}_{\mathcal{M},f}(X), \dots I^{(k-1)}_{\mathcal{M},f}(X)\}$$

as the *k-cycle of* X. Notice that every iterate-multiset in a k-cycle is also k-periodic. The following proposition demonstrates that the image under the attractor mapping of any k-periodic iterate-multiset is the union of the lowest f-valued elements of each iterate-multiset in its k-cycle. $\square$

Proposition 3.2. *If* $X \in P^{(k)}_{\mathcal{M},f}$, *then*

$$A_{I_{\mathcal{M},f}}(X) = \bigcup_{j \in \mathbb{N} \cap [0,k-1]} \operatorname{argmin}_f \left(I^{(j)}_{\mathcal{M},f}(X) \right).$$

Proof. First we notice that the inclusion

$$A_{I_{\mathcal{M},f}}(X) \subseteq \bigcup_{j \in \mathbb{N} \cap [0,k-1]} \operatorname{argmin}_f \left(I^{(j)}_{\mathcal{M},f}(X) \right)$$

follows immediately from the fact that for any index $i \in \mathbb{N}$,

$$I^{(i)}_{\mathcal{M},f}(X) = I^{(j)}_{\mathcal{M},f}(X)$$

for some $j \in \mathbb{N} \cap [0, k-1]$ since $X \in P^{(k)}_{\mathcal{M},f}$.

To get the opposite inclusion, we consider any index $\bar{j} \in \mathbb{N} \cap [0, k-1]$, define $\bar{X} := I^{(\bar{j})}_{\mathcal{M},f}(X)$, and notice that we have $\bar{X}^k := I^{(k)}_{\mathcal{M},f}(\bar{X}) = \bar{X}$. From this, we obtain

$$\bar{X}^{2k} := I^{(k)}_{\mathcal{M},f}\left(\bar{X}^k\right) = I^{(k)}_{\mathcal{M},f}(\bar{X}) = \bar{X},$$

and likewise that

$$\bar{X}^{jk} := I^{(k)}_{\mathcal{M},f}\left(\bar{X}^{(j-1)k}\right) = I^{(k)}_{\mathcal{M},f}(\bar{X}) = \bar{X}$$

for all indices $j \geq 3$ (using induction to get the second equality).

We then have that $\quad \operatorname{argmin}_f(\bar{X}) = \operatorname{argmin}_f(\bar{X}^{j\,k})$

for any $j \in \mathbb{N}$, which implies that $\operatorname{argmin}_f(\bar{X}) \subseteq A_{I_{\mathcal{M},f}}(X)$ (using the infinite set of indices $N := \{\bar{j}\,j\,k : j \in \mathbb{N}\}$ for the outer limit). □

Remark 3.3. Notice that one consequence of Proposition 3.2 is that every member of a k-cycle $C^{(k)}_{\mathcal{M},f}(X)$ has the same image $A_{I_{\mathcal{M},f}}(X)$ under the attractor mapping.

The following immediate corollary to Proposition 3.2 identifies precisely when periodic iterate-multisets are in the basin of attraction associated with an attractor x^∞.

Corollary 3.4. *If $X \in P^{(k)}_{\mathcal{M},f}$, then*

$$X \in \operatorname{Basin}_{\mathcal{M},f}(x^\infty) \quad \Longleftrightarrow \quad x^\infty \in \bigcup_{j \in \mathbb{N} \cap [0,k-1]} \operatorname{argmin}_f \left(I^{(j)}_{\mathcal{M},f}(X) \right).$$

Remark 3.5. A k-periodic iterate-multiset X is thus only in the basin of attraction associated with attractors that are elements of the iterate-multisets in its k-cycle $C^{(k)}_{\mathcal{M},f}(X)$.

Proposition 3.2 also rules out periodic iterate-multisets when there are no attractors.

Corollary 3.6. *When there are no attractors for $I_{\mathcal{M},f}$, there are no periodic iterate-multisets:*

$$\operatorname{dom}\left(A_{I_{\mathcal{M},f}}\right) = \emptyset \quad \Longrightarrow \quad P^{(k)}_{\mathcal{M},f} = \emptyset \text{ for all } k \in \mathbb{N}^+. \tag{4}$$

Proof. The formula from Proposition 3.2 ensures the inclusion

$$P^{(k)}_{\mathcal{M},f} \subseteq \operatorname{dom}\left(A_{I_{\mathcal{M},f}}\right) \text{ for all } k \in \mathbb{N}^+, \tag{5}$$

from which the implication (4) follows immediately. □

We illustrate more a subtle use of Proposition 3.2 in the following example.

Example 3.7. Consider the objective function

$$f_3(x_1, x_2) = \sqrt{x_1^2 + x_2^2} = \|x\|$$

from [9] whose gradient satisfies $\nabla f_3(x) = \frac{x}{\|x\|}$ for $x \neq (0,0)$. The iteration mapping (2) without a line-search is simply

$$I_{\mathrm{SD}_1,f_3}(x) = \left(1 - \frac{1}{\|x\|}\right) x = \frac{\|x\| - 1}{\|x\|}\, x$$

for any $x \in \mathbb{R}^2 \backslash (0,0)$. Using the convention from [8], we define the iteration mapping I_{SD_1,f_3} at the attractor (and minimizer) $x^\infty = (0,0)$ to be $I_{\mathrm{SD}_1,f_3}\big((0,0)\big) = (0,0)$, and as a result have the formula

$$I_{\mathrm{SD}_1,f_3}(x) = \begin{cases} (0,0) & \text{if } x = (0,0) \\ \frac{\|x\|-1}{\|x\|}\, x & \text{otherwise.} \end{cases} \tag{6}$$

To determine fixed iterates, we solve $I_{\mathrm{SD}_1,f_3}(x) = x$. Evidently the origin $(0,0)$ is a fixed iterate, and any others must satisfy

$$\frac{\|x\|-1}{\|x\|} = 1 \quad \iff \quad \|x\| - 1 = \|x\| \quad \iff \quad -1 = 0$$

which is impossible. We conclude that the origin is the lone fixed iterate for I_{SD_1,f_3}

$$P^{(1)}_{\mathrm{SD}_1,f_3} = \{(0,0)\}.$$

To identify 2-periodic iterates, we solve $I^{(2)}_{\mathrm{SD}_1,f_3}(x) = x$. According to (6), for any $x \in \mathbb{R}^2 \setminus (0,0)$, the norm of the iteration mapping satisfies

$$\|I_{\mathrm{SD}_1,f_3}(x)\| = \big|\|x\| - 1\big|.$$

It follows that the set of attractors for this iteration mapping is the open unit ball about the origin:

$$\mathrm{range}\left(A_{I_{\mathcal{M},f}}\right) = \mathbb{B}\left((0,0);1\right), \tag{7}$$

where the notation indicates the center (the origin $(0,0)$ in this case) and then the radius. According to Proposition 3.2, k-periodic iterates are always attractors (since, in this case, $\mathrm{argmin}_f(X) = X$ because the iterate-multisets X are single-valued). Thus we conclude that the 2-periodic iterates x in this case must satisfy $x \in \mathbb{B}\left((0,0);1\right)$, and we can rule out the origin as a 2-periodic iterate since we have already determined that it is a fixed point. This information is consolidated via the set inclusion

$$P^{(2)}_{\mathrm{SD}_1,f_3} \subseteq \mathbb{B}\big((0,0);1\big) \setminus (0,0). \tag{8}$$

Appealing to (6), we have $I^{(2)}_{\mathrm{SD}_1,f_3}(x) = x$ if

$$\frac{|\|x\|-1|-1}{|\|x\|-1|}\,\frac{\|x\|-1}{\|x\|} = 1,$$

which holds for all $x \in \mathbb{B}\left((0,0);1\right) \setminus (0,0)$. It follows that we have the set inclusion

$$P^{(2)}_{\mathrm{SD}_1,f_3} \supseteq \mathbb{B}\big((0,0);1\big) \setminus (0,0).$$

We conclude from this and (8) that the 2-periodic iterates for I_{SD_1,f_3} are given by:

$$P^{(2)}_{\mathrm{SD}_1,f_3} = \mathbb{B}\big((0,0);1\big) \setminus (0,0).$$

According to Proposition 3.2, there are no other k-periodic iterates for I_{SD_1,f_3} because we have determined that every element of the attractor set (7) in this case is either a fixed iterate or a 2-periodic iterate.

Remark 3.8. Note that this iteration mapping I_{SD_1,f_3} demonstrates that the gap in the inclusion (5) from the proof of Corollary 3.6 can be quite large

$$\mathrm{dom}\left(A_{I_{\mathrm{SD}_1,f_3}}\right) = \mathbb{R}^2 \;\text{ but }\; P^{(k)}_{\mathrm{SD}_1,f_3} \subseteq \mathbb{B}\big((0,0);1\big) \;\text{ for all } k \in \mathbb{N}^+.$$

Example 3.9. To illustrate k-periodic iterates for $k > 2$, we can consider the family of objective functions of one-variable

$$f_\mu(x) := \mu\,\frac{x^3}{3} + \left(\frac{1-\mu}{2}\right) x^2,$$

parameterized by $\mu \in [0, 4]$. If we apply the steepest-descent iteration mapping (2) without a line-search we get a corresponding family of iteration mappings

$$I_{SD_1, f_\mu}(x) = \mu\, x\,(1 - x)$$

well-known as the logistic map. For parameter values near $\mu = 3.5$, this iteration mapping has 4-periodic iterates (e.g., when $\mu = 3.5$, the 4-periodic iterates include 0.38282, 0.826941, 0.500884, and 0.874997 since these four values form a 4-cycle of the logistic map). Increasing the parameter values from there will generate iteration mappings with 2^j-periodic iterates for indices $j > 2$.

3.1. Consequences of minvalue-monotonicity

Definition 3.10. [8] The collection $\mathcal{V}_{\mathcal{M},f}$ of all *viable* iterate-multisets is defined by

$$\mathcal{V}_{\mathcal{M},f} := \{X \in \operatorname{dom}(I_{\mathcal{M},f}) \;:\; I^{(k)}_{\mathcal{M},f}(X) \in \operatorname{dom}(I_{\mathcal{M},f}) \text{ for } k \in \mathbb{N}^+\}.$$

Definition 3.11. [8] An iteration mapping $I_{\mathcal{M},f}$ is *minvalue-monotone* if for every viable iterate-multiset $X \in \mathcal{V}_{\mathcal{M},f}$, the minimum value

$$m^k := \min_{x \in I^{(k)}_{\mathcal{M},f}(X)} f(x)$$

never increases from one iteration to the next: $m^k \le m^{k-1}$ for $k \in \mathbb{N}^+$. An iteration mapping $I_{\mathcal{M},f}$ is called *strictly minvalue-monotone* if for every viable iterate-multiset $X \in \mathcal{V}_{\mathcal{M},f}$, the minimum value decreases whenever the iterate-multiset changes: $m^k < m^{k-1}$ for $k \in \mathbb{N}^+$ with $I^{(k)}_{\mathcal{M},f}(X) \ne I^{(k-1)}_{\mathcal{M},f}(X)$. □

The iteration mapping $I_{\mathrm{SD}_5,f}$ is minvalue-monotone for any objective function f, and the iteration mapping $I_{\mathrm{SD}_4,f}$ is strictly minvalue-monotone for any objective function f. The iteration mappings I_{SD_1,f_3} and I_{SD_1,f_μ} from Examples 3.7 and 3.9 respectively are not strictly minvalue-monotone, and the next result suggests that we would not have found k-periodic iterates if they had been.

Proposition 3.12. *If the iteration mapping $I_{\mathcal{M},f}$ is strictly minvalue-monotone, then there cannot be any viable k-periodic iterate-multisets*

$$P^{(k)}_{\mathcal{M},f} \cap \mathcal{V}_{\mathcal{M},f} = \emptyset \textit{ for } k > 1.$$

Proof. We suppose there is a viable k-periodic iterate-multiset X for some $k > 1$, and seek a contradiction. Then we know $I^{(k)}_{\mathcal{M},f}(X) = X$ and $I_{\mathcal{M},f}(X) \ne X$ (using the index $j = 1 < k$). Recalling the definition

$$m^k := \min_{x \in I^{(k)}_{\mathcal{M},f}(X)} f(x),$$

we note that the first condition $I^{(k)}_{\mathcal{M},f}(X) = X$ implies $m^k = m^0$. However, strict minvalue-monotonicity ensures that the second condition $I_{\mathcal{M},f}(X) \ne X$ implies

$m^1 < m^0$, and that $m^k \le m^1$ for all $k > 1$. These two inequalities together imply $m^k < m^0$, contradicting our earlier equation $m^k = m^0$. $\square$

Strict minvalue-monotonicity cannot be relaxed in Proposition 3.12 to minvalue-monotonicity, as we will see in the following example.

Example 3.13. Consider the objective function

$$f_4(x_1, x_2) = \left(x_1^2 + x_2^2\right)^2 = \|x\|^4$$

from [9] whose gradient satisfies $\nabla f_4(x) = 4\,\|x\|^2\,x$, and use the iteration mapping $I_{\mathrm{SD}_5,f_4}(x) = x - \alpha_5(x)\,\nabla f_4(x)$ from (3) which is minvalue-monotone, but not strictly minvalue-monotone. The line search in this case satisfies

$$\begin{aligned}
\alpha_5(x) &= \max\Big(\operatorname*{argmax}_{\alpha\in\{2^{-j}:j\in\mathbb{N}\}} \delta_{[0,\infty)}\left(\|x\|^4 - \left\|x - \alpha\,4\,\|x\|^2\,x\right\|^4\right)\Big) \\
&= \max\Big(\operatorname*{argmax}_{\alpha\in\{2^{-j}:j\in\mathbb{N}\}} \delta_{[0,2]}\left(\alpha\,4\,\|x\|^2\right)\Big) \\
&= \max\Big(\operatorname*{argmax}_{\alpha\in\{2^{-j}:j\in\mathbb{N}\}} \delta_{\left[0,\sqrt{\frac{1}{2\alpha}}\right]}\left(\|x\|\right)\Big) \\
&= \delta_{\left[0,\frac{1}{2}\right]}\left(\|x\|\right) + \sum_{j=0}^{\infty} 2^{-j}\,\delta_{\left(2^{\frac{j-2}{2}},2^{\frac{j-1}{2}}\right]}\left(\|x\|\right).
\end{aligned}$$

It follows that the iteration mapping (1) for any $x \in \mathcal{V}_{\mathrm{SD}_4,f_4} = \mathbb{R}^2$ reduces to

$$I_{\mathrm{SD}_5,f_4}(x) = \begin{cases} \left(1 - 4\,\|x\|^2\right)\,x & \text{if } \|x\| \in \left[0, \frac{1}{2}\right] \\ \left(1 - 2^{2-j}\,\|x\|^2\right)\,x & \text{if } \|x\| \in \left(2^{\frac{j-2}{2}}, 2^{\frac{j-1}{2}}\right] \quad j \in \mathbb{N}, \end{cases} \tag{9}$$

from which we conclude that the origin is the lone fixed iterate $P^{(1)}_{\mathrm{SD}_5,f_4} = \{(0,0)\}$. For all points $x \ne (0,0)$, we conclude from (9) that

$$\begin{aligned}
x \in U \quad &\Longrightarrow \quad I_{\mathrm{SD}_5,f_4}(x) = -x \in U \\
x \notin U \quad &\Longrightarrow \quad \|I_{\mathrm{SD}_5,f_4}(x)\| < \|x\|,
\end{aligned}$$

in terms of the set $\quad U := \bigcup_{j\in\mathbb{N}} \left\{x : \|x\| = 2^{\frac{j-1}{2}}\right\}.$

It follows that the 2-periodic iterates are given by

$$P^{(2)}_{\mathrm{SD}_5,f_4} = \bigcup_{j\in\mathbb{N}} \left\{x : \|x\| = 2^{\frac{j-1}{2}}\right\}, \tag{10}$$

and that the set of attractors for this iteration mapping is given by

$$\operatorname{range}\left(A_{I_{\mathcal{M},f}}\right) = \{(0,0)\} \cup \bigcup_{j\in\mathbb{N}} \left\{x : \|x\| = 2^{\frac{j-1}{2}}\right\}. \tag{11}$$

According to Proposition 3.2, there are no other k-periodic iterates for I_{SD_5,f_4} because we have determined that every element of the attractor set (11) in this case is either a fixed iterate or a 2-periodic iterate.

Numerical minimization methods are often designed to be strictly minvalue-monotone, and Proposition 3.12 ensures that periodic iterate-multisets will not be encountered when employing such methods. This motivates a more subtle notion of "near periodicity", which we will develop after we introduce some tools in the following section.

4. Pseudo-distance between iterate-multisets

The distance from a point $x' \in \mathbb{R}^n$ to an iterate-multiset $X \subseteq \mathbb{R}^n$ is given by

$$d(x', X) := \min_{x \in X} \|x' - x\|,$$

since any iterate-multiset has a finite number m of elements. This is used in [8] to define the *excess* of one iterate-multiset $\bar{X}$ over another X as

$$\operatorname{exc}(\bar{X},\, X) := \max_{\bar{x} \in \bar{X}} d(\bar{x}, X), \tag{12}$$

which we now use to define the *pseudo-distance between two iterate-multisets $\bar{X}$ and* X as follows:

$$d_\infty\left(\bar{X}, X\right) := \max\left\{\operatorname{exc}(\bar{X},\, X),\, \operatorname{exc}(\bar{X},\, X)\right\}. \tag{13}$$

Remark 4.1. This extends a definition of set-distance in [7] to iterate-multisets, and notice that d_∞ is not a traditional distance function since

$$d_\infty\left(\bar{X}, X\right) = 0 \not\Rightarrow \bar{X} = X.$$

For example the iterate-multisets $\bar{X} = \{1, 1, 2\}$ and $X = \{1, 2, 2\}$ are not equal even though the pseudo-distance between them is zero. □

Remark 4.2. In the case when $m = 1$ and the iterate-multisets are singleton iterates $\bar{x}$ and x, the pseudo-distance (13) reduces to the usual distance between two vectors $\|\bar{x} - x\|$. □

We use the pseudo-distance to define the *pseudo-ball* of radius $\gamma \geq 0$ about an iterate multiset $\bar{X}$ as follows:

$$\mathbb{B}_\infty\left(\bar{X}; \gamma\right) := \{X : d_\infty\left(\bar{X}, X\right) \leq \gamma\},$$

and we will say that an iterate multiset X is "near" another iterate-multiset $\bar{X}$ when $X \in \mathbb{B}_\infty\left(\bar{X}; \gamma\right)$ for a relatively small γ. Notice that, since $d_\infty\left(\bar{X}, \bar{X}\right) = 0$,

$$\bar{X} \in \mathbb{B}_\infty\left(\bar{X}; \gamma\right) \text{ for all } \gamma \geq 0. \tag{14}$$

The following lemma provides an equivalent formula for the pseudo-distance (13) that will be useful later.

Lemma 4.3. *For any iterate-multisets $X \subseteq \mathbb{R}^n$ and $\bar{X} \subseteq \mathbb{R}^n$ we have*

$$d_\infty\left(\bar{X}, X\right) = \sup_{x' \in \mathbb{R}^n} \left| d(x', \bar{X}) - d(x', X) \right|. \tag{15}$$

Proof. From the definition (12) of the excess between two iterate-multisets, we know there exist $\bar{x} \in \bar{X}$ and $x \in X$ with

$$\operatorname{exc}(\bar{X},\, X) = d(\bar{x}, X) \quad \text{and} \quad \operatorname{exc}(X,\, \bar{X}) = d(x, \bar{X}).$$

We conclude from (13) that the pseudo-distance $d_\infty\left(\bar{X}, X\right)$ satisfies

$$\begin{aligned} d_\infty\left(\bar{X}, X\right) &= \max\left\{d(\bar{x}, X), d(x, \bar{X})\right\} \\ &= \max\left\{\left|d(\bar{x}, X) - d(\bar{x}, \bar{X})\right|, \left|d(x, \bar{X}) - d(x, X)\right|\right\} \\ &\leq \sup_{x' \in \mathbb{R}^n} \left|d(x', \bar{X}) - d(x', X)\right|, \end{aligned} \tag{16}$$

where we used $d(\bar{x}, \bar{X}) = d(x, X) = 0$ to get the second equality.
Now choose an arbitrary $x' \in \mathbb{R}^n$ and prove the inequality

$$\left|d(x', \bar{X}) - d(x', X)\right| \leq d_\infty\left(\bar{X}, X\right). \tag{17}$$

From the triangle inequality, we know

$$\|x' - x\| \leq \|\bar{x} - x\| + \|x' - \bar{x}\|$$

for any x and $\bar{x}$. This implies that for any $\bar{x} \in \bar{X}$, there exists $x \in X$ with

$$\|x' - x\| \leq d(\bar{x}, X) + \|x' - \bar{x}\|,$$

where $x \in X$ achieves the distance on the right $\|\bar{x} - x\| = d(\bar{x}, X)$. Since we have $\|x' - x\| \geq d\left(x', X\right)$, we conclude that for any $\bar{x} \in \bar{X}$,

$$d\left(x', X\right) \leq d(\bar{x}, X) + \|x' - \bar{x}\|.$$

Now if we choose $\bar{x} \in \bar{X}$ to achieve the distance to x', $\|x' - \bar{x}\| = d\left(x', \bar{X}\right)$, we conclude that

$$d\left(x', X\right) \leq \operatorname{exc}(\bar{X}, X) + d\left(x', \bar{X}\right)$$

since $\operatorname{exc}(\bar{X}, X) \geq d(\bar{x}, X)$ for every $\bar{x} \in \bar{X}$. The definition (13) of the pseudo-distance between two multisets then gives the inequality

$$d\left(x', X\right) \leq d_\infty\left(\bar{X}, X\right) + d\left(x', \bar{X}\right). \tag{18}$$

Switching the multisets X and $\bar{X}$ in the argument above gives the inequality

$$d\left(x', \bar{X}\right) \leq d_\infty\left(\bar{X}, X\right) + d\left(x', X\right),$$

which, together with (18), gives the inequality (17). Since $x' \in \mathbb{R}^n$ is arbitrary, we can combine this with the inequality (16) to deduce the result (15) as claimed. $\square$

Remark 4.4. The notation d_∞ and formula (15) were used in [11, Example 4.13] to define the set-distance originally defined in [7], and the notation d_{PH} and formula (15) were used in [8] to define this pseudo-distance as a "pre-distance function" honoring Pompeiu and Hausdorff. Also, the proof above makes it clear that the supremum on the right side of the formula (15) could be restricted to $x' \in \bar{X} \cup X$ instead of $\mathbb{R}^n$.

5. Nearly periodic iterate-multisets

Definition 5.1. For $k \in \mathbb{N}^+$, we say that an iterate-multiset $\bar{X}$ is *nearly k-periodic for $I_{\mathcal{M},f}$* if for every $\gamma > 0$, there exists an iterate-multiset $X \in \mathbb{B}_\infty\left(\bar{X};\gamma\right)$ such that

$$I^{(k)}_{\mathcal{M},f}(X) \in \mathbb{B}_\infty\left(X;\gamma\right). \tag{19}$$

Remark 5.2. The terminology of "nearly" here primarily references the fact that $I^{(k)}_{\mathcal{M},f}(X)$ is near X via (19), but is not necessarily equal to X (as would be the case if X was k-periodic). The terminology also refers to the fact that the iterate-multiset X for which this holds is near $\bar{X}$.

Remark 5.3. The term "almost periodic" is used in discrete dynamical systems to describe the following property for continuous maps F: for every neighborhood U of an almost periodic point $\bar{x}$ there is an index $k \in \mathbb{N}$ such that

$$U \cap \Big(\bigcup_{i\in\mathbb{N}\cap[0,k]} F^{(j+i)}(\bar{x})\Big) \neq \emptyset \quad \text{for all } j \in \mathbb{N}^+.$$

This property is more restrictive than our notion of nearly k-periodic not only because it requires evaluation at $\bar{x}$ (and not merely at some x nearby), but also because it requires a return to the neighborhood within k steps from every index j.

Proposition 5.4. *For any $k \in \mathbb{N}^+$ and $j \in \mathbb{N}^+$, if $\bar{X}$ is k-periodic for $I_{\mathcal{M},f}$, then it is also nearly $j\,k$-periodic for $I_{\mathcal{M},f}$.*

Proof. As in the proof of Proposition 3.2, we use induction and the assumption $\bar{X} \in P^{(k)}_{\mathcal{M},f}$ to deduce that

$$I^{(j\,k)}_{\mathcal{M},f}(\bar{X}) = \bar{X}^{j\,k} = I^{(k)}_{\mathcal{M},f}\left(\bar{X}^{(j-1)\,k}\right) = I^{(k)}_{\mathcal{M},f}(\bar{X}) = \bar{X}.$$

The inclusion (19) is then satisfied with $X = \bar{X}$, and we always have $\bar{X} \in \mathbb{B}_\infty\left(\bar{X};\gamma\right)$ via (14). $\square$

Example 5.5. We consider again the objective function

$$f_4(x_1,x_2) = \left(x_1^2 + x_2^2\right)^2 = \|x\|^4$$

as in Example 3.13, but we use the iteration mapping $I_{\mathrm{SD}_4,f_4}(x) = x - \alpha_4(x)\,\nabla f_4(x)$ from (1) this time. Adapting the argument in Example 3.13 (or referencing [9] where this iteration mapping is also investigated), we can conclude that the iteration mapping for any $x \in \mathbb{R}^2$ reduces to

$$I_{\mathrm{SD}_4,f_4}(x) = \begin{cases} \left(1 - 4\,\|x\|^2\right) x & \text{if } \|x\| < \frac{1}{2} \\ \left(1 - 2^{2-j}\,\|x\|^2\right) x & \text{if } \|x\| \in \left[2^{\frac{j-2}{2}}, 2^{\frac{j-1}{2}}\right) \quad j \in \mathbb{N}. \end{cases}$$

From this we see that the iteration mapping satisfies

$$I_{\mathrm{SD}_4,f_4}(x) \in \begin{cases} \displaystyle\bigcup_{c\in(0,1)} c\,x & \text{if } \|x\| \in \left(0,\frac{1}{2}\right) \\ \displaystyle\bigcup_{c\in(-1,0)} c\,x & \text{if } \|x\| > \frac{1}{2} \text{ and } \|x\| \notin \displaystyle\bigcup_{j\in\mathbb{N}^+} 2^{\frac{j-2}{2}} \\ \{(0,0)\} & \text{if } x = (0,0) \text{ or } \|x\| \in \displaystyle\bigcup_{j\in\mathbb{N}} 2^{\frac{j-2}{2}}, \end{cases} \tag{20}$$

where the $c \in (0,1)$ in the first union correspond to $\|x\|$ closer to the left-endpoint 0 of the interval $\left(0, \frac{1}{2}\right)$, and the $c \in (-1,0)$ in the second union correspond to $\|x\|$ closer to the right-endpoint $2^{\frac{j-1}{2}}$ of the interval $\left[2^{\frac{j-2}{2}}, 2^{\frac{j-1}{2}}\right)$.

It follows from (20) that there is thus a lone fixed iterate $(0,0)$ for this iteration mapping, and that there are no k-periodic iterates since

$$\|I_{\mathrm{SD}_4,f_4}(x)\| < \|x\| \quad \text{when } x \neq (0,0). \tag{21}$$

We know from Proposition 5.4 that the fixed iterate $(0,0)$ is also nearly k-periodic for all $k \in \mathbb{N}^+$. To find other nearly 1-periodic iterates, we note that the relevant pseudo-distance for (19) in this case is simply

$$\|x - I_{\mathrm{SD}_4,f_4}(x)\| = \begin{cases} 4\,\|x\|^3 & \text{if } \|x\| < \frac{1}{2} \\ 2^{2-j}\,\|x\|^3 & \text{if } \|x\| \in \left[2^{\frac{j-2}{2}}, 2^{\frac{j-1}{2}}\right) \quad j \in \mathbb{N}. \end{cases}$$

Since $\gamma > 0$ in (19) is arbitrarily small, we conclude that $(0,0)$ is the only nearly 1-periodic iterate.

To investigate nearly 2-periodic iterates other than the fixed iterate $(0,0)$, we note that the relevant pseudo-distance for (19) in this case is $\left\|x - I^{(2)}_{\mathrm{SD}_4,f_4}(x)\right\|$; which we can write as $g(\|x\|)$ in terms of the function

$$g(r) := \begin{cases} 8r^3\,|1 - 6r^2 + 24r^4 - 32r^6| & \text{if } r < \frac{1}{2} \\ 2^{3-4j}r^3\,|8^j - 3\cdot 2^{1+2j}r^2 + 3\cdot 2^{3+j}r^4 - 32r^6| & \text{if } r \in \left[2^{\frac{j-2}{2}}, 2^{\frac{j-1}{2}}\right) \quad j \in \mathbb{N}. \end{cases}$$

This function is continuous on each interval $\left(2^{\frac{j-2}{2}}, 2^{\frac{j-1}{2}}\right)$ for $j \in \mathbb{N}$, and strictly positive except at $r = 0$. In addition, $g(r)$ approaches zero as r approaches $2^{\frac{j-1}{2}}$ from the left for any $j \in \mathbb{N}$ (or as r approaches 0 from the right), and nowhere else. We conclude that the nearly 2-periodic iterates are

$$\{(0,0)\} \cup \bigcup_{j\in\mathbb{N}} \left\{x : \|x\| = 2^{\frac{j-1}{2}}\right\}. \tag{22}$$

We can appeal to the inclusion (20) to rule out any other nearly k-periodic iterates in this case. In particular, we can conclude from (20) that there are no nearly k-periodic iterates (other than the fixed iterate $(0,0)$) for k odd. This follows because (i) the first case in (20) would require $c \in (0,1)$ to be arbitrarily close to the left-endpoint 0 (which re-identifies the fixed iterate $(0,0)$), and (ii) the second case with $c \in (-1,0)$ gives an image across the origin from x after an odd number of compositions of I_{SD_4,f_4}.

Remark 5.6. Notice that the iteration mapping I_{SD_4,f_4} is strictly minvalue-monotone, so Example 5.5 demonstrates that nearly periodic iterate-multisets can occur even when periodic iterate-multisets cannot (via Proposition 3.12). □

Remark 5.7. Note that the nearly 2-periodic iterates in (22) are the same as the fixed and 2-periodic iterates from Example 3.13, which is a consequence of the close connection between the iteration mappings I_{SD_4,f_4} and I_{SD_5,f_4}. Notice however that the iteration mapping I_{SD_4,f_4} is strictly minvalue-monotone, so Example 5.5 demonstrates that nearly periodic iterate-multisets can occur even when periodic iterate-multisets cannot (via Proposition 3.12). □

6. Attraction-stifling iterate-multisets

Definition 6.1. Given an attractor x^∞ for an iteration mapping $I_{\mathcal{M},f}$, we say that the iterate-multiset $\bar{X}$ is *attraction-stifling* if there exists a $k \in \mathbb{N}^+$ for which

$$\lim_{\gamma \searrow 0}\left(\inf_{X \in \mathbb{B}_\infty(\bar{X};\gamma)} \max_{k' \in [1,k]\cap\mathbb{N}^+} \left| d(x^\infty, X) - d\left(x^\infty, I^{(k')}_{\mathcal{M},f}(X)\right)\right|\right) = 0. \tag{23}$$

For a particular $k \in \mathbb{N}^+$, we call the collection $S^{(k)}_{\mathcal{M},f}(x^\infty)$ of all iterate-multisets satisfying (23) the *k-stifling iterate-multisets for $I_{\mathcal{M},f}$ to x^∞*.

Remark 6.2. This condition gets it label because it guarantees that there are X near $\bar{X}$ from which there is arbitrarily negligible progress toward the attractor x^∞ through k applications of the iteration mapping. □

Remark 6.3. Note it is always the case that

$$S^{(1)}_{\mathcal{M},f}(x^\infty) \supseteq S^{(2)}_{\mathcal{M},f}(x^\infty) \supseteq \ldots \supseteq S^{(k)}_{\mathcal{M},f}(x^\infty). \quad \square \tag{24}$$

6.1. Connection to periodic iterate-multisets

Since a fixed iterate-multiset never changes its distance to an attractor, it is no surprise that it is one example of an attraction-stifling iterate-multiset.

Proposition 6.4. *Any fixed iterate-multiset is k-stifling for $I_{\mathcal{M},f}$ to x^∞ for all $k \in \mathbb{N}^+$:*

$$P^{(1)}_{\mathcal{M},f} \subseteq S^{(k)}_{\mathcal{M},f}(x^\infty) \text{ for all } k \in \mathbb{N}^+.$$

Proof. This follows immediately from the definition of fixed iterate-multisets since $I_{\mathcal{M},f}(\bar{X}) = \bar{X}$ implies through induction that

$$I^{(k')}_{\mathcal{M},f}(\bar{X}) = \bar{X} \text{ for any } k' \in \mathbb{N}^+,$$

so that $d(x^\infty, \bar{X}) = d\big(x^\infty, I^{(k')}_{\mathcal{M},f}(\bar{X})\big)$. □

Other k-periodic iterate-multisets (i.e., with $k > 1$) are not necessarily k-stifling, as the next example shows.

Example 6.5. For the iteration mapping $I_{SD_1,f_{3.5}}(x) = 3.5x(1-x)$ from Example 3.9 with $\mu = 3.5$, the point $x^\infty = 0$ is an attractor since $0 = A_{SD_1,f_{3.5}}(1)$, and the point $\bar{x} := 0.38282$ is a 4-periodic iterate with $I_{SD_1,f_{3.5}}(0.38282) = 0.826941$. It follows that the absolute-value in (23) with $k' = 1$ and $x \in \mathbb{B}(\bar{x};\gamma)$ satisfies (for small γ)

$$\begin{aligned} |d\,(0,x) - d\,(0, 3.5\,x\,(1-x))| &= 2.5\,x - 3.5\,x^2 \\ &\geq 2.5\,(\bar{x}+\gamma) - 3.5\,(\bar{x}+\gamma)^2 = -3.5\,\gamma^2 - 0.17974\,\gamma + 0.444121, \end{aligned}$$

so that the limit in (23) is always at least 0.444121. (Note, the inequality above follows because the quadratic $2.5\,x - 3.5\,x^2$ is minimized on $\mathbb{B}(\bar{x};\gamma)$ at $\bar{x}+\gamma$.)

On the other hand, if the k-periodic iterate-multisets in a k-cycle happen to have the same distance from the attractor, each of these iterate-multisets is also k-stifling. For instance, the iteration mapping I_{SD_5,f_4} from Example 3.13 satisfies $I_{\mathrm{SD}_5,f_4}(\bar{x}) = -\bar{x}$ for every 2-periodic iterate $\bar{x} \in P^{(2)}_{\mathrm{SD}_5,f_4}$. Using the attractor $x^\infty = (0,0)$ for I_{SD_5,f_4} and any of these 2-periodic iterates $\bar{x}$, we have

$$\max_{k' \in [1,k]\cap\mathbb{N}^+} \left| d(x^\infty, \bar{x}) - d\left(x^\infty, I^{(k')}_{\mathrm{SD}_5,f_4}(\bar{x})\right)\right| = \big| \|\bar{x}\| - \|\bar{x}\| \big| = 0$$

for any $k \in \mathbb{N}^+$. Appealing to (14) then we conclude that each such $\bar{x}$ is k-stifling for any $k \in \mathbb{N}^+$.

6.2. Connection to nearly periodic iterate-multisets

Even *nearly* k-periodic iterate-multisets can be k-stifling, as the following example shows.

Example 6.6. Consider again the iteration mapping

$$I_{\mathrm{SD}_4,f_4}(x) = \begin{cases} (1 - 4\|x\|^2)\, x & \text{if } \|x\| < \frac{1}{2} \\ (1 - 2^{2-j}\|x\|^2)\, x & \text{if } \|x\| \in \left[2^{\frac{j-2}{2}}, 2^{\frac{j-1}{2}}\right) \quad j \in \mathbb{N}, \end{cases} \tag{25}$$

from Example 5.5 with attractor $x^\infty = (0,0)$. In this case, the absolute-value inside (23) is given by

$$\left| d(x^\infty, x) - d\left(x^\infty, I^{(k')}_{\mathcal{M},f}(x)\right)\right| = \|x\| - \left\| I^{(k')}_{\mathrm{SD}_4,f_4}(x)\right\|. \tag{26}$$

Note that there is no absolute value on the right because this iteration mapping satisfies $\|I_{\mathrm{SD}_4,f_4}(x)\| < \|x\|$ when $x \neq (0,0)$, which implies via recursion that $\left\| I^{(k')}_{\mathrm{SD}_4,f_4}(x)\right\| \leq \|x\|$ for all x. From (25) we have

$$\|x\| - \|I_{\mathrm{SD}_4,f_4}(x)\| = \begin{cases} 4\|x\|^3 & \text{if } \|x\| < \frac{1}{2} \\ (2 - 2^{2-j}\|x\|^2)\, \|x\| & \text{if } \|x\| \in \left[2^{\frac{j-2}{2}}, 2^{\frac{j-1}{2}}\right) \quad j \in \mathbb{N}. \end{cases} \tag{27}$$

From this and (26) we deduce that if

$$\|\bar{x}\| \notin \{0\} \cup \bigcup_{j\in\mathbb{N}} 2^{\frac{j-1}{2}},$$

then $$\lim_{\gamma \searrow 0} \left(\inf_{\|x\|\leq\gamma} \|x\| - \|I_{\mathrm{SD}_4,f_4}(x)\|\right) = \|\bar{x}\| - \|I_{\mathrm{SD}_4,f_4}(\bar{x})\| > 0.$$

We immediately conclude that

$$S^{(1)}_{\mathrm{SD}_4,f_4}(0,0) \subseteq \{(0,0)\} \cup \bigcup_{j\in\mathbb{N}} \left\{ \bar{x} : \|\bar{x}\| = 2^{\frac{j-1}{2}} \right\}.$$

More generally, the nested inclusions (24) then imply that

$$S^{(k)}_{\mathrm{SD}_4,f_4}(0,0) \subseteq \{(0,0)\} \cup \bigcup_{j\in\mathbb{N}} \left\{ \bar{x} : \|\bar{x}\| = 2^{\frac{j-1}{2}} \right\}. \tag{28}$$

Now, suppose $\|\bar{x}\| = 2^{\frac{j-1}{2}}$ for some $j \in \mathbb{N}$. In this case, for any $\gamma \in \left(0, \frac{\sqrt{2}-1}{\sqrt{2}}\right]$, we can define

$$x_\gamma = \left(2^{\frac{j-1}{2}} - \gamma\right) \frac{\bar{x}}{\|\bar{x}\|} \tag{29}$$

so that $\quad \|\bar{x} - x_\gamma\| = \gamma \quad$ and $\quad \|x_\gamma\| = 2^{\frac{j-1}{2}} - \gamma \in \left[2^{\frac{j-2}{2}}, 2^{\frac{j-1}{2}}\right).$

From this and (27) it follows that

$$\|x_\gamma\| - \|I_{\mathrm{SD}_4,f_4}(x_\gamma)\| = \left(2 - 2^{2-j}\,\|x_\gamma\|^2\right)\,\|x_\gamma\| = 4\,\gamma - 3\,\left(2^{\frac{3-j}{2}}\right)\,\gamma^2 + 2^{2-j}\,\gamma^3,$$

so we conclude that $\quad \lim_{\gamma \searrow 0} \left(\|x_\gamma\| - \|I_{\mathrm{SD}_4,f_4}(x_\gamma)\|\right) = 0. \qquad (30)$

This, together with the inclusion (28) using $k = 1$, gives

$$S^{(1)}_{\mathrm{SD}_4,f_4}(0,0) = \{(0,0)\} \cup \bigcup_{j\in\mathbb{N}} \left\{\bar{x} : \|\bar{x}\| = 2^{\frac{j-1}{2}}\right\}.$$

To consider the case with $k = 2$, we notice that (30) implies $\|I_{\mathrm{SD}_4,f_4}(x_\gamma)\|$ can be made arbitrarily close to $\|\bar{x}\| = 2^{\frac{j-1}{2}}$ by choosing small enough $\gamma \in \left(0, \frac{\sqrt{2}-1}{\sqrt{2}}\right]$. Since

$$\|I_{\mathrm{SD}_4,f_4}(x_\gamma)\| \le \|x_\gamma\| = 2^{\frac{j-1}{2}} - \gamma < 2^{\frac{j-1}{2}},$$

we conclude that $\quad a_\gamma := \|I_{\mathrm{SD}_4,f_4}(x_\gamma)\| \in \left[2^{\frac{j-2}{2}}, 2^{\frac{j-1}{2}}\right)$

for all such γ. Then it follows from (25) that

$$\|I^{(2)}_{\mathrm{SD}_4,f_4}(x_\gamma)\| = \left(2^{2-j}\,a_\gamma^2 - 1\right)\,a_\gamma,$$

and from (30), since $\|x_\gamma\| = 2^{\frac{j-1}{2}} - \gamma$, we know that $\lim_{\gamma\searrow 0} a_\gamma = 2^{\frac{j-1}{2}}$;

so we conclude that

$$\lim_{\gamma\searrow 0} \|I^{(2)}_{\mathrm{SD}_4,f_4}(x_\gamma)\| = \left(2^{2-j}\,2^{j-1} - 1\right)\,2^{\frac{j-1}{2}} = 2^{\frac{j-1}{2}}.$$

Thus we have $\quad \lim_{\gamma\searrow 0} \left(\|x_\gamma\| - \|I^{(2)}_{\mathrm{SD}_4,f_4}(x_\gamma)\|\right) = 0,$

which, together with the inclusion (28) using $k = 2$, gives

$$S^{(2)}_{\mathrm{SD}_4,f_4}(0,0) = \{(0,0)\} \cup \bigcup_{j\in\mathbb{N}} \left\{\bar{x} : \|\bar{x}\| = 2^{\frac{j-1}{2}}\right\}.$$

A similar argument can be used to show that

$$S^{(k)}_{\mathrm{SD}_4,f_4}(0,0) = \{(0,0)\} \cup \bigcup_{j\in\mathbb{N}} \left\{\bar{x} : \|\bar{x}\| = 2^{\frac{j-1}{2}}\right\}$$

for any $k \in \mathbb{N}^+$. This collection of iterates is the same as the collection of nearly 2-periodic iterates (22), so we have demonstrated that nearly 2-periodic iterates can be k-stifling.

6.3. Other attraction-stifling iterate-multisets

Periodic or nearly periodic iterate-multisets are not the only source of attraction-stifling iterate-multisets, as the following example shows.

Example 6.7. Consider the objective function

$$f_1(x_1, x_2) = \frac{x_1^2 + x_2^2}{2}$$

from [9] and a quasi-Newton method QN_θ using approximate Hessian inverse matrix $I - R_\theta$, the difference of the identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and the rotation matrix

$$R_\theta := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

which rotates vectors by the angle θ counterclockwise about the origin. For this example, we will assume that θ is *not* a rational multiple of π (in radians). The resulting iteration mapping is, since $\nabla f_1(x) = x$,

$$I_{\mathrm{QN}_\theta, f_1}(x) = x - [I - R_\theta] \cdot \nabla f_1(x) = R_\theta \cdot x.$$

This iteration mapping has an attractor $x^\infty = (0,0)$ at the origin since we have $(0,0) = A_{\mathrm{QN}_\theta, f_1}(0,0)$. Further, the k^{th} composition of this iteration mapping satisfies

$$I^{(k)}_{\mathrm{QN}_\theta, f_1}(x) = R_{k\,\theta} \cdot x, \tag{31}$$

so a k-periodic iterate would require $x = R_{k\,\theta} \cdot x$. Evidently, the attractor $x^\infty = (0,0)$ is a fixed iterate, but for all other x this matrix equation only has solutions when $\theta = \frac{2j}{k}\pi$ for $j \in \mathbb{Z}$. Our assumption that θ is not a rational multiple of π thus ensures that $x^\infty = (0,0)$ is the only k-periodic iterate. We can also rule out nearly k-periodic iterates (other than $x^\infty = (0,0)$) by applying (31) to deduce

$$I^{(k)}_{\mathrm{QN}_\theta, f_1}(x) \in \mathbb{B}_\infty(x; \gamma) \iff 2\left|\sin\left(\frac{k\,\theta}{2}\right)\right| \|x\| \le \gamma. \tag{32}$$

Since θ is not a rational multiple of π, we know that $\left|\sin\left(\frac{k\,\theta}{2}\right)\right|$ is non-zero. Under the condition $x \in \mathbb{B}_\infty(\bar{x}; \gamma)$, the left-side of the inequality in (32) approaches $2\left|\sin\left(\frac{k\,\theta}{2}\right)\right| \|\bar{x}\|$ as γ diminishes to zero. Thus, $\bar{x} = x^\infty = (0,0)$ is the only point for which there exists $x \in \mathbb{B}_\infty(\bar{x}; \gamma)$ with $I^{(k)}_{\mathrm{QN}_\theta, f_1}(x) \in \mathbb{B}_\infty(x; \gamma)$ for arbitrarily small γ. This iteration mapping does however maintain distance to the origin

$$\left\| I^{(k)}_{\mathrm{QN}_\theta, f_1}(x) \right\| = \|x\| \quad \text{for all } k \in \mathbb{N}^+,$$

so every non-zero vector $\bar{x} \neq (0,0)$ is k-stifling for $I_{\mathrm{QN}_\theta, f_1}$ to $(0,0)$, for all $k \in \mathbb{N}^+$.

6.4. Connection to catastrophic radii

The paper [9] defines the following notion of "catastrophic radius" to categorize the effectiveness of minimization methods.

Definition 6.8. [9, Definition 4] For an attractor x^∞, a radius $r \geq 0$ is *catastrophic for x^∞ within $\epsilon > 0$* if and only if for every $\gamma' > 0$ and index $k \in \mathbb{N}^+$, there exists $r' \in (r - \gamma', r + \gamma')$ with

$$\sup_{X: d(x^\infty, X) = r'} \min \left\{ k' \in \mathbb{N} : d\left(x^\infty, I^{(k')}_{\mathcal{M},f}(X)\right) \leq \epsilon \right\} > k; \tag{33}$$

and the set of all catastrophic radii for x^∞ within ϵ is denoted by $CR(x^\infty, \epsilon)$.

Proposition 6.9. *If $\bar{X}$ is k-stifling for $I_{\mathcal{M},f}$ to x^∞ for all $k \in \mathbb{N}^+$, then the radius $r = d(x^\infty, \bar{X})$ is a catastrophic radius for x^∞ within any $\epsilon \in (0, r)$:*

$$\bar{X} \in S^{(k)}_{\mathcal{M},f}(x^\infty) \text{ for all } k \in \mathbb{N}^+ \implies d\left(x^\infty, \bar{X}\right) \in \bigcap_{\epsilon < d\left(x^\infty, \bar{X}\right)} CR(x^\infty, \epsilon).$$

Proof. Set $r = d\left(x^\infty, \bar{X}\right)$, consider any $\epsilon \in (0, r)$, and define $\bar{\epsilon} := r - \epsilon$. For any $\gamma' > 0$ and index $k \in \mathbb{N}^+$, choose a small enough

$$\gamma \in (0, \gamma') \cap \left(0, \frac{\bar{\epsilon}}{2}\right)$$

so that the assumption that $\bar{X}$ is k-stifling guarantees the existence of $X \in \mathbb{B}_\infty\left(\bar{X}; \gamma\right)$ that satisfies

$$\max_{k' \in [1,k] \cap \mathbb{N}^+} \left| d\left(x^\infty, X\right) - d\left(x^\infty, I^{(k')}_{\mathcal{M},f}(X)\right) \right| \leq \gamma < \frac{\bar{\epsilon}}{2}. \tag{34}$$

Applying Lemma 4.3 with $x' = x^\infty$ and appealing to the fact that $d_\infty\left(\bar{X}, X\right) \in [0, \gamma')$, we conclude that

$$r' := d\left(x^\infty, X\right) \in \left(r - \gamma',\, r + \gamma'\right).$$

Applying Lemma 4.3 with $x' = x^\infty$ again, but this time appealing to the fact that $d_\infty\left(\bar{X}, X\right) \in \left[0, \frac{\bar{\epsilon}}{2}\right)$, we conclude that

$$|d(x^\infty, X) - r| < \frac{\bar{\epsilon}}{2}.$$

This in particular means that

$$d(x^\infty, X) - r > -\frac{\bar{\epsilon}}{2} \implies d(x^\infty, X) > r - \frac{\bar{\epsilon}}{2},$$

which, together with the bound (34), guarantees that

$$d\left(x^\infty, I^{(k')}_{\mathcal{M},f}(X)\right) > r - \bar{\epsilon} = \epsilon \quad \text{for all } k' \in [1, k] \cap \mathbb{N}^+.$$

Thus we have confirmed (33). $\square$

Remark 6.10. The implication in Proposition 6.9 cannot be reversed in general. For instance, [9, Example 4.4] investigates the same iteration mapping I_{SD_4, f_4} and attractor $x^\infty = (0, 0)$ as in our Example 6.6, and defines a set $R(0)$ containing catastrophic radii that happen to have arbitrarily large distance-reductions to the target on the first iteration: for every $\beta > 0$ and every $\epsilon > 0$, there exists some $r \in R(0) \cap CR(x^\infty, \epsilon)$ such that for every iterate-multiset $\bar{X}$ with $r = d\left(x^\infty, \bar{X}\right)$ we have

$$\lim_{\gamma \searrow 0} \left(\inf_{X \in \mathbb{B}_\infty\left(\bar{X};\gamma\right)} \left| d(x^\infty, X) - d\left(x^\infty, I_{\mathcal{M},f}(X)\right) \right| \right) > \beta.$$

Such $\bar{X}$ are evidently not 1-stifling, and so are not k-stifling for any $k \in \mathbb{N}^+$. $\square$

Remark 6.11. For the iteration mapping I_{SD_4,f_4} from Example 6.6, we can combine the results there with Proposition 6.9 to deduce that that

$$2^{\frac{j-1}{2}} \in CR\big((0,0), \epsilon\big) \text{ as long as } j \in \left(\frac{2\,\log(\epsilon)}{\log(2)} + 1, \infty \right) \cap \mathbb{N},$$

since such indices j guarantee $\epsilon < 2^{\frac{j-1}{2}}$. This set of catastrophic radii was identified by different means in [9, Example 4.4], where it was labeled $R^{(0)}$. $\square$

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