Journal of Convex Analysis
Volume 30 (2023), No. 4, 1203–1216

On the Gauss Range of a Closed Convex Set

Juan Enrique Martínez-Legaz*

Department d'Economia i d'Història Econòmica, Universitat Autònoma de Barcelona,
Barcelona Graduate School of Mathematics (BGSMath), Spain
JuanEnrique.Martinez.Legaz@uab.es

Cornel Pintea

Faculty of Mathematics and Computer Science, Babeş-Bolyai University, Cluj-Napoca, Romania
cpintea@math.ubbcluj.ro

Dedicated to Roger J-B Wets on the occasion of his 85th birthday.

Received: July 11, 2022
Accepted: October 30, 2022

The aim of this paper is twofold. One one hand we study how the Gauss range of a closed convex subset of the Euclidean space changes when acting on the given set with an affine automorphism. On the other hand we investigate the spherical-convexity of the Gauss range of such a set. Since the Gauss range is not always spherically-convex, we focus our attention on the spherical-convexity of its spherical interior and of its closure as well. Examples of closed convex sets with spherically-convex Gauss ranges are, among others, the epigraphs of suitable lower semicontinuous convex functions.

Keywords: Closed convex set, Gauss range, spherically-convex set.

2010 Mathematics Subject Classification: 52A20, 26B25.

1. Introduction

The collection $\mathcal{C}(\mathbb{R}^n)$ of closed convex subsets of $\mathbb{R}^n$ is invariant under the action of the group of affine automorphisms of $\mathbb{R}^n$ on the whole collection $\mathcal{P}(\mathbb{R}^n)$ of subsets of $\mathbb{R}^n$. In order to distinguish some orbits of this restricted action we employ the Gauss range of the closed convex sets. Such distinctions, for two closed convex sets, are possible when the Gauss range of one of them is open and that of the other is closed. Indeed, the openness of the Gauss range happens to be preserved over the orbit of a closed convex subset of $\mathbb{R}^n$ having this property, under the action of the group of affine automorphisms of $\mathbb{R}^n$ over $\mathcal{C}(\mathbb{R}^n)$. Recall that the closed convex sets with open or closed Gauss ranges were characterized in [4]. A further action restriction, on the group side this time, from the group of affine automorphisms of $\mathbb{R}^n$ to the group of its isometries allows us to distinguish the orbits of closed convex sets with spherically-convex Gauss ranges from those with spherically non-convex Gauss ranges. Indeed, the spherical-convexity of the Gauss range happens to be preserved over the orbit of a closed convex subset of $\mathbb{R}^n$ having this property, under the action of the group of isometries of $\mathbb{R}^n$ over $\mathcal{C}(\mathbb{R}^n)$.

* This author acknowledges financial support from the Spanish Ministry of Economy and Competitiveness, through Grant PGC2018-097960-B-C21 and the Severo Ochoa Programme for Centres of Excellence in R&D (SEV-2015-0563). He is affiliated with MOVE (Markets, Organizations and Votes in Economics).

ISSN 0944-6532 / $ 2.50

After the next section, where we recall a few concepts and results proved earlier, we study, in the third section, how the Gauss range of a closed convex set changes when acting on the given set with an affine automorphism. The outcome of this investigation allows us to show that the quality of the Gauss range of a closed convex subset of $\mathbb{R}^n$ to be open is preserved all over the affine orbit of the considered closed convex set. In the fourth section we describe the Gauss range of the ice-cream cones $\mathbb{R}_+B(x_0,\rho)$, where $B(x_0,\rho)$ is the closed ball centered at $x_0 \in S^{n-1}$ of radius $0<\rho<1$. Above and in what follows, S^{n-1} denotes the unit sphere of $\mathbb{R}^n$ endowed with its Euclidean norm. The diameter of the Gauss range $G_{\mathbb{R}_+B(x_0,\rho)}(\mathbb{R}^n)$ is also shown to be inferior to π, i.e, it doesn't contain pairs of antipodal points. While this section is finally pointing out a class of closed convex subsets of $\mathbb{R}^n$ whose Gauss ranges do not contain pairs of antipodal points, the fifth section deals with the class of closed convex sets whose Gauss ranges do contain pairs of antipodal points. In the sixth section we show that the closure and the spherical interior of the Gauss range of a closed convex set are both spherically-convex. Our previous characterizations from [4] of the closed convex sets with open or closed Gauss ranges return back into the picture through the hypotheses on the sets considered in the last section. Each of these hypotheses combined with the suitable spherical-convexity (of either the closure or of the spherical interior of the Gauss range) produces several examples of closed convex sets with spherically-convex Gauss ranges.

2. Definitions and preliminary results

In this section we recall several results about the Gauss map of a closed convex set and its range, most of them taken from [4]. Recall that a vector $x^* \in \mathbb{R}^n$ is said to be *normal* to a convex set $C \subseteq \mathbb{R}^n$ at a point $x \in C$ if $\langle c-x, x^*\rangle \le 0$, for all $c \in C$, where $\langle \cdot,\cdot\rangle$ stands for the standard Euclidean product. The set of vectors normal to C at x is a closed convex cone denoted by $N_C(x)$ and is called the *normal cone* to C at x. Note that $N_C(x) = \{0\}$ whenever $x \in \operatorname{int} C$. We extend the mapping $N_C : C \rightrightarrows \mathbb{R}^n$ to the whole space $\mathbb{R}^n$ by taking $N_C(x) := \emptyset$ whenever $x \in \mathbb{R}^n \setminus C$. The mapping N_C is multivalued at any point in $C \cap \operatorname{bd} C$. Denote by $N_C(\mathbb{R}^n)$ the range of N_C, i.e. the union of the normal cones to C at all points of C, and call it the *total normal cone* of C. Note that $N_C(\mathbb{R}^n) = \{0\}$ if and only if C is open. As is well known and easy to prove, one has

$$N_C(\mathbb{R}^n) = \{x^* \in \mathbb{R}^n : \operatorname{argmax}_{x\in C} \langle x, x^*\rangle \neq \emptyset\}.$$

Note that the total normal cone need not be convex, as shows the example of the epigraph of the convex function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) := \max\{e^{-x}, |y|\}$ (see [4]). We recall that the epigraph of $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is the set

$$\operatorname{epi}(f) := \{(x,\alpha) \in \mathbb{R}^n \times \mathbb{R} : f(x) \le \alpha\}.$$

Remark 2.1. ([4], Remark 3) Clearly, for any $x^* \in N_C(\mathbb{R}^n)$, the solution set of the linear optimization problem

$$\begin{cases} \max\langle x, x^*\rangle \\ x \in C \end{cases} \tag{1}$$

is the exposed face $C \cap H_{x^*}$, where $H_{x^*} := \{x \in \mathbb{R}^n \ : \ \langle x, x^*\rangle = \max_{c\in C}\langle c, x^*\rangle\}$.

Note that the optimization problems (1) associated with x^* and αx^* have the same nonempty solution set $C \cap H_{\alpha x^*} = C \cap H_{x^*}$ whenever $x^* \in N_C(\mathbb{R}^n)$ and $\alpha > 0$. Thus, the significantly much smaller set $N_C(\mathbb{R}^n) \cap S^{n-1}$ is enough to parameterize all nonempty solution sets of the optimization problems (1), as (1) has a nonempty solution set if and only if $\frac{x^*}{\|x^*\|} \in N_C(\mathbb{R}^n) \cap S^{n-1}$. In such a case, the common nonempty solution set associated with x^* and $\frac{x^*}{\|x^*\|} \in N_C(\mathbb{R}^n) \cap S^{n-1}$ is the set $C \cap H_{x^*} = C \cap H_{\frac{x^*}{\|x^*\|}}$. □

Definition 2.2. Let $C \subseteq \mathbb{R}^n$ be a closed convex set. We call the set-valued mapping

$$G_C : \mathbb{R}^n \rightrightarrows S^{n-1},\ G_C(x) := N_C(x) \cap S^{n-1}$$

the *Gauss map* of C. The range $G_C(\mathbb{R}^n)$ of this set-valued mapping is called the *Gauss range* of C. □

Clearly, the Gauss range of a compact convex set is the whole of S^{n-1}; therefore, only the Gauss ranges of unbounded closed convex sets are interesting to study.

Lemma 2.3. *Let $C \subseteq \mathbb{R}^n$ be an unbounded closed convex set. Then the set $G_C(\mathbb{R}^n) \backslash (C - C)^{\perp}$ is open in S^{n-1} if and only if $N_C(\mathbb{R}^n) \backslash (C - C)^{\perp}$ is open. In particular, when* int $C \neq \emptyset$*, the set $G_C(\mathbb{R}^n)$ is open in S^{n-1} if and only if $N_C(\mathbb{R}^n) \setminus \{0\}$ is open, in which case one has*

$$N_C(\mathbb{R}^n) \setminus \{0\} = \operatorname{int} N_C(\mathbb{R}^n). \tag{2}$$

Note that Lemma 2.3 is based on [4, Lemma 7], but the equality (2) fails when the closed convex set C is bounded, i.e. compact, as $N_C(\mathbb{R}^n) = \mathbb{R}^n$ in such a case.

Theorem 2.4. [4] *Let $C \subseteq \mathbb{R}^n$ be a closed convex set. Then $G_C(\mathbb{R}^n)$ is open in S^{n-1} if and only if $C \cap H$ is compact for every supporting hyperplane H of C.*

Remark 2.5. ([4], Remark 10) Let $x^* \in \mathbb{R}^n \backslash \{0\}$ be such that the linear optimization problem (1) has a solution, i.e. $\frac{x^*}{\|x^*\|} \in G_C(\mathbb{R}^n)$. Then the solution set of (1) is the exposed face $C \cap H_{x^*}$. By using similar arguments as those in the proof of Theorem 2.4, one can deduce that the solution set of problem (1) is a compact exposed face of C if and only if $\frac{x^*}{\|x^*\|} \in \operatorname{int}_{S^{n-1}} G_C(\mathbb{R}^n)$. Equivalently, the solution set of the linear optimization problem (1) is an unbounded exposed face of C if and only if $\frac{x^*}{\|x^*\|} \in G_C(\mathbb{R}^n) \cap \operatorname{bd}_{S^{n-1}} G_C(\mathbb{R}^n)$. □

Although the main topic of our previous paper [4] concerns closed, unbounded, convex sets with open or closed ranges, it also touches slightly the closed convex sets with Gauss ranges which are neither open nor closed. An example of such a set is

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq x^2 + y^2 \text{ and } y \leq 0\}.$$

Its Gauss range is $G_C(\mathbb{R}^n) = \{(0, 1, 0)\} \cup \{(u, v, w) \in S^2 \mid w < 0\}$. Note that $\{(0, 1, 0)\} = G_C(\mathbb{R}^3) \cap \operatorname{bd}_{S^2} G_C(\mathbb{R}^3)$ is the only spherical boundary point of $G_C(\mathbb{R}^3)$; it is the unit normal vector of the only unbounded facet of C, i.e.

$$F = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq x^2 + y^2 \text{ and } y = 0\}.$$

3. Gauss ranges of affine images of closed convex sets

In this section, we study the behavior of Gauss ranges under the action of affine transformations. We recall that the adjoint of a linear isomorphism $A\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the linear isomorphism $A^* : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $\langle A^*x, y\rangle = \langle x, Ay\rangle$.

Proposition 3.1. *Let $C \subseteq \mathbb{R}^n$ be a closed convex set and $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear isomorphism. Then*

$$G_{A(C)}(\mathbb{R}^n) = \left(\mathrm{pr}_{S^{n-1}} \circ \left(A^{-1}\right)^*\right)(G_C(\mathbb{R}^n)),$$

where $\mathrm{pr}_{S^{n-1}}$ stands for the radial projection of $\mathbb{R}^n \setminus \{0\}$ over S^{n-1}, i.e.

$$\mathrm{pr}_{S^{n-1}} : \mathbb{R}^n \setminus \{0\} \longrightarrow S^{n-1}, \quad \mathrm{pr}_{S^{n-1}}(x) = \frac{x}{\|x\|}.$$

Proof. We have

$$G_{A(C)}(\mathbb{R}^n) = N_{A(C)}(\mathbb{R}^n) \cap S^{n-1}, \tag{3}$$

$$N_{A(C)}(\mathbb{R}^n) = \bigcup_{x\in C} N_{A(C)}(Ax) \tag{4}$$

and

$$\begin{aligned} N_{A(C)}(Ax) &= \{x^* \in \mathbb{R}^n : \langle Ac - Ax, x^*\rangle \le 0,\ \forall c \in C\} \\ &= \{x^* \in \mathbb{R}^n : \langle A(c - x), x^*\rangle \le 0,\ \forall c \in C\} \\ &= \{x^* \in \mathbb{R}^n : \langle c - x, A^*x^*\rangle \le 0,\ \forall c \in C\} \\ &= \left\{(A^*)^{-1} y^* \in \mathbb{R}^n : \langle c - x, y^*\rangle \le 0,\ \forall c \in C\right\} \\ &= \left\{\left(A^{-1}\right)^* y^* \in \mathbb{R}^n : \langle c - x, y^*\rangle \le 0,\ \forall c \in C\right\} \\ &= \left(A^{-1}\right)^*(N_C(x)) \quad \text{for } x \in C. \end{aligned} \tag{5}$$

From (4) and (5), it follows that

$$N_{A(C)}(\mathbb{R}^n) = \bigcup_{x\in C} \left(A^{-1}\right)^*(N_C(x)) = \left(A^{-1}\right)^*\Big(\bigcup_{x\in C} N_C(x)\Big) = \left(A^{-1}\right)^*(N_C(\mathbb{R}^n));$$

hence, using (3), we obtain

$$G_{A(C)}(\mathbb{R}^n) = S^{n-1} \cap \left(A^{-1}\right)^*(N_C(\mathbb{R}^n)). \tag{6}$$

Moreover,

$$\begin{aligned} G_{A(C)}(\mathbb{R}^n) &= S^{n-1} \cap \left(A^{-1}\right)^*(N_C(\mathbb{R}^n)) = S^{n-1} \cap \left(\left(A^{-1}\right)^*(N_C(\mathbb{R}^n) \setminus \{0\})\right) \\ &= \left\{\frac{\left(A^{-1}\right)^* y}{\|\left(A^{-1}\right)^* y\|} \ :\ y \in N_C(\mathbb{R}^n) \setminus \{0\}\right\} \\ &= \left\{\mathrm{pr}_{S^{n-1}}\left(\left(A^{-1}\right)^* y\right) \ :\ y \in N_C(\mathbb{R}^n) \setminus \{0\}\right\} \\ &= \left\{\mathrm{pr}_{S^{n-1}}\left(\left(A^{-1}\right)^* (y/\|y\|)\right) \ :\ y \in N_C(\mathbb{R}^n) \setminus \{0\}\right\} \\ &= \left\{\mathrm{pr}_{S^{n-1}}\left(\left(A^{-1}\right)^* y\right) \ :\ y \in S^{n-1} \cap (N_C(\mathbb{R}^n) \setminus \{0\})\right\} \\ &= \left\{\left(\mathrm{pr}_{S^{n-1}} \circ \left(A^{-1}\right)^*\right)(y) \ :\ y \in S^{n-1} \cap N_C(\mathbb{R}^n)\right\} \\ &= \left\{\left(\mathrm{pr}_{S^{n-1}} \circ \left(A^{-1}\right)^*\right)(y) \ :\ y \in G_C(\mathbb{R}^n)\right\} \\ &= \left(\mathrm{pr}_{S^{n-1}} \circ \left(A^{-1}\right)^*\right)(G_C(\mathbb{R}^n)). \end{aligned}$$

$\square$

Using the obvious fact that the Gauss range is invariant under translations, we obtain the following corollary.

Corollary 3.2. *Let $C \subseteq \mathbb{R}^n$ be a closed convex set and $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an affine automorphism, namely*

$$Tx := Ax + b \qquad (x \in \mathbb{R}^n),$$

with $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ being a linear isomorphism and $b \in \mathbb{R}^n$. Then

$$G_{T(C)}(\mathbb{R}^n) = \left(\mathrm{pr}_{S^{n-1}} \circ \left(A^{-1}\right)^*\right)(G_C(\mathbb{R}^n)).$$

Proof. We first observe that $T = t_b \circ g$, where $t_b : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ stands for the translation of vector b, namely $t_b(p) := p+b$ for all $p \in \mathbb{R}^n$. Therefore, by Proposition 3.1 we obtain

$$\begin{aligned} G_{T(C)}(\mathbb{R}^n) &= G_{(t_b \circ A)(C)}(\mathbb{R}^n) = G_{t_b(A(C))}(\mathbb{R}^n) \\ &= G_{A(C)}(\mathbb{R}^n) = \left(\mathrm{pr}_{S^{n-1}} \circ \left(A^{-1}\right)^*\right)(G_C(\mathbb{R}^n)) \end{aligned} \tag{7}$$

and the statement is now completely proved. □

The following lemma is very likely to be well known, but we include its proof in order to make the paper self contained.

Lemma 3.3. *The radial projection $\mathrm{pr}_{S^{n-1}} : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ is an open mapping.*

Proof. It will suffice to prove that, for $x_0 \in \mathbb{R}^n \setminus \{0\}$ and $r \in]0, \|x_0\|[$, the following inclusion holds:

$$\mathrm{int}\ B_{S^{n-1}}\left(\frac{x_0}{\|x_0\|}, \arcsin\frac{r}{\|x_0\|}\right) \subseteq \mathrm{pr}_{S^{n-1}}(\mathrm{int}\ B(x_0, r)). \tag{8}$$

Let $x \in \mathrm{int}\ B_{S^{n-1}}\left(\frac{x_0}{\|x_0\|}, \arcsin\frac{r}{\|x_0\|}\right)$, and observe that

$$\langle x, x_0\rangle x \in \mathrm{int}\ B(x_0, r).$$

Indeed, since $\arccos\left\langle x, \frac{x_0}{\|x_0\|}\right\rangle < \arcsin\frac{r}{\|x_0\|}$, one has

$$\left\langle x, \frac{x_0}{\|x_0\|}\right\rangle > \cos\arcsin\frac{r}{\|x_0\|} = \sqrt{1 - \frac{r^2}{\|x_0\|^2}},$$

and hence

$$\|\langle x, x_0\rangle x - x_0\|^2 = \|x_0\|^2 - \langle x, x_0\rangle^2 < \|x_0\|^2 - \|x_0\|^2\left(1 - \frac{r^2}{\|x_0\|^2}\right) = r^2.$$

Therefore $x = \mathrm{pr}_{S^{n-1}}(\langle x, x_0\rangle x) \subseteq \mathrm{pr}_{S^{n-1}}(\mathrm{int}\ B(x_0, r))$. This proves (8). □

Proposition 3.4. *Let $C \subseteq \mathbb{R}^n$ be a closed convex set, and $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an affine automorphism. If $G_C(\mathbb{R}^n)$ is spherically open (closed), then $G_{T(C)}(\mathbb{R}^n)$ is spherically open (closed, respectively), too.*

Proof. For the openness result, combine Corollary 3.2 with Lemma 3.3, observing that $(A^{-1})^*$ is a homeomorphism. Assume now that $G_C(\mathbb{R}^n)$ is closed. Then it is compact, which implies that $(A^{-1})^*(G_C(\mathbb{R}^n))$ is compact in $\mathbb{R}^n \setminus \{0\}$. Hence, by Corollary 3.2, the set $G_{T(C)}(\mathbb{R}^n)$ is compact in S^{n-1}, which implies that it is closed. $\square$

Remark 3.5. According to Proposition 3.4, the quality of the Gauss range of a closed convex set $C \subseteq \mathbb{R}^n$ to be open (closed) is preserved all over the orbit $\mathrm{Aut}_{\mathrm{aff}}(\mathbb{R}^n)C$ of C with respect to the action

$$\mathrm{Aut}_{\mathrm{aff}}(\mathbb{R}^n) \times \mathcal{C}(\mathbb{R}^n) \longrightarrow \mathcal{C}(\mathbb{R}^n),\ (T, C) \mapsto T \cdot C := T(C) \tag{9}$$

of the group of affine automorphisms $\mathrm{Aut}_{\mathrm{aff}}(\mathbb{R}^n)$ on the collection $\mathcal{C}(\mathbb{R}^n)$ of all closed convex subsets of $\mathbb{R}^n$. Note that (9) is the restriction of the action

$$\mathrm{Aut}_{\mathrm{aff}}(\mathbb{R}^n) \times \mathcal{P}(\mathbb{R}^n) \longrightarrow \mathcal{P}(\mathbb{R}^n),\ (T, C) \mapsto T(C), \tag{10}$$

as $\mathcal{C}(\mathbb{R}^n)$ is invariant under the action (10). $\square$

Remark 3.6. An unbounded closed convex set $C \subseteq \mathbb{R}^n$ whose all proper faces are compact and such that $\mathrm{int}(C) \neq \emptyset$ is not affinely equivalent with any unbounded hyperbolic closed convex set $K \subseteq \mathbb{R}^n$ that does not admit any asymptotic hyperplane. In other words, no affine automorphism $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ maps C to K. Indeed, $G_C(\mathbb{R}^n)$ is, according to [4, Corollary 11 (2)], spherically open, and the Gauss range of K is, according to [4, Theorem 18)], spherically closed. By using Proposition 3.4, one deduces the spherical openness of $G_{T(C)}(\mathbb{R}^n)$ for any affine automorphism $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. If $T(C) = K$ for some affine automorphism $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, then $G_{T(C)}(\mathbb{R}^n) = G_K(\mathbb{R}^n)$ would be both spherically open and closed, i.e. it would be the whole sphere S^{n-1}, due to the connectedness of S^{n-1}. This is obviously not the case, as C is unbounded. $\square$

Corollary 3.7. *Let $C \subseteq \mathbb{R}^n$ be a closed convex set and $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an affine isometry, namely*

$$Tx := Ax + b \qquad (x \in \mathbb{R}^n),$$

with A being a linear isometry and $b \in \mathbb{R}^n$. Then

$$G_{T(C)}(\mathbb{R}^n) = A(G_C(\mathbb{R}^n)).$$

It is an immediate consequence of Corollary 3.2, since A being a linear isometry implies that $(A^{-1})^* = A$ and $A(G_C(\mathbb{R}^n)) \subseteq A(S^{n-1}) = S^{n-1}$.

Remark 3.8. If $K \subseteq C \subseteq \mathbb{R}^n$ are two closed convex sets such that K does not admit asymptotic hyperplanes, then $N_C(\mathbb{R}^n) \subseteq N_K(\mathbb{R}^n)$, which proves the inclusion $G_C(\mathbb{R}^n) \subseteq G_K(\mathbb{R}^n)$ in such a case. Indeed, for $x^* \in N_C(\mathbb{R}^n)$ there exists $x_0 \in C$ such that $\langle x, x^* \rangle \leq \langle x_0, x^* \rangle$ for all $x \in C$. In particular $\langle x, x^* \rangle \leq \langle x_0, x^* \rangle$ for all $x \in K$, which proves that $\sup_{x \in K} \langle x, x^* \rangle \leq \langle x_0, x^* \rangle$ and the existence of an element $y_0 \in K$ such that $\operatorname{argmax}_{y \in K} \langle y, x^* \rangle \neq \emptyset$, i.e. $x^* \in N_K(\mathbb{R}^n)$, as the hyperplane

$$\left\{ z \in \mathbb{R}^n : \langle z, x^* \rangle = \sup_{x \in K} \langle x, x^* \rangle \right\}$$

cannot be asymptotic for K. $\square$

If C is convex, closed and unbounded, then its Gauss range is contained in a closed hemisphere. Indeed, the halfline $x+\mathbb{R}_+d$ is contained in C for every $x \in C$ and $d \in 0^+(C) \setminus \{0\}$ (by $0^+(C)$ we denote the recession cone of C, see [5]) and does not admit any asymptotic hyperplane. Thus,

$$G_C\left(\mathbb{R}^n\right) \subseteq G_{x+\mathbb{R}_+d}\left(\mathbb{R}^n\right) = \{x^* \in \mathbb{R}^n \ : \ \langle d, x^* \rangle \leq 0\} \cap S^{n-1}.$$

4. Gauss ranges containing no antipodal points

While Proposition 5.1 is a source of examples of closed convex and unbounded subsets of $\mathbb{R}^n$ whose Gauss ranges are not contained in any open hemisphere, in this section we shall deal with examples of closed convex and unbounded subsets of $\mathbb{R}^n$ whose Gauss ranges are contained in open hemispheres, which is equivalent to saying that they do not contain antipodal points. We recall that two points $x^*, y^* \in S^{n-1}$ are said to be *antipodal* if $x^* + y^* = 0$.

The first results in this section deal with ice-cream cones, an important example of closed convex sets whose Gauss ranges, as we will see, do not contain antipodal points.

Proposition 4.1. *If $x_0 \in S^{n-1}$ and $0 < \rho < 1$, then*

$$\mathbb{R}_+ B\left(x_0, \rho\right) = \left\{x \in \mathbb{R}^n : \|x\| \sqrt{1-\rho^2} \leq \langle x, x_0 \rangle\right\}. \tag{11}$$

Proof. We first prove the inclusion $\subseteq$. Since the right hand side is a cone, we may disregard $\mathbb{R}_+$ in the left hand side. Let $x \in B\left(x_0, \rho\right)$. From the inequality $\|x - x_0\|^2 \leq \rho^2$, we get $\|x\|^2 \leq 2\langle x_0, x\rangle - (1-\rho^2)$. Multiplying both sides of this inequality by $1-\rho^2$ yields

$$\|x\|^2\left(1-\rho^2\right) \leq 2\langle x_0, x\rangle\left(1-\rho^2\right) - \left(1-\rho^2\right)^2 \leq \langle x_0, x\rangle^2;$$

taking square roots in these inequalities and observing that

$$\langle x_0, x\rangle \geq \frac{\|x\|^2 + 1 - \rho^2}{2} \geq 0,$$

we obtain the desired inclusion. To prove the converse inclusion, let $x \neq 0$ belong to the right hand side of (11). We have

$$\left\|\frac{\langle x_0, x\rangle}{\|x\|^2}x - x_0\right\|^2 = \frac{\langle x_0, x\rangle^2}{\|x\|^2} - 2\frac{\langle x_0, x\rangle^2}{\|x\|^2} + 1 = 1 - \frac{\langle x_0, x\rangle^2}{\|x\|^2} \leq 1 - \left(1-\rho^2\right) = \rho^2;$$

hence $$x = \frac{\|x\|^2}{\langle x_0, x\rangle}\frac{\langle x_0, x\rangle}{\|x\|^2}x \in \frac{\|x\|^2}{\langle x_0, x\rangle}B\left(x_0, \rho\right) \subseteq \mathbb{R}_+ B\left(x_0, \rho\right). \quad \square$$

We recall that the standard metric $d_{S^{n-1}}$ on S^{n-1} is given by

$$d_{S^{n-1}}(x, y) := \arccos\langle x, y\rangle.$$

We will denote by $B_{S^{n-1}}(x, \alpha)$ the closed ball in S^{n-1} with center $x \in S^{n-1}$ and radius $\alpha > 0$ with respect to the metric $d_{S^{n-1}}$.

Corollary 4.2. *If* $x_0 \in S^{n-1}$ *and* $0 < \rho < 1$, *then*

$$\mathbb{R}_+ B(x_0, \rho) \cap S^{n-1} = B_{S^{n-1}}(x_0, \arcsin \rho).$$

Proof. By Proposition 4.1, we have

$$\begin{aligned}
\mathbb{R}_+ B(x_0, \rho) \cap S^{n-1} &= \left\{ x \in S^{n-1} : \sqrt{1-\rho^2} \leq \langle x, x_0 \rangle \right\} \\
&= \left\{ x \in S^{n-1} : \arccos \langle x, x_0 \rangle \leq \arccos \sqrt{1-\rho^2} \right\} \\
&= \left\{ x \in S^{n-1} : d_{S^{n-1}}(x, x_0) \leq \arcsin \rho \right\} \\
&= B_{S^{n-1}}(x_0, \arcsin \rho).
\end{aligned}$$ □

We recall that the *polar cone* of $K \subseteq \mathbb{R}^n$ is the closed convex cone

$$K^0 := \{x^* : \langle x, x^* \rangle \leq 0 \qquad \forall x \in K\}.$$

K^{00}, the second polar of $K \subseteq \mathbb{R}^n$, coincides with the smallest closed convex cone that contains K; in particular, if K is a closed convex cone, one has $K^{00} = K$.

Corollary 4.3. *If* $x_0 \in S^{n-1}$ *and* $0 < \rho < 1$, *then*

$$(\mathbb{R}_+ B(x_0, \rho))^0 = \mathbb{R}_+ B\left(-x_0, \sqrt{1-\rho^2}\right).$$

Proof. Since $\|x\| = \sup_{x^* \in B(0,1)} \langle x, x^* \rangle$, from (11) we get

$$\begin{aligned}
\mathbb{R}_+ B(x_0, \rho) &= \left\{ x \in \mathbb{R}^n : \left\langle x, \sqrt{1-\rho^2} x^* - x_0 \right\rangle \leq 0 \qquad \forall x^* \in B(0,1) \right\} \\
&= \left\{ x \in \mathbb{R}^n : \langle x, y^* \rangle \leq 0 \qquad \forall x^* \in B\left(x_0, \sqrt{1-\rho^2}\right) \right\} \\
&= B\left(x_0, \sqrt{1-\rho^2}\right)^0.
\end{aligned}$$

Taking polars, we deduce that

$$(\mathbb{R}_+ B(x_0, \rho))^0 = B\left(x_0, \sqrt{1-\rho^2}\right)^{00} = \mathbb{R}_+ B\left(x_0, \sqrt{1-\rho^2}\right).$$ □

Corollary 4.4. *If* $x_0 \in S^{n-1}$ *and* $0 < \rho < 1$, *then*

$$G_{\mathbb{R}_+ B(x_0,\rho)}(\mathbb{R}^n) = B_{S^{n-1}}(-x_0, \arccos \rho).$$

Proof. Using Corollaries 4.3 and 4.2, we obtain

$$\begin{aligned}
G_{\mathbb{R}_+ B(x_0,\rho)}(\mathbb{R}^n) &= (\mathbb{R}_+ B(x_0, \rho))^0 \cap S^{n-1} = \mathbb{R}_+ B\left(-x_0, \sqrt{1-\rho^2}\right) \cap S^{n-1} \\
&= B_{S^{n-1}}\left(-x_0. \arcsin \sqrt{1-\rho^2}\right) = B_{S^{n-1}}(-x_0, \arccos \rho).
\end{aligned}$$ □

Below, the diameter of a set $Q \subseteq S^{n-1}$ with respect to the metric $d_{S^{n-1}}$will be denoted $\operatorname{diam}_{S^{n-1}} Q$.

Theorem 4.5. *If $C \subseteq \mathbb{R}^n$ is an unbounded closed convex set and* int $0^+(C) \neq \emptyset$*, then* $\operatorname{diam}_{S^{n-1}}(G_C(\mathbb{R}^n)) < \pi$*, and hence $G_C(\mathbb{R}^n)$ contains no pair of antipodal points.*

Proof. Take $x_0 \in 0^+(C) \cap S^{n-1}$ and $\rho \in]0,1[$ such that $B(x_0, \rho) \subseteq 0^+(C)$, and $x \in C$. Since $x + \mathbb{R}_+ B(x_0, \rho) \subseteq x + 0^+(C) \subseteq C$, and cones do not admit any asymptotic hyperplane, it follows, via Remark 3.8 and Corollary 4.4, that

$$G_{\mathbb{C}}(\mathbb{R}^n) \subseteq G_{x+\mathbb{R}_+ B(x_0,\rho)}(\mathbb{R}^n) = G_{\mathbb{R}_+ B(x_0,\rho)}(\mathbb{R}^n) = B_{S^{n-1}}(-x_0, \arccos\rho).$$

Thus, $\operatorname{diam}_{S^{n-1}}(G_{\mathbb{C}}(\mathbb{R}^n)) \leq 2\arccos\rho < \pi$. □

Corollary 4.6. *If $C \subseteq \mathbb{R}^n$ is an unbounded closed convex set and* int $0^+(C) \neq \emptyset$*, then $G_{\mathbb{C}}(\mathbb{R}^n)$ is contained in an open hemisphere.*

The convex hull of a parabola in the plane shows that the converse of Corollary 4.6 does not hold.

Another example of Gauss ranges containing no antipodal points will be provided by the next proposition. We recall that the barrier cone of a closed convex set C is

$$\operatorname{barr}(C) := \left\{ x^* \in \mathbb{R}^n : \sup_{x \in C} \langle x, x^* \rangle < +\infty \right\}.$$

One has $[\operatorname{barr}(C)]^0 = 0^+(C)$, and hence

$$G_{\mathbb{C}}(\mathbb{R}^n) \subseteq \operatorname{barr}(C) \subseteq [\operatorname{barr}(C)]^{00} = 0^+(C)^0. \tag{12}$$

We refer to [5] for a thorough study of convexity in finite dimensions.

Proposition 4.7. *If $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous proper convex function such that* $\operatorname{dom}(f)$ *is open, then $G_{\operatorname{epi}(f)}(\mathbb{R}^{n+1})$ is contained in the hemisphere $\{(x, \alpha) \in (\mathbb{R}^n \times \mathbb{R}) \cap S^n : \alpha < 0\}$, and hence it contains no pair of antipodal points.*

Proof. Since the vector $(0, 1) \in \mathbb{R}^n \times \mathbb{R}$ belongs to $0^+(C)$, by (12) we have

$$G_{\operatorname{epi}(f)}(\mathbb{R}^{n+1}) \subseteq (\operatorname{epi}(f)^0 \subseteq \{(0,1)\}^0 = \{(x^*, \alpha^*) \in \mathbb{R}^n \times \mathbb{R} : \alpha^* \leq 0\}.$$

If $(x^*, 0) \in G_{\operatorname{epi}(f)}(\mathbb{R}^{n+1})$ for some $x^* \in \mathbb{R}^n$, then it is easy to prove that this point x^* defines a supporting hyperplane of $\operatorname{dom}(f)$, which is impossible because $\operatorname{dom}(f)$ is open. This proves the inclusion asserted in the statement. □

The epigraph of the indicator function of a compact convex set with nonempty interior shows that the assumptions int $0^+(C) \neq \emptyset$ in Corollary 4.6 and that $\operatorname{dom}(f)$ is open in Proposition 4.7 are not superfluous.

5. Gauss ranges containing antipodal points

In this section we consider Gauss ranges containing antipodal points.

One says that a closed convex set is Motzkin decomposable if it can be expressed as the Minkowski sum of a compact convex set and a closed convex cone. We refer to [2, 3] for detailed studies of Motzkin decomposable sets.

Theorem 5.1. *Let $C \subseteq \mathbb{R}^n$ be an unbounded closed convex set with* int $C \neq \emptyset$. *If $G_{\mathbb{C}}(\mathbb{R}^n)$ has a pair of antipodal points, then* int $0^+(C) = \emptyset$ *and the corresponding exposed faces F_i $(i = 1, 2)$ of C satisfy $0^+(F_i) = 0^+(C)$ $(i = 1, 2)$. Conversely, if C is a Motzkin decomposable set and* int $0^+(C) = \emptyset$, *then, for every $x^* \in [0^+(C)]^\perp \cap S^{n-1}$, one has a pair $x^*, -x^* \in G_{\mathbb{C}}(\mathbb{R}^n)$ of antipodal points.*

Proof. If $G_{\mathbb{C}}(\mathbb{R}^n)$ has a pair of antipodal points, then $\operatorname{diam}_{S^{n-1}}(G_{\mathbb{C}}(\mathbb{R}^n)) \geq \pi$; hence, by Theorem 4.5, int $0^+(C) = \emptyset$. Let x^* and $-x^*$ be one such pair. Then there exist $\overline{x}, \overline{y} \in C$ such that

$$\langle \overline{x}, x^* \rangle \leq \langle c, x^* \rangle \leq \langle \overline{y}, x^* \rangle, \ \forall c \in C. \tag{13}$$

Let $d \in 0^+(C)$. From $\overline{x} + d, \overline{y} + d \in C$ and (13), it easily follows that $\langle d, x^* \rangle = 0$, and hence $d \in 0^+(F_i)$ $(i = 1, 2)$ for the exposed faces

$$F_1 = \{c \in C : \langle c, x^* \rangle = \langle \overline{x}, x^* \rangle\}, \ F_2 = \{c \in C : \langle c, x^* \rangle = \langle \overline{y}, x^* \rangle\},$$

which are proper because $\langle \overline{x}, x^* \rangle < \langle \overline{y}, x^* \rangle$ (by the nonemptiness of int C).

Conversely, assume that C is Motzkin decomposable and int $0^+(C) = \emptyset$, and let $x^* \in [0^+(C)]^\perp \cap S^{n-1}$. Notice that $[0^+(C)]^\perp \cap S^{n-1} \neq \emptyset$, because int $0^+(C) = \emptyset$ implies that $0^+(C)$ is contained in a proper subspace of $\mathbb{R}^n$. We have

$$x^*, -x^* \in \left[0^+(C)\right]^0 = [\operatorname{barr}(C)]^{00} = \operatorname{cl}(\operatorname{barr}(C)) = N_C(\mathbb{R}^n),$$

the latter equality being a consequence of the Motzkin decomposable character of C. Therefore, the antipodal points x^* and $-x^*$belong to $G_{\mathbb{C}}(\mathbb{R}^n)$. □

The epigraph of a parabola in the plane shows that the assumption that C is Motzkin decomposable in Theorem 5.1 is not superfluous.

6. Spherical-convexity and the Gauss range

A curve contained in the intersection of a plane through the origin of $\mathbb{R}^n$ with the sphere S^{n-1} is a geodesic in S^{n-1}. A geodesic segment $\gamma : [a, b] \longrightarrow S^{n-1}$ is said to be *minimal* if its arc length is equal to the intrinsic distance between its end points, i.e., if $l(\gamma) := \arccos\langle \gamma(a), \gamma(b) \rangle$.

Definition 6.1. [1] The set $P \subseteq S^{n-1}$ is said to be *spherically-convex* if, for any $p, q \in P$, all the minimal geodesic segments joining p and q are contained in P. □

Proposition 6.2. [1] *Let $P \subseteq S^{n-1}$ be spherically-convex. If P has a pair of antipodal points, then $P = S^{n-1}$.*

If $p, q \in S^{n-1}$ are such that $q \neq p$ and $q \neq -p$, then the geodesic segment between p and q, i.e. the smallest geodesic arc $S^{n-1} \cap \operatorname{cone}\{p, q\}$ (here and in the sequel, "cone" denotes "convex cone generated by") joining p and q can be parameterized both by the arc-length parameterization, given by

$$\gamma_{pq}(t) = \left(\cos t - \frac{\langle p, q \rangle \sin t}{\sqrt{1 - \langle p, q \rangle^2}}\right) p + \frac{\sin t}{\sqrt{1 - \langle p, q \rangle^2}} q, \quad t \in [0, d_{S^{n-1}}(p, q)]$$

(see [1]) or by the differentiable curve $c_{pq} : [0,1] \longrightarrow S^{n-1}$ defined by

$$c_{pq}(s) = \frac{(1-s)\,p + sq}{\|(1-s)\,p + sq\|}\,.$$

Since the definition of spherical-convexity only depends on the geodesics, we can handle *spherical-convexity* by using the most convenient parameterizations of the geodesics of the sphere S^{n-1}. Therefore a set $P \subseteq S^{n-1}$ containing no pair of antipodal points is spherically-convex in S^{n-1} if the following implication holds:

$$x, y \in P,\ \lambda \in [0,1] \Rightarrow c_{xy}(\lambda) \in P.$$

Our next results provide useful characterizations of spherical-convexity.

Proposition 6.3. *A set $P \subseteq S^{n-1}$ containing no pair of antipodal points is spherically-convex if and only if $\mathbb{R}_+P$ is convex. In particular, the Gauss range $G_C(\mathbb{R}^n)$ of an unbounded closed convex set $C \subseteq \mathbb{R}^n$ is spherically-convex if and only if its total normal cone $N_C(\mathbb{R}^n) = \mathbb{R}_+G_C(\mathbb{R}^n)$ is convex.*

Proof. Assume first that $G_C(\mathbb{R}^n)$ is spherically-convex. Let $\lambda, \mu \in \mathbb{R}_+$, $p, q \in G_C(\mathbb{R}^n)$ and $\alpha \in [0,1]$. If $(1-\alpha)\,\lambda p + \alpha\mu q \neq 0$, then, setting

$$u := \frac{(1-\alpha)\,\lambda p + \alpha\mu q}{(1-\alpha)\,\lambda + \alpha\mu},$$

we have $\frac{u}{\|u\|} \in G_C(\mathbb{R}^n)$, and hence

$$(1-\alpha)\,\lambda p + \alpha\mu q = ((1-\alpha)\,\lambda + \alpha\mu)\,\|u\|\,\frac{u}{\|u\|} \in \mathbb{R}_+G_C(\mathbb{R}^n).$$

This proves that $\mathbb{R}_+G_C(\mathbb{R}^n)$ is convex. Conversely, assume that $N_C(\mathbb{R}^n) = \mathbb{R}_+G_C(\mathbb{R}^n)$ is convex, and let $p.q \in G_C(\mathbb{R}^n)$ and $\alpha \in [0,1]$. Then

$$\frac{(1-\alpha)\,p + \alpha q}{\|(1-\alpha)\,p + \alpha q\|} \in N_C(\mathbb{R}^n) \cap S^{n-1} = G_C(\mathbb{R}^n),$$

which proves that $G_C(\mathbb{R}^n)$ is spherically-convex. □

Proposition 6.4. *For a set $P \subseteq S^{n-1}$, the following statements are equivalent:*

(a) *The set P is spherically-convex.*

(b) $P = (\mathrm{cone}P) \cap S^{n-1}$.

(c) *There exists a convex cone $K \subseteq \mathbb{R}^n$ such that $P = K \cap S^{n-1}$.*

Proof. (a) $\Rightarrow$ (b). One has $P = (\mathbb{R}_+P) \cap S^{n-1} = (\mathit{cone}\ P) \cap S^{n-1}$, the second equality being a consequence of Proposition 6.3.

The implication (b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (a). One can easily prove that $\mathbb{R}_+P = K$. From this equality and Proposition 6.3, it follows that P is spherically-convex. □

Remark 6.5. Let $C \subseteq \mathbb{R}^n$ be a closed convex set such that $G_C(\mathbb{R}^n)$ is spherically-convex, and T and A be as in Corollary 3.2. If $G_C(\mathbb{R}^n)$ has a pair of antipodal points, then, by Proposition 6.2, one has $G_C(\mathbb{R}^n) = S^{n-1}$, that is, C is compact, and hence $T(C)$ is compact, too, so that $G_{T(C)}(\mathbb{R}^n)$ is the whole of S^{n-1} and is therefore a spherically-convex set. Assume now that C is unbounded. By (7) and (6), one has

$$G_{T(C)}(\mathbb{R}^n) = G_{A(C)}(\mathbb{R}^n) = S^{n-1} \cap \left(A^{-1}\right)^* \left(N_C\left(\mathbb{R}^n\right)\right);$$

hence, by Propositions 6.3 and 6.4, the set $G_{T(C)}(\mathbb{R}^n)$ is spherically-convex. Thus, the quality of the Gauss range of a closed convex set $C \subseteq \mathbb{R}^n$ to be spherically-convex is preserved all over the orbit $\mathrm{Iso}(\mathbb{R}^n)C$ of C with respect to the action

$$\mathrm{Iso}(\mathbb{R}^n) \times \mathcal{C}(\mathbb{R}^n) \longrightarrow \mathcal{C}(\mathbb{R}^n),\ (T, C) \mapsto T \cdot C := T(C) \tag{14}$$

of the group $\mathrm{Iso}(\mathbb{R}^n)$ of isometries of $\mathbb{R}^n$ on the collection $\mathcal{C}(\mathbb{R}^n)$ of all closed convex subsets of $\mathbb{R}^n$. Note that (14) is the restriction, on the left hand (group) side, of the action $\mathrm{Aut}_{\mathrm{aff}}(\mathbb{R}^n) \times \mathcal{C}(\mathbb{R}^n) \longrightarrow \mathcal{C}(\mathbb{R}^n),\ (T, C) \mapsto T(C)$. □

Lemma 6.6. *If $K \subseteq \mathbb{R}^n$ is a cone, then*

$$\mathrm{cl}\left(K \cap S^{n-1}\right) = (\mathrm{cl}\ K) \cap S^{n-1} \tag{15}$$

and

$$\mathrm{int}_{S^{n-1}}\left(K \cap S^{n-1}\right) = (\mathrm{int}\ K) \cap S^{n-1}. \tag{16}$$

Proof. The inclusion $\subseteq$ in (15) is obvious.

To prove the oposite inclusion, let $x \in (\mathrm{cl}\ K) \cap S^{n-1}$. We have $x = \lim x_k$, with $x_k \in K$. From the equality $\lim \|x_k\| = \|x\| = 1$, we deduce that $\lim \frac{x_k}{\|x_k\|} = x$, which, since K is a cone, shows that $x \in \mathrm{cl}\left(K \cap S^{n-1}\right)$.

The inclusion $\supseteq$ in (16) is obvious. To prove the opposite inclusion, assume that $x \in \mathrm{int}_{S^{n-1}}\left(K \cap S^{n-1}\right)$. There exists $r \in]0, 2[$ such that

$$B(x, r) \cap S^{n-1} \subseteq K. \tag{17}$$

Take a positive number $\delta < \min\left\{1, \frac{4}{4-r^2} - 1\right\}$. For $y \in B(x, \delta)$, since $\|x\| = 1$ we have $1 - \delta \leq \|y\| \leq 1 + \delta$ and $\|y\|^2 - 2\langle x, y\rangle + 1 \leq \delta^2$; hence

$$\begin{aligned} \left\|\frac{y}{\|y\|} - x\right\|^2 &= 2 - 2\frac{\langle x, y\rangle}{\|y\|} \leq 2 + \frac{\delta^2 - 1 - \|y\|^2}{\|y\|} \leq 2 + \frac{\delta^2 - 1 - (1-\delta)^2}{1+\delta} \\ &= 4 - \frac{4}{1+\delta} < r^2, \end{aligned}$$

which shows that $\frac{y}{\|y\|} \in B(x, r)$. Therefore, from (17) we deduce that $\frac{y}{\|y\|} \in K$, so that, since K is a cone, we have $y \in K$, thus proving the inclusion $B(x, \delta) \subseteq K$, which shows that $x \in \mathrm{int}\ K$. □

Proposition 6.7. *The sets* cl $G_{\mathbb{C}}(\mathbb{R}^n)$ *and* $\mathrm{int}_{S^{n-1}} G_C(\mathbb{R}^n)$ *are spherically-convex.*

Proof. Since $G_{\mathbb{C}}(\mathbb{R}^n) = N_C(\mathbb{R}^n) \cap S^{n-1}$ and $N_C(\mathbb{R}^n)$ is a cone, from Lemma 6.6 we obtain

$$\mathrm{cl}\ G_{\mathbb{C}}(\mathbb{R}^n) = (\mathrm{cl}\ N_C(\mathbb{R}^n)) \cap S^{n-1}$$

and

$$\mathrm{int}_{S^{n-1}} \mathrm{cl}\ G_{\mathbb{C}}(\mathbb{R}^n) = (\mathrm{int}\ N_C(\mathbb{R}^n)) \cap S^{n-1}. \tag{18}$$

Given that N_C is a maximally monotone operator, its range is nearly convex (see, e.g., [6, 12.4.1]), which implies that both cl $N_C(\mathbb{R}^n)$ and int $N_C(\mathbb{R}^n)$ are convex; therefore, from (18) and Proposition 6.4, the statement follows. □

Corollary 6.8. *If $C \subseteq \mathbb{R}^n$ is a closed convex set such that $C \cap H$ is compact for every supporting hyperplane H of C, then $G_C(\mathbb{R}^n)$ is spherically-convex.*

Proof. This follows from combining Proposition 6.7 with [4, Theorem 8]. □

Corollary 6.9. *If $C \subseteq \mathbb{R}^n$ is a closed convex hyperbolic set with no asymptotic hyperplanes, then $G_C(\mathbb{R}^n)$ is spherically-convex.*

Proof. This follows from combining Proposition 6.7 with [4, Theorem 18]. □

Remark 6.10. Since the Gauss range of the epigraph of the convex function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) := \max\{e^{-x}, |y|\}$ is not spherically-convex, by Remark 6.5 and Corollary 6.8 the set epi(f) is not isometrically equivalent with any closed convex set $C \subseteq \mathbb{R}^n$ such that $C \cap H$ is compact for every supporting hyperplane H of C. By Remark 6.5 and Corollary 6.9, it is also not isometrically equivalent with any closed convex hyperbolic set having no asymptotic hyperplanes. □

Our last two corollaries deal with Gauss ranges of epigraphs of lower semicontinuous proper convex functions as well as with their subdifferential operators. We recall that the domain of $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is the set

$$\mathrm{dom}(f) := f^{-1}(\mathbb{R}).$$

The subdifferential operator $\partial f : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ of f, a fundamental notion in convex analysis, is defined by

$$\partial f(x) := \{x^* \in \mathbb{R}^n : f(y) \geq f(x) + \langle y - x, x^* \rangle \qquad \forall y \in \mathbb{R}^n\}.$$

Corollary 6.11. *If $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous proper convex function such that* dom(f) *and* $\partial f(\mathbb{R}^n)$ *are open, then $G_{\mathrm{epi}(f)}(\mathbb{R}^{n+1})$ is spherically-convex.*

Proof. It follows from combining Proposition 6.7 with [4, Theorem 26]. □

Remark 6.12. Since the Gauss range of the epigraph of the convex function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) := \max\{e^{-x}, |y|\}$ is not spherically-convex, by Remark 6.5 and Corollary 6.11 the set epi(f) is not isometrically equivalent with the epigraph of any lower semicontinuous proper convex function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ such that dom(f) and $\partial f(\mathbb{R}^n)$ are open. □

Corollary 6.13. *If* $f\colon \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ *is a lower semicontinuous proper convex function with closed convex domain such that* $G_{\mathrm{dom}(f)}(\mathbb{R}^n)$ *and* $\partial f(\mathbb{R}^n)$ *are closed and* $0^+(\mathrm{dom}(f^*)) \subseteq \mathrm{barr}(\mathrm{dom}(f))$*, then* $G_{\mathrm{epi}}(f)(\mathbb{R}^{n+1})$ *is spherically-convex.*

Proof. It follows from combining Proposition 6.7 with [4, Theorem 30]. □

The assumption $0^+(\mathrm{dom}(f^*)) \subseteq \mathrm{barr}(\mathrm{dom}(f))$ in Corollary 6.13 may look a bit strange at first sight; however, it is not a too strong assumption. It is satisfied, for instance, in the case when f has full domain, since in such a case the right hand side of the inclusion is the whole of $\mathbb{R}^n$. Another easy example is the one variable function defined by

$$f(x) := \begin{cases} \frac{1}{x} \text{ if } x > 0 \\ +\infty \text{ if } x \leq 0 \end{cases};$$

indeed, for this function the sets $0^+(\mathrm{dom}(f^*))$ and $\mathrm{barr}(\mathrm{dom}(f))$ coincide with the set of nonpositive real numbers.

Remark 6.14. Like in Remark 6.12 above, it is worth noticing the following: Since the Gauss range of the epigraph of the convex function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) := \max\{e^{-x}, |y|\}$ is not spherically-convex, by Remark 6.5 and Corollary 6.13 the set $\mathrm{epi}(f)$ is not isometrically equivalent with the epigraph of any lower semicontinuous proper convex function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ with closed convex domain such that $G_{\mathrm{dom}(f)}(\mathbb{R}^n)$ and $\partial f(\mathbb{R}^n)$ are closed and $0^+(\mathrm{dom}(f^*)) \subseteq \mathrm{barr}(\mathrm{dom}(f))$. □

Acknowledgments. We are grateful to an anonymous referee for his/her careful reading of our manuscript and useful suggestions that have helped us to polish and improve the presentation.

References

[1] O. P. Ferreira, A. N. Iusem, S. Z. Németh: *Projections onto convex sets on the sphere*, J. Global Optim. 57 (2013) 663–676.

[2] M. A. Goberna, E. González, J. E. Martínez-Legaz, M. I. Todorov: *Motzkin decomposition of closed convex sets*, J. Math. Analysis Appl. 364 (2010) 209–221.

[3] M. A. Goberna, J. E. Martínez-Legaz, M. I. Todorov: *On Motzkin decomposable sets and functions*, J. Math. Analysis Appl. 372 (2010) 525–537.

[4] J. E. Martínez-Legaz, C. Pintea: *Closed convex sets with an open or closed Gauss range*, Math. Program. Ser. B 189/1-2 (2021) 433–454.

[5] R. T. Rockafellar: *Convex Analysis*, Princeton University Press, Princeton (1970).

[6] R. T. Rockafellar, R. J.-B. Wets: *Variational Analysis*, Springer, Berlin (1998).