

Primal Characterizations of Error Bounds for Composite-Convex Inequalities

Zhou Wei

*Hebei Key Laboratory of Machine Learning and Computational Intelligence, College of
Mathematics and Information Science, Hebei University, Baoding, China
weizhou@hbu.edu.cn*

Michel Théra

*XLIM UMR – CNRS 7252, Université de Limoges, France;
and: Federation University Australia, Ballarat, Australia
michel.thera@unilim.fr*

Jen-Chih Yao

*Dept. of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, Taiwan
yaojc@math.nsysu.edu.tw*

Dedicated to Roger J-B Wets on the occasion of his 85th birthday.

Received: April 26, 2022

Accepted: October 13, 2022

This paper is devoted to primal conditions of error bounds for a general function. In terms of Bouligand tangent cones, lower Hadamard directional derivatives and the Hausdorff-Pompeiu excess of subsets, we provide several necessary and/or sufficient conditions for error bounds with mild assumptions. Then we use these primal results to characterize error bounds for composite-convex functions (i.e. the composition of a convex function with a continuously differentiable mapping). It is proved that the primal characterization of error bounds can be established via Bouligand tangent cones, directional derivatives and the Hausdorff-Pompeiu excess if the mapping is metrically regular at the given point. The accurate estimate on the error bound modulus is also obtained.

Keywords: Error bound, composite-convex inequality, Bouligand tangent cone, lower Hadamard directional derivative, Hausdorff-Pompeiu excess.

2000 Mathematics Subject Classification: 90C31, 90C25, 49J52, 46B20.

1. Introduction

The main goal of this paper is to study error bounds for an inequality defined by a composite-convex function; i.e. the composition of a convex function with a continuously differentiable mapping. Error bounds of extended real-valued functions have been intensively studied for more than half a century. The starting point of

Research of the first author was supported by the National Natural Science Foundations of China (Grant Nos. 11971422 and 12171419), and funded by Science and Technology Project of Hebei Education Department (No. ZD2022037) and the Natural Science Foundation of Hebei Province (A2022201002). Research of the second author benefited from the support of the FMJH Program PGMO and from the support of EDF as well as from the FJMH Program Gaspard Monge for optimization and operation research and their interactions with data sciences, and was supported by a public grant as part of the investissement d'avenir project, reference ANR-11-LABEX-0056-LMH, LabEx LMH.

the theory of error bounds goes back to the fundamental works by Hoffman [20] and Lojasiewicz [31]. Their results were extensively studied by many authors (cf. [1, 2, 4, 25, 28, 32, 44, 45, 49]) and there have been significant developments on error bounds for convex and non-convex functions in recent years. The readers are invited to consult bibliographies [5, 8, 9, 13, 12, 16, 23, 29, 30, 33, 38, 40, 41, 42, 56, 60] and references therein for more details concerning the theory and applications of error bounds.

It is now well established that error bounds have played an important role in various aspects of optimization and variational analysis. These aspects include for instance, sensitivity analysis of linear programming (cf. [49, 50]), convergence analysis of descent methods (cf. [18, 19, 26, 33, 34, 55]), the so-called feasibility problems (cf. [6, 7, 8]), the domain of image reconstruction (cf. [11]) and many others. Error bounds are closely related with other notions well known and used in convex analysis and approximation theory such as the basic constraint qualification, the strong conical hull intersection property, the linear regularity, the Abadie constraint qualification and optimality conditions (cf. [14, 32, 35, 38]). Also error bounds are extensively discussed with weak sharp minima of functions, metric subregularity as well as calmness of multifunctions (cf. [5, 9, 10, 17, 22, 23, 36] and references therein for more details).

When dealing with error bounds, a large literature is devoted to provide dual characterizations and criteria in terms of subdifferentials or normal cones. To the best of our knowledge, [21] was among of the first papers of such kind to state sufficient conditions for error bounds of a constraint system in terms of the Clarke subdifferential. In 1997 Lewis and Pang [32] studied error bounds for convex inequality systems and provided necessary conditions via subdifferentials and normal cones. In 2003 Ngai and Théra [40] provided an error bound estimate and an implicit multifunction theorem in terms of smooth subdifferentials and abstract subdifferentials. In 2004 Zheng and Ng [61] proved dual characterizations of error bounds for convex inequalities in terms of subdifferentials and normal cones. In 2010 subdifferential characterizations of stability of error bounds for convex constraint inequalities were given in [30, 39]. In 2018 Kruger, López and Théra [29] extended results in [30, 39] and provided subdifferential characterizations of stability of error bounds for convex inequalities in the Banach space setting. It is noted that a pretty natural idea is to study error bounds in terms of various primal derivative-like objects such as directional derivatives, contingent cones or slopes. Several criteria for error bounds were worked out in [13, 12, 16, 41, 42] based on the primal-type estimate. In terms of contingent cones and directional derivatives, Wei, Yao and Zheng [59] proved primal characterizations of error bounds for a convex inequality (see [59, Proposition 5.3]). Recently, the authors [56] further studied error bounds of the convex inequality in terms of the Hausdorff-Pompeiu excess (of subsets), Bouligand tangent cones and directional derivatives, and provided the accurate primal estimate on the error bound modulus (see [56, Theorem 5.1]). Based on the works in [59, 56], a natural issue is to extend primal results on characterizations of error bounds and on the error bound modulus by dropping the convexity assumption.

Inspired by this issue, our goal in this article is to discuss error bounds of the inequality defined by a composite-convex function. Our work is to provide primal

characterizations of error bounds and the accurate estimate on the error bound modulus in terms of the notions of Hausdorff-Pompeiu excess of a set beyond another one, of the Bouligand tangent cone and of the lower Hadamard directional derivative. The paper is organized as follows. In Section 2, we give some definitions and preliminary results. In Section 3, we consider two concepts of metric regularity (of multifunctions) and the Shapiro first order contact property that are used in our analysis. Section 4 is devoted to the study of error bounds for the inequality defined by a composite-convex function. We first consider error bounds for a general inequality defined by a proper lower semicontinuous (not necessarily convex) function with the Shapiro first order contact property, and provide sufficient and/or necessary primal conditions of error bounds in terms of the Bouligand tangent cone, the lower Hadamard directional derivative and the Hausdorff-Pompeiu excess of subsets (see Theorems 4.1 and 4.2). When these results are applied to error bounds of composite-convex inequalities, the primal results on error bounds and on the error bound modulus can be obtained (see Theorems 4.4 and 4.5).

2. Preliminaries

Let $\mathbb{X}$ be a Banach space (or Euclidean space). Let $\mathbf{B}_{\mathbb{X}}$ denote the closed unit ball of $\mathbb{X}$. For $\bar{x} \in \mathbb{X}$ and $\delta > 0$, let $\mathbf{B}(\bar{x}, \delta)$ denote the open ball with center $\bar{x}$ and radius δ . For a subset Ω of $\mathbb{X}$, we denote by $\text{cl}(\Omega)$, $\text{int}(\Omega)$ and $\text{bd}(\Omega)$ the *closure*, the *interior* and the *boundary* of Ω , respectively.

Let A be a closed subset of $\mathbb{X}$ and $\bar{x} \in A$. We denote by

$$\mathbf{T}^{\mathbf{B}}(A, \bar{x}) := \text{Limsup}_{t \rightarrow 0^+} \frac{A - \bar{x}}{t}$$

the *Bouligand tangent cone* (also called *contingent cone*) of A at $\bar{x}$. Thus we have $v \in \mathbf{T}^{\mathbf{B}}(A, \bar{x})$ if and only if there exist a sequence $\{v_n\}$ in X converging to v and a sequence $\{t_n\}$ in $(0, +\infty)$ decreasing to 0 such that $\bar{x} + t_n v_n \in A$ for all $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all natural numbers.

For any subsets C and D of $\mathbb{X}$, the *excess* of C beyond D is defined as:

$$\mathbf{e}(C, D) := \sup_{x \in C} \mathbf{d}(x, D), \quad (1)$$

where $\mathbf{d}(x, D) := \inf\{\|x - y\| : y \in D\}$ and the convention is used that

$$\mathbf{e}(\emptyset, D) := \begin{cases} 0, & \text{if } D \neq \emptyset, \\ \infty, & \text{otherwise.} \end{cases} \quad (2)$$

Remark 2.1. For general subsets C and D , the excesses of $\mathbf{e}(C, D)$ and $\mathbf{e}(D, C)$ may not be necessarily equal. For example, let $\mathbb{X} := \mathbb{R}$, $C := [1, 3]$ and $D := [6, 7]$.

Then $\mathbf{e}(C, D) = 5$ and $\mathbf{e}(D, C) = 4$.

Further, the excess is used to define the Hausdorff distance $\mathbf{H}(C, D)$; that is,

$$\mathbf{H}(C, D) := \max\{\mathbf{e}(C, D), \mathbf{e}(D, C)\}.$$

We refer to the book [14] for more details on distances between subsets. $\square$

The following proposition provides a characterization for the Hausdorff-Pompeiu excess. We refer the reader to [14, Page 138] (or [58, Proposition 3.1]) for the proof in details.

Proposition 2.2. *Let C and D be subsets of $\mathbb{X}$. Then*

$$\mathbf{e}(C, D) = \inf\{\tau \geq 0 : C \subseteq D + \tau \mathbf{B}_{\mathbb{X}}\}. \quad (3)$$

Given an extended real-valued lower semicontinuous function $\varphi : \mathbb{X} \rightarrow \mathbb{R} \cup \{+\infty\}$, we denote by $\text{dom}(\varphi) := \{u \in \mathbb{X} : \varphi(u) < +\infty\}$ its *domain* and by

$$\text{epi}(\varphi) := \{(x, \alpha) \in \mathbb{X} \times \mathbb{R} : \varphi(x) \leq \alpha\}$$

its *epigraph*. φ is said to be *proper* if its domain is nonempty.

The *lower Hadamard directional derivative* of φ at x along direction h is defined as

$$\varphi'_{\mathbf{H}}(x; h) := \liminf_{t \rightarrow 0^+, h' \rightarrow h} \frac{\varphi(x + th') - \varphi(x)}{t}. \quad (4)$$

We close this section with the following lemma cited from [53, Theorem 4.1].

Lemma 2.3. *Let $\mathbb{X}$ be a Banach space and Ω be a nonempty closed subset of $\mathbb{X}$. Let $\gamma \in (0, 1)$. Then for any $x \notin \Omega$ there exists $z \in \Omega$ such that*

$$\gamma \|x - z\| < \min\{\mathbf{d}(x, \Omega), \mathbf{d}(x - z, \mathbf{T}^{\mathbf{B}}(\Omega, z))\}.$$

In particular, if $\mathbb{X}$ is of finite dimension, z can be chosen as the projection of x onto Ω .

3. Metric regularity and the Shapiro first order contact property

This section recalls the two important concepts of *metric regularity* and of *Shapiro first order contact property* that are used in our analysis. It is known that metric regularity of multifunctions occurs to be closely related to Lipschitzian properties of inverse mappings. This well-known and significant property is an extension of surjectivity to nonlinear/set-valued mappings and goes back to the Banach-Schauder open mapping theorem and to the Lyusternik-Graves theorem. We first recall the definition of metric regularity.

Definition 3.1. (i) Let $F : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a multifunction between two Banach spaces and $\bar{x} \in \mathbb{X}$. Recall that F is said to be *metrically regular* at $\bar{x}$ for $\bar{y} \in F(\bar{x})$, if there exists a constant $\kappa \in (0, +\infty)$ along with neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$\mathbf{d}(x, F^{-1}(y)) \leq \kappa \mathbf{d}(y, F(x)) \quad \text{for all } (x, y) \in U \times V. \quad (5)$$

(ii) A single-value mapping $\Psi : \mathbb{X} \rightarrow \mathbb{Y}$ is said to be *metrically regular* at $\bar{x}$ if Ψ is metrically regular at $\bar{x}$ for $\Psi(\bar{x})$. $\square$

It is known from [15, 24, 37, 46, 54] that metric regularity of a multifunction is proved to be equivalent to the covering property. We refer for instance the readers to [37, Theorem 1.52] for more details on this equivalence relationships and the modulus estimates for metric regularity and the covering property. Further the readers are

invited to consult [22, 23] for a survey and the development on metric regularity and [43] for some regular properties of graphical tangent and normal cones to paraconvex multifunctions.

From [37, Lemma 1.56], we always have the automatic closedness of the derivative image for single-valued metrically regular mappings. The following proposition, as a result of independent interest, weakens the assumption of metric regularity therein.

Proposition 3.2. *Let $\varphi : \mathbb{X} \rightarrow \mathbb{Y}$ be a mapping between two Banach spaces and $\bar{x} \in \mathbb{X}$. Suppose that φ is Fréchet differentiable at $\bar{x}$ and that there exist $\kappa, r > 0$ such that*

$$\mathbf{d}(\bar{x}, \varphi^{-1}(y)) \leq \kappa \|\varphi(\bar{x}) - y\| \quad \forall y \in \mathbb{Y} : \|y - \varphi(\bar{x})\| < r. \quad (6)$$

Then $\nabla\varphi(\bar{x})(\mathbb{X})$ is a closed subspace of $\mathbb{Y}$.

Proof. Let $y_0 \in \text{cl}(\nabla\varphi(\bar{x})(\mathbb{X}))$. Then we can find a sequence $\{y_k\}$ in $\nabla\varphi(\bar{x})(\mathbb{X})$ such

$$\text{that} \quad y_k \rightarrow y_0 \quad \text{and} \quad \|y_{k+1} - y_k\| < \frac{1}{2^k} \quad \text{for all } k. \quad (7)$$

We claim that there exists a sequence $\{x_k\}$ in X such that

$$\|x_{k+1} - x_k\| < \frac{3\kappa}{2^k} \quad \text{and} \quad \|y_k - \nabla\varphi(\bar{x})(x_k)\| < \frac{1}{2^k} \quad \text{for all } k. \quad (8)$$

Granting this, one has $\{x_k\}$ is a Cauchy sequence in $\mathbb{X}$ that converges to some $x_0 \in \mathbb{X}$ and it follows from (8) that $\nabla\varphi(\bar{x})(x_k) \rightarrow y_0$, which gives $\nabla\varphi(\bar{x})(x_0) = y_0$.

We define x_k iteratively. Let $x_1 \in \mathbb{X}$ be such that $\nabla\varphi(\bar{x})(x_1) = y_1$. Suppose $x_1, \dots, x_k$ have been given to satisfy (8) and construct x_{k+1} as follows.

Choose $u_{k+1} \in \nabla\varphi(\bar{x})^{-1}(y_{k+1}) - x_k$. Let $\varepsilon > 0$ be sufficiently small such that

$$3\varepsilon\kappa < \frac{1}{2^2}, \quad \varepsilon < \frac{1}{2^{k+2}} \quad \text{and} \quad \varepsilon\|u_{k+1}\| < \frac{1}{2^{k+2}}. \quad (9)$$

Since φ is Fréchet differentiable at $\bar{x}$, then there exists $\delta \in (0, r)$ such that

$$\|\varphi(\bar{x} + w) - \varphi(\bar{x}) - \nabla\varphi(\bar{x})(w)\| < \varepsilon\|w\| \quad \forall w : \|w\| < \delta. \quad (10)$$

Take $t_k > 0$ sufficiently small such that

$$\max \left\{ \frac{3t_k}{2^k}, \frac{3\kappa}{2^k}t_k, t_k\|u_{k+1}\| \right\} < \delta.$$

Then by (9) and (10), one has

$$\begin{aligned} & \|\varphi(\bar{x} + t_k u_{k+1}) - \varphi(\bar{x})\| \\ & \leq \|\varphi(\bar{x} + t_k u_{k+1}) - \varphi(\bar{x}) - \nabla\varphi(\bar{x})(t_k u_{k+1})\| + \|t_k \nabla\varphi(\bar{x})(u_{k+1})\| \\ & < \varepsilon t_k \|u_{k+1}\| + t_k (\|y_{k+1} - y_k\| + \|y_k - \nabla\varphi(\bar{x})(x_k)\|) \\ & < t_k \left(\frac{1}{2^{k+2}} + \frac{1}{2^k} + \frac{1}{2^k} \right) < \frac{3t_k}{2^k} < \delta < r. \end{aligned}$$

This and (6) imply that

$$\mathbf{d}(\bar{x}, \varphi^{-1}(\varphi(\bar{x} + t_k u_{k+1}))) \leq \kappa \|\varphi(\bar{x} + t_k u_{k+1}) - \varphi(\bar{x})\| < \frac{3\kappa}{2^k} t_k$$

and thus there is $w_k \in \varphi^{-1}(\varphi(\bar{x} + t_k u_{k+1}))$ such that

$$\|\bar{x} - w_k\| < \frac{3\kappa}{2^k} t_k.$$

Let $v_k := \frac{w_k - \bar{x}}{t_k}$ and $x_{k+1} := x_k + v_k$. Then

$$\|x_{k+1} - x_k\| = \|v_k\| < \frac{3\kappa}{2^k}.$$

To complete the proof, it remains to show that

$$\|y_{k+1} - \nabla\varphi(\bar{x})(x_{k+1})\| < \frac{1}{2^k}. \quad (11)$$

Combining (9) with (10), one has

$$\|\varphi(\bar{x} + t_k v_k) - \varphi(\bar{x}) - \nabla\varphi(\bar{x})(t_k v_k)\| < \varepsilon \|t_k v_k\| < \varepsilon t_k \frac{3\kappa}{2^k} < \frac{t_k}{2^{k+2}} \quad (12)$$

$$\text{and} \quad \|\varphi(\bar{x} + t_k u_{k+1}) - \varphi(\bar{x}) - \nabla\varphi(\bar{x})(t_k u_{k+1})\| < \varepsilon \|t_k u_{k+1}\| < \frac{t_k}{2^{k+2}}. \quad (13)$$

Note that $\varphi(\bar{x} + t_k v_k) = \varphi(\bar{x} + t_k u_{k+1})$ and it follows from (12) and (13) that

$$\|\nabla\varphi(\bar{x})(v_k) - \nabla\varphi(\bar{x})(u_{k+1})\| < \frac{1}{2^{k+2}} + \frac{1}{2^{k+2}} = \frac{1}{2^{k+1}}.$$

This means that

$$\|y_{k+1} - \nabla\varphi(\bar{x})(x_{k+1})\| = \|\nabla\varphi(\bar{x})(u_{k+1} + x_k) - \nabla\varphi(\bar{x})(x_k + v_k)\| < \frac{1}{2^{k+1}}.$$

The proof is complete. $\square$

As an appropriate substitute of convexity, we consider a tangential concept introduced in [51, Definition 2.1] under the name of $o(p)$ -convexity and known in the literature as the Shapiro p -order contact property.

Definition 3.3. Let A be a closed subset of $\mathbb{X}$ and $p \in \mathbb{N}$. Recall that A is said to have the p -order Shapiro contact property at $a \in A$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mathbf{d}(x - u, \mathbf{T}^{\mathbf{B}}(A, u)) \leq \varepsilon \|x - u\|^p \quad \forall x, u \in A \cap \mathbf{B}(a, \delta). \quad (14)$$

In the remainder of this paper, we will use the terminology Shapiro first order contact property in place of 1-order Shapiro contact property.

In [3], Aussel, Daniilidis and Thibault established the links between the Shapiro first order contact property and the notions of subsmoothness and semismoothness. We refer the reader to [3, Theorem 3.16] and to Thibault's book [54, subsection 8.3.2] for more details. In 2019 Shen, Yao and Zheng [53] studied the Shapiro property

and the C-Shapiro property, and showed that the Shapiro property is an extension of convexity and smoothness (see [53, Propositions 3.8 and 3.9]). Further, they considered the Shapiro property of a general multifunction and used it to study calmness for closed multifunctions. Recently, inspired by the Shapiro property of a multifunction, the authors [58] consider the epigraphical Shapiro property of a function; that is,

Definition 3.4. Let $\varphi : \mathbb{X} \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper function and $\bar{x} \in \text{dom}(\varphi)$. We say that φ has the epigraphical Shapiro first order contact property at $\bar{x}$, if the epigraph $\text{epi}(\varphi)$ has the Shapiro first order contact property at $(\bar{x}, \varphi(\bar{x}))$. $\square$

The following proposition is on the epigraphical Shapiro property which was proved in [58, Proposition 2.1].

Proposition 3.5. Let $\varphi : \mathbb{X} \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper extended real-valued lower semicontinuous function and $\bar{x} \in \text{dom}(\varphi)$. Consider the following statements:

- (i) φ has the epigraphical Shapiro first order contact property at $\bar{x}$;
- (ii) for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mathbf{d}((x-u, \varphi(x)-\varphi(u)), \mathbf{T}^{\mathbf{B}}(\text{epi}(\varphi), (u, \varphi(u)))) \leq \varepsilon(\|x-u\| + |\varphi(x)-\varphi(u)|) \quad (15)$$
 holds for all $x, u \in \mathbf{B}_{\varphi}(\bar{x}, \delta)$;
- (iii) for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varphi'_{\mathbf{H}}(u; x-u) \leq \varphi(x) - \varphi(u) + \varepsilon(\|x-u\| + |\varphi(x) - \varphi(u)|) \quad (16)$$
 holds for all $x, u \in \mathbf{B}_{\varphi}(\bar{x}, \delta)$,

where $\mathbf{B}_{\varphi}(\bar{x}, \delta) := \{x \in \mathbf{B}(\bar{x}, \delta) : |\varphi(x) - \varphi(\bar{x})| < \delta\}$.

Then (i) $\Rightarrow$ (ii) $\Leftarrow$ (iii). Further, assume that φ is continuous around $\bar{x}$. Then (i) $\Leftrightarrow$ (ii).

Remark 3.6. It is noted that the implication of (ii) $\Rightarrow$ (iii) or (ii) $\Rightarrow$ (i) in Proposition 3.5 may not be necessarily true. The readers are invited to consult the counterexample in [58, Remark 2.1] for more details. $\square$

The following result was proved by Shapiro and Al-Khayyal in [52].

Proposition 3.7. Let $\varphi : \mathbb{X} \rightarrow \mathbb{Y}$ be a continuously differentiable mapping and A be a closed convex cone of $\mathbb{Y}$. Suppose that $\bar{x} \in \varphi^{-1}(A)$ satisfies the following Robinson qualification:

$$0 \in \text{int}(\varphi(\bar{x}) + \nabla\varphi(\bar{x})\mathbb{X} - A).$$

Then $\varphi^{-1}(A)$ has the Shapiro first order contact property at $\bar{x}$.

The following proposition improves Proposition 3.7 by weakening the Robinson qualification. This proposition is a key tool to prove main results in the paper.

Proposition 3.8. Let $\varphi : \mathbb{X} \rightarrow \mathbb{Y}$ be a continuously differentiable mapping, A be a closed subset of $\mathbb{Y}$ and $\bar{x} \in \varphi^{-1}(A)$. Suppose that A has the Shapiro first order contact property at $\varphi(\bar{x})$ and φ is metrically regular around $\bar{x}$. Then $\varphi^{-1}(A)$ has the Shapiro first order contact property at $\bar{x}$.

To prove Proposition 3.8, we need the following two lemmata which are of independent interest.

Lemma 3.9. *Let $\varphi : \mathbb{X} \rightarrow \mathbb{Y}$ and $\bar{x} \in \mathbb{X}$. Suppose that φ is Fréchet differentiable at $\bar{x}$ and image $\nabla\varphi(\bar{x})(\mathbb{X})$ is of the second category. Then there exists $\mu_0 > 0$ such that*

$$\mathbf{d}(u, \nabla\varphi(\bar{x})^{-1}(v)) \leq \mu_0 \|\nabla\varphi(\bar{x})(u) - v\| \quad \forall (u, v) \in \mathbb{X} \times \mathbb{Y}. \quad (17)$$

Further, assume that φ is continuously differentiable at $\bar{x}$. Then there exist $\mu, \delta > 0$ such that

$$\mathbf{d}(u, \nabla\varphi(x)^{-1}(v)) \leq \mu \|\nabla\varphi(x)(u) - v\| \quad \forall (x, u, v) \in \mathbf{B}(\bar{x}, \delta) \times \mathbb{X} \times \mathbb{Y}. \quad (18)$$

Proof. Since $\nabla\varphi(\bar{x})(\mathbb{X})$ is of the second category, it follows from the Open Mapping Theorem that there exists $l > 0$ such that

$$2l\mathbf{B}_{\mathbb{Y}} \subseteq \nabla\varphi(\bar{x})(\mathbf{B}_{\mathbb{X}}). \quad (19)$$

Let $(u, v) \in \mathbb{X} \times \mathbb{Y}$. By virtue of (19), one has

$$\nabla\varphi(\bar{x})(u) - v \in \|\nabla\varphi(\bar{x})(u) - v\|\mathbf{B}_{\mathbb{Y}} \subseteq \nabla\varphi(\bar{x})\left(\frac{\|\nabla\varphi(\bar{x})(u) - v\|}{2l}\mathbf{B}_{\mathbb{X}}\right). \quad (20)$$

Note that $\nabla\varphi(\bar{x})^{-1}(\nabla\varphi(\bar{x})(u) - v) = u - \nabla\varphi(\bar{x})^{-1}(v)$ by the linearity of $\nabla\varphi(\bar{x})$ and then (20) gives that

$$\begin{aligned} \mathbf{d}(u, \nabla\varphi(\bar{x})^{-1}(v)) &= \mathbf{d}(0, u - \nabla\varphi(\bar{x})^{-1}(v)) = \mathbf{d}(0, \nabla\varphi(\bar{x})^{-1}(\nabla\varphi(\bar{x})(u) - v)) \\ &\leq \frac{1}{2l} \|\nabla\varphi(\bar{x})(u) - v\|. \end{aligned}$$

This means that (17) holds with $\mu_0 := \frac{1}{2l}$.

To prove (18), we next show that there exists $\delta > 0$ such that

$$l\mathbf{B}_{\mathbb{Y}} \subseteq \nabla\varphi(x)(\mathbf{B}_{\mathbb{X}}) \quad \forall x \in \mathbf{B}(\bar{x}, \delta). \quad (21)$$

Noting that φ is continuously differentiable at $\bar{x}$, then there exists $\delta > 0$ such that φ is Fréchet differentiable on $\mathbf{B}(\bar{x}, \delta)$ and

$$\|\nabla\varphi(x) - \nabla\varphi(\bar{x})\| < \frac{l}{2} \quad \forall x \in \mathbf{B}(\bar{x}, \delta) \quad (22)$$

(taking a smaller δ if necessary). Then by (19) and (22), one has

$$2l\mathbf{B}_{\mathbb{Y}} \subseteq (\nabla\varphi(x) + \nabla\varphi(\bar{x}) - \nabla\varphi(x))(\mathbf{B}_{\mathbb{X}}) \subseteq \nabla\varphi(x)(\mathbf{B}_{\mathbb{X}}) + \frac{l}{2}\mathbf{B}_{\mathbb{Y}} \quad \forall x \in \mathbf{B}(\bar{x}, \delta).$$

By the Rådström cancellation lemma [48, Lemma 2.3], this gives that

$$\frac{3l}{2}\mathbf{B}_{\mathbb{Y}} \subseteq \text{cl}(\nabla\varphi(x)(\mathbf{B}_{\mathbb{X}})) \quad \forall x \in \mathbf{B}(\bar{x}, \delta). \quad (23)$$

Applying [27, P.183, Theorem A.1] again gives that $\nabla\varphi(x)(\mathbf{B}_{\mathbb{X}})$ and $\text{cl}(\nabla\varphi(x)(\mathbf{B}_{\mathbb{X}}))$ have the same interior and thus that (21) holds by (23). The proof is complete. $\square$

Lemma 3.10. *Let $\varphi : \mathbb{X} \rightarrow \mathbb{Y}$ be a mapping and A be a closed subset of $\mathbb{Y}$. Suppose that $\bar{x} \in \varphi^{-1}(A)$ is such that φ is Fréchet differentiable at $\bar{x}$ and metrically regular around $\bar{x}$. Then there exists $\delta > 0$ such that*

$$\mathbf{T}^{\mathbf{B}}(\varphi^{-1}(A), x) = \nabla\varphi(x)^{-1}(\mathbf{T}^{\mathbf{B}}(A, \varphi(x))) \quad \forall x \in \mathbf{B}(\bar{x}, \delta) \cap \varphi^{-1}(A). \quad (24)$$

Proof. Since φ is metrically regular around $\bar{x}$, there exist $\kappa > 0$ along with neighborhoods U of $\bar{x}$ and V of $\varphi(\bar{x})$ such that

$$\mathbf{d}(x, \varphi^{-1}(y)) \leq \kappa \|\varphi(x) - y\| \quad \forall (x, y) \in U \times V. \quad (25)$$

Take $\delta > 0$ sufficiently small such that

$$\mathbf{B}(\bar{x}, \delta) \subseteq U \quad \text{and} \quad \varphi(\mathbf{B}(\bar{x}, \delta)) \subseteq V. \quad (26)$$

Let $x \in \mathbf{B}(\bar{x}, \delta) \cap \varphi^{-1}(A)$. Choose any $u \in \nabla\varphi(x)^{-1}(\mathbf{T}^{\mathbf{B}}(A, \varphi(x)))$. Then we have $\nabla\varphi(x)(u) \in \mathbf{T}^{\mathbf{B}}(A, \varphi(x))$ and thus there exist $t_k \rightarrow 0^+$ and $v_k \rightarrow \nabla\varphi(x)(u)$ such that

$$\varphi(x) + t_k v_k \in A \quad \text{for all } k.$$

By virtue of (25) and (26), for any k sufficiently large, one has

$$\mathbf{d}(x + t_k u, \varphi^{-1}(\varphi(x) + t_k v_k)) \leq \kappa \|\varphi(x + t_k u) - \varphi(x) - t_k v_k\|$$

and consequently there is $x_k \in \varphi^{-1}(\varphi(x) + t_k v_k) \subseteq \varphi^{-1}(A)$ such that

$$\|x + t_k u - x_k\| < 2\kappa \|\varphi(x + t_k u) - \varphi(x) - t_k v_k\|. \quad (27)$$

Denote $u_k := \frac{x_k - x}{t_k}$ for all k . Note that

$$\varphi(x + t_k u) = \varphi(x) + \nabla\varphi(x)(t_k u) + o(t_k)$$

and then (27) gives that

$$\|u_k - u\| < 2\kappa \left\| \nabla\varphi(x)(u) - v_k + \frac{o(t_k)}{t_k} \right\|,$$

which implies that $u_k \rightarrow u$ as $v_k \rightarrow \nabla\varphi(x)(u)$. Noting that $x + t_k u_k = x_k \in \varphi^{-1}(A)$, it follows that $u \in \mathbf{T}^{\mathbf{B}}(\varphi^{-1}(A), x)$ and thus

$$\mathbf{T}^{\mathbf{B}}(\varphi^{-1}(A), x) \supseteq \nabla\varphi(x)^{-1}(\mathbf{T}^{\mathbf{B}}(A, \varphi(x))). \quad (28)$$

On the other hand, let $h \in \mathbf{T}^{\mathbf{B}}(\varphi^{-1}(A), x)$. Then there exist $t_k \rightarrow 0^+$ and $h_k \rightarrow h$ such that $x + t_k h_k \in \varphi^{-1}(A)$ for all k . Using the Fréchet differentiability of φ at x , one has

$$\varphi(x + t_k h_k) = \varphi(x) + \nabla\varphi(x)(t_k h_k) + o(\|t_k h_k\|).$$

Denote $v_k := \nabla\varphi(x)(h_k) + \frac{o(\|t_k h_k\|)}{t_k}$ for all k .

Then $v_k \rightarrow \nabla\varphi(x)(h)$ and $\varphi(x) + t_k v_k = \varphi(x + t_k h_k) \in A$, which implies that $\nabla\varphi(x)(h) \in \mathbf{T}^{\mathbf{B}}(A, \varphi(x))$. Hence

$$\mathbf{T}^{\mathbf{B}}(\varphi^{-1}(A), x) \subseteq \nabla\varphi(x)^{-1}(\mathbf{T}^{\mathbf{B}}(A, \varphi(x)))$$

and (24) holds by (28). The proof is complete. $\square$

Proof of Proposition 3.8. Since φ is continuously differentiable and metrically regular around $\bar{x}$, it follows from [37, Theorem 1.57] that $\nabla\varphi(\bar{x})$ is surjective. By virtue of Lemmas 3.9 and 3.10, there exist $\mu, \delta > 0$ such that (18) and (24) hold.

Let $\varepsilon > 0$. Since φ is continuously differentiable at $\bar{x}$, there exists $\delta_1 \in (0, \delta)$ such

$$\text{that} \quad \|\varphi(x) - \varphi(u) - \nabla\varphi(u)(x - u)\| < \frac{\varepsilon}{2\mu} \|x - u\| \quad \forall x, u \in \mathbf{B}(\bar{x}, \delta_1) \quad (29)$$

$$\text{and} \quad \|\varphi(x) - \varphi(u)\| \leq K\|x - u\| \quad \forall x, u \in \mathbf{B}(\bar{x}, \delta_1) \quad (30)$$

where $K := \|\nabla\varphi(\bar{x})\| + 1$.

Noting A has the Shapiro first order contact property at $\varphi(\bar{x})$, it follows that there exists $\delta_2 \in (0, \delta_1)$ such that

$$\mathbf{d}(y - v, \mathbf{T}^{\mathbf{B}}(A, v)) \leq \frac{\varepsilon}{2K\mu} \|y - v\| \quad \forall y, v \in A \cap \mathbf{B}(\varphi(\bar{x}), \delta_2). \quad (31)$$

Take $\delta_3 \in (0, \delta_2)$ such that $\varphi(\mathbf{B}(\bar{x}, \delta_3)) \subseteq \mathbf{B}(\varphi(\bar{x}), \delta_2)$. Let $x, u \in \varphi^{-1}(A) \cap \mathbf{B}(\bar{x}, \delta_3)$. Take a sequence $\{v_n\}$ in $\mathbf{T}^{\mathbf{B}}(A, \varphi(u))$ such that

$$\|\varphi(x) - \varphi(u) - v_n\| \rightarrow \mathbf{d}(\varphi(x) - \varphi(u), \mathbf{T}^{\mathbf{B}}(A, \varphi(u))).$$

By virtue of (18) and (24), one has

$$\begin{aligned} \mathbf{d}(x - u, \mathbf{T}^{\mathbf{B}}(\varphi^{-1}(A), u)) &= \mathbf{d}(x - u, \nabla\varphi(u)^{-1}(\mathbf{T}^{\mathbf{B}}(A, \varphi(u)))) \\ &\leq \mathbf{d}(x - u, \nabla\varphi(u)^{-1}(v_n)) \leq \mu\|\nabla\varphi(u)(x - u) - v_n\| \\ &\leq \mu(\|\nabla\varphi(u)(x - u) + \varphi(u) - \varphi(x)\| + \|\varphi(x) - \varphi(u) - v_n\|). \end{aligned}$$

By letting $n \rightarrow \infty$, one gets

$$\begin{aligned} \mathbf{d}(x - u, \mathbf{T}^{\mathbf{B}}(\varphi^{-1}(A), u)) \\ \leq \mu(\|\varphi(x) - \varphi(u) - \nabla\varphi(u)(x - u)\| + \mathbf{d}(\varphi(x) - \varphi(u), \mathbf{T}^{\mathbf{B}}(A, \varphi(u)))). \end{aligned}$$

This together with (29), (30) and (31) gives

$$\begin{aligned} \mathbf{d}(x - u, \mathbf{T}^{\mathbf{B}}(\varphi^{-1}(A), u)) \\ \leq \mu(\|\varphi(x) - \varphi(u) - \nabla\varphi(u)(x - u)\| + \mathbf{d}(\varphi(x) - \varphi(u), \mathbf{T}^{\mathbf{B}}(A, \varphi(u)))) \\ \leq \mu\left(\frac{\varepsilon}{2\mu}\|x - u\| + \frac{\varepsilon}{2K\mu}\|\varphi(x) - \varphi(u)\|\right) \leq \mu\left(\frac{\varepsilon}{2\mu}\|x - u\| + \frac{\varepsilon}{2K\mu}K\|x - u\|\right) \\ = \varepsilon\|x - u\|. \end{aligned}$$

Hence $\varphi^{-1}(A)$ has the Shapiro first order contact property at $\bar{x}$. The proof is complete. $\square$

4. Main results

In this section, we study error bounds of the inequality defined by a general function and aim to provide primal criteria of error bounds via Bouligand tangent cones, lower Hadamard directional derivatives and the Hausdorff-Pompeiu excess of subsets. Then we apply these primal results to error bounds of the composite-convex

function, and establish primal characterizations of error bounds and an accurate estimate of the error bound modulus in terms of Bouligand tangent cones, directional derivatives of convex functions and the Hausdorff-Pompeiu excess. We first consider error bounds of the inequality defined by a general lower semicontinuous function.

Let $\varphi : \mathbb{X} \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous extended real-valued function. We consider the following inequality:

$$\varphi(x) \leq 0. \quad (32)$$

We denote by $\mathbf{S}_\varphi := \{x \in \mathbb{X} : \varphi(x) \leq 0\}$ the solution set. Recall that inequality (32) is said to have a local error bound at $\bar{x} \in \mathbf{S}_\varphi$, if there exist $\tau, \delta \in (0, +\infty)$ such that

$$\mathbf{d}(x, \mathbf{S}_\varphi) \leq \tau[\varphi(x)]_+, \quad \forall x \in \mathbf{B}(\bar{x}, \delta), \quad (33)$$

where $[\varphi(x)]_+ := \max\{\varphi(x), 0\}$. We denote by

$$\tau(\mathbf{S}_\varphi, \bar{x}) := \inf\{\tau > 0 : \text{there exists } \delta > 0 \text{ such that (33) holds}\} \quad (34)$$

the local error bound modulus of $\mathbf{S}_\varphi$ at $\bar{x}$.

The theory of error bounds has been a subject of intense study during many years due to its numerous applications in optimization and variational analysis. They include for instance areas like sensitivity analysis of linear programming, convergence analysis of descent methods, the feasibility problems and the domain of image reconstruction. This notion has been proved to have close connections with several notions in convex analysis and approximation like the basic constraint qualification (BCQ), the Abadie constraint qualification (ACQ), the strong conical hull intersection property (CHIP), the linear regularity and normal property, the metric subregularity as well as the calmness of multifunctions.

The following theorem gives necessary and/or sufficient criteria of local error bounds for the inequality (32) in terms of the Bouligand tangent cone, the lower Hadamard directional derivative and the Hausdorff-Pompeiu excess of a set beyond another set.

Theorem 4.1. *Let $\varphi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function and $\bar{x} \in \mathbf{S}_\varphi$.*

- (i) *Suppose that $\mathbf{S}_\varphi$ has the Shapiro first order contact property around $\bar{x}$ and inequality (32) has a local error bound at $\bar{x}$. Then there exist $\tau, r \in (0, +\infty)$ such that*

$$\mathbf{e}(\varphi'_H(x, \cdot)^{-1}(-\infty, 1], \mathbf{T}^B(\mathbf{S}_\varphi, x)) \leq \tau \quad (35)$$

holds for all $x \in \mathbf{S}_\varphi \cap \mathbf{B}(\bar{x}, r)$ with $\varphi(x) = 0$.

- (ii) *Suppose that $\text{bd}(\mathbf{S}_\varphi) \subseteq \varphi^{-1}(0)$, φ has the epigraphical Shapiro first order contact property at $\bar{x}$ and that there exist $\tau, r \in (0, +\infty)$ such that (35) holds for all $x \in \text{bd}(\mathbf{S}_\varphi) \cap \mathbf{B}(\bar{x}, r)$. Then inequality (32) has a local error bound at $\bar{x}$.*

Proof. (i) By the local error bound of inequality (32) at $\bar{x}$, there exist $\tau, \delta \in (0, +\infty)$ such that (33) holds. Note that $\mathbf{S}_\varphi$ has the Shapiro first order contact property around $\bar{x}$ and thus there exists $r \in (0, \delta)$ such that $\mathbf{S}_\varphi$ has the Shapiro first order contact property on $\mathbf{S}_\varphi \cap \mathbf{B}(\bar{x}, r)$.

Let $x \in \mathbf{S}_\varphi \cap \mathbf{B}(\bar{x}, r)$ with $\varphi(x) = 0$. Take any $h \in \mathbb{X}$ such that $\varphi'_\mathbf{H}(x, h) \leq 1$. We need to show that

$$\mathbf{d}(h, \mathbf{T}^\mathbf{B}(\mathbf{S}_\varphi, x)) \leq \tau. \quad (36)$$

Note that $\varphi'_\mathbf{H}(x, h) \leq 1$ and then there exist $t_n \rightarrow 0^+$ and $h_n \rightarrow h$ such that

$$\frac{\varphi(x + t_n h_n) - \varphi(x)}{t_n} < 1 + \frac{1}{n}. \quad (37)$$

If there exists a subsequence $\{n_k\} \subseteq \mathbb{N}$ such that $\varphi(x + t_{n_k} h_{n_k}) \leq 0$ for all k , then one has $h \in \mathbf{T}^\mathbf{B}(\mathbf{S}_\varphi, x)$ and thus (36) holds by $\mathbf{d}(h, \mathbf{T}^\mathbf{B}(\mathbf{S}_\varphi, x)) = 0$.

We next consider the case that $\varphi(x + t_n h_n) > 0$ for all n . Since $\mathbf{S}_\varphi$ has the Shapiro first order contact property at x , by [53, Proposition 3.4], for any $k \in \mathbb{N}$, there exist $r_k \in (0, +\infty)$ such that $\mathbf{B}(x, r_k) \subseteq \mathbf{B}(\bar{x}, \delta)$ and

$$\mathbf{d}(u - x, \mathbf{T}^\mathbf{B}(\mathbf{S}_\varphi, x)) \leq \mathbf{d}(u, \mathbf{S}_\varphi) + \frac{1}{k} \|u - x\| \quad \forall u \in \mathbf{B}(x, r_k). \quad (38)$$

Take a subsequence $\{n_k\} \subseteq \mathbb{N}$ such that $\|t_{n_k} h_{n_k}\| < \frac{1}{k}$. Then by virtue of (33), (37) and (38), one has

$$\begin{aligned} \mathbf{d}(x + t_{n_k} h_{n_k} - x, \mathbf{T}^\mathbf{B}(\mathbf{S}_\varphi, x)) &\leq \mathbf{d}(x + t_{n_k} h_{n_k}, \mathbf{S}_\varphi) + \frac{1}{k} t_{n_k} \|h_{n_k}\| \\ &\leq \tau \varphi(x + t_{n_k} h_{n_k}) + \frac{1}{k} t_{n_k} \|h_{n_k}\| < \tau \left(1 + \frac{1}{n_k}\right) t_{n_k} + \frac{1}{k} t_{n_k} \|h_{n_k}\| \end{aligned}$$

thanks to $\varphi(x) = 0$. This implies that

$$\mathbf{d}(h_{n_k}, \mathbf{T}^\mathbf{B}(\mathbf{S}_\varphi, x)) \leq \tau \left(1 + \frac{1}{n_k}\right) + \frac{1}{k} \|h_{n_k}\|$$

and consequently (36) holds by letting $k \rightarrow \infty$.

(ii) Let $\varepsilon > 0$ such that $(1 + \tau)\varepsilon < 1$. Since φ has the Shapiro first order contact property at $\bar{x}$, then there exists $\delta_1 \in (0, r)$ such that

$$\mathbf{d}((x - u, \alpha - \beta), \mathbf{T}^\mathbf{B}(\text{epi}(\varphi), (u, \beta))) < \varepsilon (\|x - u\| + \|\alpha - \beta\|) \quad (39)$$

holds for all $(x, \alpha), (u, \beta) \in \text{epi}(\varphi) \cap (\mathbf{B}(\bar{x}, \delta_1) \times (\varphi(\bar{x}) - \delta_1, \varphi(\bar{x}) + \delta_1))$.

Take $\delta \in (0, \frac{\delta_1}{2})$ and let $x \in \mathbf{B}(\bar{x}, \delta) \setminus \mathbf{S}_\varphi$. Then $\mathbf{d}(x, \mathbf{S}_\varphi) \leq \|x - \bar{x}\| < \delta$. Choose any $\gamma \in (0, 1)$ such that

$$\gamma > \max \left\{ \frac{\mathbf{d}(x, \mathbf{S}_\varphi)}{\delta}, (1 + \tau)\varepsilon \right\}. \quad (40)$$

By using Lemma 2.3, there exists $a \in \text{bd}(\mathbf{S}_\varphi)$ such that

$$\gamma \|x - a\| \leq \min \{ \mathbf{d}(x - a, \mathbf{T}^\mathbf{B}(\mathbf{S}_\varphi, a)), \mathbf{d}(x, \mathbf{S}_\varphi) \}. \quad (41)$$

Then $\|\bar{x} - a\| \leq \|\bar{x} - x\| + \|x - a\| < \delta + \frac{\mathbf{d}(x, \mathbf{S}_\varphi)}{\gamma} < 2\delta < \delta_1$

and it follows from (39) that

$$\mathbf{d}((x - a, \varphi(x) - \varphi(a)), \mathbf{T}^{\mathbf{B}}(\text{epi}(\varphi), (a, \varphi(a)))) < \varepsilon(\|x - a\| + |\varphi(x) - \varphi(a)|).$$

Thus, there is $(h, \alpha) \in \mathbf{T}^{\mathbf{B}}(\text{epi}(\varphi), (a, \varphi(a))) = \text{epi}(\varphi'_{\mathbf{H}}(a, \cdot))$ such that

$$\varphi'_{\mathbf{H}}(a, h) \leq \alpha \quad \text{and} \quad \|x - a - h\| + |\varphi(x) - \varphi(a) - \alpha| < \varepsilon(\|x - a\| + |\varphi(x) - \varphi(a)|). \quad (42)$$

Case 1: $\alpha = 0$. Then for any $\lambda > 0$, one has $\varphi'_{\mathbf{H}}(a, \lambda h) \leq 0$ and then (35) implies that

$$\mathbf{d}(\lambda h, \mathbf{T}^{\mathbf{B}}(\mathbf{S}_{\varphi}, a)) \leq 1.$$

This means $h \in \mathbf{T}^{\mathbf{B}}(\mathbf{S}_{\varphi}, a)$ by letting $\lambda \rightarrow +\infty$. By virtue of (41) and (42), one has

$$\begin{aligned} \gamma\|x - a\| \leq \mathbf{d}(x - a, \mathbf{T}^{\mathbf{B}}(\mathbf{S}_{\varphi}, a)) &\leq \|x - a - h\| < \varepsilon(\|x - a\| + |\varphi(x) - \varphi(a)|) \\ &= \varepsilon\|x - a\| + \varepsilon\varphi(x) \end{aligned}$$

the equality follows by $\varphi(x) > 0$ and $\varphi(a) = 0$. This implies that

$$(\gamma - \varepsilon)\mathbf{d}(x, \mathbf{S}_{\varphi}) \leq (\gamma - \varepsilon)\|x - a\| < \varepsilon\varphi(x).$$

By letting $\gamma \rightarrow 1^-$, one has

$$\mathbf{d}(x, \mathbf{S}_{\varphi}) \leq \frac{\varepsilon}{1 - \varepsilon} \varphi(x). \quad (43)$$

Case 2: $\alpha \neq 0$. Note that $\varphi_{\mathbf{H}}H'(a, \cdot)$ is positively homogeneous and then by (42),

one has
$$\varphi'_{\mathbf{H}}\left(a, \frac{h}{|\alpha|}\right) \leq 1.$$

This and (35) imply that
$$\mathbf{d}\left(\frac{h}{|\alpha|}, \mathbf{T}^{\mathbf{B}}(\mathbf{S}_{\varphi}, a)\right) \leq \tau$$

and consequently
$$\mathbf{d}(h, \mathbf{T}^{\mathbf{B}}(\mathbf{S}_{\varphi}, a)) \leq \tau|\alpha|.$$

By virtue of (41) and (42), one has

$$\begin{aligned} \gamma\|x - a\| &\leq \mathbf{d}(x - a, \mathbf{T}^{\mathbf{B}}(\mathbf{S}_{\varphi}, a)) \\ &\leq \|x - a - h\| + \mathbf{d}(h, \mathbf{T}^{\mathbf{B}}(\mathbf{S}_{\varphi}, a)) \\ &\leq \|x - a - h\| + \tau(|\varphi(x) - \varphi(a) - \alpha| + |\varphi(x) - \varphi(a)|) \\ &< \varepsilon(\|x - a\| + |\varphi(x) - \varphi(a)|) + \tau(\varepsilon\|x - a\| + \varepsilon|\varphi(x) - \varphi(a)| + |\varphi(x) - \varphi(a)|) \end{aligned}$$

and thus

$$\begin{aligned} (\gamma - \varepsilon - \tau\varepsilon)\mathbf{d}(x, \mathbf{S}_{\varphi}) &\leq (\gamma - \varepsilon - \tau\varepsilon)\|x - a\| < (\tau + \varepsilon + \tau\varepsilon)|\varphi(x) - \varphi(a)| \\ &= (\tau + \varepsilon + \tau\varepsilon)\varphi(x) \end{aligned}$$

the equality holds by $\varphi(x) > 0$ and $\varphi(a) = 0$. By letting $\gamma \rightarrow 1^-$, one has

$$(1 - \varepsilon - \tau\varepsilon)\mathbf{d}(x, \mathbf{S}_{\varphi}) < (\tau + \varepsilon + \tau\varepsilon)\varphi(x)$$

and consequently
$$\mathbf{d}(x, \mathbf{S}_{\varphi}) \leq \frac{\tau + \varepsilon + \tau\varepsilon}{1 - \varepsilon - \tau\varepsilon} \varphi(x).$$

This and (43) imply that inequality (32) has the local error bound at $\bar{x}$ (with the constant $\max\{\frac{\varepsilon}{1 - \varepsilon}, \frac{\tau + \varepsilon + \tau\varepsilon}{1 - \varepsilon - \tau\varepsilon}\}$). The proof is complete. $\square$

The following theorem, immediate from (i) and (ii) in Theorem 4.1 provides an equivalent primal condition for the local error bound under the Shapiro first order contact property.

Theorem 4.2. *Let $\bar{x} \in \mathbf{S}_\varphi$. Suppose that $\text{bd}(\mathbf{S}_\varphi) \subseteq \varphi^{-1}(0)$, φ has the epigraphical Shapiro first order contact property at $\bar{x}$ and that $\mathbf{S}_\varphi$ has the Shapiro first order contact property around $\bar{x}$. Then inequality (32) has the local error bound at $\bar{x}$ if and only if*

$$\limsup_{x \xrightarrow{\text{bd}(\mathbf{S}_\varphi)} \bar{x}} \mathbf{e}(\varphi'_\mathbf{H}(x, \cdot)^{-1}(-\infty, 1], \mathbf{T}^\mathbf{B}(\mathbf{S}_\varphi, x)) < +\infty.$$

Further, one has the following accurate estimate for the local error bound modulus:

$$\tau(\mathbf{S}_\varphi, \bar{x}) = \limsup_{x \xrightarrow{\text{bd}(\mathbf{S}_\varphi)} \bar{x}} \mathbf{e}(\varphi'_\mathbf{H}(x, \cdot)^{-1}(-\infty, 1], \mathbf{T}^\mathbf{B}(\mathbf{S}_\varphi, x)). \quad (44)$$

Proof. Thanks to (i) and (ii) in Theorem 4.1, it suffices to prove (44). We denote

$$\alpha := \limsup_{x \xrightarrow{\text{bd}(\mathbf{S}_\varphi)} \bar{x}} \mathbf{e}(\varphi'_\mathbf{H}(x, \cdot)^{-1}(-\infty, 1], \mathbf{T}^\mathbf{B}(\mathbf{S}_\varphi, x)).$$

We first consider the case $\tau(\mathbf{S}_\varphi, \bar{x}) < +\infty$. Let $\tau \in (0, \tau(\mathbf{S}_\varphi, \bar{x}))$. Then by using the proof of (i) in Theorem 4.1, there exists $\delta > 0$ such that

$$\mathbf{e}(\varphi'_\mathbf{H}(x, \cdot)^{-1}(-\infty, 1], \mathbf{T}^\mathbf{B}(\mathbf{S}_\varphi, x)) \leq \tau \quad \forall x \in \mathbf{B}(\bar{x}, \delta) \cap \text{bd}(\mathbf{S}_\varphi).$$

This means that $\alpha \leq \tau$ and thus $\alpha \leq \tau(\mathbf{S}_\varphi, \bar{x})$ by letting $\tau \downarrow \tau(\mathbf{S}_\varphi, \bar{x})$.

Note that $\alpha < +\infty$ and for any $\nu > 0$ there exists $\delta > 0$ such that

$$\mathbf{e}(\varphi'_\mathbf{H}(x, \cdot)^{-1}(-\infty, 1], \mathbf{T}^\mathbf{B}(\mathbf{S}_\varphi, x)) \leq \alpha + \nu \quad \forall x \in \mathbf{B}(\bar{x}, \delta) \cap \text{bd}(\mathbf{S}_\varphi).$$

Applying the proof of (ii) in Theorem 4.1, for any $\varepsilon > 0$ sufficiently small, one has

$$\tau(\mathbf{S}_\varphi, \bar{x}) \leq \frac{(\alpha + \nu) + \varepsilon + (\alpha + \nu)\varepsilon}{1 - \varepsilon - (\alpha + \nu)\varepsilon}.$$

By letting $\varepsilon \downarrow 0$, one has

$$\tau(\mathbf{S}_\varphi, \bar{x}) \leq \alpha + \nu \quad \forall \nu > 0,$$

and consequently $\tau(\mathbf{S}_\varphi, \bar{x}) \leq \alpha$ by letting $\nu \downarrow 0$. Hence (44) holds.

We next consider the case $\tau(\mathbf{S}_\varphi, \bar{x}) = +\infty$. We claim that $\alpha = +\infty$ (otherwise, $\alpha < +\infty$ and by using the proof of (ii) in Theorem 4.1 again, one can obtain that $\tau(\mathbf{S}_\varphi, \bar{x}) < +\infty$, a contradiction). The proof is complete. $\square$

Remark 4.3. For the case that φ is convex, the Shapiro first order contact property holds automatically and thus Theorem 4.2 reduces to [56, Theorem 5.3], which means that Theorem 4.2 is an extension of [56, Theorem 5.3] from the convex case to the non-convex one. $\square$

It is noted that (35) is a key inequality to characterize the local error bound of inequality (32). We are now in a position to give a characterization of (35) via the following proposition.

Proposition 4.4. *Let $\tau > 0$ and $x \in \mathbf{S}_\varphi$. Then (35) holds if and only if*

$$\mathbf{d}(h, \mathbf{T}^{\mathbf{B}}(\mathbf{S}_\varphi, x)) \leq \tau \max\{\varphi'_{\mathbf{H}}(x, h), 0\} \quad \forall h \in \mathbb{X}. \quad (45)$$

Proof. The necessity part. Suppose that (35) holds. Let $h \in \mathbb{X}$ be such that $\varphi'_{\mathbf{H}}(x, h) \leq 0$. Then for any $\lambda > 0$, one has $\varphi'_{\mathbf{H}}(x, \lambda h) \leq 0$ and it follows from (35) that

$$\mathbf{d}(\lambda h, \mathbf{T}^{\mathbf{B}}(\mathbf{S}_\varphi, x)) \leq \tau$$

This implies that $\mathbf{d}(h, \mathbf{T}^{\mathbf{B}}(\mathbf{S}_\varphi, x)) \leq \frac{\tau}{\lambda}$ for any $\lambda > 0$.

By letting $\lambda \rightarrow +\infty$, one has $h \in \mathbf{T}^{\mathbf{B}}(\mathbf{S}_\varphi, x)$ as $\mathbf{T}^{\mathbf{B}}(\mathbf{S}_\varphi, x)$ is closed, which implies that (45) holds.

Let $h \in \mathbb{X}$ be such that $\varphi'_{\mathbf{H}}(x, h) > 0$. Note that $\varphi'_{\mathbf{H}}(a, \cdot)$ is positively homogeneous and then

$$\varphi'_{\mathbf{H}}\left(x, \frac{h}{\varphi'_{\mathbf{H}}(x, h)}\right) = 1.$$

By virtue of (35), one has

$$\mathbf{d}\left(\frac{h}{\varphi'_{\mathbf{H}}(x, h)}, \mathbf{T}^{\mathbf{B}}(\mathbf{S}_\varphi, x)\right) \leq \tau$$

and consequently $\mathbf{d}(h, \mathbf{T}^{\mathbf{B}}(\mathbf{S}_\varphi, x)) \leq \tau \varphi'_{\mathbf{H}}(x, h)$, which implies that (45) holds.

The sufficiency part. Suppose that (45) holds. Let $h \in \mathbb{X}$ be such that $\varphi'_{\mathbf{H}}(x, h) \leq 1$. Then (45) implies that

$$\mathbf{d}(h, \mathbf{T}^{\mathbf{B}}(\mathbf{S}_\varphi, x)) \leq \tau \max\{\varphi'_{\mathbf{H}}(x, h), 0\} \leq \tau.$$

Hence (35) holds. The proof is complete. $\square$

The following theorem follows immediately from Theorem 4.1 and Proposition 4.4.

Theorem 4.5. *Let $\bar{x} \in \mathbf{S}_\varphi$ and $\tau > 0$. Suppose that $\text{bd}(\mathbf{S}_\varphi) \subseteq \varphi^{-1}(0)$, φ has the epigraphical Shapiro first order contact property at $\bar{x}$ and that $\mathbf{S}_\varphi$ has the Shapiro first order contact property around $\bar{x}$. Then the following statements are equivalent:*

- (i) *Inequality (32) has the local error bound at $\bar{x}$ with constant $\tau > 0$;*
- (ii) *There exists $\delta > 0$ such that (35) holds for all $x \in \text{bd}(\mathbf{S}_\varphi) \cap \mathbf{B}(\bar{x}, \delta)$;*
- (ii) *There exists $\delta > 0$ such that (45) holds for all $x \in \text{bd}(\mathbf{S}_\varphi) \cap \mathbf{B}(\bar{x}, \delta)$.*

Given $x \in X$, we can consider the inequality $\varphi'_{\mathbf{H}}(x, \cdot) \leq 0$ defined by the lower Hadamard directional derivative $\varphi'_{\mathbf{H}}(x, \cdot)$ and study global error bounds of the inequality $\varphi'_{\mathbf{H}}(x, \cdot) \leq 0$; that is, there exists $\tau > 0$ such that

$$\mathbf{d}(h, S_{\varphi'_{\mathbf{H}}(x, \cdot)}) \leq \tau \max\{\varphi'_{\mathbf{H}}(x, h), 0\} \quad \text{for all } h \in \mathbb{X}, \quad (46)$$

where $S_{\varphi'_{\mathbf{H}}(x, \cdot)} := \{u \in \mathbb{X} : \varphi'_{\mathbf{H}}(x, u) \leq 0\}$.

The following corollary, immediate from Theorem 4.1, shows that the local error bound of $\mathbf{S}_\varphi$ at $\bar{x}$ is, to some degree, equivalent to the global error bound of the inequality $\varphi'_{\mathbf{H}}(x, \cdot) \leq 0$ for all x close to $\bar{x}$ with the same constant. This corollary

is inspired from [57, Theorem 4.4] which studies the convex inequality defined by the Clarke directional derivative of a locally Lipschitz function and its global error bound.

Corollary 4.6. *Let $\bar{x} \in \mathbf{S}_\varphi$ and $\tau > 0$. Assume that $\text{bd}(\mathbf{S}_\varphi) \subseteq \varphi^{-1}(0)$, φ has the epigraphical Shapiro first order contact property at $\bar{x}$ and that $\mathbf{S}_\varphi$ has the Shapiro first order contact property around $\bar{x}$.*

- (i) *Suppose that inequality (32) has the local error bound at $\bar{x}$. Then there exists $\delta > 0$ such that for any $x \in \text{bd}(\mathbf{S}_\varphi) \cap \mathbf{B}(\bar{x}, \delta)$, the inequality $\varphi'_\mathbf{H}(x, \cdot) \leq 0$ has the global error bound with the same constant.*
- (ii) *Suppose there exists a neighborhood U of $\bar{x}$ such that $\ker \varphi'_\mathbf{H}(x, \cdot) \subseteq \mathbf{T}^\mathbf{B}(\mathbf{S}_\varphi, x)$ for all $x \in U \cap \text{bd}(\mathbf{S}_\varphi)$. Then inequality (32) has the local error bound at $\bar{x}$ if and only if there exists $\delta > 0$ such that for any $x \in \text{bd}(\mathbf{S}_\varphi) \cap \mathbf{B}(\bar{x}, \delta)$, the inequality $\varphi'_\mathbf{H}(x, \cdot) \leq 0$ has the global error bound with the same constant.*

Proof. By the definition of the lower Hadamard directional derivative, for any $x \in \text{bd}(\mathbf{S}_\varphi)$, one can verify that

$$\{h \in \mathbb{X} : \varphi'_\mathbf{H}(x, h) < 0\} \subseteq \mathbf{T}^\mathbf{B}(\mathbf{S}_\varphi, x) \subseteq \{h \in \mathbb{X} : \varphi'_\mathbf{H}(x, h) \leq 0\}$$

and thus the conclusions follow from Theorem 4.5. The proof is complete. $\square$

Now, we are in a position to study error bounds of a composite-convex function. We apply main results obtained above to establish primal characterizations of error bounds and give an accurate estimate of the error bound modulus in terms of Bouligand tangent cones, directional derivatives of convex functions and the Hausdorff-Pompeiu excess.

Throughout the rest of this section, we always assume that $\mathbb{Y}$ is a Banach space, $g : \mathbb{X} \rightarrow \mathbb{Y}$ is a continuously differentiable mapping and that $f : \mathbb{Y} \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous and convex function.

We consider the following composite-convex inequality:

$$(f \circ g)(x) \leq 0. \tag{47}$$

We denote by $\mathcal{S} := \{x \in \mathbb{X} : (f \circ g)(x) \leq 0\}$ the solution set of (47). For any given $\bar{x} \in \mathcal{S}$, we denote by $\tau(\mathcal{S}, \bar{x})$, defined as said in (34), the local error bound modulus of $\mathcal{S}$ at $\bar{x}$.

Theorem 4.7. *Denote $S_f := \{y \in \mathbb{Y} : f(y) \leq 0\}$ and assume that $\text{bd}(S_f) \subseteq f^{-1}(0)$. Let $\bar{x} \in \mathcal{S}$ be such that $g(\bar{x}) \in \text{int}(\text{dom} f)$ and g is metrically regular around $\bar{x}$. Then composite-convex inequality (47) has a local error bound at $\bar{x}$ if and only if*

$$\limsup_{x \xrightarrow{\text{bd}(\mathcal{S})} \bar{x}} \mathbf{e}(\{h \in \mathbb{X} : \mathbf{d}^+ f(g(x), \nabla g(x)(h)) \leq 1\}, \nabla g(x)^{-1}(\mathbf{T}^\mathbf{B}(S_f, g(x)))) < +\infty.$$

Further, one has the following accurate estimate for the local error bound modulus:

$$\tau(\mathcal{S}, \bar{x}) = \limsup_{x \xrightarrow{\text{bd}(\mathcal{S})} \bar{x}} \mathbf{e}(\{h \in \mathbb{X} : \mathbf{d}^+ f(g(x), \nabla g(x)(h)) \leq 1\}, \nabla g(x)^{-1}(\mathbf{T}^\mathbf{B}(S_f, g(x)))).$$

To prove Theorem 4.7, we need the following lemma which is of some independent interest.

Lemma 4.8. *Let $\varphi := f \circ g$ and $\bar{x} \in \mathbb{X}$ be such that $g(\bar{x}) \in \text{int}(\text{dom} f)$. Then there exists $\delta > 0$ such that*

$$\varphi'_{\mathbf{H}}(x, h) = \mathbf{d}^+ f(g(x), \nabla g(x)(h)) \quad \forall h \in \mathbb{X} \quad (48)$$

holds for all $x \in \mathbf{B}(\bar{x}, \delta)$.

Proof. Note that $g(\bar{x}) \in \text{int}(\text{dom} f)$ and then [47, Proposition 1.6] implies that f is locally Lipschitzian around $g(\bar{x})$; that is, there exist $L, r > 0$ such that

$$|f(y_1) - f(y_2)| \leq L \|y_1 - y_2\| \quad \forall y_1, y_2 \in \mathbf{B}(g(\bar{x}), r). \quad (49)$$

Using the continuity of g , there is $\delta > 0$ such that

$$g(\mathbf{B}(\bar{x}, \delta)) \subseteq \mathbf{B}(g(\bar{x}), r). \quad (50)$$

Let $x \in \mathbf{B}(\bar{x}, \delta)$ and take any $h \in \mathbb{X}$. Then for any $t > 0$ sufficiently small and h' close to h , one has

$$\begin{aligned} \frac{1}{t}[\varphi(x, th') - \varphi(x)] &= \frac{1}{t}[f(g(x, th')) - f(g(x))] \\ &= \frac{1}{t}[f(g(x) + \nabla g(x)(th') + o(t)) - f(g(x))] \\ &= \frac{1}{t}[f(g(x) + \nabla g(x)(th') + o(t)) - f(g(x) + \nabla g(x)(th))] \\ &\quad + \frac{1}{t}[f(g(x) + \nabla g(x)(th)) - f(g(x))]. \end{aligned}$$

Using (49) and (50), for any $t > 0$ sufficiently small and h' close to h , one has

$$\begin{aligned} &\left\| \frac{1}{t}[f(g(x) + \nabla g(x)(th') + o(t)) - f(g(x) + \nabla g(x)(th))] \right\| \\ &\leq \frac{L}{t} \|t \nabla g(x)(h' - h) + o(t)\| \rightarrow 0 \quad (t \rightarrow 0^+, h' \rightarrow h). \end{aligned}$$

Then

$$\begin{aligned} \liminf_{t \rightarrow 0^+, h' \rightarrow h} \frac{1}{t}[\varphi(x, th') - \varphi(x)] &= \liminf_{t \rightarrow 0^+} \frac{1}{t}[f(g(x) + t \nabla g(x)(h)) - f(g(x))] \\ &= \mathbf{d}^+ f(g(x), \nabla g(x)(h)). \end{aligned}$$

This means that (48) holds. The proof is complete. $\square$

Proof of Theorem 4.7. Let $\varphi := f \circ g$. By the metric regularity of g around $\bar{x}$, there exist $\kappa, \delta_0 > 0$ such that

$$\mathbf{d}(x, g^{-1}(y)) \leq \kappa \|g(x) - y\| \quad \forall (x, y) \in \mathbf{B}(\bar{x}, \delta_0) \times \mathbf{B}(g(\bar{x}), \delta_0). \quad (51)$$

Note that $\text{bd}(S_f) \subseteq f^{-1}(0)$ and then

$$\text{bd}(\mathcal{S}) \subseteq g^{-1}(\text{bd}(S_f)) \subseteq g^{-1}(f^{-1}(0)) = \varphi^{-1}(0).$$

By virtue of Lemma 3.10 and Proposition 3.8, there is $\delta \in (0, \delta_0)$ sufficiently small such that $g^{-1}(S_f)$ has the Shapiro first order contact property on $\mathbf{B}(\bar{x}, \delta)$ and

$$\mathbf{T}^{\mathbf{B}}(g^{-1}(S_f), x) = \nabla g(x)^{-1}(\mathbf{T}^{\mathbf{B}}(S_f, g(x))) \quad \forall x \in \mathbf{B}(\bar{x}, \delta) \cap g^{-1}(S_f). \quad (52)$$

This means that $\mathcal{S}$ has the Shapiro first order contact property around the point $\bar{x}$ as $\mathcal{S} = g^{-1}(S_f)$.

Define $\Psi : \mathbb{X} \times \mathbb{R} \rightarrow \mathbb{Y} \times \mathbb{R}$ as

$$\Psi(x, r) := (g(x), r) \quad \forall (x, r) \in \mathbb{X} \times \mathbb{R}.$$

Then one can verify that

$$\text{epi}(\varphi) = \Psi^{-1}(\text{epi}(f)).$$

For any $(x, y) \in \mathbf{B}(\bar{x}, \delta) \times \mathbf{B}(g(\bar{x}), \delta)$ and any $r, s \in (\varphi(\bar{x}) - \delta, \varphi(\bar{x}) + \delta)$, one has

$$\Psi^{-1}(y, s) = g^{-1}(y) \times \{s\}$$

and it follows from (51) that

$$\begin{aligned} \mathbf{d}((x, r), \Psi^{-1}(y, s)) &= \mathbf{d}((x, r), g^{-1}(y) \times \{s\}) = \mathbf{d}(x, g^{-1}(y)) + |r - s| \\ &\leq \kappa \|g(x) - y\| + |r - s| \leq (\kappa + 1) \|\Psi(x, r) - (y, s)\|, \end{aligned}$$

which implies that Ψ is metrically regular around $(\bar{x}, \varphi(\bar{x}))$. Thus, Proposition 3.8 gives that $\Psi^{-1}(\text{epi}(f))$ has the Shapiro first order contact property at $(\bar{x}, \varphi(\bar{x}))$ and so φ has the epigraphical Shapiro first order contact property at $\bar{x}$ (thanks to $\text{epi}(\varphi) = \Psi^{-1}(\text{epi}(f))$).

Finally, by applying Theorem 4.2 and Lemma 4.8, and (52), one can obtain the proof of Theorem 4.7. The proof is complete. $\square$

The following theorem follows immediately from Theorem 4.7.

Theorem 4.9. *Suppose that f is continuous and $\bar{x} \in \mathcal{S}$ is such that g is metrically regular around $\bar{x}$. Then composite-convex inequality (47) has a local error bound at $\bar{x}$ if and only if*

$$\limsup_{x \xrightarrow{\text{bd}(\mathcal{S})} \bar{x}} \mathbf{e}(\{h \in \mathbb{X} : \mathbf{d}^+ f(g(x), \nabla g(x)(h)) \leq 1\}, \nabla g(x)^{-1}(\mathbf{T}^{\mathbf{B}}(S_f, g(x)))) < +\infty.$$

Further, one has the following accurate estimate for the local error bound modulus:

$$\tau(\mathcal{S}, \bar{x}) = \limsup_{x \xrightarrow{\text{bd}(\mathcal{S})} \bar{x}} \mathbf{e}(\{h \in \mathbb{X} : \mathbf{d}^+ f(g(x), \nabla g(x)(h)) \leq 1\}, \nabla g(x)^{-1}(\mathbf{T}^{\mathbf{B}}(S_f, g(x)))).$$

As an application of Theorem 4.9 let us give an example where we can calculate the error bound modulus of the composite-convex inequality.

Example 4.10. Let $f(y) := y - 1$ for all $y \in \mathbb{R}$, $g(x) := x^3$ for all $x \in \mathbb{R}$ and $\bar{x} := 1$. We consider the composite-convex inequality:

$$(f \circ g)(x) \leq 0.$$

Then one can verify that $\mathcal{S} = S_f = (-\infty, 1]$, $\mathbf{T}^{\mathbf{B}}(S_f, g(\bar{x})) = (-\infty, 0]$ and

$$\{h : \mathbf{d}^+ f(g(\bar{x}), \nabla g(\bar{x})(h)) \leq 1\} = (-\infty, \frac{1}{3}].$$

This implies that

$$\limsup_{x \xrightarrow{\text{bd}(\mathcal{S})} \bar{x}} \mathbf{e}(\{h \in X : \mathbf{d}^+ f(g(x), \nabla g(x)(h)) \leq 1\}, \nabla g(x)^{-1}(\mathbf{T}^{\mathbf{B}}(S_f, g(x)))) = \frac{1}{3}.$$

Hence Theorem 4.9 implies that $\mathcal{S}$ has a local error bound at $\bar{x}$ and moreover $\tau(\mathcal{S}, \bar{x}) = \frac{1}{3}$. $\square$

Finally, let us show that the local error bound may not be satisfied if the metric regularity assumption is dropped in Theorem 4.9.

Example 4.11. Let $f(y) := y$ for all $y \in \mathbb{R}$, $g(x) := x^3$ for all $x \in \mathbb{R}$ and $\bar{x} := 0$. We consider the composite-convex inequality:

$$(f \circ g)(x) \leq 0.$$

Then $\mathcal{S} = S_f = (-\infty, 0]$, and one can verify that g is not metrically regular at $\bar{x}$ since $\nabla g(\bar{x}) = 0$ is not surjective. However, for any $x > 0$ sufficiently small, one has

$$\frac{d(x, \mathcal{S})}{f(g(x))} = \frac{x}{x^3} \rightarrow +\infty, \text{ as } x \rightarrow 0^+.$$

This means that $\mathcal{S}$ has no local error bound at $\bar{x}$, and thus the conclusions in Theorem 4.9 do not hold. $\square$

Acknowledgements. The authors would like to thank two anonymous reviewers for their suggestions and comments that helped to improve the presentation of the paper.

5. Conclusions

We have presented primal characterizations of error bounds for a composite-convex inequality. To this aim, several primal necessary and/or sufficient conditions for a general inequality, under the assumption of Shapiro contact property, are given in terms of Bouligand tangent cones, lower Hadamard directional derivatives and the Hausdorff-Pompeiu excess. Then it is proved that the composite-convex inequality satisfies the Shapiro contact property at the given point where the continuously differentiable mapping is metrically regular and thus primal characterizations of error bounds can be obtained. This work actually extends the existing primal results on error bounds for a convex inequality to the non-convex case. The future work would be to investigate a broader class of non-convex inequality for which these primal results on error bounds are valid.

References

- [1] M. Abassi, M. Théra: *Strongly regular points of mappings*, Fixed Point Theory Algorithms Sci. Eng. (2021), art. no. 14.
- [2] M. Abassi, M. Théra: *About error bounds in metrizable topological vector spaces*, Set-Valued Var. Analysis 30 (2022) 1291–1311.
- [3] D. Aussel, A. Daniilidis, L. Thibault: *Subsmooth sets: Functional characterizations and related concepts*, Trans. Amer. Math. Soc. 357 (2005) 1275–1301.
- [4] A. Auslender, J. P. Crouzeix: *Global regularity theorems*, Math. Oper. Res. 13/2 (1988) 243–253.
- [5] D. Azé, J.-N. Corvellec: *Characterizations of error bounds for lower semicontinuous functions on metric spaces*, ESAIM Control Optim. Calc. Var. 10 (2004) 409–425.
- [6] H. H. Bauschke, J. M. Borwein: *On projection algorithms for solving convex feasibility problems*, SIAM Rev. 38/3 (1996) 367–426.
- [7] A. Beck, M. Teboulle: *Convergence rate analysis and error bounds for projection algorithms in convex feasibility problems*, Optim. Methods Software 18/4 (2003) 377–394.
- [8] E. M. Bednarczuk, A. Y. Kruger: *Error bounds for vector-valued functions: Necessary and sufficient conditions*, Nonlinear Analysis 75/3 (2012) 1124–1140.
- [9] J. V. Burke, S. Deng: *Weak sharp minima revisited. I: Basic theory*, Control Cyber. 31/3 (2002) 439–469.
- [10] J. V. Burke, S. Deng: *Weak sharp minima revisited. II: Application to linear regularity and error bounds*, Math. Program., 104/2-3 (2005) 235–261.
- [11] P. L. Combettes: *Hilbertian convex feasibility problem: convergence of projection methods*, Appl. Math. Optimization 35/3 (1997) 311–330.
- [12] J.-N. Corvellec, V. V. Motreanu: *Nonlinear error bounds for lower semicontinuous functions on metric spaces*, Math. Program. 114/2 (2008) 291–319.
- [13] N. D. Cuong, A. Y. Kruger: *Error bounds revisited*, Optimization 71/4 (2022) 1021–1053.
- [14] A. L. Dontchev, R. T. Rockafellar: *Implicit Functions and Solution Mappings*, Springer, Berlin (2009).
- [15] A. L. Dontchev: *Lectures on Variational Analysis*, Applied Mathematical Sciences 205, Springer, Cham (2021).
- [16] M. J. Fabian, R. Henrion, A. Y. Kruger, J. V. Outrata: *Error bounds: Necessary and sufficient conditions*, Set-Valued Var. Analysis 18/2 (2010) 121–149.
- [17] H. Gfrerer: *First order and second order characterizations of metric subregularity and calmness of constraint set mapping*, SIAM J. Optim. 21/4 (2011) 1439–1474.
- [18] O. Güler: *Augmented Lagrangian algorithms for linear programming*, J. Optimization Theory Appl. 75/3 (1992) 445–478.
- [19] R. Hesse, D. R. Luke: *Nonconvex notions of regularity and convergence of fundamental algorithms for feasibility problems*, SIAM J. Optimization 23/4 (2013) 2397–2419.
- [20] A. J. Hoffman: *On approximate solutions of systems of linear inequalities*, J. Res. Nat. Bur. Standards 49 (1952) 263–265.
- [21] A. D. Ioffe: *Regular points of Lipschitz functions*, Trans. Amer. Math. Soc. 251 (1979) 61–69.

- [22] A. D. Ioffe: *Metric regularity-a survey. I: Theory*, J. Aust. Math. Soc. 101/2 (2016) 188–243.
- [23] A. D. Ioffe: *Metric regularity-a survey. II: Applications*, J. Aust. Math. Soc. 101/3 (2016) 376–417.
- [24] A. D. Ioffe: *Variational Analysis of Regular Mappings. Theory and Applications*, Springer Monographs in Mathematics, Springer, Cham (2017).
- [25] A. Jourani: *Hoffman's error bound, local controllability, and sensitivity analysis*, SIAM J. Control Optimization 38/3 (2000) 947–970.
- [26] A. N. Iusem, A. R. De Pierro: *On the convergence properties of Hildreth's quadratic programming algorithm*, Math. Program. 47 (1990) 37–51.
- [27] G. Jameson: *Ordered Linear Spaces*, Springer, Berlin (1970).
- [28] D. Klatte, W. Li: *Asymptotic constraint qualifications and global error bounds for convex inequalities*, Math. Program. 84/1 (1999) 137–160.
- [29] A. Y. Kruger, M. A. López, M. A. Théra: *Perturbation of error bounds*, Math. Program. 168 (2018) 533–554.
- [30] A. Y. Kruger, H. V. Ngai, M. Théra: *Stability of error bounds for convex constraint systems in Banach spaces*, SIAM J. Optimization 20/6 (2010) 3280–3296.
- [31] S. Łojasiewicz: *Sur le problème de la division*, Studia Math. 18 (1959) 87–136.
- [32] A. S. Lewis, J. S. Pang: *Error bounds for convex inequality systems*, in: *Generalized Convexity, Generalized Monotonicity: Recent Results*, Proc. Fifth Symp. on Generalized Convexity, Luminy 1996, J.-P. Crouzeix et al. (eds.), Kluwer Academic Publishers, Dordrecht (1997) 75–110.
- [33] Z.-Q. Luo, P. Tseng: *On a global error bound for a class of monotone affine variational inequality problems*, Oper. Res. Letters 11/3 (1992) 159–165.
- [34] Z.-Q. Luo, P. Tseng: *Error bounds and convergence analysis of feasible descent methods: a general approach*, Ann. Oper. Res. 46 (1993) 157–178.
- [35] W. Li: *Abadie's constraint qualification, metric regularity, and error bounds for differentiable convex inequalities*, SIAM J. Optimization 7 (1997) 966–978.
- [36] B. S. Mordukhovich: *Complete characterization of openness, metric regularity, and Lipschitzian properties of set-valued mappings*, Trans. Amer. Math. Soc. 340 (1993) 1–35.
- [37] B. S. Mordukhovich: *Variational Analysis and Generalized Differentiation I*, Springer, Berlin (2006).
- [38] K. F. Ng, X. Y. Zheng: *Error bounds for lower semicontinuous functions in normed spaces*, SIAM J. Optimization 12/1 (2001) 1–17.
- [39] H. V. Ngai, A. Y. Kruger, M. Théra: *Stability of error bounds for semi-infinite convex constraint systems*, SIAM J. Optimization 20 (2010) 2080–2096.
- [40] H. V. Ngai, M. Théra: *Error bounds and implicit multifunction theorem in smooth Banach spaces and applications to optimization*, Set-Valued Analysis 12/1-2 (2004) 195–223.
- [41] H. V. Ngai, M. Théra: *Error bounds in metric spaces and applications to the perturbation stability of metric regularity*, SIAM J. Optimization 19/1 (2008) 1–20.
- [42] H. V. Ngai, M. Théra: *Error bounds for systems of lower semicontinuous functions in Asplund spaces*, Math. Program. 116/1-2 (2009) 397–427.

- [43] H. V. Ngai, N. H. Tron, N. V. Vu, M. Théra: *Variational analysis of paraconvex multifunctions*, J. Optim. Theory Appl. 193 (2022) 180–218.
- [44] J. S. Pang: *Error bounds in mathematical programming*, Math. Program. 79/1-3 (1997) 299–332.
- [45] J.-P. Penot: *Error bounds, calmness and their applications in nonsmooth analysis*, in: *Nonlinear Analysis and Optimization. II: Optimization*, Contemporary Mathematics 514, American Mathematical Society, Providence (2010) 225–247.
- [46] J.-P. Penot: *Calculus Without Derivatives*, Graduate Texts in Mathematics 266, Springer, New York (2013).
- [47] R. R. Phelps: *Convex Functions, Monotone Operators and Differentiability*, Lecture Notes in Mathematics 1364, Springer, New York (1989).
- [48] H. Rådström: *An embedding theorem for spaces of convex sets*, Proc. Amer. Math. Soc. 3 (1952) 165–169.
- [49] S. M. Robinson: *Bounds for error in the solution set of a perturbed linear program*, Linear Algebra Appl. 6 (1973) 69–81.
- [50] S. M. Robinson: *A characterization of stability in linear programming*, Oper. Res. 25 (1977) 435–447.
- [51] A. Shapiro: *Existence and differentiability of metric projections in Hilbert spaces*, SIAM J. Optimization 4 (1994) 130–141.
- [52] A. Shapiro, F. Al-Khayyal: *First order conditions for isolated locally optimal solutions*, J. Optim. Theory Appl. 77 (1993) 189–196.
- [53] Z. Shen, J.-C. Yao, X. Y. Zheng: *Calmness and the Abadie CQ for multifunctions and linear regularity for a collection of closed sets*, SIAM J. Optim. 29/3 (2019) 2291–2319.
- [54] L. Thibault: *Unilateral Variational Analysis in Banach Spaces*, World Scientific, Singapore (2023).
- [55] P. Tseng, D. P. Bertsekas: *On the convergence of the exponential multiplier method for convex programming*, Math. Program. 60/1 (1993) 1–19.
- [56] Z. Wei, C. Tammer, J.-C. Yao: *Characterizations for strong Abadie constraint qualification and applications to calmness*, J. Optimization Theory Appl. 189 (2021) 1–18.
- [57] Z. Wei, J.-C. Yao: *On constraint qualifications of a nonconvex inequality*, Optimization Letters 12 (2018) 1117–1139.
- [58] Z. Wei, J.-C. Yao: *On applications of the calmness moduli for multifunctions to error bounds*, Optimization 71/12 (2022) 3647–3668.
- [59] Z. Wei, J.-C. Yao, X. Y. Zheng: *Strong Abadie CQ, ACQ, calmness and linear regularity*, Math. Program. 145 (2014) 97–131.
- [60] C. Zălinescu: *Weak sharp minima, well-behaving functions and global error bounds for convex inequalities in Banach spaces*, in: Proc. 12th Baikal Int. Conf. on Optimization Methods and Their Applications, Irkutsk (2001) 272–284.
- [61] X. Y. Zheng, K. F. Ng: *Metric regularity and constraint qualifications for convex inequalities on Banach spaces*, SIAM J. Optimization 14 (2004) 757–772.